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When homology equivalence implies homotopy equivalence for 2-complexes

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Abstract

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Let K and L be two finite 2-dimensional CW-complexes with the same fundamental group. If the fundamental group is finite and abelian, it is well known that K and L are homotopy equivalent and simple homotopy equivalent if and only if they are homology equivalent via a map that induces an isomorphism on the fundamental group. We generalize this result to left-iterated semi-direct products of finite cyclic groups. We must add other hypotheses which were automatically satisfied in the abelian case. Example of groups that satisfy the hypotheses include dihedral groups of order $4n$.

Introduction

Let K and L be finite 2-dimensional CW-complexes with the same fundamental group and the same Euler characteristic. There is a total obstruction, called bias, to K and L being homology equivalent via a map which is an isomorphism on fundamental groups (see [5, 9, 12]). This obstruction lies in a quotient group of the group of units of $\mathbb{Z}/m\mathbb{Z}$, where $m = 1$ or m is the g.c.d. of the torsions of the elements of $H_2(\pi)$, where $\pi \cong \pi_1(K) \cong \pi_1(L)$.

Let π be a finite group satisfying Eichler's condition. There is a total obstruction, we call the Browning obstruction, to two finite 2-complexes, K and L being homotopy equivalent. This obstruction lies in a quotient group of a subgroup of $K_1(\mathbb{Z}_u\pi)/K_1(\mathbb{Z}\pi)$, where $\mathbb{Z}_u\pi$ is the localized group ring and u is the set of prime divisors of the order of π. The Browning obstruction is only defined if the fundamental group π is finite and satisfies Eichler's condition.

Browning was able to show (see [3]) that when π is also abelian, representative

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2-complexes realizing all of the bias obstruction actually realized all of the Browning obstruction. He showed therefore, that homotopy equivalence for finite 2-complexes with finite abelian fundamental group depends only on homology (with isomorphism on π_1).

In this paper, we exhibit a wider class of groups for which homology equivalence of finite 2-complexes with the same fundamental group implies homotopy equivalence of those 2-complexes.

1. The Browning obstruction and bias

Definition 1.0. Following the notation of Dyer [5], a $(\pi, 2)$-*complex* is a finite 2-dimensional CW-complex with fundamental group π.

Definition 1.1. A *minimal* $(\pi, 2)$-complex is a $(\pi, 2)$-complex with minimal Euler characteristic.

Since finite 2-complexes with the same finite fundamental group and the same Euler characteristic are all homotopy equivalent above minimal Euler level by Browning [2–4], we will only be interested in minimal $(\pi, 2)$-complexes. So let K be a $(\pi, 2)$-complex.

Let G_u be the grothendieck group generated by all $\mathbb{Z}\pi$-modules X such that $X_u = X \otimes_{\mathbb{Z}} \mathbb{Z}_u = 0$. Let $G_u(\pi_2(K))$ be the grothendieck group generated by all quotient modules X of $\pi_2(K)$, such that $X_u = 0$. By Bass [1, p. 494], we have an exact sequence

$$K_1(\mathbb{Z}\pi) \xrightarrow{i_*} K_1(\mathbb{Z}_u\pi) \xrightarrow{\omega} G_u \xrightarrow{\sigma} \tilde{K}_0(\mathbb{Z}\pi) \rightarrow 0\,, \tag{1}$$

where i_* is induced by the inclusion map $\mathbb{Z}\pi \hookrightarrow \mathbb{Z}_u\pi$. Define $\mathrm{cl}_u(\pi_2(K)) \subset G_u(\pi_2(K))$ to be the subgroup which is also in the kernel of σ above. So

$$\mathrm{cl}_u(\pi_2(K)) \equiv \omega(K_1(\mathbb{Z}_u\pi)) \cap G_u(M)\,.$$

From the exactness of equation (1), we may identify the kernel of σ, with

$$K_1(\mathbb{Z}_u\pi)/i_*(K_1(\mathbb{Z}\pi)) \equiv K_1(\mathbb{Z}\pi, u)\,.$$

So we may think of $\mathrm{cl}_u(\pi_2(K))$ as $K_1(\mathbb{Z}\pi, u) \cap G_u(\pi_2(K))$, by identifying $K_1(\mathbb{Z}\pi, u)$ with the image of $K_1(\mathbb{Z}\pi)$ in G_u.

Let $\mathrm{Aut}_u(C_*(\tilde{K}))$ be the group of $\mathbb{Z}_u\pi$-chain equivalences on $C_*(\tilde{K})_u$, the localized chain complex of the universal cover of K. Given an element $\phi \in \mathrm{Aut}_u(C_*(\tilde{K}))$, we may take its Whitehead torsion, $\tau(\phi) \in K_1(\mathbb{Z}_u\pi)$.

Definition 1.2. The *Browning obstruction group* $B(\pi, \chi)$ is defined to be the group

$$B(\pi, \chi) \equiv \mathrm{cl}_u(\pi_2(K))/\omega\tau(\mathrm{Aut}_u(C_*(\tilde{K})))$$
$$\subset K_1(\mathbb{Z}\pi, u)/\tau(\mathrm{Aut}_u(C_*(\tilde{K}))) .$$

The group $B(\pi, \chi)$ does not depend on the complex K, but on the fundamental group $\pi = \pi_1(K)$ and the Euler characteristic $\chi = \chi(K)$ (see [2–4], or [6]).

Now let L be a finite 2-complex as above. That is, let $\pi_1(L) \cong \pi$ and $\chi(L) = \chi(K)$. Let $\phi : \pi_1(K) \to \pi_1(L)$ be an isomorphism. Using [6, Lemma 6.2, p. 371], there is a map $f : K \to L$ with $f_* = \phi$ and $[\tilde{f}_*]_u : [C_*(\tilde{K})]_u \to [C_*(\tilde{L})]_u$ is a chain equivalence.

Definition 1.3. The *Browning obstruction*, $B[K, L]$, is defined to be $[\tau([\tilde{f}_*]_u)] \in B(\pi, \chi)$, where $f : K \to L$ is any map which induces an isomorphism of fundamental groups and a $\mathbb{Z}_u\pi$-chain equivalence from $[C_*(\tilde{K})]_u$ to $[C_*(\tilde{L})]_u$.

It is known (see [2–4], or [6]) that $B[K, L]$ is well defined for the above restrictions on K and L, and that K and L are homotopy equivalent if and only if $[\tau([\tilde{f}_*]_u)] \equiv 0 \in B(\pi, \chi)$.

Now suppose K and L are finite 2-complexes with $\pi \cong \pi_1(K) \cong \pi_1(L)$ and $\chi(K) = \chi(L)$. Let $\phi : \pi_1(K) \cong \pi_1(L)$ be a group isomorphism. Let $f : K \to L$ be a map with $f_* = \phi : \pi_1(K) \cong \pi_1(L)$. We may alter L within its simple homotopy class, so that on the 1-skeletons, $f^{(1)} : K^{(1)} \to L^{(1)}$ is the identity.

If we choose a preferred lift of a given 0-cell in K to the universal cover $\tilde{K}$, this provides preferred lifts of the other cells. It also provides bases for the free $\mathbb{Z}\pi$-modules of the cellular chain complex $C_*(\tilde{K})$. These bases will be referred to as the *usual* bases.

The above f, with $f^{(1)} : K^{(1)} \to L^{(1)}$ the identity, will induce the identity on the zero and one levels of the cellular chain complexes of the universal covers, with the usual basis. That is,

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \pi_2(K) & \longrightarrow & C_2(\tilde{K}) & \longrightarrow & C_1(\tilde{K}) & \longrightarrow & C_0(\tilde{K}) & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f_{2*}} & & \downarrow{\scriptstyle \tilde{f}_2} & & \| & & \| & & \| & & \\
0 & \longrightarrow & \pi_2(L) & \longrightarrow & C_2(\tilde{L}) & \longrightarrow & C_1(\tilde{L}) & \longrightarrow & C_0(\tilde{L}) & \longrightarrow & \mathbb{Z} & \longrightarrow & 0
\end{array}$$

commutes. Let $\tilde{M}_f$ represent the matrix representative of $\tilde{f}_2$ with respect to the usual bases.

Remark. If, by chance, $\tilde{f}_2$ localizes to an isomorphism, $[\tilde{f}_2]_u : [C_2(\tilde{K})]_u \cong [C_2(\tilde{L})]_u$, then the matrix representative $\tilde{M}_f$ will represent the Browning obstruction, $B[K, L]$.

Definition 1.4. Let $\Sigma_2(K)$ represent the spherical elements in $H_2(K)$, that is, the image of the Hurewitz map $\pi_2(K) \to H_2(K)$. Define the *bias modulus*, m, to be

$$m = \begin{cases} 1, & \text{if } \operatorname{rank}(H_2(K)) > \operatorname{rank}(H_2(\pi)), \\ \text{g.c.d.}\{\text{torsion of } H_2(K)/\Sigma_2(K) = H_2(\pi)\}, \\ \qquad \text{otherwise}. \end{cases}$$

Let R be any commutative ring with unit. We will denote by $\varepsilon : R\pi \to R$ the augmentation map, the map that is defined by sending the entire group π to 1. We will abuse notation by letting ε refer also to induced maps, e.g. We will also refer to the maps $\varepsilon : \mathrm{GL}_n(R\pi) \to \mathrm{GL}_n(R)$ and $\varepsilon : M_n(\mathbb{Z}\pi) \to M_n(\mathbb{Z})$ as the maps on matrices induced by augmentation. If $\tilde{M}_f$ is the representative matrix for $\tilde{f}_2$, with respect to the usual basis, let $M_f = \varepsilon(\tilde{f}_2)$, the matrix with entries in $\mathbb{Z}$. Note that M_f is the representative matrix for the induced map from $C_2(K)$ to $C_2(L)$.

For any ring R, let R^* refer to the units in R; e.g., $(\mathbb{Z}/m\mathbb{Z})^*$ is the multiplicative group of integers mod m, relatively prime to m.

Definition 1.5. For any isomorphism $\phi : \pi_1(K) \to \pi_1(L)$, define $b(\phi) \equiv [\mathrm{Det}[M_f]] \in (\mathbb{Z}/m\mathbb{Z})^*$, where m is the bias modulus as defined above. We will also denote $b(\phi)$ by $b(f)$, when we are referring to a specific map $f : K \to L$.

It can be shown (see [5] or [10]) that the integer $[\mathrm{Det}[M_f]]$ is indeed relatively prime to m.

Definition 1.6. Let $D(\pi, \chi)$ represent the subgroup $\{b(\phi) \mid \phi \in \mathrm{Aut}(\pi_1(K))\}$. As noted above, $D(\pi, \chi) \subset (\mathbb{Z}/m\mathbb{Z})^*$.

Again, it can be shown (see [5] or [10]) that $D(\pi, \chi)$ depends only on the fundamental group and Euler characteristic.

Definition 1.7. Let $b(K, L)$ be defined as

$$b(K, L) \equiv [\mathrm{Det}[M_f]] \in (\mathbb{Z}/m\mathbb{Z})^* / \pm D(\pi, \chi).$$

It was shown by Dyer [5] (see also [10]), that $b(K, L) \equiv 1 \in (\mathbb{Z}/m\mathbb{Z})^* / \pm D(\pi, \chi)$ if and only if there is a homology equivalence $g : K \to L$ with $g_* = \phi$.

Since it does not matter which map we choose to measure bias, we may choose a map $f : K \to L$ which induces a local equivalence on the chain complexes of the universal covers. Consequently, the same matrix may be used to compute both invariants. In the one case we have

$$[\tilde{M}_f] \in B(\pi, \chi) \subset K_1(\mathbb{Z}\pi, u)/\tau(\mathrm{Aut}_u(C_*(\tilde{K}))),$$

and in the other we have $[\mathrm{Det}[M_f]]$.

We will show under sufficient hypotheses, that there is a surjective map,

$$\eta : K_1(\mathbb{Z}\pi, u)/\tau(\mathrm{Aut}_u(C_*(\tilde{K}))) \to (\mathbb{Z}/m\mathbb{Z})^* / \pm D(\pi, \chi)$$

which factors through a quotient of $\mathbb{Z}/\mu\mathbb{Z}$, where μ is the order of π. Under still further conditions, we can show that $K_1(\mathbb{Z}\pi, u)/\tau(\mathrm{Aut}_u(C_*(\tilde{K})))$ is isomorphic to this quotient of $\mathbb{Z}/\mu\mathbb{Z}$.

2. Definitions and statements of theorems

Let $f: K \to L$ be a local equivalence that is an isomorphism on fundamental groups. Consider $\mathrm{Det}[M_f] \in (\mathbb{Z}/\mu\mathbb{Z})^*$, where μ is the order of the group π.

Definition 2.1. Let $D_\mu(\pi)$ represent the image of $\mathrm{Aut}_u(K)$ in $(\mathbb{Z}/\mu\mathbb{Z})^*$. That is, $\mathrm{Aut}_u(K)$ is the set of all maps $f: K \to K$ which induce isomorphisms on π and are local equivalences on the universal cover.

Definition 2.2. A group π is a *left-iterated semi-direct product* of groups $\{G_i\}$, if π can be expressed as a semi-direct product of groups which are themselves semi-direct products, etc., ending in the groups $\{G_i\}$ in the pairwise grouping left to right, e.g., $(G_1 \rtimes G_2) \rtimes G_3$. Note that we have not uniquely defined the group.

Definition 2.3. Call a group presentation P *normal* if

$$P = \langle a_1, a_2, \dots, a_m \mid R_1, R_2, \dots, R_m, S_1, \dots, S_k \rangle ,$$

where $R_1 = a_1^{n_1}$, $R_2 = a_2^{n_2}, \dots, R_m = a_m^{n_m}$, and $\{n_1, n_2, \dots, n_m\}$ are the respective orders of the generators $\{a_1, a_2, \dots, a_m\}$ of the group π.

Definition 2.4. Let the *normal deficiency*, d, represent the largest deficiency for which a normal presentation exists.

Note. Deficiency is the number of generators minus the number of relators. A presentation has deficiency d if and only if the corresponding standard 2-complex has Euler characteristic $\chi = 1 - d$.

Definition 2.5. Call π *minimally presentable*, if there exists a normal presentation of maximal deficiency d, i.e., a presentation of the group π which has maximal deficiency, which is also a normal presentation. If a presentation has maximal deficiency, its corresponding standard 2-complex has minimal Euler characteristic.

Theorem 2.6. *Let π be a left-iterated semi-direct product of finite cyclic groups which is minimally presentable with respect to the generators of the finite cyclic factors, with π satisfying Eichler's condition. Let K and L be a minimal $(\pi, 2)$-complex. Then K and L are homotopy equivalent if and only if $b_\mu[K, L] = 1 \in (\mathbb{Z}/\mu\mathbb{Z})^* / \pm D_\mu(\pi, x)$.*

Theorem 2.7. *Let K be a minimal $(\pi, 2)$-complex, where π is any finite group that satisfies Eichler's condition. Suppose there exist units $\theta_k \in i_*(\mathbb{Z}_u\pi)^* \cap \tau(\mathrm{Aut}_u(C_*(\tilde{K})))$ of augmentation $1 + km$, for each integer k. Suppose also that $D(\pi, \chi) \subset D_m(\pi, \chi)$, then a complex L is homotopy equivalent to K if and only if L is homology equivalent to K via a map that induces an isomorphism of fundamental groups.*

It is known by the work of Sieradski [13] and of Browning [4], that the hypotheses of the above theorem are satisfied by complexes with fundamental group finite abelian.

3. Proofs

Let $\eta_1 : K_1(\mathbb{Z}_u\pi) \to (\mathbb{Z}_u)^*$ be the map induced by the composition $(\eta_0 \circ \varepsilon)$, where $\varepsilon : (\mathbb{Z}_u\pi)^* \to (\mathbb{Z}_u)^*$ is augmentation and $\eta_0 : (\mathbb{Z}_u)^* \to (\mathbb{Z}/\mu\mathbb{Z})^*$ is defined by $\eta_0(a/b) = ab^{-1}$. The above η_1 is well defined, since $\mathbb{Z}_u\pi$ is semi-local [6, (3.7)], so the map induced by inclusion, $i^* : (\mathbb{Z}_u\pi)^* \to K_1(\mathbb{Z}_u\pi)$, is surjective and has kernel which maps to 1 under augmentation (see [14, Section 6.5]).

Let $[1 + N] = \{1 + kN \mid k \in \mathbb{Z}\}$, where $N = \sum_{g_i \in \pi} g_i$. Note that for any $k \in \mathbb{Z}$, $1 + kN \in (\mathbb{Z}_u)^*$, since $(1 + kN)(1 - (1/(1 + k\mu))N) = 1$.

Let $A = \{\theta \mid \theta \in K_1(\mathbb{Z}_uG)$ and $\varepsilon(\theta) = 1\}$.

Define $[1 + N]A \subset K_1(\mathbb{Z}G, u)$ to be the subgroup generated by $[1 + N] \cup A$.

Lemma 3.1. *The induced map $\eta : K_1(\mathbb{Z}G, u)/[1 + N]A \to (\mathbb{Z}/\mu\mathbb{Z})^*/\{\pm 1\}$ is an isomorphism.*

Proof. η is induced from η_1. Since

$$\begin{tikzcd}
\mathrm{GL}_n(\mathbb{Z}\pi) \arrow[r, hook] \arrow[d, "\varepsilon"'] & \mathrm{GL}_n(\mathbb{Z}_u\pi) \arrow[d, "\varepsilon"'] & (\mathbb{Z}_u\pi)^* \arrow[l, hook'] \arrow[dl, "\varepsilon"] \\
\mathrm{GL}_n(\mathbb{Z}) \arrow[r, hook] \arrow[d, "\det"'] & \mathrm{GL}_n(\mathbb{Z}_u) \arrow[d, "\det"'] & \\
\{\pm 1\} \arrow[r, hook] & (\mathbb{Z}_u)^* &
\end{tikzcd}$$

commutes, we have $\eta_1(K_1(\mathbb{Z}\pi)) = \{\pm 1\}$. So the induced map

$$\bar{\eta}_1 : K_1(\mathbb{Z}G, u) \to (\mathbb{Z}_u)^*/\pm 1 \to (\mathbb{Z}/\mu\mathbb{Z})^*/\pm 1$$

is well defined. Since $\bar{\eta}_1$ sends $[1 + N]A$ to 1 mod μ, η is also well defined.

η is onto, since for any $r \in \mathbb{Z}$, with $(r, \mu) = 1$, r^{-1} is an element of $(\mathbb{Z}_u)^* \subset (\mathbb{Z}_u\pi)^*$. So $r \in (\mathbb{Z}_u\pi)^*$ and $\eta(r) = r$.

To show that η is injective, let $\theta \in (\mathbb{Z}_u\pi)^*$ with $[\theta] \in K_1(\mathbb{Z}G, u)/[1+N]A$ and $\varepsilon(\theta) = r \in (\mathbb{Z}_u)^*$. So $\varepsilon(\theta r^{-1}) = 1$ and $\theta r^{-1} \in A$. So $\theta \equiv r$ in $K_1(\mathbb{Z}G, u)/[1+N]A$. Since $\varepsilon(1+kN) \equiv 1 + k\mu$, anything that augments to 1 mod μ is equivalent to 1 in $K_1(\mathbb{Z}G, u)/[1+N]A$. In particular, if $r = a/b$, we have $1/b \equiv b^{-1} \in \mathbb{Z}$, so $r \equiv ab^{-1}$. This implies that every element of $K_1(\mathbb{Z}G, u)/[1+N]A$ is equivalent to an integer. Moreover, since we are modding out by $1 + k\mu$, for any $k \in \mathbb{Z}$, we have that any element is equivalent to an integer mod μ. So if $\eta([\theta]) = 1$, then θ is equivalent to 1. $\square$

Lemma 3.2 (compare with [4, (4.2)]). *Let π be a left-iterated semi-direct product of finite cyclic groups and let R be any commutative ring, then for any unit $\theta \in R\pi$ with $\varepsilon(\theta) = 1$ there are units θ_i such that $\theta = \theta_1\theta_2\cdots\theta_m$, and $\theta_i \sum_{j=0}^{n_i-1} a_i^j = \sum_{j=0}^{n_i-1} a_i^j$, where the a_i are the generators of the cyclic factors of π, and n_i are torsions of the a_i.*

Proof. We will proceed by induction. Let $H \subset \pi$ be a left-iterated sub-semi-direct product of at most k finite cyclic factors. Let $\sum_{h_i \in H} h_i$ be the sum of the group elements in H, and let $\theta \in R\pi$ be a unit such that $\theta \sum_{h_i \in H} h_i = \sum_{h_i \in H} h_i$. The induction hypothesis will be that under the above conditions, θ can be written as a product of $\theta_k \cdots \theta_2\theta_1$ such that $\theta_i \sum_{j=0}^{n_i-1} a_i^j = \sum_{j=0}^{n_i-1} a_i^j$, where the a_i are the generators of the cyclic factors of H, and n_i are torsions of the a_i.

To start the induction, suppose $H \subset \pi$ is of the form $H = \mathbb{Z}/t\mathbb{Z}$. If $\theta \in R\pi$ is a unit such that $\theta \sum_{i=0}^{t-1} x^i = \sum_{i=0}^{t-1} x^i$, where x is the generator of H, then θ already satisfies the conclusion.

Now assume the induction hypothesis for k. Let $G \subset \pi$ be a left-iterated sub-semi-direct product of finite cyclic groups, such that there is a split exact sequence

$$0 \to H \to G \to Q \to 0,$$

where H is a left-iterated semi-direct products of k finite cyclic groups and $Q = \mathbb{Z}/s\mathbb{Z}$. The induced map $\eta : RG \to RQ$ is a ring homomorphism, so that $\eta(\theta) = \theta_2$ is a unit, and $\varepsilon(\theta_2) = 1$. Note that $\theta_2 \sum_{i=0}^{s-1} y^i = \sum_{i=0}^{s-1} y^i$, where y is the generator of Q.

Since G is a semi-direct product of H and Q, θ can be written in the form $\theta = \sum_{i,j} n_{ij}q_ih_j$, where $n_{ij} \in R$, $q_i \in Q$ and $h_j \in H$. Notice that $\sum_{i,j} n_{ij} = 1$ and $\sum_i (\sum_j n_{ij})q_i = \theta_2$ (just drop the h_i's). Notice also that

$$\theta \sum_{h_k \in H} h_k = \Big(\sum_{i,j} n_{ij}q_ih_j\Big) \sum_{h_k \in H} h_k = \Big(\sum_{i,j}\Big(\sum_j n_{ij}\Big)q_i\Big) \sum_{h_k \in H} h_k = \theta_2 \sum_{h_k \in H} h_k .$$

Let $\nu = \theta^{-1}$. So

$$\sum_{h_k \in H} h_k = \nu\theta \sum_{h_k \in H} h_k = \nu\theta_2 \sum_{h_k \in H} h_k .$$

But

$$\sum_{h_k \in H} h_k = \nu\theta_2 \sum_{h_k \in H} h_k \quad \text{implies} \quad \theta_2^{-1}\theta \sum_{h_k \in H} h_k = \sum_{h_k \in H} h_k \,.$$

By letting $\theta_1 = \theta_2^{-1}\theta$, we have that θ_1 now satisfies the induction hypothesis and $\theta = \theta_2\theta_1$. □

Lemma 3.3. *Let K be the standard complex of the normal presentation*

$$P = \langle a_1, a_2, \ldots, a_m \mid a_1^{n_1}, a_2^{n_2}, \ldots, a_m^{n_m}, S_1, \ldots, S_k \rangle \,.$$

Then the subgroup $[1+N]A \subset K_1(\mathbb{Z}G, u)$ is contained in $\tau(\mathrm{Aut}_u(C_*(\tilde{K})))$.

Proof. Let $\{\tilde{R}_i, \tilde{S}_j\}$ represent the standard generators of $C_2(\tilde{K})$, where $C_*(\tilde{K})$ is the cellular chain complex of the universal cover $\tilde{K}$ of K. We will first show that $[1+N] \subset \tau(\mathrm{Aut}_u(C_*(\tilde{K})))$. By [11, Proposition II.2.3], there is a simple equivalence $f: K \to K'$ leaving the 1-skeleton fixed from K to a 2-complex K' such that the S_j' have zero coefficient sum. Consequently, $\partial(N\tilde{S}_j') = N\partial(\tilde{S}_j') = 0$. So if we define the map $\phi: C_2(\tilde{K}') \to C_2(\tilde{K}')$ as the identity on all the 2-cells except $\tilde{S}_1'$, sending $S_1' \mapsto (1+kN)\tilde{S}_1$ for some $k \in \mathbb{Z}$, the torsion of this map in $K_1(\mathbb{Z}_u\pi)$ will be $(1+kN)$. So $[1+N] \subset \tau(\mathrm{Aut}_u(C_*(\tilde{K}')))$. But since f fixes the 1-skeleton, the induced map $f^*: K_1(\mathbb{Z}G, u) \to K_1(\mathbb{Z}G, u)$ is the identify. So $[1+N] \subset \tau(\mathrm{Aut}_u(C_*(\tilde{K})))$.

Let us now show that the subset A is realizable. We may realize any unit in $(\mathbb{Z}_u\pi)^*$ of augmentation 1, by Lemma 3.2. Given any unit θ that augments to 1, then θ may be expressed as $\theta_m \cdots \theta_2\theta_1$ where each θ_i has the property that $\partial(\theta_i\tilde{R}_i) = \theta_i\partial(\tilde{R}_i) = \mu_i \sum_{j=0}^{n_i-1} a_i^j$. Since each θ_i can be realized by a map $C_2(\tilde{K}) \mapsto C_2(\tilde{K})$ which is the identity on each 2-cell except $\tilde{R}_i$, the product of the θ_i's is realizable. So any unit of augmentation 1 is realizable. □

Proof of Theorem 2.6. It is sufficient to let K be the standard complex of the normal presentation

$$P = \langle a_1, a_2, \ldots, a_m \mid a_1^{n_1}, a_2^{n_2}, \ldots, a_m^{n_m}, S_1, \ldots, S_k \rangle \,.$$

By Lemma 3.3 the subgroup $[1+N]A$ is contained in $\tau(\mathrm{Aut}_u(C_*(\tilde{K})))$. So, by Lemma 3.1 the map

$$K_1(\mathbb{Z}G, u)/[1+N]A \to (\mathbb{Z}/\mu\mathbb{Z})^*/\{\pm 1\}$$

is an isomorphism. This implies that the map

$$K_1(\mathbb{Z}\pi, u)/\tau(\mathrm{Aut}_u(C_*(\tilde{K}))) \to (\mathbb{Z}/\mu\mathbb{Z})^*/\pm D_\mu(\pi, \chi)$$

is also an isomorphism. This is sufficient, since it implies that, $b_\mu[K, L] = 1$, if and only if $B[K, L] = 0$. □

Proof of Theorem 2.7. The hypotheses imply that the map $K_1(\mathbb{Z}G, u) \to (\mathbb{Z}/m\mathbb{Z})^*/\{\pm 1\}$ induces an isomorphism on

$$K_1(\mathbb{Z}\pi, u)/\tau(\mathrm{Aut}_u(C_*(\tilde{K}))) \to (\mathbb{Z}/m\mathbb{Z})^*/\pm D(\pi, \chi).$$

Consequently $B[K, L] = 0$ if and only if $b[K, L] = 1$. The obstruction to homotopy equivalence vanishes if and only if the obstruction to homology equivalence vanishes. □

4. Examples

Very little is known at present about the maximal deficiency of groups. It is a topic of ongoing research. This author is familiar with some of the results relevant to this paper. Applying the results of Wamsley [15], we get the following example:

Example 4.1. Given the finite group presentation

$$P = \{a, b \mid a^m, b^n, aba^{-s}b^{-1}\}$$

with $s \geq -1$ and m,n nonnegative, such that the order of the group is mn, then P is a minimal presentation if and only if

$$\text{g.c.d.}\left(s - 1, m, \frac{s^n - 1}{m}, \frac{s^n - 1}{s - 1}\right) \neq 1.$$

Example 4.2. By setting, in the above presentation, $n = 2$ and $s = -1$ we get the dihedral group of order $2m$. The normal presentation for the dihedral group of order m

$$P = \{a, b \mid a^m, b^2, aba^s b^{-1}\}$$

is minimal (has maximum deficiency) if and only if m is even.

Let D_{4n} be the dihedral group of order $4n$, which has the group presentation

$$\{a, b \mid a^{2n}, b^2, abab^{-1}\}.$$

Comparing Example 4.2 with results of Jajodia and Magurn [7, 8] we get the following:

Theorem 4.3. *Any minimal $(D_{4n}, 2)$-complex is homotopy and simple homotopy equivalent to the standard complex of a presentation of the form*

$$\{a, b \mid a^{2n}, b^2, a^rba^rb^{-1}\},$$

where $(r, 2n) = 1$.

Proof. Let K_r be the standard complex of the presentation

$$\{a, b \mid a^{2n}, b^2, a^rba^rb^{-1}\}.$$

By [6, Proposition 4.9], composition of local equivalences commutes with addition of the obstruction elements. So if we can realize all of the obstruction group by maps from the K_r, we are done. Let R_a, R_b and R_{ab} be the standard generators of $C_2(\tilde{K})$ corresponding to the relations a^{2n}, b^2, and $abab^{-1}$. LetS_a, S_b and S_ab be the standard generators of $C_2(\tilde{K}_r)$ corresponding to the relations a^{2n}, b^2, *and* $a^rba^rb^{-1}$. Then the map $\phi : C_2(\tilde{K}) \to C_2(\tilde{K}_r)$ given by the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sum_{i=1}^{r-1} a^i \end{pmatrix}$$

commutes with the boundary operator and therefore can be realized as a map $f : K \to K_r$ which induces the identity on the one skeletons. Note that since $\sum_{i=1}^{r-1} a^i$ is a unit in $\mathbb{Z}_u\pi$, then f is a local equivalence. When we augment, the torsion of f in $\mathbb{Z}/\mu\mathbb{Z}$ will be r. Consequently, all torsion has been realized and we are done. □

More general results similar to Theorem 4.3, with fundamental group a metacyclic group, can most likely be garnered from [7] and [8]. Such results are scheduled for a later paper.

Corollary 4.5. *The homotopy type and simple homotopy type of $(D_{4n}, 2)$-complexes is determined by their Euler characteristic.*

Proof. We merely need to show that the complexes of Theorem 4.3 are all homotopy equivalent. To show they are all homotopy equivalent, we merely need to show that for any integer of the form $1 + 2k$, where k is any integer such that $1 + 2k$ is relatively prime to $2n$, there is a map $f : K \to K$, with

$$\tau(f) = [1 + 2k] \in K_1(\mathbb{Z}\pi, u)/\tau(\mathrm{Aut}_u(C_*(\tilde{K}))).$$

This would imply that $K_1(\mathbb{Z}\pi, u)/\tau(\mathrm{Aut}_u(C_*(\tilde{K})))$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^*$, and our result would follow.

It is straightforward to check, using Fox calculus, that $-(b + 1)R_a +$

$\sum_{i=0}^{2n-1} a^i R_{ab}$ is an element of $\pi_2(K)$ $(= H_2(\tilde{K}))$. Consequently, the matrix

$$\begin{pmatrix} 1+k(b+1) & 0 & -k\sum_{i=0}^{2n-1} a^i \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

commutes in place of the identity matrix $\mathrm{id}_2 : C_2(\tilde{K}) \to C_2(\tilde{K})$ with the identity at all other levels. Using Puppe modifications, there is a map $f: K \to K$ with $\tau(f) \in K_1(\mathbb{Z}_n\pi)$ equal to that of the above matrix, provided the above matrix is invertible over $\mathbb{Z}_n\pi$. The above matrix is equivalent, via a row operation, to the matrix

$$\begin{pmatrix} 1+k(b+1) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

If $1 + k(b+1)$ is a unit in $\mathbb{Z}_n\pi$, we are done. But

$$(1+k(b+1))\left(1 - \frac{1}{1+2k}(b+1)\right) = 1\,.$$

$(1-(b+1)/(1+2k))$ is in $\mathbb{Z}_n\pi$ precisely when $1+2K$ is relatively prime to $2n$. □

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