1. a) isotrop, homogen

b) $$\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda c^2}{3} - \frac{kc^2}{a^2}$$

set $$\boxed{\rho' = \rho + \frac{\Lambda c^2}{8\pi G}}$$

$$\frac{\ddot a}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) + \frac{\Lambda c^2}{3}$$

$$= -\frac{4\pi G}{3}\left(\rho' - \frac{\Lambda c^2}{8\pi G} + \frac{3p}{c^2}\right) + \frac{\Lambda c^2}{3}$$

$$= -\frac{4\pi G}{3}\left(\rho' + \frac{3p'}{c^2}\right)$$

$$\Rightarrow \boxed{p' = p - \frac{\Lambda c^2}{8\pi G}}$$

$$\Rightarrow \left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\rho' - \frac{kc^2}{a^2}$$

$$\frac{\ddot a}{a} = -\frac{4\pi G}{3}\left(\rho' + \frac{3p'}{c^2}\right) = -\frac{4\pi G}{3}\rho'(1+3w')$$

c) $$\frac{d}{dt}\left(\frac{\dot a}{a}\right)^2 = 2\frac{\dot a}{a}\cdot\left[\frac{\ddot a}{a} - \frac{\dot a^2}{a^2}\right]$$

$$= 2\frac{\dot a}{a}\left[-\frac{4\pi G}{3}\rho(1+3w) - \frac{8\pi G}{3}\rho + \frac{kc^2}{a^2}\right]$$

$$\frac{d}{dt}\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\dot\rho + \frac{2kc^2\dot a}{a^3}$$

$$\Rightarrow \frac{8\pi G}{3}\dot\rho = -\frac{\dot a}{a}\frac{8\pi G}{3}\rho(3+3w)$$

$$\boxed{\dot\rho = \rho\left(-\frac{3\dot a}{a}(1+w)\right)}$$

d) $w = \text{const.}$

$$\frac{d\rho}{\rho} = -\frac{3\dot a}{a}(1+w)\,dt \qquad \frac{da}{dt} = \dot a \Rightarrow dt = \frac{da}{\dot a}$$

$$= -3(1+w)\frac{da}{a}$$

$$\ln\rho\Big|_{\rho_0}^{\rho} = -3(1+w)\ln a\Big|_{a_0}^{a} \qquad a_0 = 1$$

$$\rho = \rho_0\, a^{-3(1+w)}$$

e) "ordinary" $\rho = \rho_0 a^{-3}$

photons $\rho = \rho_0 a^{-4}$

2a) $H^2 = \frac{8\pi G}{3}\rho_{crit} \Rightarrow \rho_{crit} = \frac{3H^2}{8\pi G} = 8{,}67\cdot 10^{-27}\,\frac{kg}{m^3}$

b) $\Omega > 1 \Rightarrow k > 0$ hyperbolic

$\Omega = 1 \Rightarrow k = 0$ euclidic

$\Omega < 1 \Rightarrow k < 0$ parabolic

$$k = 0 \Rightarrow \begin{cases} \text{unknown energy with negativ density} \\ \text{everything is ok} \\ \text{unknown energy with positiv density} \end{cases}$$

c)
$$\begin{aligned} H^2(a) &= \frac{8\pi G}{3}\rho \qquad (k=0) \\ &= \frac{8\pi G}{3}\rho_0\, a^{-3(1+w)} \\ &= \frac{8\pi G}{3}\rho_{crit}\cdot\Omega_0\, a^{-3(1+w)} \\ &= H_0^2\cdot\Omega_0\, a^{-3(1+w)} \\ &\equiv H_0^2 E^2(a) \end{aligned}$$

d)
$$H^2(a) = \left(\frac{\dot a}{a}\right)^2 = H_0^2 E^2(a)$$

$$\Leftrightarrow \quad \dot a = H_0 E(a)\cdot a = \frac{da}{dt}$$

$$dt = \frac{da}{a H_0 E(a)} = \frac{da}{a H_0}\left[\sum \Omega_{0,i}\, a^{-3(1+w_i)}\right]^{-\frac{1}{2}}$$

$$= \frac{da}{a H_0}\left[\Omega_{0,m}\, a^{-3} + \Omega_{0,\Lambda}\right]^{-\frac{1}{2}}$$

3

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