Toric Varieties

Notes by Mateusz Michałek
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Toric varieties form one of the most accessible classes of algebraic varieties. They appear often in both theoretical mathematics and in applications. We start directly with a definition. Recall that nonnegative integer vectors $\mathbf{b} = (b_1, \dots, b_n)$ are identified with monomials

$$\mathbf{x}^{\mathbf{b}} := x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n}$$

In the following definition we allow $\mathbf{b} \in \mathbb{Z}^n$ to have negative entries. This means that $\mathbf{x}^{\mathbf{b}}$ is a *Laurent monomial*, i.e. a monomial with possibly negative exponents.

Definition 1 (Toric variety)**.** An *affine toric variety* is the closed image of a monomial map

$$(K^*)^n \to K^N, \ \mathbf{x} \mapsto \left(\mathbf{x}^{\mathbf{a}_1}, \mathbf{x}^{\mathbf{a}_2} \dots, \mathbf{x}^{\mathbf{a}_N}\right),$$

where $\mathbf{a}_i \in \mathbb{Z}^n$ and $K^* = K \setminus \{0\}$. In the same way we define a *projective toric variety* as the closed image of the same monomial map into projective space $\mathbb{P}^{N-1}$.

Example 2.

1. *The affine and projective spaces are toric varieties.*
2. *The cuspidal curve $x^3 - y^2$ is a toric variety, as it is the image of $z \to (z^2, z^3)$.*
3. *Veronese reembeddings and Segre products of projective spaces are toric varieties.*

The name *toric variety* comes from the fact that $(K^*)^n = \operatorname{Spec} K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ is known as the *algebraic torus*. It is an algebraic group with the action given by coordinatewise multiplication. If $n = 2$ and $K = \mathbb{C}$ then the algebraic torus $(\mathbb{C}^*)^2 \simeq (\mathbb{R}_+ \times \mathbb{S}^1)^2$ coincides with the usual topological torus $\mathbb{S}^1 \times \mathbb{S}^1$ up to multiplication with the contractible factor $\mathbb{R}_+^2$.

Definition 3 (Character of a torus)**.** *A* character *of a torus $T = (K^*)^n$ is an algebraic map $T \to K^*$ that is also a group morphism.*

In Exercise 1 the reader is asked to prove that all characters are given by Laurent monomials. The characters of T are hence the elements of $\mathbb{Z}^n$. Hence, to specify a toric variety, we need to specify N characters of a torus, equivalently N integer points in $\mathbb{Z}^n$. Characters of a torus T can be identified with $\mathbb{Z}^n$ not only as a set but also as a group $(\mathbb{Z}^n, +)$ with the action given by:

$$(\chi_1 + \chi_2)(t) := \chi_1(t)\chi_2(t).$$

A group isomorphic to $\mathbb{Z}^n$ is called *a lattice*. The lattice of characters of T will be denoted by M_T or simply M. Toric geometry relates the geometric properties of a toric variety X with combinatorics of a finite set of lattice points defining X. As a subgroup of a free abelian group is free, the characters defining X generate a sublattice $\tilde{M} \subset M$.

Proposition 4. *Fix characters $a_1, \dots, a_N \in M_T$ generating a sublattice $\tilde{M}$. The image of T in $(K^*)^N$ by the map $x \to (x^{a_1}, \dots, x^{a_N})$ is also a torus $\tilde{T}$ with the character lattice equal to $\tilde{M}$. In particular, the dimension of the associated toric variety equals the rank of $\tilde{M}$.*

Proof. Consider the map of rings associated with the monomial map $f : T \mapsto (K^*)^N$, i.e.

$$f^* : K[y_1^{\pm 1}, \dots, y_N^{\pm 1}] \to K[x_1^{\pm 1}, \dots, x_n^{\pm 1}], \ y_i \mapsto \mathbf{x}^{\mathbf{a}_i}.$$

The spectrum of the image of the ring map f^* is the image $\tilde{T}$ we are interested in. Note that $im f^*$ equals the group algebra $K[\tilde{M}]$. By definition, this is the vector space over K with basis given by elements of $\tilde{M}$ and multiplication induced from addition in M_T. The lattice $\tilde{M}$ is isomorphic to the group $\mathbb{Z}^d$ for some $d \in \mathbb{N}$. We have $\tilde{T} = \operatorname{Spec} K[\tilde{M}] = (K^*)^d$. The associated toric variety in K^N is the Zariski closure of $\tilde{T}$. As the Zariski closure does not change the dimension, the toric variety is also of dimension d. □

We see that we may equivalently define toric varieties as closures of a subtorus of the torus $(K^*)^N \subset K^N$. Further, in analogy to the proof of Proposition 4 we see that the toric variety equals $\operatorname{Spec} K[S]$, where S is the monoid in M_T generated by the distinguished characters, i.e. the smallest set containing 0, the chosen characters and closed under addition.

Example 5.

1. *The cuspidal curve defined by the equation $x^3 - y^2$ equals $\operatorname{Spec} K[z^2, z^3]$. Here, the associated monoid equals $\{0, 2, 3, 4, \dots\}$.*

2. *The affine line is the closure of the image of the map*

$$K^* \ni x \to x \in K.$$

Here the character lattice is $M = \mathbb{Z}$, the distinguished character corresponds to $1 \in M$ and the monoid equals $\{0, 1, 2, \dots\}$.

There is a fundamental difference between the example of the cuspidal curve and affine line. When we look at the monoid for the cuspidal curve, there is a 'hole' in it: the character corresponding to 1.

Definition 6. *A submonoid S in a lattice M is called* saturated *if and only if for any $x \in M$ and $k \in \mathbb{Z}_+$ the following implication holds:*

$$kx \in S \Rightarrow x \in S.$$

Affine toric varieties for which S is saturated (in the lattice $\tilde{M}$ that it generates) are called *normal*. For the algebraic definition of normal varieties we refer to [1, Chapter 5]. Nonnormal varieties are always singular and for curves the two notions coincide. Hence, Example 5 shows one nonnormal (equivalently singular) curve - as seen in Figure 1 - and one normal (equivalently smooth) curve.

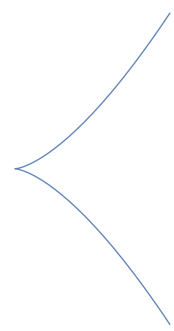


Figure 1: The cuspidal curve

Further, we can find the generators of the ideal of the variety X from the characters that define it. In general, given a variety defined as a Zariski closure of the image of a map, finding the defining equations is a hard problem, known as *implicitization*. We discussed this in Lecture 3. The implicitization problem greatly simplifies when the variety is toric. Recall that a *binomial* is a polynomial that is a difference of two monomials.

Lemma 7. *Let X be a toric variety defined by characters $\mathbf{a}_1, \dots, \mathbf{a}_N \in \mathbb{Z}^n$. Then:*

1. *any relation $\sum_i b_i \mathbf{a}_i = \sum_j c_j \mathbf{a}_j$, with positive integral coefficients $b_i, c_j \in \mathbb{Z}_+$ provides a binomial $\prod y_i^{b_i} - \prod y_j^{c_j}$ in the prime ideal I_X of X;*
2. *every binomial in the ideal I_X is of the form described in point 1;*
3. *the ideal I_X is generated by binomials.*

Sketch of the proof: Properties 1 and 2 follow from the fact that a polynomial vanishes on the toric variety X if and only if we obtain zero after substituting y_i by $\mathbf{x}^{\mathbf{a}_i}$. However, such a substitution turns monomials (in variables y) to monomials (in variables x). The fact that the monomials in $\mathbf{x}$ cancel is precisely encoded by the integral relations in point 1. Property 3 follows similarly, by induction on the support of a polynomial in the ideal of X. $\square$

Example 8. Let $n = 3$, $N = 7$ and take $\mathbf{a}_1, \dots, \mathbf{a}_7$ to be the column vectors of the matrix

$$A \quad = \quad \begin{pmatrix} 2 & 2 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 2 & 2 & 1 \\ 0 & 1 & 2 & 2 & 1 & 0 & 1 \end{pmatrix}$$

The associated toric variety X is a threefold in K^7. Its ideal I_X is the binomial ideal

$$\langle y_1y_3 - y_2y_7, y_1y_4 - y_7^2, y_1y_5 - y_6y_7, y_2y_4 - y_3y_7, y_2y_5 - y_7^2, y_2y_6 - y_1y_7, y_3y_5 - y_4y_7, y_3y_6 - y_7^2, y_4y_6 - y_5y_7 \rangle$$

Since these binomials are homogeneous, the variety is a cone in K^7. It can thus also be regarded as projective toric variety in in $\mathbb{P}^6$. That variety is a smooth surface of degree six.

Theorem 9. *A prime ideal generated by binomials defines a toric variety.*

Proof. This follows from the fact that binomials may be translated to Laurent monomials on $(K^*)^N$, where they have to define a torus. For details see [2, Proposition 1.1.11]. $\square$

Definition 10. *A* convex polyhedral cone *in a real vector space V is a subset of elements of the form $\lambda_1 v_1 + \cdots + \lambda_k v_k$ for some fixed integer k, $v_1, \ldots, v_k \in V$ and $\lambda_1, \ldots, \lambda_k \in \mathbb{R}_{\geq 0}$. If we identify V with $\mathbb{R}^d$ we call a convex polyhedral cone* rational *if all the v_i's can be chosen as rational vectors.*

We will refer to rational convex polyhedral cones simply as cones.

In Exercise 4 the reader is asked to show that a finitely generated, saturated monoid in a lattice $\mathbb{Z}^n$ is the same as the set of integral points in a cone in the corresponding $\mathbb{R}^n$.

Definition 11. *A* face *of a cone $C \subset V$ is a subset $F \subset C$ defined by:*

$$F = \{c \in C : f(c) = 0\},$$

where f is such a linear function $f \in V^*$ that for any $p \in C$ we have $f(c) \geq 0$. If $\dim C = \dim V = \dim F + 1$, then f is uniquely determined, up to scalar. In such a case F is called a facet and the hyperplane defined by f is called a supporting hyperplane of C.

We point out that if $f = 0$ we obtain $F = C$. Further, any face of a cone is also a cone.

Example 12. *Consider C equal to the positive quadrant in $\mathbb{R}^2$. It has one two dimensional face - the whole cone, two one dimensional facets and one zero dimensional face $\{0\} \subset C$.*

By Proposition 4 a toric variety X is a closure of a torus $T \subset X \subset \mathbb{C}^N$, where

$$T = \{t \in X : \text{all coordinates of } t \text{ are nonzero}\}.$$

As T is a group that acts both on itself and $\mathbb{C}^n$, it must also act on $X = \overline{T}$. Our next aim is to provide a combinatorial and geometric description of the orbits of this action.

Let us make the following assumptions. The toric variety X is defined by characters A that generate a saturated monoid. Let C be the corresponding cone and $T \subset X$ the torus dense in X.

Theorem 13. *Using the above notation, the T-orbits in X are in bijection with the faces of the cone C. The orbit corresponding to a face F consists exactly of those points $x \in X$ that have a nonzero coordinate corresponding to a character $a \in A$ if and only if $a \in F$.*

Further, the closure of the orbit corresponding to F is the toric variety $\operatorname{Spec} \mathbb{C}[F \cap A]$*, where formally $F \cap A$ represents the monoid generated by $F \cap A$. In particular, the dimension of F equals the dimension of the orbit. Moreover, an orbit corresponding to face F_1 belongs to the closure of the orbit corresponding to face F_2 if and only if $F_1 \subset F_2$.*

Example 14. *Consider the toric variety associated to characters* $(1,0,0), (1,0,1), (1,1,0), (1,1,1)$. *It is the affine cone over the quadric $xt - yz$. The four two dimensional facets of the cone correspond to four two dimensional tori contained in it. For example the face generated by* $(1,0,0), (1,0,1)$ *corresponds to the set of points of the type* $(*, *, 0, 0)$*, where $*$ are nonzero.*

The four one dimensional faces correspond to coordinate axis (minus $\{0\}$*). Note that the intersection of faces in the cone and intersection of the corresponding closures of orbits agree.*

As we can see one can 'read off' the geometry of X from the cone C representing it.

Let us now pass to projective toric varieties. We note that given a set A of N monomials, we obtain the same projective variety if we multiply every monomial by a new variable x_0. In many aspects such a description is better, as it also parameterizes the affine cone over the projective variety. Thus, instead of working in lattice $\mathbb{Z}^n$, we will be working in the lattice $\mathbb{Z}^{n+1} = \mathbb{Z} \times \mathbb{Z}^n$ assuming that the defining set of characters/monomials A is contained in $\{1\} \times \mathbb{Z}^n$.

Definition 15. *A* polytope *in a vector space V is a convex hull of a finite set of vectors. A polytope is called* a lattice polytope *if it is a convex hull of points of a lattice $M \subset V$.*

If we want A to generate a saturated monoid a necessary condition is that $A = \operatorname{conv}(A) \cap \tilde{M}$ where $\tilde{M}$ is the lattice generated by A. In other words, A is the set of integral points of an integral polytope. However, in general this is not enough.

Definition 16. *A lattice polytope P (in a lattice M) is called* normal *if and only if for any integer k and any point $p \in kP \cap M$ there exist $p_1, \dots, p_k \in P \cap M$ such that $p = \sum_{i=1}^{k} p_i$.*

In Exercise 5 the reader can find an example of a nonnormal polytope. In Exercise 6 the reader is asked to prove that a lattice polytope P is normal if and only if $(\{1\} \times P) \cap M$ generates a saturated monoid. The orbit cone correspondence from Theorem 13 in a trivial way generalizes to projective toric varieties and polytopes. As we will see below there is another way to explain why the geometry of the polytope coincides with the geometry of the toric variety. Our aim is to define a map, called *the moment map*, that takes the toric variety X onto the associated polytope P.

Let $X \subset \mathbb{P}(\mathbb{C}^N)$ be a toric variety defined by a set of characters $A \subset \mathbb{Z}^n$. In particular, $|A| = N$ and the coordinates of $\mathbb{C}^N$ correspond to elements of A. For a point $y \in \mathbb{C}^N$ and $a \in A$ we denote by $a(y) \in \mathbb{C}$ the coordinate of y corresponding to a. In other words $y = (y_a)_{a \in A}$.

Definition 17. *The* algebraic moment map $\mu_A : X \to \mathbb{R}^n$ *is defined by:*

$$\mu_A(x) = \frac{\sum_{a \in A} |a(x)| a}{\sum_{a \in A} |a(x)|}.$$

Here, as $x \in \mathbb{P}(\mathbb{C}^N)$, the value $a(x)$ is defined only up to a scalar. However, as $\mu_A(x)$ is a fraction it does not depend on the choice of the scalar.

The numerator in the above definition is a nonnegative combination of integral points A defining X. The denominator assures that $\mu_A(x) \in \operatorname{conv}(A)$. Consider a torus fixed point $x_0 \in X$. By Theorem 13 it must have all coordinates equal to zero, apart from one, corresponding to a vertex $a_0 \in A$ of $\operatorname{conv} A$. In particular, $\mu_A(x_0) = a_0$. Our aim is to present a vast generalization of the above fact, which explains why the geometry of X is related to the geometry of $\operatorname{conv} A$. We start with a definition.

Definition 18. *For a set of characters A we define* the nonnegative *(resp.* positive*)* part *of a related toric variety $X \subset \mathbb{C}^{|A|}$ as $X_{\geq 0} := X \cap (\mathbb{R}_{\geq 0})^{|A|}$ (resp. $X_{>0} := X \cap (\mathbb{R}_{>0})^{|A|}$).*

Let $T \to \mathbb{C}^{|A|}$ be the map defining X. By Proposition 4 T maps surjectively to a torus $\tilde{T}$ that can be identified with those points of X that have all coordinates nonzero. Further, $T_{>0}$ maps surjectively to $X_{>0} = \tilde{T}_{>0}$. This is especially useful in statistics, where our defining map can be interpreted as a statistical model and coordinates as probabilities - cf. Lecture 2 and many more examples in [5, Chapter 5, Chapter 14]. More generally, we have a map $X \to X_{\geq 0}$ given by $r : (x_1, \ldots, x_{|A|}) \to (|x_1|, \ldots, |x_{|A|}|)$. Note that $\tilde{T} = (\mathbb{C}^*)^d$ contains a *topological torus* S^d, by taking points with coordinates of module one. Further, S^d is a subgroup of $\tilde{T}$ that acts transitivly on each fiber of r. Thus r may be regarded as a quotient map $X \to X/(S^d) = X_{\geq 0}$. Hence $r : X \to X_{\geq 0}$ has fibers that are real tori, with dimension equal to the dimension of the orbit of $\tilde{T}$ they belong to. A formal statement and a proof can be found for example in [2, Proposition 12.2.3]. We have now related the geometry of X with the geometry of $X_{\geq 0}$. We note that we can make the same definitions when X is projective, where a point is positive if and only if it has a positive representative.

Theorem 19. *Let A be the set of lattice points in a lattice polytope $P \subset \mathbb{R}^n$ and let X be the associated toric variety. The moment map: $\mu_A : X_{\geq 0} \to \mathbb{R}^n$ is a homeomorphism onto P.*

The proof, along with many more interesting facts, can be found in [4, Theorem 8.4].

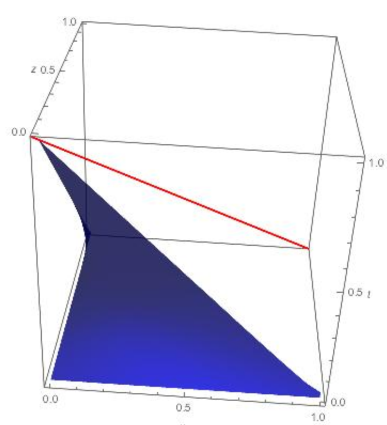
Example 20. *Let us continue the statistically motivated Example 3 from Lecture 2, in the case $n = 2$. We obtain the Segre embedding:*

$$\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3,$$

where our toric variety is represented as a unit square and is defined as a quadric $xt - yz$. If we consider the affine set $\mathbb{R}^3 \subset \mathbb{P}^3$ defined by $x + y + z + t = 1$, then the moment map, restricted to $X_{\geq 0}$, becomes simply a linear projection $\mu : \mathbb{R}^3 \to \mathbb{R}^2$:

$$[x : y : z : t] \to x(0,0) + y(1,0) + z(0,1) + t(1,1) = (y + t, z + t).$$

Below we present the picture of $X_{\geq 0}$ in coordinates y, z, t. The red line shows the direction of the projection of the moment map:



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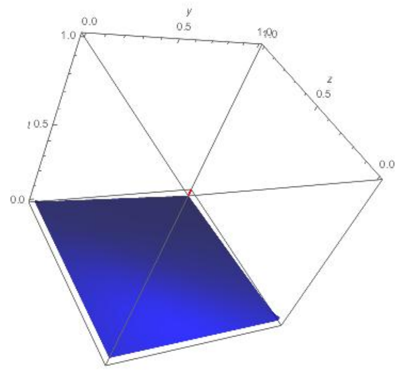
If we rotate the picture so that the red line becomes (nearly) a point, we see that the projection is indeed a square:



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Example 21. *Suppose we throw two (possibly biased) coins* 1024 *times (each time two coins at once) and observe:*

- 128 *times both heads,*
- 128 *times the first coin gives heads, the second tails,*
- 384 *times both tails,*
- 384 *times the first coin gives tails, the second heads.*

This can be translated to:

- $1/8$ *times* $(0, 0)$,
- $1/8$ *times* $(0, 1)$,
- $3/8$ *times* $(1, 1)$,
- $3/8$ *times* $(1, 0)$.

The data can be represented as a point in the square:

$$p := 1/8(0, 0) + 1/8(0, 1) + 3/8(1, 1) + 3/8(1, 0) = (3/4, 1/2).$$

The unique preimage of the point p *by the moment map, as in Example 20, has coordinates* $(y, z, t) = (1/8, 3/8, 3/8)$ *and translates to* $(x, y, z, t) = (1/8, 1/8, 3/8, 3/8)$*. Looking at the associated toric map:*

$$(a, b) \times (c, d) \to (ac, ad, bc, bd),$$

under the assumption $a + b = c + d = 1$*, we obtain* $a = 1/4$*,* $b = 3/4$*,* $c = d = 1/2$*. This is the correct estimate of the probability distribution for the coins: the first coin is biased (with probability of tails* $3/4$ *and heads* $1/4$*) and the second one is fair.*

The previous example is a very special case of a general theorem in algebraic statistics. Toric varieties correspond to discrete statistical models. The inverse image of any point p in the polytope, by the moment map, is known as the *Birch point* or *Maximum Likelihood Estimator*. For further reading on toric models see [3, Section 1.2.2].

For a more theoretical application of toric varieties notice that:

- $\mathbb{P}^n$ has a representation as a(n n-dimensional) simplex,
- the product $\mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_n}$ has a representation as a product of simplices,
- more generally a product of projective toric varieties represented by polytopes is a projective toric variety represented by the product of the polytopes.

As an application of toric geometry we see that a product of projective spaces is a projective variety, naturally embedded in another projective space. Hence, if we are given (possibly nontoric) projective algebraic varieties $X \subset \mathbb{P}^a$, $Y \subset \mathbb{P}^b$, we see that the product $X \times Y \subset \mathbb{P}^a \times \mathbb{P}^b$ is also a projective variety. Notice however, that the natural embedding is *not* in $\mathbb{P}^{a+b}$, as one could expect from the affine case. In fact $\mathbb{P}^a \times \mathbb{P}^b \subset \mathbb{P}^{ab+a+b}$.

Exercises

1. Prove that every character $(\mathbb{C}^*)^n \to \mathbb{C}^*$ is given by $x \to x^a$ for some $a \in \mathbb{Z}^n$.
2. Prove that every polynomial in the ideal of an affine toric variety is a linear combination of binomials - cf. point 3. in Lemma 7.
3. Describe the ideals of the Segre product $\mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_n}$ and of the (aribtrary) Veronese reembeding of $\mathbb{P}^a$.
4. Prove that for a fixed lattice $\mathbb{Z}^d \subset \mathbb{R}^d$ there is a natural bijection between (convex, rational, polyhedral) cones (in $\mathbb{R}^d$) and finitely generated saturated monoids (in $\mathbb{Z}^d$).
5. Prove that the convex hull of points $(0,0,0), (0,1,0), (1,0,0), (1,1,2), (1,1,3)$ is not a normal polytope (in the lattice $\mathbb{Z}^3$).
6. Prove that a lattice polytope P is normal if and only if $(\{1\} \times P) \cap M$ generates a saturated monoid.
7. (a) An f-vector $(f_0, \ldots, f_m) \in \mathbb{Z}^{m+1}$ for an m-dimensional polytope P is a sequence of positive integers, where f_i equals the number of i dimensional faces of P. Compute the number of points of a projective toric variety X defined by lattice points of a polytope P over a finite field, in terms of the f-vector.

   (b) * Assuming X is smooth, use Weil conjectures (now proved due to work of Grothendieck and Deligne), to give a formula for Betti numbers of X, again in terms of the f-vector.
8. Prove that for any lattice polytope P of dimension d, the polytope $(d-1)P$ is normal.

9. Prove a theorem of Mumford, in the case of toric varieties; Let X be a projective toric variety. For r large enough the r-th Veronese reembeding $v_r(X)$ of X is defined by quadratic equations.

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