general relativity 2015: Minkowski-geometry

o geometry of Minkowski-space

$$ds^2 = c^2 d\tau^2 = c^2 dt^2 - d\vec{x}^2$$

combine all coordinates into a 4-vector

$$x^0 = ct,\ x^1 = x,\ x^2 = y,\ x^3 = z$$

$\rightarrow ds^2 = \eta_{\mu\nu} \cdot dx^\mu dx^\nu$ with summation convention

invariance of ds^2 under Lorentz-transformations

$$\rightarrow ds'^2 = \eta_{\mu\nu} dx'^\mu dx'^\nu = \eta_{\mu\nu} dx^\mu dx^\nu = ds^2$$

$\rightarrow \eta_{\mu\nu} = \underbrace{\frac{\partial x'^\rho}{\partial x^\mu}}_{=\text{const}} \underbrace{\frac{\partial x'^\sigma}{\partial x^\nu}}_{=\text{const}} \eta_{\rho\sigma}$ for a change $x^\mu \rightarrow x'^\mu$

$=$const, otherwise homogeneity is broken $\rightarrow$ transform must be linear

$$= \Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu \eta_{\rho\sigma}$$

o relativistic mechanics

4-velocity of a particle ~ tangent vector to the world line

$u^\mu = \frac{dx^\mu}{d\tau}$ τ ~ proper time, measured on the particle

$$u^2 = \eta_{\mu\nu} u^\mu u^\nu = \eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = \frac{ds^2}{d\tau^2} = c^2$$

$\rightarrow$ length of the 4-velocity is constant and $= c$

$$u^\mu = \frac{dx^\mu}{d\tau} = \gamma \cdot \frac{dx^\mu}{dt} = \gamma (c, \vec{u})$$

change to coordinate time.

transformation of u^μ: just like that of $x^\mu \rightarrow u'^\mu = \Lambda^\mu{}_\nu u^\nu$

4-momentum of a particle

$$p^\mu = m_0 \cdot u^\mu$$

Lorentz-invariant: rest mass m_0!

$$\eta_{\mu\nu} p^\mu p^\nu = m_0^2 \cdot \eta_{\mu\nu} u^\mu u^\nu = m_0^2 c^2 = E^2 - c^2 p^2$$

if $p^\mu = \left(\frac{E}{c}, \vec{p}\right)$

$\rightarrow$ relativistic dispersion relation

general relativity 2015: Minkowski-geometry + Lorentz-force

○ relativistic mechanics, cont'ed

4-force ~ change of 4-momentum with proper time

$$\frac{dp^\mu}{d\tau} = f^\mu$$

○ relativistic mechanics from a variational principle

→ a Lorentz invariant would be a great starting point

$$S = m\int_a^b ds = mc\int_a^b d\tau$$

~ unit of an energy

~ interpretation of proper time along trajectory

with $ds^2 = \eta_{\mu\nu}\, dx^\mu dx^\nu = c^2 d\tau^2$

→ variation : replace Hamilton's principle by the principle of least proper time: $\delta S = 0$

$$\delta S = \delta\int_a^b mc\, d\tau = mc\,\delta\int_a^b \sqrt{\eta_{\mu\nu}\, dx^\mu dx^\nu}$$

$$= mc\int_a^b \frac{\eta_{\mu\nu}(dx^\mu\cdot\delta dx^\nu + \delta dx^\mu\cdot dx^\nu)}{2\cdot\underbrace{\sqrt{\eta_{\mu\nu}\, dx^\mu dx^\nu}}_{=d\tau}}$$

$$= mc\int_a^b \eta_{\mu\nu}\frac{dx^\mu}{d\tau}\cdot\underbrace{\delta dx^\nu}_{=d\,\delta x^\nu} = -mc\int_a^b \eta_{\mu\nu}\cdot du^\mu\cdot\delta x^\nu$$

+ partial integration

$$= -mc\int_a^b \eta_{\mu\nu}\underbrace{\frac{du^\mu}{d\tau}}_{=0}\cdot d\tau\cdot\delta x^\nu = 0$$

→ equation of motion $m\frac{du^\mu}{d\tau} = \frac{dp^\mu}{d\tau} = 0$ ~ inertial motion

○ introduce a coupling to an electromagnetic field

minimal coupling, with coupling constant q ~ charge

with 4-potential $A^\mu = (\phi, \vec{A})$.

$$S = -\int_a^b \left(mc\,d\tau + \frac{q}{c}\,\eta_{\mu\nu} A^\mu\cdot dx^\mu\right)$$

principle of least proper time: $\delta S = 0$

general relativity 2015: Lorentz-force

o coupling to electromagnetic field, cont'ed

$$\delta S = \delta \int_a^b \left(-mc\,d\tau - \frac{q}{c}\eta_{\mu\nu}\cdot A^\mu dx^\nu\right)$$

$$= -\eta_{\mu\nu}\int_a^b mc\,\frac{dx^\mu}{d\tau}\,d\delta x^\nu + \frac{q}{c}A^\mu d\delta x^\nu + \frac{q}{c}\cdot\delta A^\mu dx^\nu$$

$$= +\eta_{\mu\nu}\int_a^b mc\,du^\mu\delta x^\nu + \frac{q}{c}\,dA^\mu\delta x^\nu - \frac{q}{c}\,\delta A^\mu dx^\nu$$

use $\delta A^\mu = \frac{\partial A^\mu}{\partial x^\nu}\delta x^\nu$ and $dA^\mu = \frac{\partial A^\mu}{\partial x^\nu}dx^\nu$

$$= +\eta_{\mu\nu}\int mc\,du^\mu\delta x^\nu + \frac{q}{c}\frac{\partial A^\mu}{\partial x^\sigma}dx^\sigma\delta x^\nu - \frac{q}{c}\frac{\partial A^\mu}{\partial x^\sigma}\delta x^\sigma dx^\nu$$

substitute $du^\mu = \frac{du^\mu}{d\tau}\cdot d\tau$ and $dx^\mu = u^\mu d\tau$

$$= +\eta_{\mu\nu}\int\left(mc\frac{du^\mu}{d\tau} + \frac{q}{c}\frac{\partial A^\mu}{\partial x^\sigma}\cdot u^\sigma - \frac{q}{c}\frac{\partial A^\sigma}{\partial x^\mu}\cdot u^\sigma\right)d\tau\,\delta x^\nu$$

after swapping indices in the last term

→ equation of motion

$$mc\frac{du^\mu}{d\tau} = \frac{q}{c}\left(\frac{\partial A^\sigma}{\partial x^\mu} - \frac{\partial A^\mu}{\partial x^\sigma}\right)\cdot u^\sigma$$

fully relativistic, includes Lorentz-force

compare observer at rest relative to charge vs. motion!
(1) E + B - fields
(2) Lorentz-contraction / time dilation

o field strength tensor $F^{\mu\nu} = \frac{\partial A^\mu}{\partial x^\nu} - \frac{\partial A^\nu}{\partial x^\mu}$ (antisymmetric)

$$mc\frac{du^\mu}{d\tau} = \frac{q}{c}\cdot F^{\mu\nu}\cdot u^\nu$$

substitution of the derivatives of $\vec{A}$ and ϕ gives

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & +B_y \\ E_y & +B_z & 0 & -B_x \\ E_z & -B_y & +B_x & 0 \end{pmatrix}$$

general relativity 2015: geometry + physics

O there are a <u>number</u> of important observations

(1) relativistic equation of motion:

- links 4-vectors and can be Lorentz-boosted

  → problem does not arise in CM: $m\ddot{x} = -\nabla\phi$

- relativity: observers measure E and B depending on their state of motion

- principle of minimised proper time replaces Lagrange:

$$m\int ds = mc\int d\tau = mc^2\int\sqrt{1-\beta^2}\,dt,$$

because $d\tau = \sqrt{c^2-v^2}\,dt$, $v = \frac{dx}{dt}$

$$\cong \int \Big(\underbrace{mc^2}_{\text{const.}} + \underbrace{\tfrac{m}{2}v^2}_{\text{kin energy}}\Big)\,dt = \int T\,dt \quad ✓$$

- $d\tau$ is a Lorentz-scalar, identical in all frames

- extremal principle $\delta S = 0$ works as well for relativistic physics

- don't ever write $m\ddot{x} = q(\vec{E} + \vec{v}\times\vec{B})$: it mixes Galilean and relativistic physics, only valid if $\beta \ll 1$ and $\gamma \simeq 1$.

(2) $\eta_{\mu\nu}$ defines a product for 4-vectors, which is invariant under Lorentz-transformations.

- $ds^2 = \eta_{\mu\nu}\,dx^\mu dx^\nu = dx_\mu dx^\nu$ → index raising/lowering

- generalise to projections $A_\mu dx^\mu$ between two 4-vectors ~ Lorentz-invariant

- $\left.\begin{array}{ll} x_\mu & \text{covariant} \\ x^\mu & \text{contravariant} \end{array}\right\}$ 4-vector

(3) Lorentz-scalars are nice quantities to work with because they're of course constant in all frames

- classification time/null/spacelike Δs.

- principle of extremised proper time $S = -mc\int d\tau$

- equation of continuity $\partial_\mu J^\mu = 0$ for charge general relativity 2015: geometry + physics

o important observations: Lorentz-scalars

- continuity $\partial_\mu j^\mu = 0$ is a frame indep. statement, because $\partial_\mu j^\mu$ is a Lorentz-scalar

  but please be careful with the interpretation!

Lorentz	Galilei
$\cdot\ \partial_\mu j^\mu = 0$ is frame independent	in every frame:
choose frame: $\partial_t \rho + \nabla \vec{j} = 0$	$\partial_t + \nabla \vec{j} = 0$
$\partial_t \int_V dV \rho + \int_{\partial V} d\vec{A} \cdot \vec{j} = 0$	$\frac{d}{dt} \int_V dV \rho + \int_{\partial V} d\vec{A} \cdot \vec{j} = 0$
time dilation in j and t, length contraction in ρ and V	all quantities are equal in all frames

- Lorentz-transforms + orthogonality with $\eta_{\mu\nu}$

  from invariance $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = \eta_{\mu\nu} dx'^\mu dx'^\nu = ds'^2$

  $$\rightarrow \eta_{\mu\nu} = \underbrace{\frac{\partial x'^\rho}{\partial x^\mu}}_{\Lambda^\rho{}_\mu} \underbrace{\frac{\partial x'^\sigma}{\partial x^\nu}}_{\Lambda^\sigma{}_\nu} \cdot \eta_{\rho\sigma} \quad \text{because } x'^\mu = \Lambda^\mu{}_\sigma x^\sigma$$

  $$\Lambda^\mu{}_\nu = \frac{\partial x'^\mu}{\partial x^\nu} \longrightarrow \text{inverse transform } \Lambda_\mu{}^\nu = \frac{\partial x^\nu}{\partial x'^\mu}$$

  $$\rightarrow \Lambda_\mu{}^\nu = \eta_{\mu\rho} \eta^{\nu\sigma} \Lambda^\rho{}_\sigma \quad \text{is the inverse:}$$

  $$\rightarrow \Lambda^\mu{}_\nu \cdot \Lambda_\mu{}^\sigma = \Lambda^\mu{}_\nu \eta_{\mu\rho} \eta^{\sigma\tau} \Lambda^\rho{}_\tau$$
  $$= \eta_{\nu\tau} \eta^{\sigma\tau} = \delta^\sigma_\nu \quad \text{orthogonal!}$$

o why is now gravity not just a 4-force like A_μ?

pro: $\Delta \phi = 4\pi G \rho \quad \sim \quad \Delta \phi = 4\pi \rho$

perhaps $\Delta \vec{A} = \frac{4\pi}{c} G \vec{J}$ is a new effect?

pro: could be formulated in a covariant way

pro: $\frac{1}{r}$-potentials are well understood.

general relativity 2015: gravity and geometry

○ ideas for a geometric theory of gravity

- Minkowski-space for electrodynamics is very elegant, keep principle of minimised proper time
- equivalence: trajectories of particles in grav. fields don't depend on mass

CM: $\frac{d^2\vec{x}}{dt^2} = -\underbrace{\frac{m_g}{m_i}}_{\hat{=}}\nabla\phi$ $\cong 1$, experimental bounds $\lesssim 10^{-11}$

2 properties of mass → coupling to grav. field m_g → inertia 'resisting' acceleration m_i

have to be identical for all objects, no matter what they are made of

(many more reasons for a relativistic geometric field equ.!)

○ equivalence principle

→ in a freely-falling (+ non rotating) small laboratory one observes that Nature follows the laws of special relativity

- small: gravitational fields are not uniform in general, so objects at different places fall at different rates

  → tidal fields

- any force can be made to vanish by chosing an accelerated frame, but for gravity this is universal

  ED: acceleration set by $\frac{q}{m}$ → depends on particle

  GR: acceleration set by $\frac{m_g}{m_i} \equiv 1$ for <u>all</u> particles

- equivalence of a particle moving freely along a straight line in the freely falling frame and a curved trajectory in a stationary frame.
- gravity is not just a 4-force.
- if particles follow lines determined by $\delta\int mc\,d\tau = 0$ and if τ contains the geometry of space-time → all particles would follow the same trajectories

general relativity 2015: geodesic equation

Einstein's grand idea: stick to the principle of minimised proper time $\delta\int mc\,d\tau = 0$, but encode gravity in a non-Minkowskian geometry:
$ds^2 = g_{\mu\nu}\,dx^\mu dx^\nu$ with the metric tensor $g_{\mu\nu}(x^\mu)$.

→ time varying geometry: "differential geometry"

→ geodesic equation: minimising $mc\,d\tau$ in a general geometry described by $g_{\mu\nu}(x^\mu)$.