general relativity 2015 : classical mechanics

(*) where's the kinetic energy now?

<u>Newton</u>: use actio = reactio, and perform work for accelerating a body, and integrate up the work

$$T = \int dx \cdot F = m \int dx \cdot \ddot{x} = m \int \underbrace{dt\, \dot{x}}_{dx = dt \cdot \frac{dx}{dt}} \ddot{x}$$

$$= m \int dt \, \frac{1}{2} \frac{d}{dt} \dot{x}^2 = \frac{m}{2} \dot{x}^2$$

because reactio is always in the opposite direction as actio, you need to perform work for accelerating

<u>Lagrange</u>: be very careful → L contains $\frac{m}{2}\dot{x}^2$

because it's the square of a derivative, not because it's the kinetic energy (we see that later!)

in fact, inertial motion and accelerated motion is already described by L <u>without</u> any energy interpretation

L needs to be proportional to a velocity^2: Look at Galilei-transforms again

$$x' = x + vt \longrightarrow \dot{x}' = \dot{x} + v, \quad \text{assume } v \ll \dot{x}$$

$$L(\dot{x}'^2) = L(\dot{x}^2 + 2\dot{x}v + v^2)$$

$$\simeq L(\dot{x}^2) + \underline{\frac{\partial L}{\partial \dot{x}^2} \cdot 2\dot{x}v}$$

only a total time derivative <u>if</u> $L \propto \dot{x}^2 \rightarrow \frac{\partial L}{\partial \dot{x}^2} = \text{const.}$

then $= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^2} 2xv\right)$

→ $L \propto \dot{x}^2$ from invariance + linearity of the equation of motion

(*) can I use this? yes! mechanical similarity

Hamilton's principle is linear, so the Lagrange formalism is invariant under a transformation of the type $L \rightarrow q \cdot L$

this invariance looks like nothing at all, but it is immensely helpful: mechanical similarity

general relativity 2015 : classical mechanics

(*) mechanical similarity: classical mechanics works in exactly the same way on all scales

scaling $x \to \alpha x$, $t \to \beta t$

implies $T \to \left(\frac{\alpha}{\beta}\right)^2 T$ and $\phi \to \alpha^n \phi$ for $\phi \propto x^n$

$\to$ $L = T - \phi \to \left(\frac{\alpha^2}{\beta^2}\right) T - \alpha^n \phi$

can only be done if $\frac{\alpha^2}{\beta^2} \sim \alpha^n$

$\to$ $t^2 \sim x^{2-n}$

$n = 2$ $t \sim \text{const}$ isochronism of the harmonic oscillator

$n = 1$ $t^2 \sim x$ $x = \frac{1}{2}at^2$ uniform acceleration potential $\phi = ax$

$n = 0$ $t^2 \sim x^2$ $v = \frac{x}{t} = \text{const}$ inertial motion no energy conservation needed

$n = -1$ $t^2 \sim x^3$ Kepler's 3rd law

(*) look at Newton's gravitational law

$$g = -G \frac{M}{r^2}$$

M: mass of field generating body

$-$: attractive

r^2: inverse square

$$G \sim 10^{-11} \frac{m^3}{kg\, s^2}$$

look! Kepler's law is built in!

similarity implies, that there are no fundamentally 'new' effects if gravity could become strong

$\to$ orbits of planets are scaled versions of each other general relativity : classical mechanics

(*) Newton's gravitational law

origin of $g \propto \frac{1}{r^2}$: geometry

use Gauß-law for describing the gravitational field

$\operatorname{div} \vec{g} = 4\pi G \rho$ in analogy to $\operatorname{div} \vec{E} = 4\pi \rho$

$$4\pi G \cdot \underbrace{\int dV \rho}_{=M} = \int_V dV \operatorname{div} \vec{g} = \int_{\partial V} d\vec{A} \cdot \vec{g} = 4\pi r^2 g$$

$\rightarrow \quad g = \frac{GM}{r^2}$, enforced by growth of the area with r^2

but: Mercury behaves differently

- orbits are not closed
- tiny violation of $t^2 \propto r^3$
- perihelion precession: 43'' / 100 years

→ looks as if the grav. field is a tiny bit stronger close to the sun.

→ today we think that the geometry is no longer Euclidean

→ sneak preview: units of $\frac{G}{c^2}$ are $\frac{m}{kg}$: change in geometry for every unit of mass!

(*) structure of modern physics

k_B statistics + thermodynamics

correlations ↑ ↓ equilibria

$\hbar$ matter + fields

connection Γ, metric g ↑ ↓ energy-momentum $T_{\mu\nu}$

G, Λ gravity + geometry

light cones $ds^2 = 0$, metric η ↑ ↓ geodesics

c causality

general relativity 2015: change of reference frames

o linearity of transforms between inertial frames

2 frames $S:(\vec{x},t)$ and $S':(\vec{x}',t')$ ~ need to be related by a linear transform

- a clock moves freely (no acceleration) through S, trajectory $\vec{x}(t)$ $\rightarrow$ velocity $\frac{d}{dt}\vec{x}$ = const.

- τ is the time indicated by the clock "proper time"
- homogeneity $\rightarrow$ constant $\frac{dt}{d\tau}$, or equal measurements of time intervals at each point.

$\frac{dx_\mu}{d\tau}$ = const , with $x_\mu = (\vec{x},t)$ ~ 4-vector

$\frac{d^2x_\mu}{d\tau^2} = 0$, and the same argument holds in S': $\frac{d^2x'_\mu}{d\tau^2} = 0$

expand:

if this term $\neq 0$, the transform becomes quadratic

if this term $= 0$ $\rightarrow$ the transformation is linear

$$\frac{dx'_\mu}{d\tau} = \frac{\partial x'_\mu}{\partial x_\nu}\frac{dx_\nu}{d\tau}$$

$$\frac{d^2x'_\mu}{d\tau^2} = \frac{\partial x'_\mu}{\partial x_\nu}\frac{d^2x_\nu}{d\tau^2} + \frac{\partial^2 x'_\mu}{\partial x_\nu \partial x_\sigma}\frac{dx_\nu}{d\tau}\frac{dx_\sigma}{d\tau}$$

introduce Einstein summation convention: automatic summation over double indices.

(actually, the term $\partial^2 x_\mu / \partial x_\nu \partial x_\sigma$ will become important)

$\rightarrow$ transformations between frames can only obey the homogeneity of space if they're linear.

o linear transforms $\rightarrow$ matrix relation

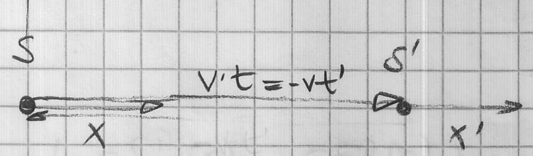
$$x'_\mu = \sum_\nu A_{\mu\nu} \cdot x'_\nu + B_\mu$$

(1) it's always possible to align the coordinate frames

(2) and to make the origins coincide at $t = t' = 0$.

general relativity 2015: change of reference frames

o construction of the most general transformation



event → (x,t) in S
event → (x',t') in S'
*

start with a linear transform

- $x' = ax + bt$, but $x = vt$ must imply $x' = 0$

$x' = 0 = avt + bt = (av+b)t$, $b = -av$

$\rightarrow x' = a \cdot (x - vt)$

- reverse roles: $x = ax' + bt'$, but $x' = -vt'$ must imply $x = 0$

$x = 0 = -avt + bt' = (-av+b)t'$, $b = +av$

$\rightarrow x = a(x' + vt')$

① Galilei: universal time $t = t'$

$x' = a(x - vt)$

$x = a(x' + vt)$

only consistent if $a = 1$

Galilei - transform

② Lorentz-transform

every observer measures the same speed of light

$x = ct$ and $x' = ct'$

$\rightarrow ct' = a \cdot (ct - vt)$

$ct = a(ct' + vt')$

$\rightarrow a = \frac{1}{\sqrt{1 - (\frac{v}{c})^2}}$

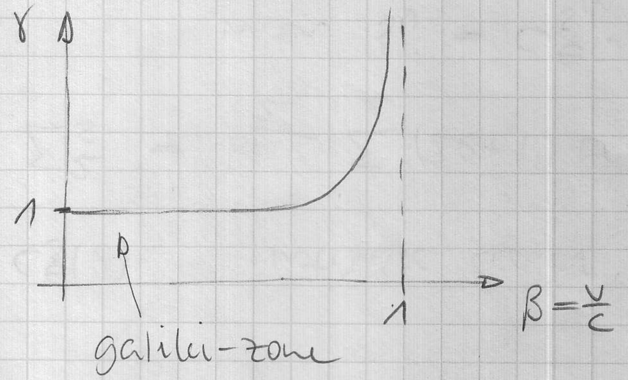
$\gamma = \frac{1}{\sqrt{1-\beta^2}}$ Lorentz-factor , $\beta = \frac{v}{c}$ velocity

(derivation: $c^2 \cancel{t}t' = a^2 \cancel{tt'}(c+v)(c-v) = a^2(c^2 - v^2)$

$\rightarrow a = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$, from multiplication of the two transformation laws)

general relativity 2015: change of reference frames

- Lorentz-factor



2 types of transforms

$t = t' \longrightarrow \gamma = 1$ Galilei

$c = c' \longrightarrow \gamma = \frac{1}{\sqrt{1-\beta^2}}$ Lorentz

t does not "take part" in the transform → parameter for motion

t now becomes a coordinate transforms along with x

reduction for $\beta \ll 1$: $x = x' + vt$, $x' = x - vt$,

$$\gamma \simeq 1 + \frac{\partial^2 \gamma}{\partial \beta^2}\Big|_{\beta=0} \cdot \frac{\beta^2}{2} = 1 \left(+ \frac{\beta^2}{2}\right)$$

unphysical for $\beta \geqslant 1$: transformed coordinates become imaginary

- standard form of the Lorentz-transforms

$$\begin{cases} x' = \gamma(x - vt) \\ t' = \gamma\left(t - \frac{vx}{c^2}\right) \end{cases} \longrightarrow \begin{aligned} x' &= \gamma(x - \beta ct) \\ ct' &= \gamma(ct - \beta x) \end{aligned}$$

→ by eliminating x'

<u>common</u> transformation of x and t into x' and t'

→ source of almost all 'paradoxes' in special relativity

c is the speed of light, $c \sim 3 \cdot 10^8 \frac{m}{s}$

but actually it's a fundamental parameter needed for establishing a geometric description of causality