

Framed quiver moduli, cohomology, and quantum groups

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Abstract

Framed quiver moduli parametrize stable pairs consisting of a quiver representation and a map to a fixed graded vector space. Geometric properties and explicit realizations of framed quiver moduli for quivers without oriented cycles are derived, with emphasis on their cohomology. Their use for quantum group constructions is discussed.

1 Introduction

In formulating interesting moduli problems for objects in an abelian category, one is usually confronted with the problem of either restricting the class of objects to be parametrized to certain rigid, or stable, objects, or to endow them with some additional structure, which rigidifies them; see [19] for a thorough discussion of this problem.

Moduli spaces for representations of quivers, introduced in [15], provide an interesting testing ground for techniques of moduli theory. This class of moduli spaces relates to many moduli problems arising in Algebraic Geometry (in particular, in connection to vector bundles). It is also interesting in itself, both in view of the classification problem of quiver representations, and, via the Hall algebra approach of [24], to quantum group realizations (see, for example, [21]).

While the first approach to the problem of formulating interesting moduli problems for representations of quivers was used in [15] by introducing a concept of (semi-)stable representations, the second one was first considered in [20], as a preliminary step in the construction of Nakajima quiver varieties. In the latter case, the resulting moduli spaces are called framed quiver moduli; their construction will be recalled in section 2.

The aim of this note is to study the geometry of framed quiver moduli in the case of quivers without oriented cycles. This study is based on two realizations,

the first in terms of the ordinary quiver moduli of [15], the second in terms of Grassmannians of subrepresentations; they are developed in section 3.

In section 4, after first characterising non-vanishing of framed quiver moduli (Proposition 4.3), we construct an embedding into a product of Grassmannians (Proposition 4.4). This gives rise to an explicit description of framed quiver moduli as iterated Grassmann bundles (Theorem 4.10).

Most of the results of the present paper arose out of the attempt to study the cohomology of these framed quiver moduli, with the aim of giving more precise versions of some general statements ([10, 16, 21]). The paper's main result in this direction, Theorem 5.2, in fact gives very detailed information even on the Chow ring of framed quiver moduli, in terms of Schubert calculus (see [11]). This follows from the realization of framed moduli as a tower of Grassmann bundles.

In section 6, a natural action of a linear algebraic group on framed quiver moduli is defined. The main result of this section, Theorem 6.4, gives a description of the orbit structure of this action in terms of the orbit structure of representation varieties of quivers with respect to the base change group, which is rather well-understood.

Finally, section 7.2 relates the subject of framed quiver moduli to quantum group constructions. The point of view advocated there is that framed quiver moduli provide a candidate for a generalization of the approach of [3] to quantum groups of arbitrary (symmetric) Kac-Moody type. The main result, Theorem 7.11, at least provides a realization of quotients of modified quantized Borel algebras in terms of a convolution algebra defined using framed quiver moduli.

All the results of this note strongly depend on the hypothesis that the underlying quiver has no oriented cycles, since this allows for inductive constructions. For quivers with oriented cycles, the geometry of framed moduli is expected to be of a quite different nature (for the case of the one-loop quiver, resp. the multiple-loop quiver, see [12, 17] and [22], respectively).

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2 Definition of framed quiver moduli

Let Q be a finite quiver with set of vertices I . The arrows in Q will be written as $\alpha : i \rightarrow j$ for $i, j \in I$. Throughout the paper we assume that Q does not contain oriented cycles. Therefore, we can consider the partial order induced on I by the arrows, i.e. defined by $i < j$ if there exists an arrow $\alpha : i \rightarrow j$.

We denote by $\text{rep}_k Q$ the abelian k -linear category of finite dimensional k -representations of Q . The homomorphisms between objects M and N of $\text{rep}_k Q$

will be denoted by $\text{Hom}_Q(M, N)$. For all basic notions on representations of quivers, the reader is referred to [2, 23].

Let $\mathbf{Z}I$ be the free abelian group in I ; its elements will be written as $d = \sum_{i \in I} d_i i$. Let $\mathbf{N}I$ be the subsemigroup of $\mathbf{Z}I$ of nonnegative linear combinations of the elements of I ; this will be viewed as the space of dimension types of representations of Q . We have a non-symmetric inner product (the Euler form) on $\mathbf{Z}I$ defined by

$$\langle i, j \rangle = \delta_{i,j} - \#\{\text{arrows from } i \text{ to } j\}$$

for $i, j \in I$. It has the property

$$\langle \underline{\dim} M, \underline{\dim} N \rangle = \dim \text{Hom}_Q(M, N) - \dim \text{Ext}_Q^1(M, N)$$

for any two representations M and N of Q . Later on, we will use the symmetrization $(d, e) = \langle d, e \rangle + \langle e, d \rangle$. We also consider the standard inner product $i \cdot j = \delta_{i,j}$ on $\mathbf{N}I$.

Let k be an algebraically closed field, and let d, n be dimension types, that is, elements of $\mathbf{N}I$. For $i \in I$, let M_i (resp. V_i) be a k -vector space of dimension d_i (resp. n_i). In the following, we will frequently consider $M = \bigoplus_{i \in I} M_i$ as an I -graded vector space. In particular, we consider I -tuples of linear maps like $(f_i : M_i \rightarrow V_i)_{i \in I}$ as linear maps $f : M \rightarrow V$ of I -graded vector spaces.

We define the variety of k -representations of Q of dimension type d by

$$R_d = R_d(Q) = \bigoplus_{\alpha: i \rightarrow j} \text{Hom}_k(M_i, M_j),$$

where the sum runs over all arrows in Q . The group $G_d = \prod_{i \in I} \text{GL}(M_i)$ acts on R_d via

$$g \cdot M = (g_i)_{i \in I} \cdot (M_\alpha)_{\alpha: i \rightarrow j} = (g_j M_\alpha g_i^{-1})_{\alpha: i \rightarrow j}.$$

The variety R_d is an affine space of dimension $\dim R_d = \sum_{\alpha: i \rightarrow j} d_i d_j$, and the group G_d is reductive and algebraic of dimension $\dim G_d = \sum_{i \in I} d_i^2$. In particular, note that

$$\dim G_d - \dim R_d = \langle d, d \rangle.$$

By definition, the orbits $\mathcal{O}_M$ of G_d in R_d are in bijection with the isomorphism classes $[M]$ of k -representations of Q of dimension type d . An orbit $\mathcal{O}_M$ is closed in R_d if and only if the representation M is semisimple. Since Q has no oriented cycles, there exists a unique such representation (up to isomorphism) for each dimension vector. Thus, G_d acts on R_d with a unique closed orbit, for each $d \in \mathbf{N}I$.

Now we consider the extended representation variety

$$R_{d,n} = R_d \times \bigoplus_{i \in I} \text{Hom}_k(M_i, V_i).$$

Again, it carries a natural action of G_d given by

$$g \cdot (M, (f_i)_{i \in I}) = (g \cdot M, (f_i g_i^{-1})_{i \in I}),$$

extending the action of G_d on R_d .

We introduce a notion of stability for points of $R_{d,n}$:

Definition 2.1 *A point $(M, f = (f_i)_{i \in I})$ of $R_{d,n}$ is called stable if there is no non-zero subrepresentation U of M which is contained in $\text{Ker } f \subset M$. The set of all stable points of $R_{d,n}$ is denoted by $R_{d,n}^s$.*

Now we can define the framed quiver moduli:

Theorem 2.2 ([20]) *There exists a geometric quotient $\mathcal{M}_{d,n} = \mathcal{M}_{d,n}(Q)$ of $R_{d,n}^s$ by G_d .*

The spaces $\mathcal{M}_{d,n}$ are called framed quiver moduli. They can be defined explicitly in terms of semi-invariants on the extended representation varieties as follows (see [20]): define a character χ on the group G_d by

$$\chi((g_i)_{i \in I}) := \prod_{i \in I} \det(g_i).$$

Let $A = k[R_{d,n}]$ be the coordinate ring of the affine variety $R_{d,n}$. For $n \in \mathbb{N}$, let A^{G_d, χ^n} be the space of semi-invariant functions with respect to the character χ^n , that is, regular functions $f \in A$ such that

$$f(g \cdot x) = \chi(g)^n \cdot f(x) \text{ for all } g \in G_d \text{ and } x \in R_{d,n}.$$

Then

$$\mathcal{M}_{d,n} = \mathbf{Proj}(\bigoplus_{n \in \mathbb{N}} A^{G_d, \chi^n}).$$

In particular, since 0 is the unique closed G_d -orbit in R_d , we have $A^{G_d} = k$. Thus $\mathcal{M}_{d,n}$ is a projective variety. From Theorem 2.2, we also get immediately:

Lemma 2.3 *If $\mathcal{M}_{d,n}$ is nonempty, it is a smooth projective variety of dimension*

$$\dim \mathcal{M}_{d,n} = d \cdot n - \langle d, d \rangle.$$

Proof: By definition of $\mathcal{M}_{d,n}$ as a geometric quotient of $R_{d,n}^s$ by G_d , we have:

$$\dim \mathcal{M}_{d,n} = \dim R_{d,n} - \dim G_d = \sum_{\alpha: i \rightarrow j} d_i d_j + \sum_{i \in I} d_i n_i - \sum_{i \in I} d_i^2 = d \cdot n - \langle d, d \rangle.$$

■

Example: As an easy (but nevertheless typical) example of framed quiver moduli, we consider the case of the quiver of type A_1 , that is, $I = \{i\}$ is a one-element set, and there are no arrows. In this case, $R_{d,n} = \text{Hom}_k(M_i, V_i)$

for k -vector spaces M_i and V_i of dimension d_i and n_i , respectively, and $R_{d,n}$ consists precisely of the injective homomorphisms. Thus, we see immediately that $\mathcal{M}_{d,n}$ is empty if $d > n$, and that $\mathcal{M}_{d,n}$ is isomorphic to the Grassmannian $\text{Gr}_d(V_i)$ of d -dimensional subspaces of V_i , by associating to a point $f \in R_{d,n}^s$ the subspace $\text{Im}(f)$ of V_i .

Remark: The construction of framed quiver moduli has a dual version by defining the extended representation variety as $R_d \times \bigoplus_{i \in I} \text{Hom}_k(V_i, M_i)$ and considering stable pairs (M, f) such that there is no proper subrepresentation U of M containing $\text{Im} f$. The interested reader will easily dualize all the results of this paper to this situation.

3 Two realizations of framed quiver moduli

In this section, we give two realizations of the framed quiver moduli $\mathcal{M}_{d,n}$. The first gives an interpretation in terms of the ordinary quiver moduli introduced by A. King in [15], the second in terms of Grassmannians of subrepresentations of a quiver representation.

3.1 Realization in terms of ordinary quiver moduli - deframing

We first recall A. King's construction of (ordinary) quiver moduli [15], with a slightly different notion of stability used in [21]. As before, let Q be a finite quiver without oriented cycles. Choose a linear function $\Theta : \mathbf{Z}I \rightarrow \mathbf{Z}$, called a stability for Q , and denote by $\dim$ the linear function on $\mathbf{Z}I$ defined by $\dim i = 1$ for all $i \in I$. Define the slope of a non-zero dimension type $d \in \mathbf{N}I \setminus 0$ by $\mu(d) = \frac{\Theta(d)}{\dim d}$. Finally, define the slope $\mu(X) = \mu(\underline{\dim} X)$ of a representation X of Q as the slope of its dimension type.

Definition 3.1 *A representation X of Q is called semistable (resp. stable) if $\mu(U) \leq \mu(X)$ (resp. $\mu(U) < \mu(X)$) for all non-zero proper subrepresentations U of X . Denote by R_d^{ss} (resp. R_d^s) the set of points of R_d corresponding to semistable (resp. stable) representations.*

Theorem 3.2 ([15]) *The semistable locus R_d^{ss} is an open subset of R_d , and the stable locus R_d^s is an open subset of R_d^{ss} . There exists an algebraic quotient $\mathcal{M}_d^{ss}$ of R_d^{ss} by G_d . The variety $\mathcal{M}_d^{ss}$ is projective, of dimension $1 - \langle d, d \rangle$ if it is non-empty. There exists a geometric quotient $\mathcal{M}_d^s$ of R_d^s by G_d . The variety $\mathcal{M}_d^s$ is smooth and embeds as an open subset of $\mathcal{M}_d^{ss}$.*

In particular, consider the case where $R_d^{ss} = R_d^s$, that is, all semistables are already stable. Then $\mathcal{M}_d^{ss} = \mathcal{M}_d^s$ is a smooth projective variety, parametrizing the isomorphism classes of (semi-)stable representations of Q of dimension type d (since $\mathcal{M}_d^s$ is a geometric quotient).

We introduce a new quiver $\tilde{Q}$ with set of vertices $\tilde{I} = I \dot{\cup} \{\infty\}$, the arrows in $\tilde{Q}$ being those of Q , together with n_i arrows from i to ∞ , for each $i \in I$. We also extend the dimension type to $\tilde{d} = d + \infty \in \mathbf{N}\tilde{I}$. We consider the stability $\Theta = -\infty^*$.

By definition, we have a G_d -equivariant isomorphism

$$R_{\tilde{d}}(\tilde{Q}) = \bigoplus_{\alpha: i \rightarrow j} \mathrm{Hom}_k(k^{d_i}, k^{d_j}) \oplus \bigoplus_{i \in I} \mathrm{Hom}_k(k^{d_i}, k)^{n_i} \simeq R_{d,n}(Q)$$

via the identification $\mathrm{Hom}_k(k^{d_i}, k)^{n_i} \simeq \mathrm{Hom}_k(k^{d_i}, k^{n_i})$.

Proposition 3.3 *The moduli space $\mathcal{M}_d^{ss}(\tilde{Q})$, defined by the above stability, is isomorphic to $\mathcal{M}_{d,n}$.*

Proof: Using the above identification $R_{\tilde{d}}(\tilde{Q})$ with $R_{d,n}(Q)$, we will show that the set of (semi-)stable points $R_d^{ss}(\tilde{Q})$ identifies with the set of stable points $R_{d,n}^s(Q)$ as in Definition 2.1. Let X be a representation of $\tilde{Q}$ of dimension type $\tilde{d}$, viewed as a pair (M, f) consisting of a representation M of Q of dimension type d , together with an I -graded linear map $f = (f_i)_{i \in I} : M \rightarrow V$. By definition, X is semistable if and only if $\mu(Y) \leq \mu(X)$ for all non-zero proper subrepresentations Y of X .

We have $\mu(X) = -\frac{1}{\dim X + 1}$. If $(\underline{\dim} Y)_\infty = 1$, then $\mu(Y) = -\frac{1}{\dim Y} < \mu(X)$. If $(\underline{\dim} Y)_\infty = 0$, then $\mu(Y) = 0 > \mu(X)$. Thus, X is semistable if and only if it is stable if and only if there is no non-zero subrepresentation Y of X such that $(\underline{\dim} Y)_\infty = 0$. But giving such a subrepresentation Y is equivalent to giving a subrepresentation U of M such that U is contained in $\mathrm{Ker} f$. We arrive at Definition 2.1 of stability for points of $R_{d,n}(Q)$. ■

Corollary 3.4 *The set $R_{d,n}^s$ is an open subset of $R_{d,n}$. The quotient map $R_{d,n}^s \rightarrow \mathcal{M}_{d,n}$ is a principal G_d -bundle. In particular, it is a smooth morphism.*

Proof: By [21, Lemma 6.5], the action of G_d on $R_{d,n}^s$ is free in the sense of [19, 0.8.iv], using the above realization of $\mathcal{M}_{d,n}$. By [19, Proposition 0.9], this implies the claimed property. ■

The technique of extending the quiver used in this section was called “deframing” by W. Crawley-Boevey in [9].

3.2 Representation-theoretic interpretation

First we recall the Grassmannians of subrepresentations introduced in [25]. Let X be a representation of a quiver Q , and let $e \leq d = \underline{\dim} X$ be a dimension type. Inside the product of Grassmannians $\prod_{i \in I} \mathrm{Gr}_{e_i}(X_i)$, we can consider the closed subset $\mathrm{Gr}_e(X)$ of tuples of subspaces $(U_i)_{i \in I}$ defining a subrepresentation of

X , that is, fulfilling $X_\alpha(U_i) \subset U_j$ for all $\alpha : i \rightarrow j$. The subset $\text{Gr}_e(X)$ is called the Grassmannian of e -dimensional subrepresentations of M .

To a point $U = (U_i)_{i \in I} \in \text{Gr}_e(X)$ we can thus associate a representation of Q of dimension type e , again denoted by U , in a natural way. Later on, we need to know that this association can also be done on the level of the corresponding varieties:

The product of Grassmannians $\prod_{i \in I} \text{Gr}_{e_i}(X_i)$ naturally arises as the quotient of $\text{IHom}_k(M, X) := \prod_{i \in I} \text{IHom}_k(M_i, X_i)$ by the group $\prod_{i \in I} \text{GL}(M_i)$, where $\text{IHom}_k(M_i, X_i)$ denotes the variety of injective k -linear maps from M_i to X_i . Denote by $\text{IHom}_e(X)$ the inverse image of $\text{Gr}_e(X)$ under this quotient map π .

Lemma 3.5 *There exists a morphism $\sigma : \text{IHom}_e(X) \rightarrow R_e$ such that, for any point $f \in \text{IHom}_e(X)$, the quiver representation $\sigma(f)$ is isomorphic to the quiver representation associated to $\pi(f)$ as above.*

Proof: By definition, a point $f = (f_i)_{i \in I}$ of $\text{IHom}_k(M, X)$ belongs to $\text{IHom}_e(X)$ if and only if there exist maps $(M_\alpha)_\alpha$ such that $X_\alpha f_i = f_j M_\alpha$ for all arrows $\alpha : i \rightarrow j$ in Q . Viewing the f_i as $d_i \times e_i$ -matrices, we can cover $\text{IHom}_e(X)$ by open subsets where various tuples of $d_i \times d_i$ -minors of these matrices do not vanish. On each of these open subsets, the maps M_α , whose existence and uniqueness are guaranteed, can be recovered algebraically via multiplication by inverses of matrices. ■

For each vertex $i \in I$, denote by E_i the simple representation at i , defined by $\dim E_i = i$. Denote by P_i (resp. I_i) the projective cover (resp. the injective hull) of E_i . These representations can be described explicitly as follows:

By definition, a path in Q from a vertex i to a vertex j is a sequence of arrows $p = (\alpha_1, \dots, \alpha_k)$ in Q such that

$$i = i_0 \xrightarrow{\alpha_1} i_1 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_k} i_k = j;$$

we also have one path of length $k = 0$ at any vertex $i \in I$. Then the space $(P_i)_j$ (resp. $(I_i)_j$) has a basis consisting of the paths p from i to j (resp. from j to i). Furthermore, for any representation M of Q , there are functorial isomorphisms

$$\text{Hom}_Q(P_i, M) \simeq M_i \text{ and } \text{Hom}_Q(M, I_i) \simeq M_i^*,$$

where M_i^* denotes the k -linear dual of the vector space M_i .

Denoting by $I \otimes V$ the injective representation $I \otimes V = \bigoplus_{i \in I} I_i \otimes_k V_i$, we have

$$(I \otimes V)_i \simeq \bigoplus_{j \in I} (I_j)_i \otimes_k V_j \simeq \bigoplus_{j \in I} \bigoplus_{p: i \rightsquigarrow j} V_j \simeq \bigoplus_{i \rightsquigarrow j} V_j,$$

where $i \rightsquigarrow j$ indicates a path from i to j .

Definition 3.6 Given a point $(M, f) \in R_{d,n}$, define a map $\Phi_{(M,f)} = \varphi : M \rightarrow I \otimes V$ recursively as follows:

If i is a sink in Q , then $\varphi_i = f_i : M_i \rightarrow V_i$. For an arbitrary vertex $i \in I$ define

$$\varphi_i = f_i \oplus \bigoplus_{\alpha: i \rightarrow k} \varphi_k M_\alpha : M_i \rightarrow V_i \oplus \bigoplus_{i \rightarrow k} \bigoplus_{k \rightsquigarrow j} V_j \simeq \bigoplus_{i \rightsquigarrow j} V_j.$$

Remark: We see that we can recover f from $\Phi_{(M,f)}$ by defining f_i as $\varphi_i : M_i \rightarrow \bigoplus_{i \rightsquigarrow j} V_j$, followed by the projection to V_i .

Lemma 3.7 The subspace $\text{Ker}\Phi_{(M,f)}$ is the maximal subrepresentation of M contained in $\text{Ker}f$.

Proof: Suppose $\varphi_i m = 0$ for some $m \in M_i$. Then $\varphi_j M_\alpha m = 0$ for all $\alpha : i \rightarrow j$, thus $M_\alpha m \in \text{Ker}\varphi_j$. This proves that $M_\alpha(\text{Ker}\varphi_i) \subset \text{Ker}\varphi_j$ for all $\alpha : i \rightarrow j$, so that $\text{Ker}\varphi$ is a subrepresentation of M .

Now suppose $U \subset M$ is a subrepresentation contained in the kernel of f . We prove by “downward induction over Q ” that $U_i \subset \text{Ker}\varphi_i$ for all $i \in I$. If $i \in I$ is a sink in Q , then $\varphi_i = f_i$ by definition, thus $U \subset \text{Ker}f_i = \text{Ker}\varphi_i$. For arbitrary $i \in I$, we have $f_i(U) = 0$ and $M_\alpha(U_i) \subset U_j \subset \text{Ker}\varphi_j$ by induction, thus $(\varphi_j M_\alpha)(U_i) = 0$. By definition of φ_i , this means $\varphi_i(U_i) = 0$. ■

Corollary 3.8 The map $\Phi_{(M,f)} : M \rightarrow I \otimes V$ is injective if and only if the pair (M, f) is stable.

Proof: The pair (M, f) is stable if and only if there is no non-zero subrepresentation U of M such that $U \subset \text{Ker}f$. But $\text{Ker}\Phi_{(M,f)}$ is the maximal such subrepresentation. ■

Remark: Under the chain of isomorphisms

$$\text{Hom}_Q(M, I \otimes V) \simeq \bigoplus_{i \in I} \text{Hom}_Q(M, I_i) \otimes_k V_i \simeq \bigoplus_{i \in I} M_i^* \otimes_k V_i \simeq \bigoplus_{i \in I} \text{Hom}_k(M_i, V_i),$$

the map $f = (f_i)_{i \in I} \in \bigoplus_{i \in I} \text{Hom}_k(M_i, V_i)$ corresponds to the map $\Phi_{(M,f)} \in \text{Hom}_Q(M, I \otimes V)$.

Proposition 3.9 The framed moduli space $\mathcal{M}_{d,n}$ is isomorphic to the Grassmannian of subrepresentations $\text{Gr}_d(I \otimes V)$.

Proof: Consider the map $\Phi : R_{d,n} \rightarrow \text{Hom}_k(M, I \otimes V)$ given by $\Phi(M, F) = \Phi_{(M,f)}$. By Corollary 3.8, it maps $R_{d,n}^s$ to the open subset $\text{IHom}_k(M, I \otimes V)$ of injective linear maps. The map Φ is also easily seen to be G_d -equivariant, for the G_d -action on $R_{d,n}$ defined in section 2, and the action on $\text{Hom}_k(M, I \otimes V)$ given by base change in M . By Lemma 3.7, the image belongs to $\text{IHom}_d(I \otimes V)$. On the other hand, given a point in $\text{IHom}_d(I \otimes V)$, we can recover both the

representation M by Lemma 3.5, and the map $M \rightarrow V$ by the remark following Definition 3.6. Thus, we have a G_d -equivariant isomorphism $\Phi : R_{d,n}^s \xrightarrow{\sim} \mathrm{IHom}_d(I \otimes V)$. Consequently, the map Φ descends to an isomorphism of the geometric quotients

$$\overline{\Phi} : \mathcal{M}_{d,n} \simeq R_{d,n}^s / G_d \rightarrow \mathrm{IHom}_d(I \otimes V) / G_d \simeq \mathrm{Gr}_d(I \otimes V).$$

■

4 Geometry of framed moduli

4.1 Nonemptiness of $\mathcal{M}_{d,n}$

One of the problems posed by H. Nakajima in [20] on framed quiver moduli $\mathcal{M}_{d,n}$ is to give a criterion for their nonemptiness. We will derive such a criterion using the interpretation of section 3.2 and properties of injective coresolutions of representations of quivers.

Lemma 4.1 *Given a representation M , there exists a map $f : M \rightarrow V$ making the pair $(M, f) \in R_{d,n}$ stable if and only if $n_i \geq \dim \mathrm{Hom}_Q(E_i, M)$ for all $i \in I$.*

Proof: As in the preceding section, a stable pair (M, f) induces an injection of representations $0 \rightarrow M \rightarrow I \otimes V$. Application of the functor $\mathrm{Hom}_Q(E_i, _)$ yields an embedding $0 \rightarrow \mathrm{Hom}_Q(E_i, M) \rightarrow \mathrm{Hom}_Q(E_i, I \otimes V) \simeq V_i$, thus the claimed inequality is necessary for stability of (M, f) .

We thus have to prove sufficiency. The simple representation E_i admits a projective resolution $0 \rightarrow \bigoplus_{i \rightarrow j} P_j \rightarrow P_i \rightarrow E_i \rightarrow 0$. Application of the functor $\mathrm{Hom}_Q(_, M)$ gives an exact sequence

$$0 \rightarrow \mathrm{Hom}_Q(E_i, M) \rightarrow \underbrace{\mathrm{Hom}_Q(P_i, M)}_{\simeq M_i} \rightarrow \bigoplus_{i \rightarrow j} \mathrm{Hom}_Q(P_j, M) \simeq \bigoplus_{i \rightarrow j} M_j,$$

so that $\mathrm{Hom}_Q(E_i, M)$ can be identified with $\mathrm{Ker}(\bigoplus_{\alpha: i \rightarrow j} M_\alpha)$. For any $i \in I$, we can choose direct sum decompositions

$$M_i = \mathrm{Ker}(\bigoplus_{\alpha} M_\alpha) \oplus W_i \text{ and } V_i \simeq \mathrm{Ker}(\bigoplus_{\alpha} M_\alpha) \oplus V'_i,$$

since $\dim V_i = n_i \geq \dim \mathrm{Hom}_Q(E_i, M) = \dim \mathrm{Ker}(\bigoplus_{\alpha} M_\alpha)$. Define f_i as the projection along W_i , followed by the inclusion into V_i . Now suppose that $U = (U_i)_i$ is a subrepresentation of M contained in $\mathrm{Ker}(f) = W_i$. We prove $U_i = 0$ for all $i \in I$ by descending induction on the partially ordered set I . If i is a sink in Q , then $W_i = 0$, thus $U_i = 0$. For arbitrary $i \in I$, we have $M_\alpha(U_i) \subset U_j$ for all $\alpha : i \rightarrow j$. But $U_j = 0$ by induction, thus $U_i \subset \mathrm{Ker} \bigoplus_{\alpha: i \rightarrow j} M_\alpha$, which intersects W_i trivially by definition. Thus $U_i = 0$.

■

Corollary 4.2 *Any representation M admits an injective coresolution*

$$0 \rightarrow M \rightarrow \bigoplus_{i \in I} I_i \otimes \text{Hom}_Q(E_i, M) \rightarrow \bigoplus_{i \in I} I_i \otimes \text{Ext}_Q^1(E_i, M) \rightarrow 0.$$

Proof: The above lemma shows the existence of a stable pair

$$(M, f : M \rightarrow \bigoplus_{i \in I} \text{Hom}_Q(E_i, M)),$$

which corresponds to an embedding $0 \rightarrow M \rightarrow \bigoplus_{i \in I} I_i \otimes \text{Hom}_Q(E_i, M)$. Since the path algebra kQ is hereditary, the cokernel is again injective, thus of the form $I \otimes X$ for a graded space $X = (X_i)_i$. Application of $\text{Hom}_Q(E_i, -)$ to the resulting exact sequence gives

$$\begin{aligned} 0 \rightarrow \text{Hom}_Q(E_i, M) &\rightarrow \text{Hom}_Q(E_i, \underbrace{\bigoplus_{j \in I} I_j \otimes \text{Hom}_Q(E_j, M)}_{\simeq \text{Hom}_Q(E_i, M)}) \rightarrow \\ &\rightarrow \text{Hom}_Q(E_i, \underbrace{\bigoplus_{j \in I} I_j \otimes X_j}_{\simeq X_i}) \rightarrow \text{Ext}_Q^1(E_i, M) \rightarrow 0, \end{aligned}$$

and thus $X_i \simeq \text{Ext}_Q^1(E_i, M)$. ■

Proposition 4.3 *We have $\mathcal{M}_{d,n} \neq \emptyset$ if and only if $n_i \geq \langle i, d \rangle$ for all $i \in I$.*

Proof: That $n_i \geq \langle i, d \rangle$ is a necessary condition for non-emptiness of $\mathcal{M}_{d,n}$ follows from Lemma 4.1 since, for any representation M of dimension type d , we have

$$\langle i, d \rangle = \dim \text{Hom}_Q(E_i, M) - \dim \text{Ext}_Q^1(E_i, M).$$

Given $d \in \mathbf{NI}$, one can clearly define a representation M of dimension type d such that, for all $i \in I$, the rank of the map $\bigoplus_{\alpha: i \rightarrow j} M_\alpha$ is maximal. Then $\dim \text{Hom}_Q(E_i, M) = \dim \text{Ker } \bigoplus_{\alpha: i \rightarrow j} M_\alpha = \max(0, \langle i, d \rangle)$. This representation thus admits a map $f : M \rightarrow V$ yielding a stable pair (M, f) , again by Lemma 4.1. ■

4.2 An embedding of $\mathcal{M}_{d,n}$

Recall from section 3.2 the embedding $\bar{\Phi} : \mathcal{M}_{d,n} \rightarrow \prod_{i \in I} \text{Gr}_{d_i}(\bigoplus_{i \rightsquigarrow j} V_j)$ given by $\bar{\Phi}(M, f) = \text{Im } \Phi_{(M, f)}$. In this section, we give an explicit description of its image, that is, of the inclusion

$$\text{Gr}_d(I \otimes V) \subset \prod_{i \in I} \text{Gr}_{d_i}(\bigoplus_{i \rightsquigarrow j} V_j).$$

Proposition 4.4 *A point $(U_i \subset \bigoplus_{i \rightsquigarrow j} V_j)_{i \in I}$ belongs to the image of $\overline{\Phi}$ if and only if for all $i \in I$, we have $U_i \subset V_i \oplus \bigoplus_{\alpha: i \rightarrow j} U_j$.*

Proof: Given $(M, f) \in \mathcal{M}_{d,n}$, let $\Phi_{(M,f)} = \varphi = (\varphi_i)_{i \in I}$ be the tuple of injective maps constructed in Definition 3.6, and let U_i be the image of φ_i . Then

$$U_i = \text{Im}(f_i \oplus \bigoplus_{\alpha: i \rightarrow j} \varphi_j M_\alpha) \subset \text{Im} f_i \oplus \bigoplus_{\alpha: i \rightarrow j} \underbrace{\text{Im} \varphi_j M_\alpha}_{\subset \text{Im} M_\alpha} \subset V_i \oplus \bigoplus_{\alpha: i \rightarrow j} U_j.$$

Conversely, assume that a tuple of subspaces $U_i \subset \bigoplus_{i \rightsquigarrow j} V_j$ of dimensions $\dim U_i = d_i$ is given such that $U_i \subset V_i \oplus \bigoplus_{\alpha: i \rightarrow j} U_j$ for all $i \in I$. We construct a stable pair (M, f) such that U equals the image of $\text{Im} \Phi_{(M,f)}$. For each vertex $i \in I$, choose a linear map $\varphi_i : M_i \rightarrow \bigoplus_{i \rightsquigarrow j} V_j$ with image U_i . Define the linear map f_i as the composition

$$f_i : M_i \xrightarrow{\varphi_i} \bigoplus_{i \rightsquigarrow j} V_j \xrightarrow{\text{pr}_{V_i}} V_i.$$

For each arrow $\alpha : i \rightarrow j$ in Q , consider the composition

$$h_\alpha : M_i \xrightarrow{\varphi_i} \bigoplus_{i \rightsquigarrow k} V_k \simeq V_i \oplus \bigoplus_{i \rightarrow l} \bigoplus_{l \rightsquigarrow k} V_k \xrightarrow{\text{pr}_V} \bigoplus_{j \rightsquigarrow k} V_k,$$

where pr_α denotes the projection map. By assumption, we have

$$\text{Im} h_\alpha = \text{Im pr}_\alpha \varphi_i = \text{pr}_\alpha U_i \subset \text{pr}_\alpha (V_i \oplus \bigoplus_{i \rightarrow k} U_k) \subset U_j = \text{Im} \varphi_j,$$

thus h_α factors through φ_j , providing a map $M_\alpha : M_i \rightarrow M_j$ such that $h_\alpha = M_\alpha \varphi_j$. From the construction, it is now clear that

$$\varphi_i = f_i \oplus \bigoplus_{\alpha: i \rightarrow j} \varphi_j M_\alpha.$$

Moreover, the maps φ_i are injective by definition, thus Lemma 3.8 implies that the pair $(M = (M_\alpha)_\alpha, f = (f_i)_{i \in I})$ is stable. ■

Examples:

1. Let Q be the quiver $i \rightarrow j \rightarrow k$. Then $\mathcal{M}_{d,n}$ is embedded into

$$\text{Gr}_{d_1}(V_1 \oplus V_2 \oplus V_3) \times \text{Gr}_{d_2}(V_2 \oplus V_3) \times \text{Gr}_{d_3}(V_3),$$

and the image consists of those points (U_1, U_2, U_3) of this product of Grassmannians such that $U_1 \subset V_1 \oplus U_2$ and $U_2 \subset V_2 \oplus U_3$.

2. We recall the main example given in [20]. Let Q be the quiver $1 \rightarrow \dots \rightarrow r$ of type A_r . Assume $n_i = 0$ for $i \neq r$, and $d_1 < d_2 < \dots < d_r < n_r$. Then $\mathcal{M}_{d,n}$ is isomorphic to the partial flag variety of flags in V_r with successive steps of dimensions $d_1, d_2, \dots, d_r, n_r$.

3. More generally, let $\mathcal{P}$ be a finite poset with a unique maximal element ∞ . Define a quiver $Q_{\mathcal{P}}$ with set of vertices $\mathcal{P}$, and an arrow $p \rightarrow q$ if and only if $p < q$ in $\mathcal{P}$ minimally. Define n_p as 0 if $p \neq \infty$. Then, for arbitrary $d \in \mathbf{N}\mathcal{P}$, the variety $\mathcal{M}_{d,n}(Q_{\mathcal{P}})$ parametrizes tuples $(U_p)_{p \in \mathcal{P}}$ of subspaces of V_{∞} such that $U_p \subset U_q$ whenever $p < q$ in $\mathcal{P}$.
4. Let Q be the quiver $i \xrightarrow{2} j$. Then $\mathcal{M}_{d,n}$ parametrises pairs of subspaces $(U_1 \subset V_1 \oplus V_2^2, U_2 \subset V_2)$ such that $U_1 \subset V_1 \oplus U_2^2$.

4.3 $\mathcal{M}_{d,n}$ as an iterated Grassmann bundle

In this section we describe $\mathcal{M}_{d,n}$ as an iterated Grassmann bundle. We first recall their definition (see [11]). Given a vector bundle $p : E \rightarrow X$ on a variety X , we denote by E_x its fibre over a point $x \in X$.

Proposition 4.5 *Let X be a variety, let $p : E \rightarrow X$ be a vector bundle, and let d be an integer $0 < d < \text{rk} E$. Then there exists a variety (the Grassmann bundle) $\text{Gr}_d(E)$, together with a projection $\pi : \text{Gr}_d(E) \rightarrow X$ and a subbundle $S \subset \pi^* E$ (the tautological bundle), defined by the following universal property:*

For any morphism $f : T \rightarrow X$, the map $g \mapsto g^ S$ induces a functorial bijection between factorizations $g : T \rightarrow \text{Gr}_d(E)$ of f through π and rank d -subbundles of $f^* E$. In particular, we have $\pi^{-1}(x) \simeq \text{Gr}_d(E_x)$ for each $x \in X$.*

Remark: The morphism $\text{Gr}_d(E) \rightarrow X$ is thus defined as an object in the category of X -varieties representing the functor mapping an object $f : T \rightarrow X$ to the set of rank d -subbundles of $f^* E$.

We will use the construction of Grassmann bundles only in the following special situation:

Lemma 4.6 *Let X be a variety, and let E be a subbundle of a trivial vector bundle $V \times X$ on X . Then the closed subvariety Y of $\text{Gr}_d(V) \times X$ consisting of pairs (U, x) such that $U \subset E_x$ is isomorphic to $\text{Gr}_d(E)$, with tautological bundle $S = \{(v, (U, x)) \in V \times Y : v \in U \subset E_x\}$.*

Proof: The universal property defining $\text{Gr}_d(E)$ can be verified immediately. ■

To prepare the application of Grassmann bundles to the present setting, we first use the embedding of $\mathcal{M}_{d,n}$ in $\prod_{i \in I} \text{Gr}_{d_i}(\bigoplus_{i \rightsquigarrow j} V_j)$ to define certain universal bundles $\mathcal{V}_i$ for $i \in I$ on $\mathcal{M}_{d,n}$ as follows:

Definition 4.7 *Denote by $\mathcal{V}_i$ the subbundle of the trivial bundle $\bigoplus_{i \rightsquigarrow j} V_j \times \mathcal{M}_{d,n}$ on $\mathcal{M}_{d,n}$ consisting of pairs $(v, (U_i)_i)$ such that $v \in U_i$. For each arrow $\alpha : i \rightarrow j$ in Q , define a map $\mathcal{V}_{\alpha} : \mathcal{V}_i \rightarrow \mathcal{V}_j$ by $\mathcal{V}_{\alpha}(v, (U_i)_i) = (\text{pr}_{\alpha} v, (U_i)_i)$, where $\text{pr}_{\alpha} : \bigoplus_{i \rightsquigarrow k} V_k \rightarrow \bigoplus_{j \rightsquigarrow k} V_k$ denotes the natural projection.*

Remark: The discussion of the maps h_{α} in the proof of Proposition 4.4 shows that the maps $\mathcal{V}_{\alpha}$ are indeed well-defined.

Lemma 4.8 *The tuple $((\mathcal{V}_i)_{i \in I}, (\mathcal{V}_\alpha)_\alpha)$ is the universal quiver representation on $\mathcal{M}_{d,n}$, that is, for each point in $(M, f) \in \mathcal{M}_{d,n}$, the quiver representation induced by the universal one on the fibre over (M, f) is isomorphic to M .*

Proof: This follows analogously to the reconstruction of the maps M_α from the maps h_α in the proof of Proposition 4.4. ■

Now we come to the central construction in this section:

Let i_0 be a source in the quiver Q , and let $\overline{Q}$ be the full subquiver of Q supported on $\overline{I} := I \setminus \{i_0\}$. Let $\overline{d}, \overline{n}$ be the restrictions to $\overline{I}$ of d and n , respectively.

Using Proposition 4.4, we can realize $\mathcal{M}_{d,n}(Q)$ as the closed subvariety of $\text{Gr}_{d_{i_0}}(\bigoplus_{i_0 \rightsquigarrow j} V_j) \times \mathcal{M}_{\overline{d}, \overline{n}}(\overline{Q})$ consisting of pairs $(U, (U_i)_{i \in \overline{I}})$ such that $U \subset V_{i_0} \oplus \bigoplus_{i_0 \rightarrow j} U_j$. This induces a projection $\mathcal{M}_{d,n}(Q) \rightarrow \mathcal{M}_{\overline{d}, \overline{n}}(\overline{Q})$. It is also obvious from Definition 4.7 that, for all $i \in \overline{I}$, the universal bundles $\mathcal{V}_i$ for $i \in \overline{I}$ of $\mathcal{M}_{d,n}(Q)$ and of $\mathcal{M}_{\overline{d}, \overline{n}}(\overline{Q})$ are compatible with this realization. Applying Lemma 4.6, we get immediately:

Lemma 4.9 *The projection $\mathcal{M}_{d,n}(Q) \rightarrow \mathcal{M}_{\overline{d}, \overline{n}}(\overline{Q})$ is isomorphic to the Grassmann bundle $\text{Gr}_{d_{i_0}}(V_{i_0} \oplus \bigoplus_{i_0 \rightarrow j} \mathcal{V}_j)$, with tautological bundle $\mathcal{V}_{i_0}$.*

Choose an enumeration of I as $\{1, \dots, s\}$ such that $i < j$ whenever $i \rightarrow j$. Repeated application of the previous lemma yields the main result of this section.

Theorem 4.10 *Define a chain of iterated Grassmann bundles*

$$M_1 \xrightarrow{p_1} M_2 \xrightarrow{p_2} \dots \xrightarrow{p_{n-1}} M_n \xrightarrow{p_n} \text{pt}$$

by

$$M_i = \text{Gr}_{d_i}(V_i \oplus \bigoplus_{i \rightarrow j} p_{i+1}^* \dots p_{j-1}^*(S_j)),$$

where S_i denotes the tautological bundle on M_i .

Then $\mathcal{M}_{d,n} \simeq M_1$, with universal bundles $\mathcal{V}_i \simeq p_1^* \dots p_{i-1}^* S_i$ for all $i \in I$.

5 Cohomology

In this section, we will use the realization of Lemma 4.9 of the framed moduli $\mathcal{M}_{d,n}$ to determine its Chow ring. We collect some results on Chow rings of Grassmann bundles from [11].

Slightly different notation from the one in [11] will be convenient, which we introduce now. Given a smooth irreducible variety X , let $A^*(X) = \bigoplus_i A^i(X)$ be its Chow ring, where $A^i(X)$ denotes the group of codimension i cycles modulo rational equivalence, and the ring structure is given by the intersection product. Given a vector bundle E on X , its Chern classes $c_i(E)$ for $i \geq 0$ will be considered as elements of $A^i(X)$ (they are considered as degree i endomorphisms

of $A^*(X)$ in [11]; applying these to the fundamental class $[X] \in A^0(X)$ yields the elements considered here). We also consider the formal Chern polynomial $c_t(E) := \sum_{i=0}^{\infty} c_i(X)t^i \in A^*(X)[[t]]$.

Given any formal power series $P = \sum_{i=0}^{\infty} c_i t^i \in A[[t]]$ with coefficients in a commutative ring A and a partition $\lambda = (\lambda_1 \geq \dots \geq \lambda_d \geq 0)$, we define the formal Schur polynomial $\Delta_\lambda(P) \in A$ as the determinant of the $d \times d$ -matrix $(c_{\lambda_i + j - i})_{i,j}$.

Theorem 5.1 ([11]) *Let X be a smooth irreducible variety, let E be a vector bundle on X , and let $0 < d < \text{rk} E$ be an integer. We consider the corresponding Grassmann bundle $\pi : \text{Gr}_d(E) \rightarrow X$ with tautological bundle S .*

1. *For all $i \in \mathbf{N}$, there is an isomorphism $A^i(\text{Gr}_d(E)) \simeq \bigoplus_{\lambda} A^{i-|\lambda|}(X)$, where the direct sum runs over partitions $\lambda = (\text{rk} E - d \geq \lambda_1 \geq \dots \geq \lambda_d \geq 0)$, and $|\lambda| = \sum_k \lambda_k$. Each element of $A^i(\text{Gr}_d(E))$ has a unique expression in the form $\sum_{\lambda} \Delta_{\lambda}(c_t(S)^{-1}) \cdot \pi^* \alpha_{\lambda}$ for elements $\alpha_{\lambda} \in A^{i-|\lambda|}(X)$, the sum being as above.*
2. *The ring $A^*(\text{Gr}_d(E))$ is generated by $A^*(X)$, together with the Chern classes $c_1(S), \dots, c_d(S)$, with the following defining relation: the formal power series $c_t(S)^{-1}$ is a polynomial of degree at most $\text{rk} E - d$.*
3. *For the base field $k = \mathbf{C}$, the cycle map $A^*(\text{Gr}_d(E)) \rightarrow H^*(\text{Gr}_d(E))$ to singular cohomology is an isomorphism (in particular, there is no odd cohomology) if and only if the cycle map $A^*(X) \rightarrow H^*(X)$ is so.*
4. *The Poincaré polynomial $P_{\text{Gr}_d(E)}(q) := \sum_{i=0}^{\infty} \dim A^i(X) q^i$ is given by $P_X(q) \cdot \begin{bmatrix} \text{rk} E \\ d \end{bmatrix}$, where $\begin{bmatrix} n \\ d \end{bmatrix} = \prod_{k=1}^d \frac{q^{n-d+k}-1}{q^k-1}$ denotes the q -binomial coefficient.*

Proof: The first part is precisely [11, Proposition 14.6.5] translated to our conventions. Note that the Schur polynomials there are taken in a certain formal power series $c_t(Q - \pi^* E)$ in terms of the sequence of vector bundles $0 \rightarrow S \rightarrow \pi^* E \rightarrow Q \rightarrow 0$, which identifies with $c_t(S)^{-1}$ by the Whitney sum formula [11, Theorem 3.2]. The second part is a reformulation of [11, Example 14.6.6], where the ring structure is described by generators $c_1(S), \dots, c_d(S)$ and $c_1(Q), \dots, c_{\text{rk} E - d}(Q)$ (for the bundle Q defined above) subject to the relation $c_t(S) \cdot c_t(Q) = c_t(\pi^*(E))$. Thus, as a formal power series, $c_t(Q)$ is given by $c_t(\pi^*(E))/c_t(S)$, the only restriction being that this is actually a polynomial of degree $\leq \text{rk} E - d$. The third part is [11, Example 19.1.11 d)], together with Poincaré duality. The fourth part follows from the first by some standard combinatorics. ■

Repeated application of this theorem to the chain of Grassmann bundles in Theorem 4.10 easily yields the following description of the Chow ring of $\mathcal{M}_{d,n}$:

Theorem 5.2 *For all $\mathcal{M}_{d,n} \neq 0$, the following holds:*

1. The Chow ring $A^\bullet(\mathcal{M}_{d,n})$ has a linear basis

$$\left\{ \prod_{i \in I} \Delta_{\lambda^i} (c_t(\mathcal{V}_i)^{-1}), \right.$$

indexed by tuples of partitions $(\lambda^i)_{i \in I}$ such that

$$n_i - \langle i, d \rangle \geq \lambda_1^i \geq \dots \geq \lambda_{d_i}^i \geq 0.$$

2. As a ring, $A^\bullet(\mathcal{M}_{d,n})$ is generated by the Chern classes of the universal bundles $\mathcal{V}_i$ for $i \in I$, together with the following defining relations:

For each $i \in I$, the formal power series

$$\frac{\prod_{i \rightarrow j} c_t(\mathcal{V}_j)}{c_t(\mathcal{V}_i)}$$

is a polynomial of degree at most $\leq n_i - \langle i, d \rangle$.

3. Assume $k = \mathbf{C}$. Then the odd cohomology $H^{2\bullet+1}(\mathcal{M}_{d,n})$ vanishes, and the even cohomology $H^{2\bullet}(\mathcal{M}_{d,n})$ is isomorphic to the Chow ring $A^\bullet(\mathcal{M}_{d,n})$.

4. The Poincaré polynomial of $\mathcal{M}_{d,n}$ is given by

$$\sum_{k=0}^{\infty} \dim A^k(\mathcal{M}_{d,n}) q^k = \prod_{i \in I} \left[\begin{matrix} n_i + \sum_{i \rightarrow j} d_j \\ d_i \end{matrix} \right].$$

Remark: The third part of the theorem is also implied by a theorem of [16], where this property is proved in the generality of quiver moduli for which stability and semistability coincide. There it is also proved that, in this case, the Chern classes of the universal bundles generate the Chow ring (see also [10, 20]).

Note that the description of a basis of the Chow ring in terms of Schur polynomials allows application of all classical formulas of Schubert calculus, like for example the Pieri rule, Giambelli's formula and the Littlewood-Richardson rule (see [11]).

6 Orbit structure

In this section, we define a group action on the framed moduli $\mathcal{M}_{d,n}$ and relate it to the action of the group G_d on the representation variety R_d .

Recall from the proof of Proposition 3.9 the G_d -equivariant isomorphism $\Phi : R_{d,n}^s \xrightarrow{\sim} \mathrm{IHom}_d(I \otimes V)$. On the right hand side, we have a natural action of $\mathrm{Aut}_Q(I \otimes V)$, which translates into an action on $R_{d,n}^s$. This latter action obviously commutes with the G_d -action, thus it induces an action of $\mathrm{Aut}_Q(I \otimes V)$ on the quotient $\mathcal{M}_{d,n}$.

To identify this action in terms of the embedding $\mathcal{M}_{d,n} \subset \prod_{i \in I} \text{Gr}_{d_i}(\bigoplus_{i \rightsquigarrow j} V_j)$, we consider

$$\text{End}_Q(I \otimes V) \simeq \bigoplus_{i,j} \text{Hom}_Q(I_i \otimes V_i, I_j \otimes V_j) \simeq \bigoplus_{i \rightsquigarrow j} \text{Hom}_k(V_i, V_j)$$

as a subspace of

$$\bigoplus_{i \in I} \text{End}_k((I \otimes V)_i) \simeq \text{End}_k(\bigoplus_{i \rightsquigarrow j} V_j).$$

Since Q has no oriented cycles, the group $\text{Aut}_Q(I \otimes V)$ identifies with the set of tuples $(f_\omega)_\omega \in \bigoplus_{\omega: i \rightsquigarrow j} \text{Hom}_k(V_i, V_j)$ such that all components $f_{\omega: i \rightsquigarrow i}$ corresponding to the empty paths are invertible.

We thus arrive at an embedding of

$$A_n := \{(f_\omega)_\omega \in \bigoplus_{\omega: i \rightsquigarrow j} \text{Hom}_k(V_i, V_j); \text{ all } f_{\omega: i \rightsquigarrow i} \text{ invertible}\}$$

into $\prod_{i \in I} \text{GL}_k(\bigoplus_{i \rightsquigarrow j} V_j)$, which proves:

Lemma 6.1 *The canonical action of the group $\prod_{i \in I} \text{GL}_k(\bigoplus_{i \rightsquigarrow j} V_j)$ on the variety $\prod_{i \in I} \text{Gr}_{d_i}(\bigoplus_{i \rightsquigarrow j} V_j)$ induces an action of the subgroup $A_n \simeq \text{Aut}_Q(I \otimes V)$ on $\mathcal{M}_{d,n}$.*

Example: For the quiver $Q = i \rightarrow j \rightarrow k$, the group A_n is the subgroup of

$$\text{GL}(V_1 \oplus V_2 \oplus V_3) \times \text{GL}(V_2 \oplus V_3) \times \text{GL}(V_3)$$

given by block matrices of the form

$$\left(\begin{bmatrix} a & & \\ b & d & \\ c & e & f \end{bmatrix}, \begin{bmatrix} d & \\ e & f \end{bmatrix}, [f] \right)$$

such that the matrices a , d and f are invertible.

To analyse this group action on $\mathcal{M}_{d,n}$, we first record two simple representation-theoretic facts.

Lemma 6.2 *Given a representation M and two embeddings i, j of M into an injective representation I , any automorphism φ of M extends to an automorphism ψ of I , such that $j\varphi = \psi i$.*

Proof: The image of i being contained in an injective hull I' of M , we can write $I = I' \oplus I''$ and $i = \begin{bmatrix} i' \\ 0 \end{bmatrix}$. The map $j\varphi$ factors through i , yielding maps $\psi' : I' \rightarrow I$ and $\psi'' : I'' \rightarrow I$ such that $\psi' i' = j\varphi$. Since I' is an injective hull of M , injectivity of $j\varphi$ implies injectivity of ψ' . Without loss of generality we can assume ψ'' to be injective, too. The map $\psi = [\psi', \psi'']$ is the claimed extension. $\blacksquare$

Lemma 6.3 *Two subrepresentations of an injective representation I are conjugate under the action of $\text{Aut}_Q(I)$ if and only if they are isomorphic. The stabilizer $\text{Iso}_{\text{Aut}_Q(I)}(U)$ is the extension of the automorphism group $\text{Aut}_Q(U)$ by a unipotent group.*

Proof: If two subrepresentations are conjugate under $\text{Aut}_Q(I)$, they are clearly isomorphic. The converse follows from the above lemma. To analyse the stabilizer, choose an embedding $i : M \rightarrow I$ and consider the group $G := \{(\varphi, \psi) \in \text{Aut}_Q(M) \times \text{Aut}_Q(I) : i\varphi = \psi i\}$. Projection to the second component yields an isomorphism between G and $\text{Stab}_{\text{Aut}_Q(I)}(U)$, since the automorphism φ is uniquely determined by ψ , if it exists. Again by the above lemma, projection to the first component yields a surjective map to $\text{Aut}_Q(M)$, whose kernel consists of the automorphisms φ of M such that $\varphi i = i$. This condition is equivalent to φ being contained in $1 + \text{Hom}_Q(I/M, I)p$, where $p : I \rightarrow I/M$ denotes the canonical projection. This last group is clearly unipotent. ■

Denote by $R_d^{(n)}$ the image of the natural projection $p : R_{d,n}^s \rightarrow R_d$. By Lemma 4.1, $R_d^{(n)}$ consists of the representations M such that $\dim \text{Hom}_Q(E_i, M) \leq n_i$ for all $i \in I$. Since, by Corollary 3.4, the map p can be factored into an open embedding followed by a projection along an affine space, it is smooth. Furthermore, the quotient map $\pi : R_{d,n}^s \rightarrow \mathcal{M}_{d,n}$ is smooth by Corollary 3.4.

Thus, we arrive at a diagram of smooth morphisms

$$R_d^{(n)} \xleftarrow{p} R_{d,n}^s \xrightarrow{\pi} \mathcal{M}_{d,n}.$$

The map p is G_d -equivariant and A_n -invariant, whereas the map π is A_n -equivariant and a G_d -quotient. We can therefore conclude:

Theorem 6.4 *The map $X \mapsto \pi p^{-1}X$ gives a bijection between A_n -stable subvarieties of $\mathcal{M}_{d,n}$ and G_d -stable subvarieties of $R_d^{(n)}$, such that inclusions, closures, irreducibility and types of singularities are preserved.*

Proof: The map $X \mapsto \pi p^{-1}X$ maps a single G_d -orbit in R_d to a single A_n -orbit in $\mathcal{M}_{d,n}$ by Lemma 6.3. By definition of the subset $R_d^{(n)}$, all A_n -orbits in $\mathcal{M}_{d,n}$ arise in this way. This proves bijectivity of the correspondence. All the geometric properties follow from the above mentioned properties of the morphisms p and π . ■

Remark: This result is also obtained, in the generality of representations of finite dimensional algebras, in [6].

The above theorem allows us to derive several results on the A_n -orbit structure of $\mathcal{M}_{d,n}$ from corresponding results on the action of G_d on R_d . We just give two example applications.

Applying Gabriel's Theorem, we get immediately:

Corollary 6.5 *If Q is of Dynkin type, that is, the underlying unoriented graph is a disjoint union of Dynkin diagrams of type A_l, D_l, E_6, E_7, E_8 , then A_n acts on $\mathcal{M}_{d,n}$ with finitely many orbits.*

Corollary 6.6 *If the quiver Q is of type A or D , all closures of A_n -orbits in $\mathcal{M}_{d,n}$ are normal and Cohen-Macaulay.*

Proof: This follows from the corresponding properties of closures of G_d -orbits in R_d , proven in [4], [5]. ■

In view of the description of the Chow ring of $\mathcal{M}_{d,n}$ of Theorem 5.2, one can ask the following question:

What is the class $[\overline{\mathcal{O}}] \in A^*(\mathcal{M}_{d,n})$ of the closure of an A_n -orbit in $\mathcal{M}_{d,n}$, given either in terms of the linear basis of products of Schur polynomials, or in terms of the Chern classes of universal bundles?

Example: If Q is the equioriented quiver $0 \rightarrow 1 \rightarrow \dots \rightarrow n$ of type A_{n+1} , we can compute, at least in principle, the classes of orbit closures in the Chow ring, applying the formula for degeneracy loci of [7]. This can be done as follows:

Using Theorem 6.4 and the description of the closures of G_d -orbits in R_d of [1], we see that the closures of A_n -orbits are precisely of the form

$$\{((M, f) \in \mathcal{M}_{d,n} : \text{rk}(M_{ij}) \leq r_{ij} \text{ for all } 0 \leq i < j \leq n)\},$$

where M_{ij} denotes the composite map $M_i \rightarrow \dots \rightarrow M_j$, and the tuple $r = r_{ij}$ is as in [7, (1.2)]. By the description of the universal quiver representation $\mathcal{V}$ on $\mathcal{M}_{d,n}$ of Definition 4.7, we see that this orbit closure is thus precisely the degeneracy locus $\Omega_r(\mathcal{V})$ of [7, (1.1)]. Thus, by the main result of [7], its class in the Chow ring can be written in the form

$$\sum_{\lambda} c_{\lambda}(r) \Delta_{\lambda^1}(c_t(\mathcal{V}_1)/c_t(\mathcal{V}_0)) \dots \Delta_{\lambda^n}(c_t(\mathcal{V}_n)/c_t(\mathcal{V}_{n-1})),$$

the sum running over tuples $\lambda^1, \dots, \lambda^n$ of partitions, and the $c_{\lambda}(r)$ being certain integer coefficients. Furthermore, by [7, Lemma 4], we have for any two bundles E, F on a variety X the formula

$$\Delta_{\lambda}(c_t(E)/c_t(F)) = \sum_{\mu, \nu} c_{\mu, \nu}^{\lambda} \Delta_{\mu}(c_t(E)^{-1}) \Delta_{\nu}(c_{-t}(F)^{-1}),$$

the $c_{\mu, \nu}^{\lambda}$ denoting Littlewood-Richardson coefficients. These two formulas allow us to express the class of the orbit closure in terms of the basis exhibited in Theorem 5.2.

7 Realizations of quantum groups

We first recall the quantum group construction of $\mathcal{U}_q(\mathfrak{gl}_n)$ of [3]. Let k be a finite field, and let $v \in \mathbf{C}$ be a square root of the cardinality of k . Let $\mathcal{U}_v(\mathfrak{gl}_n)$ be

the quantized enveloping algebra of $\mathfrak{gl}_n$, specialized at the quantum parameter $v \in \mathbf{C}$ (see [3]).

Let V be a k -vector space of dimension $N \in \mathbf{N}$. Let $\mathcal{F}(V)$ be the variety of n -step flags

$$0 = V_0 \subset V_1 \subset \dots \subset V_d = V$$

in V , which is a non-connected projective k -variety. We have an obvious action of $\mathrm{GL}(V)$ on $\mathcal{F}(V)$, thus a diagonal action on $\mathcal{F}(V) \times \mathcal{F}(V)$. Denote by

$$A = \mathbf{C}^{\mathrm{GL}(V)}(\mathcal{F}(V) \times \mathcal{F}(V))$$

the space of (arbitrary) $\mathrm{GL}(V)$ -invariant functions from $\mathcal{F}(V) \times \mathcal{F}(V)$ to $\mathbf{C}$. The space A carries an associative multiplication given by convolution:

$$(f \cdot g)(F_1, F_3) = \sum_{F_2 \in \mathcal{F}(V)} f(F_1, F_2)g(F_2, F_3)$$

for functions $f, g \in A$, and for all $F_1, F_3 \in \mathcal{F}(V)$.

Theorem 7.1 ([3]) *There exists a homomorphism of $\mathbf{C}$ -algebras*

$$\mathcal{U}_v(\mathfrak{gl}_n) \rightarrow A(V).$$

The aim of this section is to develop a partial generalization of this result. More precisely, let $C = (\langle i, j \rangle + \langle j, i \rangle)_{i, j \in I}$ be the matrix representing the symmetrization of the Euler form. This being a symmetric generalized Cartan matrix, we have an associated Kac-Moody Lie algebra $\mathfrak{g}$ with triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$. We denote by $\mathcal{U}_v(\mathfrak{n}^+)$ the quantized enveloping algebra of $\mathfrak{n}^+$ with quantum parameter v , which is a complex algebra. Our aim is to realize certain quotients of a modified version $\tilde{\mathcal{U}}_v(\mathfrak{b})$ of $\mathcal{U}_v(\mathfrak{n}^+)$ (see below for the definition) in terms of a convolution algebra related to the varieties $\mathcal{M}_{d,n}$.

We continue to denote by k a finite field with v^2 elements. We can easily repeat the definition of R_d , $\mathcal{M}_{d,n}$ etc. over k as sets. It is then easy to see that all set-theoretic results, like the description of the embedding of $\mathcal{M}_{d,n}$ into $\prod_{i \in I} \mathrm{Gr}_{d_i}(\bigoplus_{i \rightsquigarrow j} V_j)$, the existence of the group action $A_n : \mathcal{M}_{d,n}$ etc. continue to hold. So we will use all these notions and results freely in this section.

7.1 Review of Hall algebras

We recall the Hall algebra construction of C. M. Ringel [24]. Let $\mathbf{C}^{G_d}(R_d)$ be the space of G_d -invariant functions from R_d to $\mathbf{C}$. Define

$$H = H_v(Q) = \bigoplus_{d \in \mathbf{NI}} \mathbf{C}^{G_d}(R_d).$$

We define a $\mathbf{NI}$ -graded convolution on H as follows: let $f \in \mathbf{C}^{G_d}(R_d)$, $g \in \mathbf{C}^{G_e}(R_e)$ be two functions in H . Define their convolution $f \cdot g \in \mathbf{C}^{G_{d+e}}(R_{d+e})$ by

$$(f \cdot g)(X) = v^{\langle e, d \rangle} \sum_{U \subset X} f(U)g(X/U),$$

where the sum runs over all subrepresentations U of the representation X . Note that this is well-defined, since the sum is obviously finite, and the value of f on U (resp. of g on X/U) is well-defined since it only depends on the isomorphism class of U (resp. X/U).

Let E_i be the function on R_i sending the unique point to 1. Define $C = C_v(Q)$ as the $\mathbf{C}$ -subalgebra of H generated by the functions E_i for $i \in I$. We have

Theorem 7.2 ([14, 24]) *The map η sending the Chevalley generator E_i of $\mathcal{U}_v^+(\mathfrak{g})$ to the function E_i induces an isomorphism*

$$\eta : \mathcal{U}_v^+(\mathfrak{g}) \xrightarrow{\sim} C.$$

Moreover, if Q is of Dynkin type, then $C = H$.

For the following, we need a modified version of the Hall algebra, reminiscent of the modified quantized enveloping algebra of [18].

Definition 7.3 *Define a modified Hall algebra $\dot{H} = \dot{H}_v(Q)$ by adjoining to the algebra H pairwise orthogonal idempotents 1_d for $d \in \mathbf{Z}I$ fulfilling the relation $f1_d = 1_{d+|f|}$ for all $d \in \mathbf{Z}I$ and all homogeneous elements $f \in H$. Define $\dot{C} = \dot{C}_v(Q)$ as the subalgebra of $\dot{H}$ generated by the functions E_i and all idempotents 1_d for $d \in \mathbf{Z}I$.*

Note that, by the defining relations of $\dot{H}$, any element in $\dot{H}$ can be written uniquely in the form $\sum_{d \in \mathbf{Z}I} 1_d f_d$ for elements $f_d \in H$, almost all of which are zero. The multiplication in $\dot{H}$ is then given by the rule $1_d f \cdot 1_e g = \delta_{d,e+|f|} 1_d (f \cdot g)$ for homogeneous elements $f, g \in H$. More precisely, the Hall algebra H has a basis consisting of the characteristic functions $\chi_{[M]}$ of the G_d -orbits in the spaces R_d , which correspond bijectively to the isomorphism classes $[M]$ of representations of Q . Thus, we have a corresponding basis $1_d \cdot \chi_{[M]}$ of $\dot{H}$.

In the same way as for H , we define $\dot{\mathcal{U}}_v(\mathfrak{b})$ by adjoining orthogonal idempotents to $\mathcal{U}_v(\mathfrak{n}^+)$ (see [18]). Then we have:

Corollary 7.4 *The map η above extends to an isomorphism $\eta : \dot{\mathcal{U}}_v(\mathfrak{b}) \xrightarrow{\sim} \dot{C}$. Moreover, if Q is of Dynkin type, we have $\dot{C} = \dot{H}$.*

7.2 Quantum group construction from $\mathcal{M}_{d,n}$

Definition 7.5 1. Denote by $\mathcal{M}_{d,n}^{\text{proj}}$ the set of all pairs (M, f) in $\mathcal{M}_{d,n}$, such that M is a projective representation of Q .

2. Denote by $\mathcal{M}_n^{\text{proj}}$ the disjoint union of the $\mathcal{M}_{d,n}^{\text{proj}}$, for all $d \in \mathbf{N}I$.

3. Finally, denote by $\mathcal{X}_n$ the subvariety of $\mathcal{M}_n^{\text{proj}} \times \mathcal{M}_n^{\text{proj}}$ of pairs of tuples of subspaces $((U_i)_i, (U'_i)_i)$ such that U_i contains U'_i for all $i \in I$.

Remark: Since the projective representations in R_d form a single G_d -orbit (if any), the subset $\mathcal{M}_{d,n}^{\text{proj}}$ is either empty or a single A_n -orbit by the correspondence of Theorem 6.4. By the criterion 4.3 for non-emptiness of $\mathcal{M}_{d,n}$, the set $\mathcal{M}_n$ is a finite union of the $\mathcal{M}_{d,n}^{\text{proj}}$. Therefore, A_n acts on $\mathcal{M}_n$ with finitely many orbits.

Representation-theoretically, a point in $\mathcal{X}_n$ can be viewed as a diagram

$$P' \rightarrow P \rightarrow I \otimes V$$

of embeddings of representations of Q , where P and P' are both projective.

We first define a map ξ from the set $\mathcal{X}_n/A_n$ of A_n -orbits in $\mathcal{X}_n$ to the disjoint union $\bigcup_{d \in \mathbf{NI}} R_d/G_d$ of all G_d -orbits in all R_d as follows:

Definition 7.6 *Given a point $(U_*, U'_*) = ((U_i)_i, (U'_i)_i)$ in $\mathcal{X}_n$, let $\xi(U_*, U'_*) = U_*/U'_*$ be the representation of Q defined by the following tuple of k -vector spaces and k -linear maps:*

$$U_*/U'_* = ((U_i/U'_i)_{i \in I}, (\overline{\text{pr}}_\alpha : U_i/U'_i \rightarrow U_j/U'_j)_{\alpha: i \rightarrow j}).$$

We endow $C := \mathbf{C}^{A_n}(\mathcal{X}_n)$ with an algebra structure via convolution of functions:

$$(f \cdot g)(U_*, U''_*) = \sum_{U'_*} f(U_*, U'_*) g(U'_*, U''_*).$$

This algebra is $\mathbf{NI}$ -graded by defining the homogeneous component C_d as the space of functions supported on $(\bigcup_{e \in \mathbf{NI}} \mathcal{M}_{d+e,n} \times \mathcal{M}_e) \cap \mathcal{X}_n$.

Furthermore, we define the map $\underline{\dim}$ on $\mathcal{M}_n$ by $\underline{\dim}(M, f) = \underline{\dim} M$.

Proposition 7.7 *The map ξ induces a morphism $\eta : H \rightarrow C$ of $\mathbf{NI}$ -graded $\mathbf{C}$ -algebras defined by*

$$\eta(f)(U_*, U'_*) = v^{\langle \underline{\dim} U_* - \underline{\dim} U'_*, \underline{\dim} U'_* \rangle} f(U_*/U'_*).$$

Proof: Well-definedness of the map η follows from the above definition of the map ξ . The map η preserves the $\mathbf{NI}$ -gradings by their definition. It remains to show compatibility with the respective convolution structures. So let f and g be elements of H of degree d and e , respectively, and let (U_*, U''_*) be a point in $\mathcal{X}_n$. For the calculation of $\eta(f \cdot g)(U_*, U''_*)$, we can assume without loss of generality that $\underline{\dim} U_* = \underline{\dim} U''_* + d + e$; denote $c = \underline{\dim} U''_*$. By the definitions, we have

$$\eta(f \cdot g)(U_*, U''_*) = v^{\langle d+e, c \rangle} (f \cdot g)(M) = v^{\langle d+e, c \rangle + \langle e, d \rangle} \sum_Z f(Z) g((U_*/U''_*)/Z),$$

where the sum runs over all subrepresentations Z of U_*/U''_* of dimension type $\underline{\dim} M = d$. By definition of the representation U_*/U''_* , such subrepresentations

are in bijection with points U'_* in $\mathcal{M}_{c+d}^{\text{proj}}$ such that $U_* \supset U'_* \supset U''_*$ via the identification $Z = U'_*/U''_*$. Thus, we can rewrite this last sum as

$$v^{\langle e, c+d \rangle + \langle d, c \rangle} \sum_{U_* \supset U'_* \supset U''_*} f(U_*/U'_*)g(U'_*/U''_*) = \sum_{U'_*} \eta(f)(U_*, U'_*)\eta(g)(U'_*, U''_*),$$

which equals $(\eta(f) \cdot \eta(g))(U_*, U''_*)$, as stated. $\blacksquare$

For each $d \in \mathbf{ZI}$, define a function Δ_d on $\mathcal{X}_n$ by $\Delta_d(U_*, U'_*) = 0$ if $U_* \neq U'_*$ or $\underline{\dim} U_* \neq d$, and $\Delta_d(U_*, U_*) = 1$ if $\underline{\dim} U_* = d$.

Lemma 7.8 *The map η extends to an algebra morphism, again denoted by η , from $\hat{H}$ to $\mathbf{C}^{A_n}(\mathcal{X}_n)$, mapping 1_d to the function Δ_d .*

Proof: Obviously, the functions Δ_d are pairwise orthogonal idempotents in $\mathbf{C}^{A_n}(\mathcal{X}_n)$. The relation $f1_d = 1_{d+e}f$ is immediately verified. $\blacksquare$

To understand the image of the map η , we have to analyse the set of A_n -orbits in $\mathcal{X}_n$.

Lemma 7.9 *The A_n -orbit of a point (U_*, U'_*) in $\mathcal{X}_n$ is uniquely determined by the isomorphism class of the representation U_*/U'_* , together with the dimension type $\underline{\dim} U_* \in \mathbf{NI}$.*

Proof: Clearly, both the isomorphism class U_*/U'_* and the dimension type $\underline{\dim} U_*$ are constant along the A_n -orbit of (U_*, U'_*) . Conversely, assume that for points $({}^1U_*, {}^1U'_*)$ and $({}^2U_*, {}^2U'_*)$ in $\mathcal{X}_n$, the representations ${}^1U_*/{}^1U'_*$ and ${}^2U_*/{}^2U'_*$ are isomorphic, and $\underline{\dim} {}^1U_* = \underline{\dim} {}^2U_*$. We have to prove that these two points are conjugate under A_n . We choose diagrams

$$P_1 \xrightarrow{\alpha_1} P'_1 \xrightarrow{i_1} I \otimes V \text{ and } P_2 \xrightarrow{\alpha_2} P'_2 \xrightarrow{i_2} I \otimes V,$$

respectively, representing these two points. By assumption, we have $\underline{\dim} P_1 = \underline{\dim} P_2$ and $P_1/P'_1 \simeq P_2/P'_2 =: M$. We thus have exact sequences

$$0 \rightarrow P'_1 \rightarrow P_1 \rightarrow M \rightarrow 0 \text{ and } 0 \rightarrow P'_2 \rightarrow P_2 \rightarrow M \rightarrow 0.$$

Using the obvious dual version of Lemma 6.3, we can extend the identity map on M to an isomorphism $g : P_1 \rightarrow P_2$; this in turn yields an isomorphism $f : P'_1 \rightarrow P'_2$ such that $g\alpha_1 = \alpha_2f$. Applying Lemma 6.3 again, we get an automorphism ψ of $I \otimes V$ extending g , that is, $\psi i_1 = i_2g$. With the calculation

$$\psi i_1 \alpha_1 = i_2 g \alpha_1 = i_2 \alpha_2 f$$

we see that the map ψ also extends the map f along the embeddings $i_1 \alpha_1$ and $i_2 \alpha_2$, respectively. But this means precisely that $\psi \in \text{Aut}_Q(I \otimes V) = A_n$

conjugates the two given points in $\mathcal{X}_n$. The following commutative diagrams, the left one having exact rows, illustrate this construction:

$$\begin{array}{ccccccc} 0 & \rightarrow & P'_1 & \xrightarrow{\alpha_1} & P_1 & \rightarrow & M \rightarrow 0 \\ & & f \downarrow & & g \downarrow & & \parallel \\ 0 & \rightarrow & P'_2 & \xrightarrow{\alpha_2} & P_2 & \rightarrow & M \rightarrow 0 \end{array} \quad \begin{array}{ccccccc} P'_1 & \xrightarrow{\alpha_1} & P_1 & \xrightarrow{i_1} & I \otimes V \\ f \downarrow & & g \downarrow & & \psi \downarrow \\ P'_2 & \xrightarrow{\alpha_2} & P_2 & \xrightarrow{i_2} & I \otimes V \end{array}$$

■

This characterization of the A_n -orbits in $\mathcal{X}_N$ leads us to define, for all $d \in \mathbf{Z}I$ and all isomorphism classes $[M]$ of representations of Q , the set

$$\mathcal{O}_{d,[M]} := \{(U_*, U'_*) \in \mathcal{X}_n : \underline{\dim} U_* = d, U_*/U'_* \simeq M\}.$$

Proposition 7.10 *The set $\mathcal{O}_{d,[M]}$ is non-empty if and only if $\langle i, d \rangle \leq n_i$ and $\dim \operatorname{Hom}_Q(M, E_i) \leq \langle d, i \rangle$ for all $i \in I$. In this case, it is a single A_n -orbit in $\mathcal{X}_n$, and all such orbits arise in this way.*

Proof: By Lemma 7.9, we only have to prove the criterion for non-emptiness of $\mathcal{O}_{d,[M]}$. So suppose that $\mathcal{O}_{d,[M]} \neq \emptyset$. Any point (U_*, U'_*) in this set then provides projective representations P and P' , such that there exists an exact sequence $0 \rightarrow P' \rightarrow P \rightarrow M \rightarrow 0$, and such that $\underline{\dim} P = d$. By Proposition 4.3, we necessarily have $\langle i, d \rangle \leq n_i$ for all $i \in I$. Furthermore, we have $\dim \operatorname{Hom}_Q(M, E_i) \leq \dim \operatorname{Hom}_Q(P, E_i) = \langle d, i \rangle$. This proves necessity of the numerical conditions.

To prove the converse, assume that the above two sets of conditions on d and M are satisfied. By the obvious dual version of Corollary 4.2, the representation M has a minimal projective resolution of the form

$$0 \rightarrow \bigoplus_{i \in I} P_i \otimes \operatorname{Ext}_Q^1(M, E_i) \rightarrow \bigoplus_{i \in I} P_i \otimes \operatorname{Hom}_Q(M, E_i) \rightarrow M \rightarrow 0.$$

For all $i \in I$, let Y_i be a vector space of dimension $\langle d, i \rangle - \dim \operatorname{Hom}_Q(M, E_i)$ (note that this number is non-negative by assumption). Adding the projective representation $\bigoplus_{i \in I} P_i \otimes Y_i$ to the the left and middle term of the above minimal resolution, we claim that the resulting exact sequence $0 \rightarrow P' \rightarrow P \rightarrow M \rightarrow 0$ has all the desired properties.

For any two vertices $i, j \in I$, we have $\langle \underline{\dim} P_i, j \rangle = \dim \operatorname{Hom}_Q(P_i, E_j) = \delta_{i,j}$, and thus $\langle \underline{\dim} P, i \rangle = \langle d, i \rangle$ by definition, for all $i \in I$. Since the quiver Q has no oriented cycles, the form $\langle -, \rangle$ is non-degenerate, proving $\underline{\dim} P = d$.

By the assumptions, we have $\mathcal{M}_{d,n} \neq \emptyset$. Thus, $R_{d,n}^s \neq \emptyset$ is dense in $R_{d,n}$. But the projective representation P has an open orbit in R_d , so that its inverse image under the projection $R_{d,n} \rightarrow R_d$ meets $R_{d,n}^s$. Thus, there exists a point $(P, f) \in \mathcal{M}_{d,n}$. The same argument proves that there exists a point $(P', f') \in \mathcal{M}_n$.

■

The above analysis now implies the main theorem of this section.

Theorem 7.11 *The map $\eta : \dot{H} \rightarrow \mathbf{C}^{A_n}(\mathcal{X}_n)$ is surjective. Its kernel is the linear span of the elements $1_{d\chi_{[M]}}$ for which there exists an $i \in I$ such that $\langle i, d \rangle > n_i$ or $\dim \operatorname{Hom}_Q(M, E_i) > \langle d, i \rangle$.*

Proof: The image of the basis element $1_{d\chi_{[M]}}$ is either zero or a multiple of the characteristic function of the orbit $\mathcal{O}_{d, [M]}$. The previous proposition now implies the theorem. ■

Corollary 7.12 *If Q is of Dynkin type, the algebra $\mathbf{C}^{A_n}(\mathcal{X}_n)$ is a factor of the modified quantum Borel $\dot{\mathcal{U}}_v(\mathfrak{b}^+)$.*

Proof: Using the above theorem and Corollary 7.4, we have in the case of a quiver of Dynkin type a chain of surjective algebra morphisms

$$\dot{\mathcal{U}}_v(\mathfrak{b}) \xrightarrow{\sim} \dot{C} = \dot{H} \rightarrow \mathbf{C}^{A_n}(\mathcal{X}_n).$$

Remarks: ■

1. One can see that, even for fairly simple quivers Q , like e.g. $i \rightarrow j \leftarrow k$, there is no obvious extension of the map η to a map from the whole modified quantum group $\dot{\mathcal{U}}(\mathfrak{g})$ to the convolution algebra $\mathbf{C}^{A_n}(\mathcal{M}_n \times \mathcal{M}_n)$.
2. In [3], the convolution algebra on pairs of flags gives a geometric realization of q -Schur algebras. One can therefore view the algebras $\mathbf{C}^{A_n}(\mathcal{X}_n)$ as Borel parts of q -Schur algebras “of Kac-Moody type”.

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