NOTES ON PIECEWISE-LINEAR TOPOLOGY

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Contents

1. Simplicial complexes

Throughout these notes, we will restrict our attention to topological spaces that can be encoded by simplicial complexes. A *simplicial complex* is a pair $T = \{V, S\}$ where V is a set, whose elements are called the *vertices* of T and S is a set of finite subsets of V, whose elements are called the *simplices* of T, with the following properties:

(1) For every $v \in V$, we have $\{v\} \in S$.
(2) For every simplex $\sigma \in S$, every subset $\tau \subset \sigma$ is also an element of S.

The *dimension* of a simplex $\sigma \in S$ is $dim(\sigma) = |\sigma| - 1$, where $|\sigma|$ is the number of vertices in σ, and the dimension of T is the supremum $dim(T) = \sup_{\sigma \in S} dim(\sigma)$. We say that σ is *spanned* by the vertices in σ.

We will generally assume T is a finite simiplicial complex, i.e. that V is a finite set. Let $N = |V|$ be the number of vertices, and assume the vertices are labeled $V = \{1, \ldots, N\}$. Then T encodes a topological space, which we construct as follows:

Let $\mathbf{R}^N_+ \subset \mathbf{R}^N$ be the subset of N-dimensional euclidean space such that all coordinates are non-negative. Define the *support* of a vector $x = (x_1, \ldots, x_N) \in \mathbf{R}^n$ as $supp(x) = \{i \leq N \mid x_i \neq 0\}$. Then the *realization* of a simplicial complex T is the set

$$\bar{T} = \left\{ x \in \mathbf{R}^N_+ \mid \sum_{i=1}^{N} x_i = 1, supp(x) \in S \right\}$$

In other words, $\bar{T}$ consists of all points whose coordinates sum to 1 and the set of non-zero coordinates corresponds to a simplex in T. Because $\bar{T}$ is a subset of $\mathbf{R}^N$, it inherits a subset topology from $\mathbf{R}^N$. The topological space $\bar{T}$ is called the *realization* of T.

This construction may appear rather obtuse at first, but with a little unpacking, it should seem natural. Each simplex $\sigma \in S$ defines a set of points $\bar{\sigma}$ that are non-zero on the coordinates corresponding to the vertices of σ. Therefore, we can restrict our attention to $\mathbf{R}^n \subset \mathbf{R}^N$ where $n = |v|$ is the number of vertices in σ. The points in $\mathbf{R}^n$ have non-negative entries that sum to 1, which forms the convex hull of the unit coordinate vectors $\{(0, \ldots, 1, \ldots, 0)\}$.

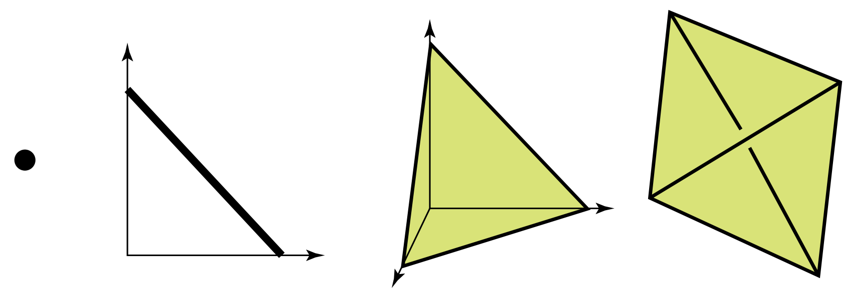

FIGURE 1. Simplices.

For the cases $n = 1, 2, 3$, these can be drawn by hand, and the reader can check that these sets form, respectively, a single point, a straight arc and a triangle. For $n = 4$, we can project the set of points $\{x \in \mathbf{R}^4 \mid \sum x_i = 1\}$ linearly onto $\mathbf{R}^3$, and check that the set of points with all positive components forms a tetrahedron. These four cases are shown in Figure 1. Notice that in each case, the set is topologically a ball of dimension $n - 1$, which is why we define the dimension of σ to be $|\sigma| - 1 = n - 1$.

Each subset $\tau \subset \sigma$ defines a convex hull in $\mathbf{R}^N_+$ which is a subset $\bar{\tau} \subset \bar{\sigma}$. We will call τ a *face* of σ and $\bar{\tau}$ a *face* of $\bar{\sigma}$. If τ is also a subset of a second simplex σ' then the intersection $\bar{\sigma} \cap \bar{\sigma}'$ will contain $\bar{\tau}$. (This will be left as an exercise.) We can therefore think of the realization $\bar{T}$ as the result of taking a collection of convex hulls and identifying them along collections of faces. The inclusion structure of S completely encodes these identifications.

A simplicial complex thus defines a topological space, but we will often want to use simplicial complexes to study pre-existing spaces.

1. **Definition.** A *triangulation* of a topological space X is a pair (T, ϕ) where T is a simplicial complex and $\phi : X \to \bar{T}$ is a homeomorphism from X to the realization of T. We will be interested in triangulations of a very specific type of topological space:

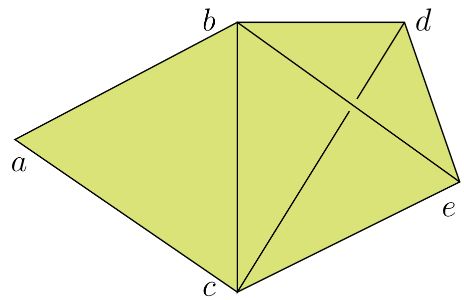

FIGURE 2. The complex in Example 2.

We will say that a simplex σ is *maximal* if it is not a proper face of another simplex. For shorthand, we will write $P(\sigma)$ for the power set of σ, i.e. the set of all subsets of σ. By the second condition on a simplicial complex, if $\sigma \in S$ then $P(\sigma) \subset S$. Therefore, we can write S as the union of the power sets of its maximal simplices.

2. **Example.** Figure 2 shows the realization $\bar{T}$ of the simplicial complex $T = (\{a,b,c,d,e\}, P(\{b,c,d,e\}) \cup P(\{a,b,c\})$.

The *interior* of a simplex $\bar{\sigma}$ is the set of points whose coordinates in $\bar{\sigma} \subset \mathbf{R}^n$ are all non-zero. The faces of $\bar{\sigma}$ are defined by setting one or more coordinates to zero, so the interior consists of precisely the points that are not in any proper faces of σ. In particular, we have:

3. **Lemma.** *The interiors of distinct simplices of T are disjoint in $\bar{T}$.*

4. **Definition.** A *subcomplex* $R \subset T$ of a simplicial complex $T = (V, S)$ is a simplicial complex $R = (V_R, S_R)$ such that $V_R \subset V$ and $S_R \subset S$. Note that the realization $\bar{R}$ can be naturally identified with a subset of $\bar{T}$, and we will abuse notation to write $\bar{R} \subset \bar{T}$. For each $k < n$, the *k-skeleton* T^k of T is the subcomplex consisting of all simplices of dimension k or less.

5. **Example.** For the simplicial complex $T = (\{a,b,c,d,e\}, P(\{b,c,d,e\}) \cup P(\{a,b,c\})$ considered above, $n = 3$. The 2-skeleton is
$T^2 = (\{a,b,c,d,e\}, P(\{b,c,d\}) \cup P(\{b,c,e\}) \cup P(\{b,d,e\}) \cup P(\{c,d,e\}) \cup P(\{a,b,c\})$. The 1-skeleton consists of all the edges that appear in these triangles and the 0-skeleton consists of all the vertices of T.

2. Manifolds

A second way to restrict our attention to a narrow class of topological spaces is as follows:

6. **Definition.** An *n-dimensional manifold* (or *n-manifold*) is a Hausdorff topological space $X = X^n$ such that every point $p \in X$ has a neighborhood $N \subset X$ with N homeomorphic to the open unit ball $\{x \in \mathbf{R}^n \mid \sum x_i^2 < 1\}$.

For example, a 2-dimensional manifold is shown in Figure 3. The arrow shows that a generic point has a neighborhood homeomorphic to the unit disk in the plane.

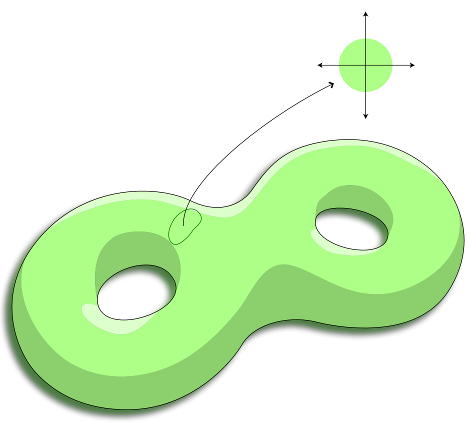

Figure 3. A 2-dimensional manifold.

7. **Definition.** The *n-sphere* is the set $S^n = \{x \in \mathbf{R}^{n+1} \mid \sum x_i^2 = 1\}$ (with the subset topology).

The 2-sphere S^2, shown on the top in Figure 4, is no doubt well understood by the reader from athletic endeavors. The 3-sphere is a little harder to visualize, but it can almost be projected into $\mathbf{R}^3$: For each point $p \neq (0,0,0,1) \in S^3$, consider the line from p to $(0,0,0,1)$ and project p to the point where this line intersects the 3-dimensional subspace $\{(x,y,z,0) \in \mathbf{R}^4\}$. This map is defined, continuous and one-to-one for all points other than $(0,0,0,1)$. We can therefore think of S^3 as $\mathbf{R}^3$ with a point added "at infinity".

In fact, this construction can be generalized to produce any S^n from $\mathbf{R}^n$, though $\mathbf{R}^n$ is harder to visualize for $n > 3$. This map sends $(0,0,0,-1)$ to the origin in $\mathbf{R}^n$ and sends a neighborhood of this point to the unit ball in $\mathbf{R}^n$. Thus a neighborhood of $(0,0,0,-1)$ is homeomorphic to the unit ball. The orthogonal group acts transitively on S^n by homeomorphisms, so every point in S^n has such a neighborhood and we see that S^n is an n-manifold.

8. **Definition.** The *n-torus* is the space $S^1 \times S^1 \times \cdots \times S^1$ with n copies of S^1.

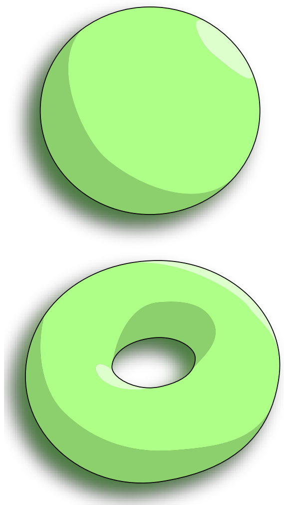

FIGURE 4. A sphere and a torus.

The 2-torus is shown on the bottom in Figure 4. Though less common than the sphere, it should still be well known to the reader from culinary endeavors. We can check that this is a manifold by repeatedly applying the following Lemma:

9. **Lemma.** *The product of an n-manifold and an m-manifold is an $(n+m)$-manifold.*

The proof of this Lemma will be left as an exercise for the reader.

10. **Definition.** An *n-manifold with boundary* is a Hausdorff topological space X in which every point $p \in X$ has a neighborhood N such that either

- N is homeomorphic to the open n-dimensional ball $\{x \in \mathbf{R}^n \mid \sum x_i^2 < 1\}$ or
- N is homeomorphic to the half-open n-dimensional ball $\{x \in \mathbf{R}^n \mid \sum x_i^2 < 1, x_1 \geq 0\}$, such that the homeomorphism sends p onto the origin.

The set of points with neighborhoods of the first type is called the *interior* of M. The set of points with neighborhoods of the second type is the *boundary* of M and is written ∂M.

A 2-manifold with boundary is shown in Figure 5. The loop of points on the left make up the boundary. A manifold without boundary, as in Figure 3, is often called a *closed* manifold to emphasize the distinction.

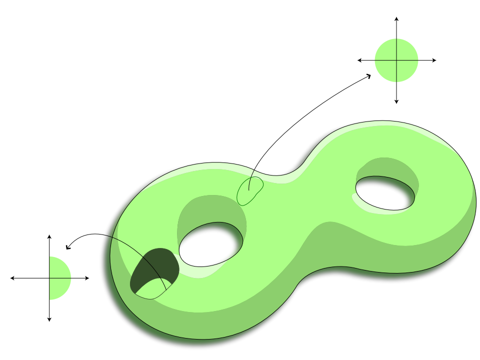

FIGURE 5. A 2-dimensional manifold with boundary.

The simplest example of an n-manifold with boundary is the closed unit ball $B^n = \{x \in \mathbf{R}^n \mid \sum x_i^2 \leq 1\}$. For every point x in the open ball, there is a ball centered at x consisting entirely of points with $\sum x_i^2 < 1$. For each point with $\sum x_i^2 = 1$, there is a neighborhood of the second type. It will be left as an exercise to find an explicit homeomorphism for such a neighborhood. This serves as a good model for the following:

11. **Lemma.** *If M is an n-manifold with boundary then*
   (1) *The interior and the boundary of M are disjoint.*
   (2) *The boundary of M is a compact $(n-1)$-manifold.*
   (3) *The interior of M is a (generally non-compact) n-manifold.*

We often distinguish between manifolds with and without boundary by saying that a manifold M is *closed* if $\partial M = \emptyset$. We can also build manifolds with boundary by direct products.

12. **Lemma.** *If M is an m-manifold with boundary and N is an n-manifold with boundary then $M \times N$ is an $(m+n)$-manifold with boundary and the boundary is $\partial(M \times N) = (\partial M \times N) \cup (\partial N \times M)$.*

The proofs of Lemmas 11 and **??** will also be left as exercises. We can build many different manifolds as products of spheres and balls of different dimensions, but this would still leave out most manifolds. We will expand our options by using simplicial complexes to build manifolds.

3. Triangulations of manifolds

The definition of a manifold is a local condition, in that it depends on checking something for each point in the space. Therefore to understand when the realization of a simplicial complex is a manifold, there should also be a local (combinatorial) condition.

The *link* of a simplex τ in a simplicial complex T is the subcomplex R defined by $S_R = \{\sigma \in S \mid \sigma \cap \tau = \emptyset \textit{ and } \sigma \cup \tau \in S\}$. In other words, the link is the set of simplices of T disjoint from τ such that there is a simplex containing both σ and τ as faces.

Equivalently, if R' is the subcomplex consisting of all simplices in T that intersect τ non-trivially, plus all their faces, then the link R is is the subcomplex consisting of simplices in R' that are disjoint from τ. We will call R' the *link neighborhood* of τ. This second formulation of the definition can be readily generalized to define the link of any subcomplex of T, but we won't consider that here.

Still, this definition may seem a bit obtuse. It should be clarified by an example. For shorthand, we will write $link(\tau) = S_R$, since the set of vertices V_R is simply the union of the sets in S_R.

13. **Example.** We can calculate the links of some of the simplices in Example 2 (Recall $T = (\{a, b, c, d, e\}, P(\{b, c, d, e\}) \cup P(\{a, b, c\}))$:

- $link(\{c\}) = P(\{a, b\}) \cup P(\{b, d, e\})$

- $link(\{b,c\}) = P(\{a\}) \cup P(\{d,e\})$
- $link(\{b,d,e\}) = \{c\}$
- $link(\{a,b,c\}) = \emptyset$

The realizations of these links are shown in Figure 6.

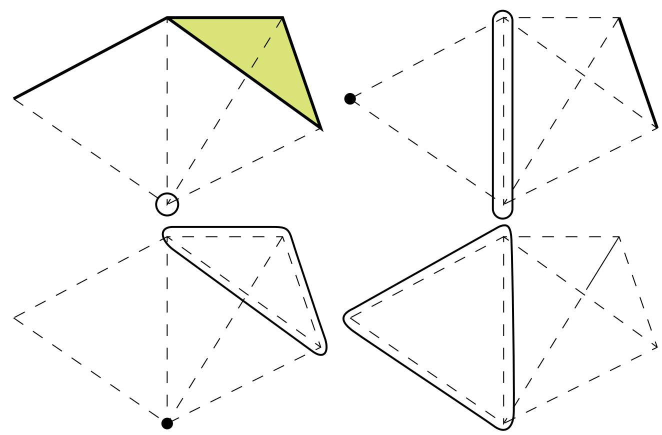

FIGURE 6. The links defined by the circled simplices are shown in bold.

We will say that a simplicial complex T is a *combinatorial n-manifold* if the link of each k-dimensional simplex is a triangulation of an $(n-k-1)$-dimensional sphere.

In order to understand the implications of this definition, let's again consider the link neighborhood R' of τ described above. This set contains both τ and the link of τ. In fact, for each simplex $\upsilon \in link(\tau)$, the union $\tau \cup \upsilon$ is a simplex $\sigma \in R'$. For each pair of points $x \in \bar{\tau}$, $y \in \bar{\upsilon}$, there is a straight arc in $\bar{\sigma}$ with endpoints x, y. Moreover, for a second pair x', y' with $x \neq x'$ or $y \neq y'$, the associated straight arcs will be disjoint. We can thus build a model for R' as follows:

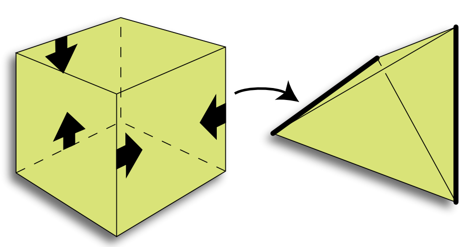

FIGURE 7. A 3-simplex is the join of opposite edges.

14. **Definition.** If X and Y are topological spaces, the *join* of X and Y is the quotient of $X \times [0,1] \times Y$ by the relation $(x,0,y) \sim (x',0,y)$ and $(x,1,y) \sim (x,1,y')$ for every $x, x' \in X$ and $y, y' \in Y$.

Figure 7 shows how to construct the join of two line segments from the cube $X \times [0,1] \times Y = [0,1] \times [0,1] \times [0,1]$. Notice that the result is a 3-dimensional simplex, with the two line segments used to define the join appearing as opposite edges. In general, we will see that the join of two simplices is a higher dimensional simplex and this is the key to Lemma 15.

15. **Lemma.** *The realization of the link neighborhood of τ is homeomorphic to the join of $\bar{\tau}$ and $\overline{link(\tau)}$.*

Proof. Let $\sigma = \{v_1, \dots, v_n\}$ be a simplex and let $\tau = \{v_1, \dots, v_k\}$ and $\rho = \{v_{k+1}, \dots, v_n\}$ be faces. Note that the realization $\bar{\sigma}$ is the convex hull of $\bar{\tau}$ and $\bar{\rho}$. In particular, a point of $\bar{\sigma}$ is in $\bar{\rho}$ if its first k components are zero and is in $\bar{\tau}$ if its last $n-k$ components are zero. Therefore every point in $\bar{\sigma}$ can be written as $tp + (1-t)r$ for $p \in \tau$, $r \in \rho$ and a unique $t \in [0,1]$. For $t \neq 0, 1$, the points p, r are uniquely determined. For $t = 0$, r is determined but p can be any point in $\bar{\tau}$, and vice versa for $t = 1$.

Conversely, we can define a map from $\bar{\tau} \times [0,1] \times \bar{\rho}$ to $\bar{\sigma}$ by sending each point (p, t, r) to $tp + (1-t)r$. By the argument above, this map is onto and is one-to-one except that it sends $\{\bar{\tau}\} \times \{0\} \times \{r\}$ to a single point for each $r \in \rho$ and sends $\{p\} \times \{0\} \times \{\bar{\rho}\}$ to a single point for $p \in \tau$. In other words, this map identifies σ with the join of $\bar{\tau}$ and $\bar{\rho}$. If we apply the same argument to each simplex ρ in the link of τ, we find that the link neighborhood of τ is the join of τ and $\overline{link(\tau)}$. $\square$

We can use Lemma 15 to construct a model of a neighborhood of each point in $\bar{T}$ based on the link of the maximal simplex that contains it. In particular, if T is a combinatorial manifold then each link neighborhood is a join of a k-dimensional ball and an $(n-k-1)$-dimensional sphere.

16. **Lemma.** *The join of a k-dimensional ball and a $(n-k-1)$-dimensional sphere is an n-dimensional ball.*

Proof. We will construct a triangulation T whose realization is a ball such that there is a k-dimensional simplex τ whose link is a sphere and such that the link neighborhood of τ is all of T. Recall that the realization $\bar{\tau}$ is defined as the subset of $\mathbf{R}^{k+1}$ consisting of vectors whose components are non-negative and sum to 1. We can project $\bar{\tau}$ into $\mathbf{R}^n$ by orthogonal projection onto the hyperplane orthogonal to the vector $(1, \dots, 1)$. Under this projection, $\bar{\tau}$ is "centered" at the origin. Let $\bar{\rho}$ be a similar projection of an $(n-k)$-dimensional simplex into $\mathbf{R}^{n-k}$.

We can think of τ and ρ as sitting in orthogonal hyperplanes of $\mathbf{R}^n = \mathbf{R}^k \times \mathbf{R}^{n-k}$. The convex hull B of the vertices in τ and ρ is an

n-dimensional ball. (It can be identified with the unit n-ball by radial scaling.) The boundary $\partial\bar{\rho}$ of $\bar{\rho}$ is an $(n-k-1)$-dimensional sphere. For each ρ' in $\partial\rho$, the convex hull of $\rho' \cup \tau$ is a simplex and the union of the simplices defined this way defines a triangulation of B. Every maximal simplex in B is the union of τ and cell in $\partial\rho$ so B is the link neighborhood of τ in this triangulation and $\partial\rho$ is its link. Thus B is the join of τ and $\partial\rho$.

The homeomorphism type of the join of two sets is completely determined by the homeomorphism type of the two sets, so this implies that the join of any triangulated ball and triangulated sphere is a ball. $\square$

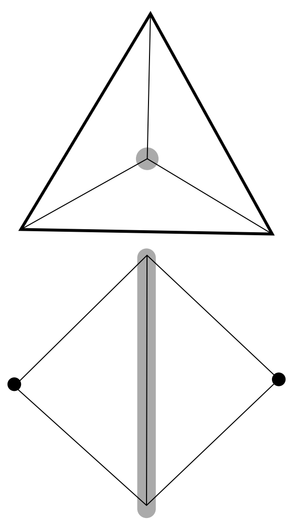

FIGURE 8. Joins that produce 2-dimensional disks.

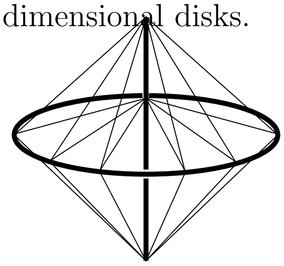

FIGURE 9. The join of a 1-ball and a circle.

This construction can be illustrated relatively easily with $n = 2$ and $n = 3$. For $n = 2$ and $k = 0$, we have $\bar{\tau}$ is a single point v (a 0-simplex) and $\partial\rho$ is a circle consisting of three edges. The join consists of three triangles centered around v as in Figure 8. For $n = 2$ and $k = 1$, we find that $\bar{\tau}$ is an edge (a 1-simplex) and $\partial\rho$ is a 0-sphere, i.e. two points. The join consists of two triangles with a common edge, which is again a disk. The 3-ball that arises as the join of a one-dimensional ball and a circle is shown in Figure 9. The construction of the other triangulations of the 3-ball that arise this way will be left as an exercise for the reader.

17. **Corollary.** *The realization of a combinatorial n-manifold is a topological n-manifold.*

Proof. Let T be a combinatorial n-manifold. Every point p is in the interior of a k-dimensional simplex τ and by the definition of a combinatorial manifold, the realization of the link of τ is an $(n-k-1)$-dimensional sphere. By Lemma 15, this implies that the link neighborhood of τ is the join of a k-dimensional ball $(\bar{\tau})$ and an $(n-k-1)$-dimensional sphere $\overline{link(\tau)}$. The homeomorphism type of the link of two spaces is determined by the homeomorphism types of the two spaces, so Lemma 16 implies that the link neighborhood is homeomorphic to an n-dimensional ball. The interior of the link neighborhood is a neighborhood of p, and p was chosen as an arbitrary point in $\bar{T}$, so $\bar{T}$ is an n-manifold. $\square$

The converse of this Lemma is not, in general, true. There are n-manifolds for $n \geq 4$ with triangulations that are not combinatorial manifolds, as well as n-manifolds that have no triangulations at all. However, in dimensions strictly below 4, things are relatively simple.

18. **Theorem** (Moise [1])**.** *Let M be a 3-dimensional or 2-dimensional manifold. Then M has a triangulation and every triangulation of M is a combinatorial manifold.*

The proof in dimension two is relatively straightforward. In dimension three, this was a long standing open problem in the early days of geometric topology, which was finally proved by Moise. Both proofs can be found in Moise's book [1].

4. Piecewise-linear maps

A *(simplicial) homomorphism* between simplicial complexes $T = (V_T, S_T)$ and $R = (V_R, S_R)$ is a function $f : V_T \to V_R$ such that for each $\sigma \in S_T$, its image $f(\sigma)$ is an element of S_R, i.e. a simplex of R. Recall that the elements of σ correspond to basis vectors for the vector space containing the realization $\bar{\sigma}$, so f determines a map from this basis onto the basis for the vector space containing the realization of $f(\sigma)$. This map extends to a unique linear map that takes $\bar{\sigma}$ onto the realization of $\phi(\sigma)$.

The function on $\bar{\sigma}$ is the restriction of a similarly defined function on the vector space containing the entire realization $\bar{T}$. By the same argument as above, this linear map defines a map $\bar{f} : \bar{T} \to \bar{R}$ that we call the *realization* of f. Any function defined this way is called a *linear function.*

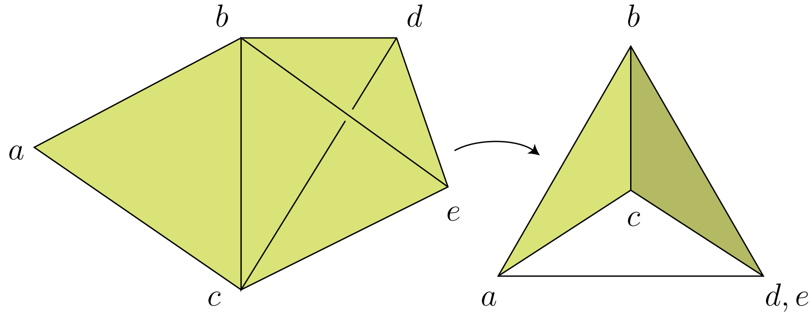

FIGURE 10. Example 19

19. **Example.** For the simplicial complex $T = (\{a, b, c, d, e\}, P(\{b, c, d, e\}) \cup P(\{a, b, c\})$ defined in Example 2, we can define a simplicial homomorphism to $R = (\{1, 2, 3, 4\}, P(\{1, 2, 3\}) \cup P(\{2, 3, 4\} \cup P(\{1, 2, 4\}))$ by sending (a, b, c, d, e) to $(1, 3, 2, 4, 4)$. This projects the tetrahedron $\{b, c, d, e\}$ to the triangle $\{2, 3, 4\}$ and sends the triangle $\{a, b, c\}$ onto the triangle $\{1, 2, 3\}$, as in Figure 10.

20. **Lemma.** *If f is a map from V_T to V_R such that the image of every maximal simplex in T is a simplex in R then ϕ is a simplicial homomorphism.*

Proof. Every simplex τ of T is a face of a maximal simplex σ. Since σ is sent to a simplex of R, τ is send to a face of a simplex in R, which is also a simplex. $\square$

Notice that the simplicial homomorphism in Example 19 is not one-to-one because it sends both vertices d and e onto vertex 4. It sends the vertices $\{a, b, c, d, e\}$ onto $\{1, 2, 3, 4\}$ but the realization $\bar{f}$ is not onto because no points are sent into the interior of the realization of $\{2, 3, 4\}$.

21. **Lemma.** *A simplicial homomorphism $f : T \to R$ defines a one-to-one realization $\bar{f}$ if and only if f is one-to-one on V_T.*

Proof. If $\bar{f}$ is one-to-one then in particular it sends the image of each vertex in $\bar{T}$ to a unique point in $\bar{R}$, which must also be a vertex, so f is one-to-one. Conversely, if f is one-to-one then it sends each simplex of T to a simplex of the same dimension in R, so $\bar{f}$ is one-to-one on each simplex $\bar{\sigma}$. It also sends each simplex of T to a distinct simplex of R, so the entire map $\bar{f}$ is one-to-one. $\square$

A *generalized simplicial homomorphism* is a map $f : V_T \to \bar{R}$ such that for each $\sigma \in S_T$, the image $f(\sigma)$ is contained in $\bar{\tau}$ for some $\tau \in S_R$.

Note that every simplicial homomorphism induces a generalized simplicial homomorphism. A generalized simplicial homomorphism also induces a linear map between the vector spaces containing $\bar{T}$ and $\bar{R}$ whose restriction is a map $\bar{f} : \bar{T} \to \bar{R}$. We will also call this map $\bar{f}$ a *linear map*.

A map $g : X \to Y$ between topological spaces X and Y is called *piecewise linear* if it is induced by a linear map between triangulations of X and Y. In other words, assume $\phi : X \to \bar{T}$ and $\psi : Y \to \bar{R}$ define triangulations of X and Y. We say that $g : X \to Y$ is piecewise linear if there is a generalized simplicial homomorphism $f : T \to R$ such that $g = \psi^{-1} \circ \bar{f} \circ \phi$.

In particular, it is sometimes useful to restrict our attention to piecewise-linear homeomorphisms between spaces. If such a map exists, we will say that X and Y are *pl-homeomorphic*. The existence of such a map may depend on the triangulation, and leads to a notion of equivalence of triangulations. However, in order to make this type of equivalence precise, we need one more idea, namely subdivision of triangulations.

5. Subdivision

Let $T = (V_T, S_T)$ be a simplicial complex and let $p \in \bar{T}$ be a point that is not the image of a vertex. The *subdivision* of T defined by p is the simplicial complex $R = div(T, p)$ such that $V_R = V_T \cup \{p\}$, and S_R consists of the set $\{p\}$ and the following set for each $\sigma \in S_T$ and each face τ of σ:

(1) If $p \notin \sigma$ then $\sigma \in S_R$.
(2) If $p \in \sigma$ and $p \notin \tau$ then $\tau \cup \{p\} \in S_R$.

Note that if p is in both τ and σ then σ and τ do not determine a simplex in $div(T, p)$. For example, consider the complex from Example 2: $T = (\{a, b, c, d, e\}, P(\{b, c, d, e\}) \cup P(\{a, b, c\})$. Assume p is a point in the edge $\{b, c\}$.

We would like to construct $R = div(T, p)$. The edge $\{b, c\}$ has two faces, $\{a\}$ and $\{b\}$, so it defines two edges in R: $\{a, p\}$ and $\{b, p\}$. The point p is also contained in $\{a, b, c\}$, which has two maximal faces that don't contain p: $\{a, b\}$ and $\{a, c\}$. Thus R contains the triangles $\{a, b, p\}$ and $\{a, c, p\}$. Similarly, the tetrahedron $\{b, c, d, e\}$ has two maximal faces $\{b, d, e\}$ and $\{c, d, e\}$ disjoint from p so this defines two tetrahedra: $\{b, d, e, p\}$ and $\{c, d, e, p\}$. The resulting complex is shown in Figure 11.

22. **Lemma.** *For any point $p \in \bar{T}$ that is not a vertex, the subdivision $R = div(T, p)$ is a simplicial complex.*

Proof. A simplicial complex must satisfy two conditions: Each vertex forms a zero-dimensional simplex and each subset of a simplex is a simplex. By definition $\{p\} \in S_R$. Moreover, since p is not the image in $\bar{T}$ of any vertex v of T, we have $\{v\} \in S_R$.

If σ is a simplex of T such that $p \notin \bar{\sigma}$ then p is not in any of the faces of $\bar{\sigma}$, so these faces are all simplices of R.

On the other hand, if $p \in \bar{\sigma}$ then we add a simplex $\tau \cup \{p\}$ for each face τ of σ that does not contain p. A face of $\tau \cup \{p\}$ is either of the form $\tau' \cup \{p\}$ for a face τ' of τ or τ' where τ' is a face of τ. Since p is not in $\bar{\tau}$, it is also disjoint from $\bar{\tau}'$. Therefore, $\tau' \cup \{p\}$ and τ' are both simplices of R and we have verified that R is a simplicial complex. □

Since we often want to describe a simplicial complex in terms of its maximal simplices, we should check where the maximal simplices of a subdivision come from.

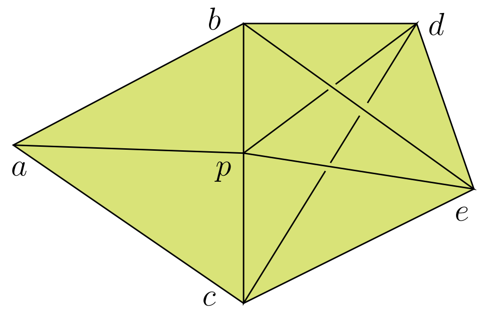

FIGURE 11. A subdivision of the complex in Example 2.

23. **Lemma.** *Every maximal simplex of the subdivision div(T, p) is either a maximal simplex σ of T such that $p \notin \bar{\sigma}$ or the union of $\{p\}$ and a maximal face τ of a maximal simplex σ such that $p \in \bar{\sigma}$ but $p \notin \bar{\tau}$.*

Before we present the proof, the reader can check that this is true for low dimensional simplices such as the two possible subdivisions of the triangle shown in Figure 12.

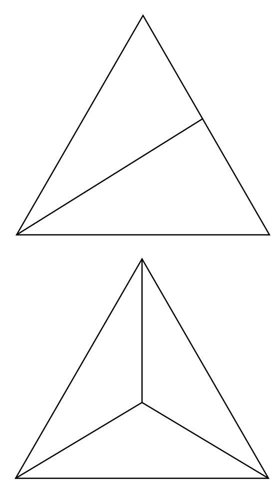

FIGURE 12. Subdivisions of a triangle.

Proof. Let σ be a maximal simplex of $R = div(T, p)$, i.e. assume σ is not a proper face of another simplex. If σ is a maximal simplex of T then $p \notin \bar{\sigma}$ since $\sigma \in V_R$ and we have the first possible conclusion.

Otherwise, assume σ is not a maximal simplex of T. If $p \notin \sigma$ then σ is a face of a simplex σ' of T. We must have that $p \in \bar{\sigma}'$ since σ' is not a simplex of R. But then $\sigma \cup \{p\}$ is a simplex of R contradicting the assumption that σ is maximal in R.

Therefore, p must be a vertex of σ and we can write $\sigma = \tau \cup \{p\}$ where τ is a simplex in

T. Moreover, τ is a face of a simplex σ' containing p. If τ is not a maximal face of σ' disjoint from p then τ is a proper face of a face τ' of σ such that $p \notin \bar{\tau}'$. Then $\tau' \cup \{p\}$ is a simplex of R, contradicting the assumption that σ is maximal. Thus we have the second possible conclusion of the Lemma. □

By repeating the subdivision construction, we can subdivide T along a sequence of vertices $(p_1, p_2, \ldots, p_k)$. We will write $div(T, (p_1, p_2, \ldots, p_k))$ for the resulting simplicial complex, which we also call a *subdivision* of T. Note that the final simplicial complex may depend on the order in which the points are added.

What makes subdivision useful is that it always produces a triangulation of the same space:

24. **Lemma.** *If R is a subdivision of T then the realization $\bar{R}$ of R is piecewise-linear homeomorphic to $\bar{T}$.*

This should appear obvious for the examples in Figures 11 and 12. The proof of Lemma 24 will rely on a careful construction of a homeomorphism.

Proof. There is a natural generalized simplicial homomorphism $f : V_R \to \bar{T}$ defined by sending each vertex of $V_T \subset V_R$ to its image in $\bar{T}$ and sending the point p to itself in $\bar{T}$. On each simplex σ of T such that $p \notin \bar{\sigma}$, the realization $\bar{f}$ sends $\bar{\sigma}$ to itself. Therefore, we need only check that $\bar{f}$ is well defined, one-to-one and onto on the realization of a simplex $\sigma \in S_T$ such that $p \in \bar{\sigma}$.

The map $\bar{f}$ is well defined because f sends every simplex of R into the realization of a single simplex in T. For each point $q \neq p \in \bar{\sigma}$, let ℓ be the infinite ray from p through q. The line segment $\ell \cap \bar{\sigma}$ has p as one endpoint and its other endpoint is in the interior of a face τ of $\bar{\sigma}$ that does not contain p. (If $\bar{\tau}$ contained p then the second endpoint would be in the boundary of τ.) Therefore, each point q is contained in the image of the simplex $\tau \cup \{p\}$, which is a simplex of R because $p \notin \bar{\tau}'$. We conclude that $\bar{f}$ is onto. On the other hand, the face τ is uniquely determined by the ray ℓ so q is in the interior of the image of exactly one simplex of R. The map $\bar{f}$ is one-to-one on each of these simplices (because the dimensions are the same) so $\bar{f}$ is both one-to-one and onto on each $\bar{\sigma}$ that contains p. □

Thus if (T, ϕ) is a triangulation of a topological space X then every subdivision of T also defines a triangulation of X. In fact, there is a canonical way to define this piecewise-linear homeomorphism, which can be a particularly useful fact. In particular, we can use subdivision to turn a generalized simplicial homomorphism into a simplicial homomorphism.

25. **Theorem.** *Let X, Y be topological spaces with triangulations (T, ϕ) and (R, ψ), respectively. If $f : X \to Y$ is a pl homeomorphism then there are subdivisions of T and R such that f is defined by a simplicial homomorphism between them.*

The proof of this Theorem is beyond the scope of the present treatement, but can be found in Rourke and Sanderson's book [2] (Theorem 2.14 on p.17.)

One particular subdivision construction turns out to be very important: The *barycenter* of an n-dimensional simplex $\bar{\sigma}$ is the point whose coordinates are all $\frac{1}{n+1}$. The *barycentric subdivision* of a simplicial complex T is the subdivision $bary(T)$ that results from subdividing along the barycenter of every positive-dimensional simplex of T. (The barycenter of a vertex is itself and we are not allowed to subdivide along a vertex.) An example of this is shown in Figure 13.

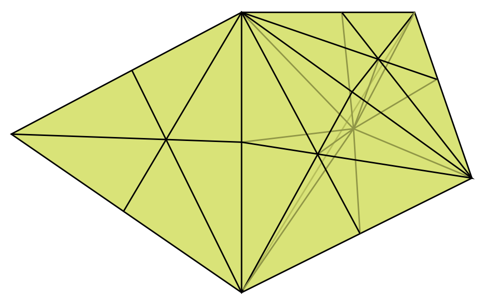

FIGURE 13. The barycentric subdivision of the complex in Example 2.

As noted above, the order in which we add the points may affect the final triangulation. We will order the points so that if p_1 and p_2 are barycenters of simplices of dimension d_1 and d_2 with $d_2 < d_1$ then we will subdivide along p_1 before we subdivide along p_2. This does not completely determine the order of subdivision, but we will see that it completely determines the structure of the resulting simplicial complex. In particular, there is a nice description of the complex that results from any such subdivision:

A sequence of simplices $\sigma_1, \sigma_2, \cdots, \sigma_k$ is *nested* if $\sigma_i < \sigma_{i+1}$ for each i. The dimensions of these a nested sequence of simplices are increasing, though they may not be consecutive integers. We also allow that σ_1 be a zero-dimensional simplex.

26. **Lemma.** *Every simplex in $bary(T)$ is spanned by the barycenters of the simplices in a nested sequence. Conversely, the barycenters of every nested sequence of simplices defines a simplex in $bary(T)$.*

Proof. Define $bary_k(T)$ to be the complex that results from subdividing T at the barycenters of all the simplices of dimension strictly greater than k, again subdividing the higher dimensional simplices before lower dimensional ones. Then $bary_0(T) = bary(T)$ and if T has dimension n then $bary_n(T) = T$. We will show that every d-dimensional simplex of $bary_k(T)$ is spanned by the barycenters of a nested sequence of simplices of dimension greater than k and the vertices of a simplex of T, of dimension at most k that is properly contained in all simplices in this sequence. Note that this implies the Lemma for $k = 0$.

We will induct on $n - k$. For the base case, $n = k$ this is trivially true since $bary_n(T) = T$ and every simplex of T has dimension at most n. For the induction step, assume that $bary_{k+1}(T)$ has the desired property and we will prove it for $bary_k(T)$.

We get from $bary_{k+1}(T)$ to $bary_k(T)$ by subdividing at the barycenter of every $k + 1$-dimensional simplex. Every simplex σ of $bary_k(T)$ is either already a simplex of $bary_{k+1}(T)$ or results from a simplex σ' of $bary_{k+1}(T)$ that contains a barycenter of a $(k+1)$-dimensional simplex in T.

If σ is already a simplex of T then by the inductive assumption, σ is spanned by a nested sequence of barycenters and a simplex of T of dimension at most $k + 1$. Since σ does not contain the barycenter of a $(k + 1)$-dimensional simplex in T, the simplex of T contained in σ has dimension at most k, so σ satisfies the desired condition for $bary_k(T)$.

Otherwise, assume σ does not come from a simplex σ' of $bary_{k+1}(T)$. Then σ' contains both a simplex of $bary_{k+1}(T)$ and a barycenter p of a k-dimensional simplex. Again, by the induction assumption, σ' is spanned by a nested sequence of barycenters and a simplex τ of T of dimension at most $k + 1$. Because σ' contains a barycenter of a $(k + 1)$-dimensional simplex, τ must have dimension exactly $k + 1$. By construction, then, σ results from σ' by replacing one of the vertices in τ with the barycenter of τ. Therefore, σ is spanned by the barycenters of a nested sequence and a simplex that is a proper subset of all the simplices in this sequence. By induction, this completes the proof. □

6. Gluing

While we can describe a large family of topological spaces as simplicial complexes, it's often much more practical to build up a space from slightly more complicated pieces. Let X be a topological space with triangulation (T, ϕ). Let A, B be disjoint subspaces such that each is the preimage of a subcomplex of T and assume $f : A \to B$ is a pl map from A to B. Define X/f to be the quotient space defined by the relation $x \sim f(x)$ as in Figure 14. We will say that X/f is the result of *gluing A to B along f*.

27. **Lemma.** *The space X/f has a triangulation naturally induced from T.*

Note that the space X/f will have many different triangulations, related by subdivision. We will construct one of these triangulations, but not by any means the most efficient.

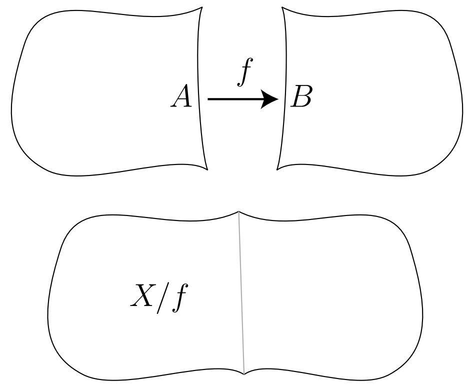

FIGURE 14. Gluing A to B along the homeomorphism f.

Proof. Let T be a triangulation of X in which A and B are subcomplexes. Because f is piecewise-linear, Lemma 25 implies that there are subdivisions of A and B after which f is a simplicial homomorphism. Because A is a subcomplex of T, each point p of $\bar{A}$ is a point in $\bar{T}$, so subdividing T along each p defines a subdivision of T that contains the subdivision of A as a subcomplex. We can similarly subdivide along the points in $\bar{B}$ to find a subdivision of T such that f is a simplicial homomorphism between subcomplexes.

To take the quotient, we need to identify corresponding vertices in A and B, but one problem remains: Consider a simplex $\sigma \subset A$ and its image $f(\sigma)$. If there is a vertex v such that $\sigma \cup \{v\}$ and $f(\sigma) \cup \{v\}$ are both simplices of T then they will become the same simplex in the quotient. (This occurs in Example 28 below.) In this case, the realization of the quotient will not be homeomorphic to the topological quotient.

Therefore, we will take the barycentric subdivision of T before we take the quotient, as in Figure 15. If $\sigma \cup \{v\}$ and $f(\sigma) \cup \{v\}$ are both simplices of $bary(T)$ then σ and $f(\sigma)$ contain vertices of codimension-one faces of a simplex in T. Such simplices have non-trivial intersection, contradicting the assumption that A and B are disjoint. Therefore each simplex σ in $bary(T)$ is identified to another simplex if and only if σ is in A, in which case it is identified with $f(A)$. $\square$

28. **Example.** We have defined the n-sphere S^n as a subset of $\mathbf{R}^n$ and constructed a simplicial triangulation of S^n as the boundary of an n-dimensional simplex. But it's also sometimes useful to define it by a gluing operation.

First note that every n-sphere is the union of two sets: Let $B_1 \subset S^n$ be the set of all points whose first coordinate (in $\mathbf{R}^n$) is non-negative and let B_2 be the set of points whose first coordinate is non-positive. The intersection $B_1 \cap B_2$ is the set of all points in $\mathbf{R^n}$ with first coordinate 0 that are unit distance from the origin, i.e. the $(n-1)$-sphere S^{n-1}. Moreover, we can project each of B_1, B_2 onto the hyperplane with first coordinate 0. This map is one-to-one and maps each of the sets onto the unit n-ball. The sets B_1 and B_2 are n-dimensional balls, so we can think of S^n as a union of the two balls glued along their boundary spheres.

To realize this construction by a pl gluing, we can identify each B_i with the realization of a simplex $\bar{\sigma}$. Since we want two copies of this simplex, we will formally have to use a trick. Let $\bar{\sigma}_1$ be the set of ordered pairs $\{(p, 1) \mid p \in \bar{\sigma}\}$ and $\bar{\sigma}_2 = \{(p, 2) \mid p \in \bar{\sigma}\}$. The sets $\bar{\sigma}_1$ and $\bar{\sigma}_2$ are now disjoint, and we call their union $X = \bar{\sigma}_1 \cup \bar{\sigma}_2$ the *disjoint union of two copies of* σ.

We will define the gluing map by $\phi(p, 1) = (p, 2)$ for each p in the boundary of σ. Since this map is induced by the identity map on σ, it is piecewise linear. Note that the induced simplicial triangulation on σ is not just the two copies of σ. Such simplices would have all the same vertices and would thus define a single simplex under the formalism of simplicial complexes. This is the reason we subdivide when forming a simplicial triangulation for the glued space.

In general, this construction in which we use ordered pairs to ensure that two or more subsets are disjoint is called a *disjoint union*.

29. **Example.** We can construct the n-dimensional torus T^n by a fairly simple sequence of gluing operations. Let C_n be the unit cube in $\mathbf{R}_n$, i.e. the set of points whose coordinates are all in the interval $[0, 1]$. For each coordinate x_i, there is a pair of faces consisting of the points where $x_i = 0$ and the points where $x_i = 1$. Let ϕ_i be the translation of $\mathbf{R}^n$ that sends $x_i = 0$ to $x_i = 1$, restricted to the appropriate face of C^n. We claim the T^n is homeomorphic to the result of sequentially gluing C^n along the maps ϕ_i. The two-dimensional case is shown in Figure 16.

To prove this, note that if we identify the Euclidean plane $\mathbf{R}^2$ with the complex plane $\mathbf{C}$ then the unit circle $S^1 \subset \mathbf{C}$ is the set of points of the form $e^{2\pi i\theta}$ for $\theta \in [0, 1]$. The values 0 and 1 define the same point

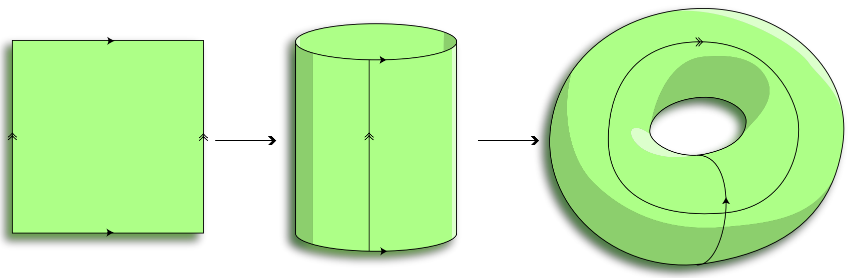

FIGURE 16. Constructing T^2 by gluing.

in $\mathbf{C}$, so the map $\phi : \theta \mapsto e^{2\pi i\theta}$ is one-to-one on $(0,1)$ and two-to-one on $\{0,1\}$. If we take the quotient space $[0,1]/(0 \sim 1)$ then ϕ defines a homeomorphism. We conclude that the 1-dimensional torus $T^1 = S^1$ is homeomorphic to the quotient of the 1-dimensional cube $C^1 = [0,1]$ by the gluing map $0 \mapsto 1$.

For larger n, consider the map $\phi \times \cdots \times \phi : C^n \to T^n$ defined by $(\theta_1, \ldots, \theta_n) \mapsto (e^{2\pi i\theta_1}, \ldots, e^{2\pi i\theta_n})$. This map is one-to-one on the interior of C^n. On the boundary, the map sends two points $p, q \in C^n$ to the same point in T^n if and only if they agree up to switching one or more 0s and 1s, i.e. there is a sequence of gluing maps and inverses of gluing maps defined above whose composition sends p to q (or vice versa). Therefore, this product of maps $\phi \times \cdots \times \phi$ defines a homeomorphism from the quotient of C^n by the gluing maps defined above to T^n.

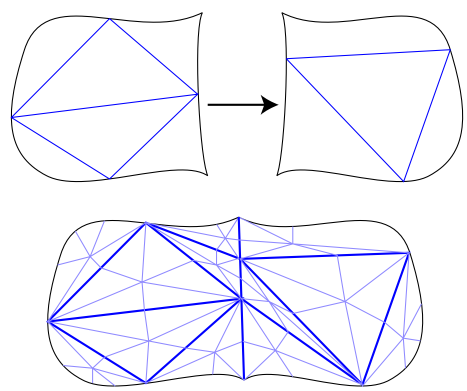

FIGURE 15. Subdividing the triangulations to allow gluing.

We next need to check that we can realize each ϕ_i by a pl map. We will define a triangulation of C^n with vertex set V consisting of points in $\mathbf{R}^n$ whose coordinates are each 0 or 1. The support of each vertex $v \in V$ (i.e. the set of non-zero coordinates) defines a subset $s_v \subset \{1, 2, \ldots, n\}$ and we will say that a set of vertices is *nested* if the support sets form a nested sequence. Let S be the set of all nested subsets of V.

30. **Lemma.** *The pair (V, S) defines a simplicial complex that is a triangulation of C^n.*

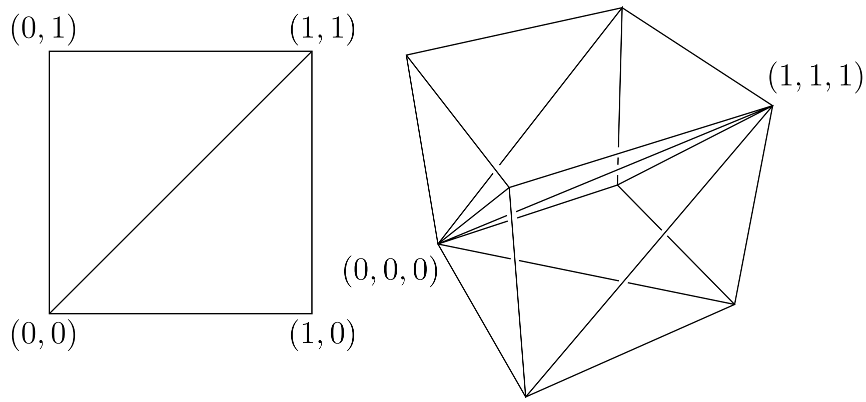

Figure 17. Triangulations of C^2 and C^3 defined by nested sequences. (Only two of the labels are shown for C^3.)

The triangulations defined this way in dimensions two and three are shown in Figure 17.

Proof. Let $x = (x_1, \dots, x_n)$ be a vector of values in $[0, 1]$. For each i between 1 and n, let A_i be the set of indices j such that $x_j \leq x_i$. Note that $A_i = A_j$ exactly when $x_i = x_j$. By construction, the set $\{A_i\}$ is a nested sequence and thus determines a simplex $\sigma \in S$. Moreover, x is a positive linear product of vectors with entries 0 and 1 with support A_i for each i. Thus x is in the interior of $\bar{\sigma}$. Conversely, every point in the interior of σ defines the nested sequence that corresponds to σ in this way. Therefore, every point of C^n is in the interior of a unique simplex and this defines a homeomorphism between C^n and the realization of (V, S). $\square$

The reader can check that the translation maps ϕ_i correspond to simplicial homomorphisms between opposite faces in this triangulation, so the gluing maps are pl. However, after all the gluing, the 2^n vertices of V are identified to a single vertex in T^n. Thus in order to find a simplicial triangulation for the glued space we again rely on the subdivision step in the proof of Lemma 27.

7. Generalized triangulations

In the gluing constructions of both S^n and T^n, we glued together simplices by linear maps in ways that did not produce simplicial complexes. This type of construction comes up quite often and generally makes for a better description of a space than the formal simplicial complex.

A *(generalized) triangulation* T is a pair (S, Φ) where S is a disjoint union of (realizations of) simplices and Φ is a collection of linear homeomorphisms between between faces of simplices in S, with the following property: If a composition of maps in Φ defines a map from a face of a simplex in S to itself then this composition is the identity map. The *realization* $\bar{T}$ is the space that results from gluing S along all the maps in Φ. Note that because we are gluing along a sequence of maps, we have to be careful about the order in which we glue.

31. **Lemma.** *The realization $\bar{T}$ does not depend on the order in which we glue along the maps in Φ.*

The proof will be left as an exercise for the reader.

A *triangulation of* X is a pair (T, h) where T is a triangulation and $h : X \to \bar{T}$ is a homeomorphism. When we want to specify that T is a simplicial complex rather than a general triangulation, we will refer to (T, h) as a *simplicial triangulation.*

32. **Example.** We can write the simplicial complex in Example 2 ($T = (\{a, b, c, d, e\}, P(\{b, c, d, e\}) \cup P(\{a, b, c\}))$ as a triangulation by letting S be the disjoint union of the realizations of a tetrahedron with vertices $\{b, c, d, e\}$ and a triangle with vertices $\{a, b, c\}$. We will let Φ consist of a single linear map from the edge $\{b, c\}$ in the triangle to the edge $\{b, c\}$ in the tetrahedron.

33. **Example.** In Example 29, we constructed each n-dimensional torus T^n by gluing a triangulated cube. As in Example 32, we can encode the triangulation of the cube C^n as a generalized triangulation. The restrictions of the gluing maps Φ to the remaining faces define a further collection of gluing maps between simplices. This construction thus defines a (generalized) triangulation of each T^n. The triangulation of T^2 defined this way is shown in Figure 18. Note that this triangulation has exactly one vertex so it cannot be a simplicial triangulation.

The quotient construction induces an inclusion map $i : S \to X$ where X is the quotient space. This map is many-to-one because of the gluing. However, the condition that the gluing maps are linear homeomorphisms implies the following:

34. **Lemma.** *The restriction of the inclusion map $i : S \to X$ to the interior of each face of each simplex in S is one-to-one.*

Proof. Let $p, p' \in \bar{\sigma}$ be points such that $i(p) = i(p')$. Then there is a sequence of gluing maps whose composition ϕ is such that $\phi(p) = p'$.

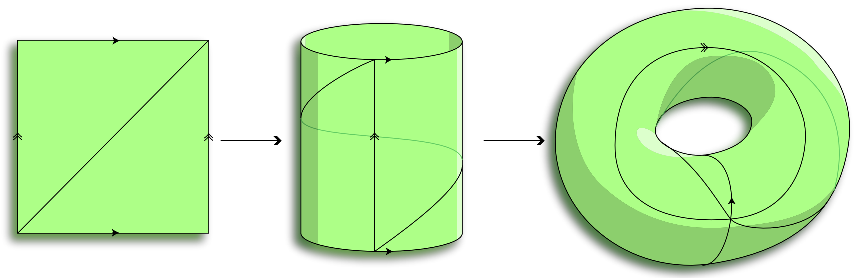

FIGURE 18. A (generalized) triangulation of T^2.

Since p and p' are in $\bar{\sigma}$, the restriction of ϕ sends $\bar{\sigma}$ to itself. The condition on gluing maps implies ϕ is the identity map, so $p = p'$ and i must be one-to-one on $\bar{\sigma}$. □

In general, every simplicial complex T can be translated to a triangulation in which S is the disjoint union of the realizations of the maximal simplices of T and the gluing maps are defined by the identity maps on the intersections of the maximal simplices.

35. **Lemma.** *The realization of the triangulation constructed this way is homeomorphic to the realization of T by a map that is the identity on the realization of each simplex.*

The proof of this Lemma is almost immediate and is left to the reader.

8. Cell complexes

We can generalize triangulations even further by removing the restriction that the "pieces" be simplices. We will define cell complexes by an inductive definition in terms of dimension.

36. **Definition.** A *0-cell* is a vertex. A *1-cell* is an edge, i.e. a 1-dimensional simplex. Notice that the boundary of an edge is two 0-cells. A *1-dimensional cell complex* is a union of edges glued together by maps between their boundary 0-cells. In other words, a 1-dimensional complex is a graph in which edges are allowed to have both end points in the same vertex.

A *2-cell* is a polygon: a 2-dimensional disk whose boundary circle is identified with a 1-complex (in this case a cycle of edges). The boundary complex can have arbitrarily many edges, or can consist of a single edge and a single vertex.

A 2-cell might not be a simplex, but it inherits a piecewise-linear structure: If we identify its boundary with the unit disk in $\mathbf{R}^2$, we can extend this to a triangulation of the unit disk with a single interior vertex at the origin and an edge from this vertex to each vertex in the boundary circle. (We have to modify this construction if there is only one edge in the boundary.)

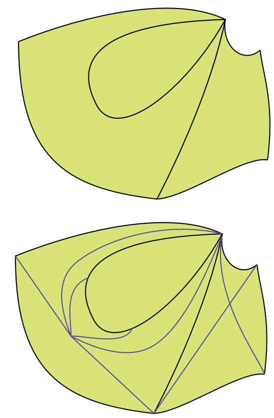

FIGURE 19. A 2-complex and its simplicial triangulation.

A *2-dimensional cell complex* is the quotient of a union of 1-cells and 2-cells by piecewise linear homeomorphisms between pairs of edges and vertices, with the condition that if a composition of gluing maps defines a map from a face of a simplex to itself then this composition is the identity map. An example is shown in Figure 19. For example, the construction of the two-dimensional torus in Example 29 suggests a cell complex as follows: Let c be a 2-cell with four edges in its boundary, i.e. a quadrilateral as on the left in Figure 16. Let ϕ_1 and ϕ_2 be linear homeomorphisms between the two pairs of opposite edges. There are two possible linear maps for each pair, and we will choose the one induced by a translation if we identify the quadrilateral with square in the plane. Then this is the gluing construction for a 2-dimensional torus T^2 as described above. The images of the 2-cells and the 1-cells in its boundary define a structure on T^2 shown in Figure 19.

Continuing the definition of a cell complex, a *3-cell* is a 3-dimensional ball whose boundary 2-sphere is identified with a 2-dimensional cell complex. This structure is essentially a polyhedron, as in Figure 20, except that the cell structure of the sphere is more flexible than is usually allowed in the definition of polyhedron. For example, there may be edges with both endpoints in the same vertex.

Each 2-cell in the boundary of the 3-cell has a triangulation as above. Since the gluing maps are piecewise-linear, there is a simplicial triangulation of the boundary of the 3-cell by Lemma 27, which extends to a simplicial triangulation of the entire 3-cell, by a coning construction similar to what we used for a 2-cell. This is shown in the lower portion of Figure 20.

A *3-dimensional cell complex* is the quotient of a collection of 3-cells by a collection of gluing maps such that each gluing map is a homeomorphism between two 0-, 1-, or 2-cells in the boundary of the balls and its restriction to the interior of each lower dimensional cell is a homeomorphism onto a face of the image cell. Moreover, if a composition of gluing maps defines a map from a face of a simplex to itself then this composition must be the identity map. In other words, the entire gluing map does not need to be one-to-one, but its restriction to lower dimensional faces must be.

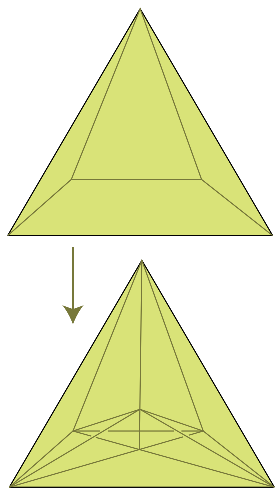

FIGURE 20. A 3-cell and its simplicial triangulation.

We have now built up enough steps that we can finish with an inductive definition. An *n-cell* is an n-dimensional ball whose boundary is identified with an $(n-1)$-dimensional cell complex. The boundary of each n-cell has a simplicial triangulation induced by the gluing which determines a simplicial triangulation on the entire cell by coning.

An *n-dimensional cell complex* is a quotient of n-cells and lower dimensional cells by a set of gluing maps such that each gluing map is a homeomorphism between cells in the boundary of maximal cells and the restriction of each gluing map to the interior of each proper face is a homeomorphism. Moreover, we also require the same condition as we had for generalized triangulations: If a composition of gluing maps defines a map from a face of a simplex in S to itself then this composition is the identity map.

We will record our cell complex as a pair $C = (S, \Phi)$ where S is a disjoint union of cells and Φ is a set of gluing maps. The *realization* $\bar{C}$ of C is the quotient of S by the maps in Φ. We will build up a simplicial triangulation for $\bar{C}$ by induction on the dimension. Define the *k-skeleton* of C as the cell complex C^k with simplices S^k consisting of all faces of cells in S of dimension less than or equal to k, and gluing maps Φ^k consisting of the restrictions of Φ to these cells.

As in the case of generalized triangulations, the pieces that we glue together together define a combinatorial structure on the final space:

37. **Lemma.** *The restriction of the inclusion map to the interior of each n-cell in S and the interior of each face is one-to-one.*

As in the proof Lemma 34, this follows from the condition on compositions of gluing maps and the details will be left to the reader.

9. Regular neighborhoods

We will use triangulations in this section to define "nice" neighborhoods of certain subsets of triangulated spaces. A subset $Y \subset X$ is *piecewise linear* if there is a triangulation T of X such that Y is the image of (the realization of) a subcomplex K of T.

Recall that $bary(T)$ is the barycentric subdivision of a simplicial complex T and that $bary^2(t) = bary(bary(T))$ is the second barycentric subdivision. The second barycentric subdivision $bary^2(T)$ contains the second barycentric subdivision $K' = bary^2(K)$ of K as a subcomplex. A *(closed) regular neighborhood* of Y is the image in X of the subcomplex of $bary^2(T)$ consisting of all simplices that intersect K' and all their faces. Notice that we don't say "the" regular neighborhood because this set depends on the triangulation T. Because it's the image of a subcomplex, a closed regular neighborhood is topologically closed as a subset of $\bar{T}$.

An *(open) regular neighborhood* of K is the image in $\bar{T}$ of the interiors of all the simplices that intersect K' (but not their faces). A regular neighborhood of three edges is shown on the right in Figure 21. The first and second barycentric subdivisions are shown in blue and green, respectively. The complement of these cells is a subcomplex of T and thus closed, so an open regular neighborhood is topologically open. Moreover, the (topological) closure of an open regular neighborhood is the closed regular neighborhood defined by the same triangulation. The difference between the two sets, i.e. the complement in the closed regular neighborhood of the open regular neighborhood is called the *frontier* of N.

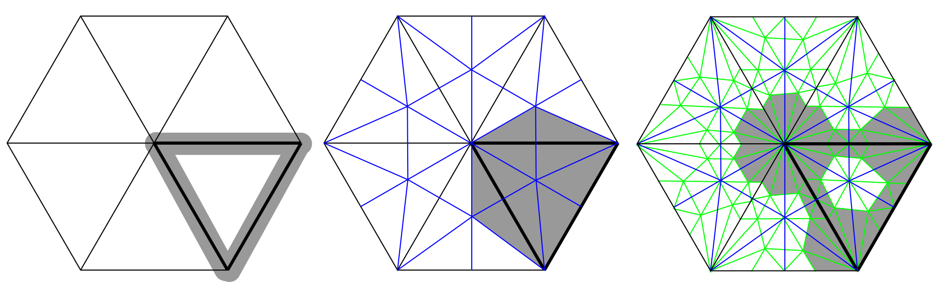

FIGURE 21. Constructing a regular neigborhood of three edges.

38. **Lemma.** *If Y and Z are disjoint piecewise linear subsets of X then closed regular neighborhoods of Y and Z are disjoint.*

Proof. Let K and L be the subcomplexes of T defining X and Y, respectively. Every simplex σ of the first barycentric subdivision $bary(T)$ corresponds to a nested sequence of simplices in T, so if σ has a vertex in $bary(K)$ and a vertex in $bary(L)$ then these correspond to a simplex in K that is a face of a simplex of L or vice versa. This contradicts the assumption that K and L are disjoint, so we conclude that no simplex of $bary(T)$ intersects both $bary(K)$ and $bary(L)$.

Let N' be the set of all simplices in $bary(T)$ that intersect $bary(K)$. The subcomplex $bary(L)$ will be disjoint from N' but there may be simplices in $bary(T)$ that intersect both $bary(L)$ and N'. However, by repeating the same argument as above, if we subdivide one more time we find that there is no simplex in $bary^2(T)$ that intersects both $bary^2(L)$ and $bary(N')$. Therefore the regular neighborhood of L defined by T is disjoint from $bary(N')$. Since the regular neighborhood of K defined by T is contained in $bary(N')$, this implies that the regular neighborhoods of K and L are disjoint. The reader can check, for example, that the regular neighborhood in Figure 21 is disjoint from the three edges in the upper left of the original hexagon. □

Regular neighborhoods have another property that is quite useful:

39. **Lemma.** *For any piecewise linear subset $Y \subset X$ with closed regular neighborhood N, there is a pl map $r : N \to Y$ that is the identity on Y.*

The map r is called a *retract* of N onto Y. The reader can check that the regular neighborhood shown in Figure 21 retracts onto the three-edge loop. However, if we have stopped with the first barycentric subdivision, the resulting neighborhood (shown shaded in the middle picture) is a disk. It is a non-trivial Theorem (which we will not prove here) that a disk does not retract onto its boundary, so this neighborhood would not satisfy the conclusion of Lemma 39.

Proof. Every regular neighborhood N of Y is defined by a triangulation T of X in which Y is a subcomplex. In the first barycentric subdivision $bary(T)$, each simplex σ has at most one maximal face τ in Y. Let R be the subcomplex of the second barycentric subdivision $b^2(T)$ that comes from subdividing σ. Let N_R be the set of all simplices in R that intersect Y and their faces. Then N_R is the intersection of R with the regular neighborhood N. We will define a piecewise-linear map ϕ from R onto itself and sends N_R onto the image of τ.

The map ϕ will be completely determined by where it sends the vertices of R in $\bar{R}$. Every vertex corresponds to a simplex of R and for each simplex α, we will write v_α for the corresponding vertex. If α is disjoint from Y then we will let $\phi(v_\alpha) = v_\alpha$). Otherwise, let β be the (unique) maximal face of α in Y. We will have ϕ send v_α to the barycenter of $\bar{\beta}$ as in Figure 22. This map is the identity on $Y \cap R$ and sends N_R onto Y. If we combine all the maps ϕ for each simplex in Y, the result is the desired map r. □

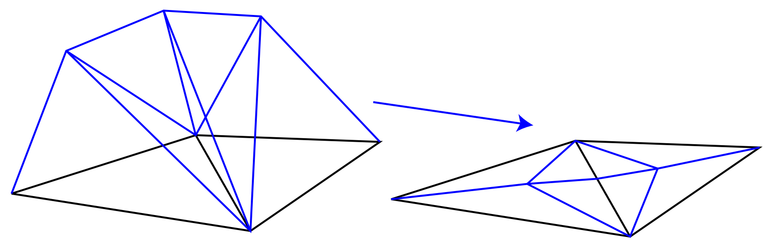

FIGURE 22. Mapping a regular neighborhood of Y onto Y.

As noted above, a regular neighborhood is not unique - it depends on the triangulation used to construct it. However, it turns out that it is very close to unique in a topological sense:

40. **Theorem.** *Let Y be a closed, piecewise linear subset of X. If N and N' are closed regular neighborhoods of Y determined by equivalent triangulations of X then there is a homeomorphism from X to itself that restricts to the identity on Y and takes N onto N'.*

The proof of this Lemma is beyond the scope of the present treatment, but we will refer the interested reader to Theorem 3.24 on p.38 of Rourke and Sanderson's book [2] for the proof.

We can use regular neighborhoods to define a construction that is roughly an inverse of gluing. Let $Y \subset X$ be a closed piecewise linear subset and let $N \subset X$ be an open regular neighborhood of Y (usually a codimension-one submanifold). The complement $X \setminus N$ is a closed piecewise-linear subset of X and we will say that $X \setminus N$ is the result of *cutting* X along Y.

Note that the complement of Y is an open set whose closure contains some or all of Y. The complement of the regular neighborhood, on the other hand, is already closed and inherits a piecewise linear structure. So in many cases where one wants to consider the complement of Y, it is often better to consider the space that results from cutting along Y.

10. Embeddings and Isotopies

41. **Definition.** An *embedding* of a topological space Y into a topological space X is a map $i : Y \to X$ such that i is a homeomorphism from Y to its image $i(Y) \subset X$. In other words, an embedding i identifies Y

with a subset of X. In particular, the inclusion map from a subset of X into X is an embedding.

This idea is particularly useful for comparing different subsets of a space X. For example if i and i' are two embeddings of Y into X then both subsets $i(Y)$ and $i'(Y)$ are homeomorphic subsets, since both are homeomorphic to Y. However, we will often want an even stronger condition:

42. **Definition.** An *isotopy* from an embedding $i : Y \to X$ to $i' : Y \to X$ is a map $I : Y \times [0, 1] \to X$ such that the restriction of I to $Y \times \{t\}$ is an embedding for each $t \in [0, 1]$, plus $I|_{Y\times\{0\}} = i$ and $I|_{Y\times\{1\}} = i'$. We say that the maps i, i' are *isotopic* if there is an isotopy between them.

Two subsets $A, B \subset X$ are *isotopic* if A and B are images of isotopic embeddings of a space Y into X.

Note that the map I need not be one-to-one, as in Figure 23 and in general an isotopy won't be. Since we are interested in piecewise-linear maps, we will often restrict our attention to piecewise-linear isotopies, i.e. piecewise-linear maps that satisfy the conditions in Definition 42. However, in order to do that, we must choose an initial triangulation for the product $Y \times [0, 1]$, which we can then subdivide to form interesting maps.

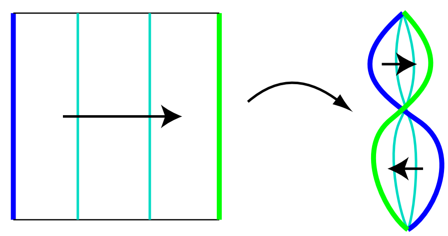

FIGURE 23. An isotopy between two arcs.

Let T be a simplicial triangulation of Y and label the vertices $V_T = \{v_1, \dots, v_N\}$. Each simplex $\sigma \in S_T$, is of the form $\sigma = \{v_{i_1}, \dots, v_{i_n}\}$ where the indices i_j are increasing with j. We will define a simplicial complex R with vertices $V_R = \{v_1, \dots, v_N, v'_1, \dots, v'_N\}$. The vertices $v_1, \dots, v_N$ will be in $Y \times \{0\}$ and the vertices $v'_1, \dots, v'_N$ will be in $Y \times \{1\}$. For each $\sigma = \{v_{i_1}, \dots, v_{i_n}\} \in S_T$, define a sequence of simplices in R of the form $\sigma_j = \{v'_{i_1}, \dots, v'_{i_j}, v_{i_j}, \dots, v_{i_n}\}$ where j ranges from 0 to n. Notice that each σ_j has dimension one greater than that of σ. We will also include in S_R all the faces of each σ_j.

We need to check that the realization $\bar{R}$ is homeomorphic to $Y \times [0, 1]$. We will check that there is a canonical homeomorphism from the subcomplex of T consisting of the union of the simplices σ_j and their faces to $\bar{\sigma} \times [0, 1]$ for each simplex σ. By combining these homeomorphisms, we will construct a homeomorphism from $\bar{R}$ to $Y \times [0, 1]$.

Write each point of $\sigma \times [0,1]$ as (x,t) where $x = (x_0, \ldots, x_n)$ is a point in $\mathbf{R}^{n+1}$ whose coordinates are non-negative and sum to 1. Because $t \leq 1$, there is an $(n+1)$-tuple $a = (a_0, \ldots, a_n)$ such that for some i, $a_j = 0$ for $j < i$, $a_j = 1$ for $j > i$, $a_i \in [0,1]$ and $\sum a_j x_j = t$. In other words, we add up the first $i-1$ coordinates of the point, then add as much as we need to of the ith coordinate to get t. We will say that (x,t) is *at step* i if there is such a value a. The tuple a is essentially unique for a given (x,t), though there may be ambiguity when one or more of the values x_j are zero. Thus (x,t) may be at step i for more than one value of i. Checking that $\bar{R}$ is homeomorphic to $\sigma \times [0,1]$ relies on the following claim:

43. **Claim.** *A point (x,t) is in the convex hull of the points $(v_0, 0), \ldots, (v_i, 0), (v_i, 1), \ldots, (v_n, 1)$ if and only of (x,t) is at step i. Moreover, the intersection of any two such convex hulls is the convex hull of their common vertices.*

The second part of the claim implies that the convex hulls of the vertices defines a triangulation of their union. The first part of the claim implies that the union of the convex hulls is all of $\sigma \times [0,1]$. This is true for each simplex σ in R and by construction, for each face τ or σ, the triangulation on $\tau \times [0,1]$ defined by τ is the same as the triangulation that comes from considering $\tau \times [0,1]$ as a subset of $\sigma \times [0,1]$. Therefore, R defines a triangulation of $Y \times [0,1]$.

In the case when X is a manifold, we change terminology slightly.

44. **Definition.** A *submanifold* of X is a subset $Y \subset X$ such that Y is a manifold (possibly with boundary) under the subset topology. By definition, if $i : Y \to X$ is an embedding where Y and X are manifolds then the image $i(Y)$ is a submanifold. An embedding $i : Y \to X$ is *proper* if the image $i(\partial Y)$ is contained in ∂X. A *proper isotopy* between proper embeddings i, i' is an isotopy such that each restriction $I|_{Y \times \{t\}}$ is proper.

A homeomorphism $h : X \to Y$ is an embedding, though the term is rarely used in this context. However, this means we can define an isotopy between homeomorphisms $h, h' : X \to Y$ as a map from $X \times [0,1]$ to Y such that the restriction to each $X \times \{t\}$ is a homeomorphism. Two subsets A, B of X are *ambient isotopic* if there is a homeomorphism $h : X \to X$ such that $h(A) = B$ and h is isotopic to the identity. For example, this idea can be used to strengthen the proof of Theorem 40:

45. **Theorem.** *Let Y be a closed, piecewise linear subsets of X. If N and N' are closed regular neighborhoods of Y determined by equivalent triangulations of X then there is a homeomorphism from X to itself*

that is isotopic to the identity on X, *restricts to the identity on* Y *and takes* N *onto* N'*.*

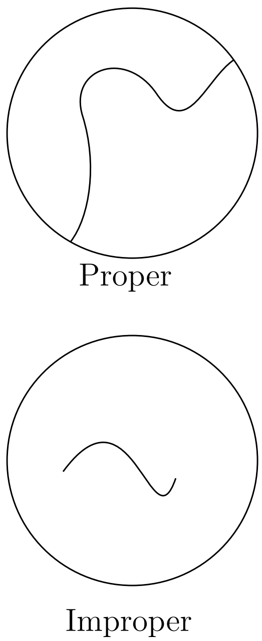

FIGURE 24. Embeddings

In other words, any two regular neighborhoods of $Y \subset X$ are ambient isotopic, by an isotopy that fixes X pointwise. We will again refer the interested reader to Theorem 3.24 on p.38 of Rourke and Sanderson's book [2] for the proof.

The idea of isotopy can also be used to better understand the gluing construction, by considering isotopic homeomorphisms between subset A, B of X. We will say that a subset $A \subset X$ has a *single product neighborhood* if a regular neighborhood N of A can be identified with $A \times [0,1]$ so that $A \times \{1\}$ is identified with A and $A \times \{0\}$ is identified with the frontier of N.

46. **Lemma.** *Let* $A, B \subset X$ *be disjoint and let* $h, h' : A \to B$ *be homeomorphisms. If* A *has a single product neighborhood and* h *is isotopic to* h' *then the quotient* X/h *is homeomorphic to* X/h'*.*

Proof. Let $I : A \times [0,1] \to B$ be an isotopy from h to h'. Because h is a homeomorphism, the composition $I' = h^{-1} \circ I$ is an isotopy from the identity on A to the homeomorphism $h^{-1} \circ h'$. Let N be a regular neighborhood of A. Because A has a single product neighborhood, there is a homeomorphism ψ that sends $A \times [0,1]$ onto the image of N in X/h and sends $A \times \{1\}$ to A. Let ψ' be the same homeomorphism, but sending $A \times [0,1]$ onto the image of N in X/h'.

Construct the map $\phi : X/h \to X/h'$ by letting ϕ be the identity outside of N and defining $\phi = \psi' \circ I' \circ \psi^{-1}$ on the image of N in X/h. As noted above, I' is the identity map on $A \times \{0\}$ and is equal to $h^{-1} \circ h'$ on $A \times \{1\}$. Thus ϕ is continuous along $A \times \{0\}$ (where it matches the identity map on X). Along $A \times \{1\}$, we should consider a point $b \in B$. In X/h, b is identified with $h^{-1}(b) \in A$. The map ϕ thus sends it to $h'^{-1}(h(h^{-1}(b)) = h'^{-1}(b)$. In X/h', the point b is identified with $h'^{-1}(b)$, so again this map matches with the identity map on $A \times \{0\}$. $\square$

11. Exercises

(1) Prove that if T is a simplicial complex with a finite number of vertices then the realization $\bar{T}$ is Hausdorff and compact.

(2) Draw the realization of the triangulation $(\{a, b, c, d, e\}, P(\{a, b, e\}) \cup P(\{c, d, e\}) \cup P(\{b, c\}))$ and write out its 1-skeleton and 2-skeleton.

(3) Show that if σ and σ' are simplices in a triangulation T then $\bar{\sigma} \cap \bar{\sigma}' = \overline{\sigma \cap \sigma'}$.

(4) Prove Lemma 9, that the product of two manifolds is a manifold.

(5) Prove Lemma 11, part 1 that the interior and boundary of a compact manifold with boundary are disjoint.

(6) Prove Lemma 11, part 2 that the boundary boundary of a compact manifold with boundary is a compact manifold.

(7) Prove Lemma 11, part 3 that the interior of a manifold with boundary is a manifold.

(8) Prove Lemma 12 that if M and N are manifolds with boundary then $\partial(M \times N) = \partial M \times N \cup M \times \partial N$.

(9) Calculate the link vertices a, b and e, and edge $\{a, b\}$ for the example in Problem 2.

(10) Draw the realization of the join of A and B where A is a tripod (a graph with three edges that share exactly one vertex) and B is a pair of points.

(11) Construct all the triangulations of the three-dimensional ball that arise as the join of a simplex and the boundary of a simplex as in the proof of Lemma 16.

(12) Find all the simplicial homomorphisms between the complex in Problem 2 and the complex $(\{a, b, c\}, P(\{a, b\}) \cup P(\{b, c\})$.

(13) Show that if T is a simplicial complex and R is a simplex (i.e. a simplicial complex with $S_R = V_R$) then every vertex map $V_T \to V_R$ defines a simplicial homomorphism.

(14) Prove that the realization of a simplicial homomorphism is always continuous.

(15) Prove that the realization of a generalized simplicial homomorphism is always continuous.

(16) Write out and draw the subdivisions of the complex in Problem 2 that arise from subdividing along a point in the triangle $\{a, b, c\}$ and then a point in the edge $\{a, b\}$.

(17) Draw realization of the first barycentric subdivision of the complex in Problem 2.

(18) Prove that if T is a simplicial complex, R is a subcomplex and $p_1, \ldots, p_k$ is a sequence of points in $\bar{R} \subset \bar{T}$ then after subdividing T along $p_1, \ldots, p_k$, the resulting simplicial complex contains the complex that results from subdividing R along $p_1, \ldots, p_k$.

(19) Prove Lemma 31, the the realization of a generalized triangulation does not depend on the order in which the cells are glued together.

(20) Prove Claim 43 that the simplicial complex constructed in Section 10 defines a triangulation of $Y \times [0, 1]$.

(21) Show that if $I : Y \times [0, 1] \to X$ is a piecewise linear isotopy then every intermediate embedding $I|_{Y \times \{t\}}$ is piecewise linear.

(22) Show that isotopy defines an equivalence relation.

(23) Show that if T is a combinatorial manifold then every subdivision of T is also a combinatorial manifold.

(24) Show that two homeomorphisms from the interval $[0, 1]$ to itself are isotopic if and only if they both fix the endpoints or both swap the endpoints.

References

1. Edwin E. Moise, *Geometric topology in dimensions* 2 *and* 3, Springer-Verlag, New York, 1977, Graduate Texts in Mathematics, Vol. 47. MR 0488059 (58 #7631)
2. C. P. Rourke and B. J. Sanderson, *Introduction to piecewise-linear topology*, Springer-Verlag, New York, 1972, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 69. MR 0350744 (50 #3236)