general relativity 2015 : causal structure

○ Lorentz-transform $(x,t) \longrightarrow (x',t')$

- linear
- combines position x and time t
- time is a coordinate
- from assumptions
  - (1) homogeneous
  - (2) speed of light is equal in all frames

geometric structure of space-time + causality

$$x' = \gamma\cdot(x - \beta ct)$$

$$ct' = \gamma\,(ct - \beta x)$$

with $\beta = \frac{v}{c}$ and the Lorentz-factor $\gamma = \frac{1}{\sqrt{1-\beta^2}}$

rewrite: $\begin{pmatrix} x' \\ ct' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix}\begin{pmatrix} x \\ ct \end{pmatrix}$

~ shearing in the (x, ct)-plane

○ invariants

- rotations leave the length of a vector $\sqrt{\vec{x}^2} = |\vec{x}|$ constant $\longrightarrow$ what about a similar invariant?

$$c^2t'^2 - x'^2 = \gamma^2\left(c^2t^2 - 2ct\beta x + \beta^2x^2 - x^2 + 2x\beta ct - \beta^2c^2t^2\right)$$

$$= \gamma^2\left(c^2t^2(1-\beta^2) - x^2(1-\beta^2)\right)$$

$$= \underbrace{\gamma^2(1-\beta^2)}_{=1}\cdot\left(c^2t^2 - x^2\right)$$

$s^2 = c^2t^2 - x^2$ is Lorentz-invariant: hyperboloid

○ rapidity and hyperbolic "rotation"

- rotations in the xy-plane leave $r^2 = x^2 + y^2$ invariant

$$x' = x\cos\alpha + y\sin\alpha$$

$$y' = -x\sin\alpha + y\cos\alpha$$

$\rightarrow$ $x'^2 + y'^2 = x^2 + y^2 = r^2$, because $\cos^2\alpha + \sin^2\alpha = 1$.

general relativity 2015 : causal structure

- which functions can fulfill $x^2 - c^2t^2 = \text{const}$?

$\rightarrow$ hyperbolic functions $\cosh^2\psi - \sinh^2\psi = 1$

$$x' = x\cosh\psi - ct\sinh\psi$$

$$ct' = -x\sinh\psi + ct\cosh\psi$$

if $\cosh\psi = \gamma$ and $\sinh\psi = \beta\gamma$:

$$\cosh^2\psi + \sinh^2\psi = \gamma^2 - \beta^2\gamma^2 = (1-\beta^2)\gamma^2 = 1.$$

$\psi \sim$ rapidity : $\tanh\psi = \frac{\sinh\psi}{\cosh\psi} = \frac{\beta\gamma}{\gamma} = \beta.$

- addition theorem: derived from hyperbolic functions

$$\tanh(\varphi+\psi) = \frac{\tanh\varphi + \tanh\psi}{1+\tanh\varphi\cdot\tanh\psi} = \frac{\beta_1+\beta_2}{1+\beta_1\cdot\beta_2} \leqslant 1$$

bounded by 1, which is the asymptote of tanh!

$\rightarrow$ succesive Lorentz-transformations can not build up velocities $> c$

- invariant $s^2 = c^2t^2 - x^2$

(or differentially $ds^2 = c^2dt^2 - dx^2$: Lorentz-transformation is linear!)

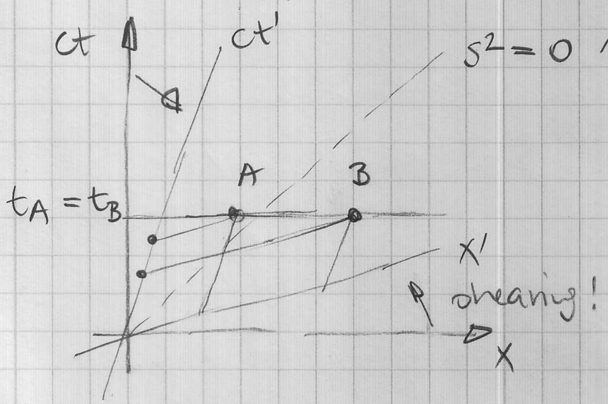
suggests a classification of events, which is <u>identical</u> in all frames.

$$\Delta s^2 \begin{cases} > 0 & \text{timelike} \\ = 0 & \text{null} \\ < 0 & \text{spacelike} \end{cases}$$

which replaces Galilei's absolute time

$\rightarrow$ we need this because of the relativity of simultaneity general relativity 2015: causal structure

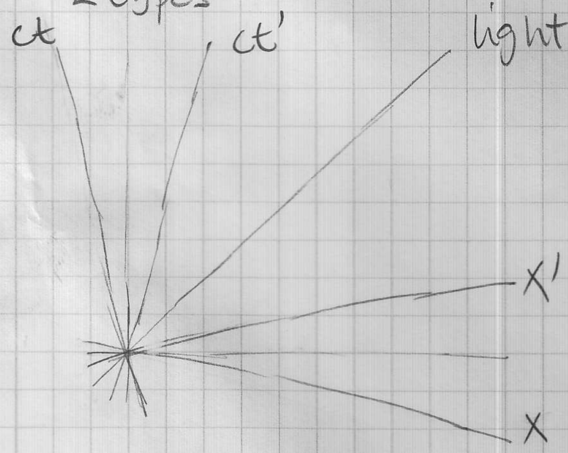
- relativity of simultaneity → "source" of many paradoxes

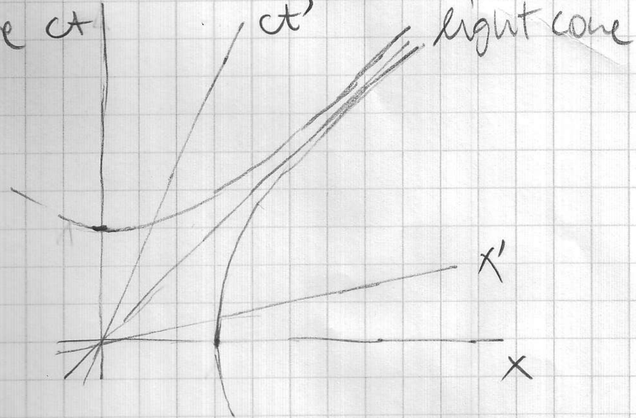


simultaneous $t_A = t_B$ in (x, ct), but
different times $t'_A \neq t'_B$ in (x', ct')!

- Minkowski-diagrams: "geometry of causality"

2 types





introduce intermediate frame "halfway" between (x, ct) and (x', ct') → both frames have correct relative scales.

invariant hyperboloids fix scaling of (x', ct') relative to (x, ct)

- length contraction: ruler at rest in the (x', ct')-system

the endpoints of a ruler are at x'_A and x'_B in (x', ct'), at times t'_A and t'_B → boost into (x, ct)-system

$$x'_A = \gamma (x_A - ct_A)$$

$$x'_B = \gamma (x_B - ct_B)$$

take measurement $x_B - x_A$ at the same time $t_A = t_B$

$$x_B - x_A = \frac{1}{\gamma}(x'_B - x'_A) < x'_B - x'_A$$

→ moving ruler appear shorter by a factor of $\frac{1}{\gamma}$

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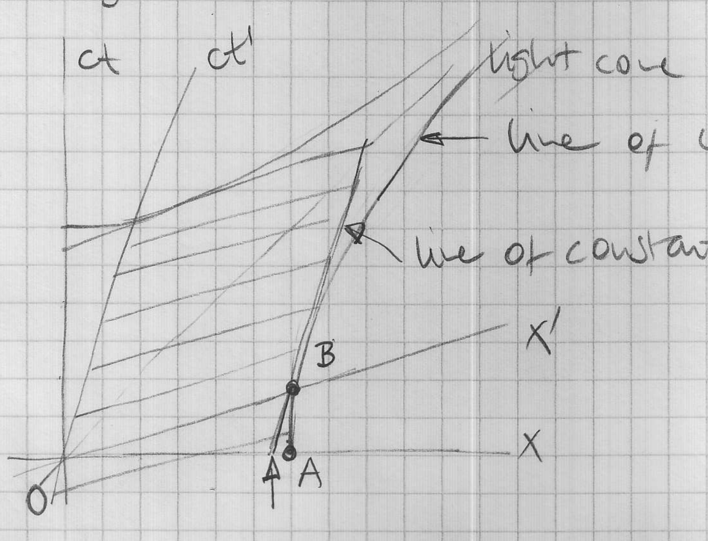
○ length contraction



- a ruler moves through the $(x'ct')$-system as time passes
- OA and OB are equivalent: they have the same s^2
- but measuring the length in the (x, ct)-system gives a result $< OA$.
- effect is reciprocal

○ time dilation: clock at rest in (x', ct')-system

two events at t'_A and t'_B in the (x', ct')-system at positions x'_A and x'_B $\longrightarrow$ boost into (x, ct)-system

$$ct_A = \gamma(ct'_A + \beta x'_A)$$

$$ct_B = \gamma(ct'_B + \beta x'_B)$$

take measurement of time interval $t_B - t_A$ at $x'_A = x'_B$

$$t_B - t_A = \gamma(t'_B - t'_A) > t'_B - t'_A \qquad \text{because } \gamma \geq 1$$

$\rightarrow$ moving clocks appear slower by a factor of γ
$\rightarrow$ real for all physical effects!

keep in mind: effects are in the transformation but are very real!

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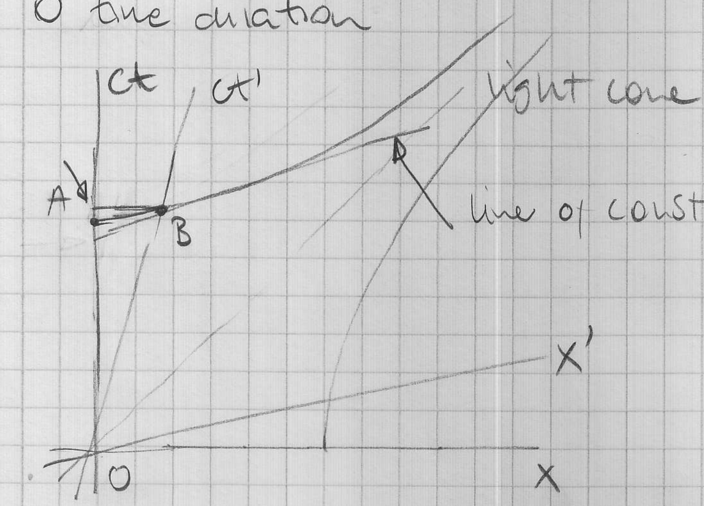
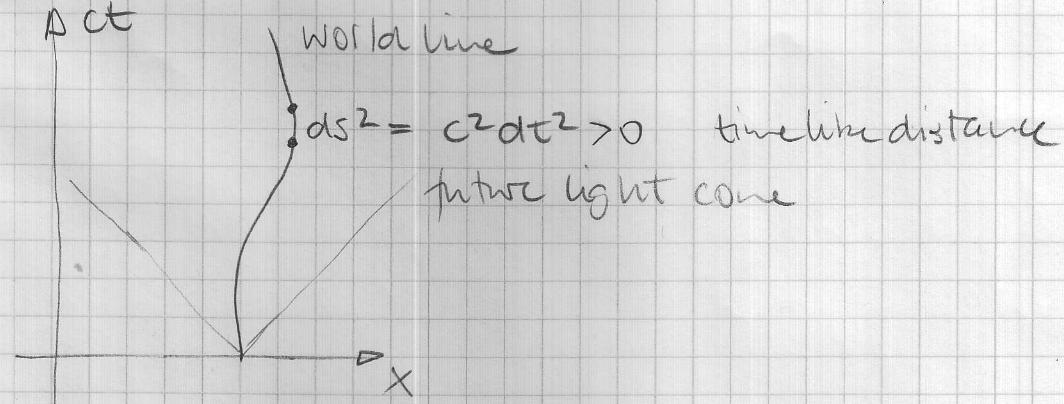
○ time dilation



- a clock is at rest in (x', ct) and displays unit time at B.
- the projection onto the (x, ct)-system at x = 0 is longer,

$0A > 0B$

○ proper time



τ ~ proper time: time displayed by a clock moving along the world line

- any parameter λ can parameterise the world line
- $c^2 d\tau^2 = ds^2$ is a choice particularly suited for massive particles, which are bound to be inside the future light cone
- $c^2 d\tau^2 = c^2 dt^2 - d\vec{x}^2$

$$\beta = \frac{1}{c}\frac{d\vec{x}}{dt} \longrightarrow d\tau = \sqrt{1-\beta^2}\, dt = \frac{dt}{\gamma}$$

- integrate: $\Delta\tau = \int d\tau = \frac{\Delta t}{\gamma}$ ~ analogous to dilation

$\longrightarrow$ τ is displayed by the moving clock.

general relativity 2015 : structure of transformations

O generation of transformations and transformation groups

Pauli-matrices

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$ 

$$\sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$ 

$$\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$  rotation $\hat{R}$

$$\sigma_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$  Lorentz-transformation Λ

- rotations: generate rotation from σ_2

$$R = \exp(\alpha \sigma_2) = \sum_n \frac{\alpha^n}{n!} \sigma_2^n$$

$$\sigma_2^0 = \sigma_0$$

$$\sigma_2^1 = \sigma_2$$

$$\sigma_2^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -\sigma_0$$

$$\sigma_2^3 = -\sigma_2$$

$$\rightarrow R = \sigma_0 \cdot \sum_n \frac{\alpha^{2n}}{(2n!)} \cdot (-1)^n + \sigma_2 \sum_n \frac{\alpha^{2n+1}}{(2n+1)!} \cdot (-1)^n$$

$$= \sigma_0 \cos\alpha + \sigma_2 \cdot \sin\alpha = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$$

- generate Lorentz-transformation from σ_3

$$\Lambda = \exp(-\psi \cdot \sigma_3)$$

$$\sigma_3^0 = \sigma_0$$

$$\sigma_3^1 = \sigma_3$$

$$\sigma_3^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \sigma_0$$

$$\sigma_3^3 = \sigma_3 \leftarrow \text{no sign change!}$$

$$\rightarrow \Lambda = \sigma_0 \cdot \sum_n \frac{\psi^{2n}}{(2n)!} + \sigma_3 \cdot \sum_n \frac{(-\psi)^{2n+1}}{2n+1}$$

$$= \sigma_0 \cosh\psi - \sigma_3 \sinh\psi = \begin{pmatrix} \cosh\psi & -\sinh\psi \\ -\sinh\psi & \cosh\psi \end{pmatrix}$$

general relativity: structure of transformations

- these are examples of a Lie-group: a continuously parameterised group with transformation R or Λ constructed from a generator σ_2 or σ_3.

- successive transformations → what happens to the parameter?

$$R(\alpha)\cdot R(\beta) = \sum_n \frac{\alpha^n}{n!}\cdot\sigma_2^n \sum_m \frac{\beta^m}{m!}\sigma_2^m$$

$$= \sum_n \sum_k \frac{\alpha^k}{k!}\cdot\frac{\beta^{n-k}}{(n-k)!}\cdot\underbrace{\sigma_2^k\cdot\sigma_2^{n-k}}_{=\sigma_2^n}$$

$$= \sum_n \frac{\sigma_2^n}{n!}\cdot\sum_k \binom{n}{k}\cdot\alpha^k\cdot\beta^{n-k}$$

$$= \sum_n \frac{\sigma_2^n}{n!}\cdot(\alpha+\beta)^n = R(\alpha+\beta)$$

$$\Lambda(\varphi)\cdot\Lambda(\psi) = \sum_n \frac{\varphi^n}{n!}\cdot\sigma_3^n \sum_m \frac{\psi^m}{m!}\sigma_3^m = \ldots = \Lambda(\varphi+\psi)$$

rapidities simply add, one can replace successive Lorentz-transformation by an effective one, parameterised by the sum $\varphi+\psi$, just like rotations with $\alpha+\beta$.

- Lorentz-transformations are abelian, $\varphi+\psi=\psi+\varphi$, just like rotations in 2 dimensions

even Lorentz-transformations in different directions commute (they share the only time coordinate)

Non-abelian groups show this in their generators → Baker-Hausdorff-formula

- when is a transformation orthogonal? $R^tR = id \rightarrow R^{-1}=R^t$

$$R^t(\alpha)\cdot R(\beta) = \sum_n \frac{\alpha^n}{n!}(\sigma_2^n)^t\cdot\sum_m \frac{\beta^m}{m!}\sigma_2^m$$

$$(\sigma_2^n)^t = (\sigma_2^t)^n = (-\sigma_2)^n$$

$$= \sum_n \frac{\sigma_2^n}{n!}\sum_k \binom{n}{k}\cdot(-\alpha)^k\cdot\beta^{n-k} = \sum_n \frac{\sigma_2^n}{n!}(\beta-\alpha)^n = \exp(\sigma_2(\beta-\alpha))$$

essential point: antisymmetry of σ_2: $\sigma_2^t = -\sigma_2$

by chosing $\alpha=\beta$ one can force $R^t(\alpha)\,R(\alpha) = id$

then, $R^t(\alpha) = R(-\alpha)$ is the inverse general relativity: structure of transformations

o orthogonal transforms $R^t \cdot R = id$: what about Λ?

$\Lambda^t \cdot \Lambda \neq id$ because $\sigma_3^t = +\sigma_3$

→ Lorentz-transformations don't conserve scalar products (Euclidean scalar products!)

but $\Lambda(-\psi) \cdot \Lambda(+\psi) = id$ → inverse transform.

o accelerations

$$\begin{cases} a = \frac{du}{dt} \\ u = \frac{dx}{dt} \end{cases}$$ in the (x, ct)-frame → boost into (x', ct')

$$dx' = \gamma \cdot (dx - \beta dt)$$
$$c dt' = \gamma (c dt - \beta dx)$$
→ $$u' = \frac{u - v}{1 - \frac{uv}{c^2}}$$

$$du' = \frac{du}{\gamma^2 (1 - \frac{uv}{c^2})^2}$$ with $c dt' = \gamma \cdot (1 - \frac{uv}{c^2}) c dt$ →

$$a' = \frac{du'}{dt'} = \frac{1}{\gamma^3 (1 - \frac{uv}{c^2})^3} \cdot a$$

acceleration is not invariant, but absolute/universal

all observers agree if a body accelerates $a = 0 \leftrightarrow a' = 0$

o invariants follow from

$$\det(R) = \cos^2\alpha + \sin^2\alpha = 1$$

$$\det(\Lambda) = \cosh^2\psi - \sinh^2\psi = 1$$

→ both enforced by off-diagonal generator