# general relativity 2015: aim of the course

(1) general relativity as a geometric theory of gravity

(2) structure of the laws of Nature

(3) understand electromagnetism better

(4) be confused by Newtonian gravity
+ develop a covariant understanding

(5) learn about the 4 great applications of GR

| | weak fields | strong fields |
|---|---|---|
| static | Newtonian gravity + gravitomagnetism | Schwartschild-solution |
| time-evolving | gravitational waves | Friedmann-Lemaître cosmologies |

(6) understand current topics
- gravitational waves
- dark energy and cosm. constant
- frame dragging

(7) experiments to test relativity

# general relativity 2015

- overview

(1) Newtonian and Lagrangian mechanics

(2) Lorentz-transformations

(3) relativistic mechanics

(4) electrodynamics: a covariant field theory

(5) differential geometry 1: geodesic motion

(6) differential geometry 2: field equation

(7) differential geometry 3: variational principles

(8) Schwarzschild-solution

(9) causal structure of black holes

(10) degenerate stars

(11) Friedmann-Lemaître-universes + dark energy

(12) gravitational waves

(13) gravitomagnetism

(14) summary

(15) open questions

# general relativity 2015 : Newtonian mechanics

- Newton's axioms for motion in inertial frames

  (1) free particles move along straight lines at const. velocity

  geodesic motion ?!

  (2) acceleration is given by specific force, $\ddot{\vec{x}} = \vec{F}/m$

  not true for Lorentz-forces ?!

  (3) actio = reactio

  need absolute time, time is a universal parameter
  (Newton thought of space as absolute as well!)

- Galilei-transformation : choosing another frame

  2 frames S, S' in relative uniform motion
  introduce cartesian coordinate systems

  for simplicity: motion ‖ x-axis:

  $\rightarrow x_1' = x_1 - vt \ , x_2' = x_2, \ x_3' = x^3$ and $t' = t$

  differentiate $\vec{u} = \frac{d}{dt}\vec{x}$ :

  $u_1' = u_1 - v \ , \ u_2' = u_2, \ u_3' = u_3$

  differentiate again $\vec{a} = \frac{d}{dt}\vec{u}$ :

  $a_1' = a_1 \ , \ a_2' = a_2 \ , \ a_3' = a_3$

  $\rightarrow$ identical accelerations in both frames

- Newtonian relativity

  (1) free particles in S are free in S'

  (2) both frames measure the same acceleration $\rightarrow$

  - Newton's axioms hold in all inertial frames
  - m must be the same in all frames
  - forces must be identical

# general relativity 2015 : classical mechanics

- lagrangian mechanics :

base classical mechanics on an extremal principle

Lagrange function $L = L(x, \dot{x}, t)$

action integral $S = \int_{t_i}^{t_f} dt\, L$

Hamilton's principle $S$ is extremised, $\delta S = 0$

$$\delta S = \delta \int_{t_i}^{t_f} dt\, L = \int_{t_i}^{t_f} dt \left( \frac{\partial L}{\partial x} \delta x + \frac{\partial L}{\partial \dot{x}} \delta \dot{x} \right)$$

$$\delta \dot{x} = \delta \frac{d}{dt} x = \frac{d}{dt} \delta x$$

$$= \int_{t_i}^{t_f} dt \underbrace{\left( \frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} \right)}_{\text{Euler-Lagrange-equations}} \delta x = 0$$

choice of $L = \frac{m}{2} \dot{x}^2 - \phi$ now leads to

Newton's equation of motion: $m\ddot{x} = -\frac{d}{dx} \phi$

Lagrange-formalism automatically generates 2nd order equations of motion.

minimum or maximum of $S$? depends!

- great advantage: find conserved quantities

careful: two types → Energy conservation
→ conservation of canonical momenta

(*) homogeneity of time (remember, time is a parameter, not a coordinate!)

$L = L(x, \dot{x})$, no explicit time dependence

no $\frac{\partial L}{\partial t}$-term

$$\frac{dL}{dt} = \frac{\partial L}{\partial x} \frac{dx}{dt} + \frac{\partial L}{\partial \dot{x}} \frac{d\dot{x}}{dt}$$

$$= \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} \cdot \dot{x} + \frac{\partial L}{\partial \dot{x}} \ddot{x}$$

substitute EL-equation for first term

$$= \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \dot{x} \right)$$

$$\rightarrow \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \dot{x} - L \right) = 0 \rightarrow \epsilon = \frac{\partial L}{\partial \dot{x}} \dot{x} - L \text{ conserved!}$$

general relativity 2015 : classical mechanics

(*) homogeneity of time $\rightarrow$ conservation of energy

· additivity of energy implied by additivity of L.

· systems with $L(x,\dot{x}) \rightarrow \frac{\partial L}{\partial t} = 0$ are called conservative

(*) homogeneity of space (position is a coordinate!)

$L = L(\dot{x})$, but not a function of $x$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x} = 0$$

$p = \frac{\partial L}{\partial \dot{x}}$ is conserved

$p$ ~ conjugate momentum, $x$ ~ cyclic variable

· additivity follows from L + linearity of $\frac{\partial}{\partial \dot{x}}(\dots)$

with this result, let's write the EL-equations like

$$\frac{d}{dt} p = F \quad \text{with} \quad p = \frac{\partial L}{\partial \dot{x}} \quad \text{and} \quad F = \frac{\partial L}{\partial x}$$

these are already two important invariants. what about

(*) adding a constant? $L + c$ gives the same equation of motion (bit boring :-) )
rewrite as $L + \frac{d}{dt}(ct)$ $\rightarrow$ scale of potential

(*) adding a total time derivative?

$L + \frac{d}{dt}M$ gives the <u>same</u> equation of motion if $M$ depends only on $x$ and $t$, but not on $\dot{x}$

use Hamilton's principle

$$\delta \int_{t_i}^{t_f} dt \; L + \frac{dM}{dt} = \delta \int_{t_i}^{t_f} dt \; L + \int dt \left(\frac{\partial}{\partial x} - \frac{d}{dt}\frac{\partial}{\partial \dot{x}}\right)\frac{dM}{dt}$$

$$= \delta \int_{t_i}^{t_f} dt \; L + \int dt \left[\frac{d}{dt}\frac{\partial M}{\partial x} - \frac{d}{dt}\frac{\partial}{\partial \dot{x}}\left(\frac{\partial M}{\partial t} + \dot{x}\frac{\partial M}{\partial x} + \underbrace{\ddot{x}\frac{\partial M}{\partial \dot{x}}}_{=0}\right)\right]$$

general relativity 2015 : classical mechanics

(*) adding a total time derivative

$$\ldots = \delta \int_{t_i}^{t_f} dt\, L + \int dt \left[ \frac{d}{dt}\frac{\partial M}{\partial x} - \frac{d}{dt}\frac{\partial}{\partial \dot{x}}\Big(\underbrace{\frac{\partial M}{\partial t}}_{= \frac{\partial}{\partial t}\frac{\partial M}{\partial \dot{x}} = 0} + \dot{x}\frac{\partial M}{\partial x}\Big)\right],$$

use $\frac{\partial}{\partial \dot{x}}\left(\dot{x}\frac{\partial M}{\partial x}\right) = \frac{\partial M}{\partial x} + \frac{\partial}{\partial x}\frac{\partial M}{\partial \dot{x}} - \frac{\partial M}{\partial x}$

$$\ldots = \delta \int_{t_i}^{t_f} dt\, L + \int dt \underbrace{\left(\frac{d}{dt}\frac{\partial M}{\partial x} - \frac{d}{dt}\frac{\partial M}{\partial x}\right)}_{=0} = \delta \int_{t_i} dt\, L$$

$$\rightarrow S + \int_{t_i}^{t_f} dt \frac{d}{dt} M = S + \underbrace{M(t_f) - M(t_i)}_{\text{drops out in variation if } M(x,t)}$$

(*) when is that helpful? Galilean relativity!

$x' = x + vt, \quad \dot{x}' = \dot{x} + v$

$$L' = \frac{m}{2}\dot{x}'^2 = \frac{m}{2}(\dot{x}^2 + 2\dot{x}v + v^2)$$
$$= L + m\dot{x}v + \frac{m}{2}v^2$$
$$= L + \frac{d}{dt}\underbrace{(mx\cdot v + \frac{m}{2}v^2\cdot t)}_{\text{identify with } M(x,t)\text{: does not depend on } \dot{x}\text{, remember: } v = \text{const.}}$$

'kinetic' part of L

invariance under addition of a total time derivative:

Lagrange principle gives identical equations of motion in all inertial frames!

check: accelerated motion, $x' = x + \frac{1}{2}at^2$, $\dot{x}' = \dot{x} + at$

→ should not leave the Lagrange function invariant

$$L' = \frac{m}{2}\dot{x}'^2 = L + \frac{m}{2}(2\dot{x}at + a^2t^2)$$
$$= L + \frac{m}{2}\cdot\frac{d}{dt}\left(\frac{a^2}{3}t^3\right) + m\frac{d}{dt}\int dt\, \dot{x}at \quad \text{(neat!)}$$
$$= L - m\frac{d}{dt}\int dt\, xa = L - max \quad \text{becomes linear potential!}$$

→ of course this acceleration needs to show up in the equation of motion!
"equivalence" of $a$ and $\phi$