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On two problems about sobriety of topological spaces ☆

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In this paper, we answer two problems concerning sobriety. One concerns the existence of a closed set lattice whose Scott space is non-sober. Another is about the reflectivity of a category of k-bounded sober spaces. We also prove some sufficient conditions for the Scott space of a dcpo to be k-bounded well-filtered.

1. Introduction

Sobriety is one of the most important and useful properties of non-Hausdorff topological spaces. This property has been extensively studied from various different perspectives. In the mean time, quite a number of weaker properties have been introduced and investigated. In their study of topologies defined by means of irreducible subsets, Zhao and Ho introduced the k-bounded sober spaces [11] and proved that a space is k-bounded sober if and only if the topology induced by the irreducible sets coincides with the original topology. One natural question is whether the category of all k-bounded sober spaces is reflective in the category of all T_0 spaces. It has long been known that the set $\mathbf{IRR}(X)$ of all irreducible closed sets of a T_0 space X equipped with the lower Vietoris topology is the sobrification of X (Exercise V-4.9 of [2]). Thus the

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subcategory of all sober spaces is reflective in the category of all T_0 spaces. Zhao and Ho ask whether the set $KB(X)$ of all irreducible closed sets of X with sups existing also severs as the k-bounded sobrification of X. A negative answer to this problem was given by Zhao, Lu and Wang [12]. Recently, Lu, Wang, Wu and Zhao further confirmed that the category of all k-bounded sober spaces is not reflective in the category of all T_0 spaces [5]. They then posed the following problem:

Problem 1.1. Is the category of all k-bounded sober spaces reflective in the category of all T_0 spaces and continuous mappings that preserve all existing sups of irreducible sets?

In domain theory, the main study of sobriety focus on the sobriety of Scott spaces of various dcpos. Johnstone constructed the first dcpo whose Scott space is not sober. Soon after, Isbell found a complete lattice with a non-sober Scott topology. Based on Isbell's example, Xu, Xi and Zhao showed that there is actually a special frame whose Scott topology is not sober, thus answered a problem posed by Achim Jung. In [7], Xu and Zhao further posed the following problem:

Problem 1.2. Is there a complete lattice with enough co-primes (i.e., a lattice isomorphic to the lattice of all closed subsets of a topological space) that has a non-sober Scott topology?

We shall answer Problem 1.1 and Problem 1.2 in Section 3 and Section 5, respectively.

One topological property which is weaker than the sobriety and has been also extensively studied in recent years is the well-filteredness. Xi and Lawson recently proved that every complete lattice has a well-filtered Scott topology [10]. In Section4, we introduce the k-bounded well-filtered spaces and prove that every L-dcpo has a k-bounded well-filtered Scott space.

2. Preliminaries

Let X be a T_0 space. The closure of a subset $A \subseteq X$ will be denoted by $cl(A)$. The set of all closed sets and the set of all open sets of X will be denoted by $\Gamma(X)$ and $\mathcal{O}(X)$, respectively. The specialization order $\leq_X$ (or just $\leq$) on a T_0 space is defined as $x \leq y$ iff $x \in cl(y)$. As usual, $\sup A$ denotes the least upper bound of A. A nonempty subset F of X is called *irreducible* if $F \subseteq B \cup C$ implies that $F \subseteq B$ or $F \subseteq C$ for any $\{B, C\} \subseteq \Gamma(X)$. A subset A of X is saturated if it is the intersection of all open sets containing it (equivalently if it is an upper set with respect to the specialization order).

The set of all irreducible closed sets of X is denoted by $\mathbf{IRR}(X)$ and the set of all compact saturated subsets of X by $Q(X)$. We write $\mathfrak{K} \subseteq_{flt} Q(X)$ to mean that $\mathfrak{K}$ is a filtered subfamily of $Q(X)$, and $F \subseteq_{fin} X$ to mean that F is a finite subset of X.

Let P be a poset. A subset D of P is *directed* if it is nonempty and every two elements of D have an upper bound in D. A poset is called a directed complete poset (dcpo, for short) if the supremum of every directed subset exists.

A complete lattice is a poset in which every subset has a supremum and an infimum. A dcpo P is called an *L-dcpo* if for each $x \in P$, the subposet $\downarrow x = \{y \in P : y \leq x\}$ is a complete lattice. A subset U of a poset P is *Scott open* if (i) it is an upper set ($U = \uparrow U = \{x \in P : u \leq x$ for some $u \in U\}$) and (ii) for every directed subset D of P with $\sup D$ existing and $\sup D \in U$, it follows that $D \cap U \neq \emptyset$. The complements of Scott open sets are called *Scott closed* sets. The collection of all Scott open subsets of P form a topology on P, called the *Scott topology* of P and is denoted by $\sigma(P)$. The collection of all Scott closed subsets of P is denoted by $\Gamma(P)$. The space $(P, \sigma(P))$, called the Scott space of P, is written as ΣP. All upper sets of P form the *Alexandroff topology* which will be denoted by $A(P)$.

Definition 2.1. [11] A T_0 space (X, τ) is *k-bounded sober* if for any irreducible closed set A with $\sup A$ existing, there is a unique point $x \in X$ such that $A = cl(\{x\})$.

A full subcategory $\mathcal{K}$ of a category $\mathcal{C}$ is called *reflective* if the inclusion functor has a left adjoint, which then is called a *reflector*.

A *retract* of a topological space Y is a topological space X such that there are two continuous maps $s : X \to Y$ (the section) and $r : Y \to X$ (the retraction) such that $r \circ s = id_X$.

Lemma 2.2. [2] *Let X, Y be two T_0 spaces. If X is a retract of Y and Y is sober, then X is sober.*

Definition 2.3. [2] Let S and T be two posets. A pair (g, d) of functions $g : S \to T$ and $d : T \to S$ is called a *Galois connection* or *an adjunction* between S and T if

(1) both g and d are monotone, and

(2) $g(s) \geq t$ if and only if $s \geq d(t)$ for all $(s, t) \in S \times T$.

We refer to [2] for more details on the relevant definitions, notations and results, which are used in this paper.

3. The non reflectivity of KBSob

First, let's recall the non-sober dcpo constructed by Johnstone [3], which is defined as $\mathbb{J} = \mathbb{N} \times (\mathbb{N} \cup \{\omega\})$, with the order defined by $(j, k) \leq (m, n)$ iff $j = m$ and $k \leq n$, or $n = \omega$ and $k \leq m$.

Now we give a negative answer to Problem 1.1.

Lemma 3.1. *Let $L = \mathbb{J} \cup \{\top_1, \top_2\}$. Define a partial order $\leq$ on L as follows:*

$x \leq y$ iff either

- *$x, y \in \mathbb{J}$ and $x \leq y$ in $\mathbb{J}$; or*
- *$x \in \mathbb{J}$, $y = \top_1$ or $y = \top_2$.*

Then ΣL is a k-bounded sober space.

Proof. It suffices to prove that $\mathbf{IRR}(L) = \{\downarrow x \mid x \in L\} \cup \{\mathbb{J}\}$.

The inclusion $\{\downarrow x \mid x \in L\} \cup \{\mathbb{J}\} \subseteq \mathbf{IRR}(L)$ is easy to verify.

Conversely, let $A \in \mathbf{IRR}(L)$ and $A \notin \{\downarrow x \mid x \in L\}$. Then we claim that $\max A$ is an infinite set, where $\max A$ denotes the set of all maximal elements of A. Suppose, on the contrary, that $\max A$ is finite. Then $A \subseteq \bigcup_{x \in \max A} \downarrow x$. This then implies that $A \subseteq \downarrow x$ for some $x \in A$ by the irreducibility of A. So $A = \downarrow x$ for some $x \in A$, which contradicts the assumption that $A \notin \{\downarrow x \mid x \in L\}$. It thus follows that $A \subseteq \mathbb{J}$.

Next, we prove that $A \cap \max \mathbb{J}$ is infinite. Otherwise $A \cap \max \mathbb{J}$ is finite. Note that $A \subseteq \downarrow(A \cap \max \mathbb{J}) \cup \downarrow(\mathbb{J} \backslash (\downarrow(A \cap \max \mathbb{J})))$, and both $\downarrow(A \cap \max \mathbb{J})$ and $\downarrow(\mathbb{J} \backslash (\downarrow(A \cap \max \mathbb{J})))$ are Scott closed subsets of L. Since A is irreducible, we have $A \subseteq \downarrow(A \cap \max \mathbb{J})$ or $A \subseteq \downarrow(\mathbb{J} \backslash (\downarrow(A \cap \max \mathbb{J})))$, both of them will imply that $A = \downarrow x$ for some $x \in \mathbb{J}$. Hence, $A \cap \max \mathbb{J}$ is infinite. Thus $A = \mathbb{J}$ because $A \subseteq \mathbb{J}$. □

Example 3.2. Let P be the subset $\mathbb{J} \cup \{\top_1\}$ of L in Lemma 3.1. Then in $(P, A(P))$, $B = \{(1, n) \mid n \in \mathbb{N}\}$ is an irreducible closed set with $\bigvee B = (1, \omega)$. But $B \neq \downarrow x$ for all $x \in P$. Hence $(P, A(P))$ is not a k-bounded sober space.

Let us denote the category of all k-bounded sober spaces and continuous mappings that preserve all existing sups of irreducible sets by **KBSob** and the category of all T_0 spaces and continuous mappings that preserve all existing sups of irreducible sets by $\mathbf{Top}_k$.

Theorem 3.3. **KBSob** *is not reflective in* $\mathbf{Top}_k$.

Proof. Assume that **KBSob** is reflective in $\mathbf{Top}_k$. Let $X = (P, A(P))$ in Example 3.2 and $Y = \Sigma L$ in Lemma 3.1. By Lemma 3.1, we have $Y \in \mathrm{Ob}(\mathbf{KBSob})$. Let $\alpha_X : X \to X^k$ be the reflection of X in **KBSob**. Define $f : X \to Y$ by $f(x) = x$ for any $x \in X$. It is obvious that f is continuous. Because P is endowed with the Alexandorff topology, a subset A of X is an irreducible subset iff A is directed. Note that $f(\sup_X D) = \sup_X D = \sup_Y D = \sup_Y f(D)$, where D is a directed subset of P. Hence, f is a homomorphism in the category $\mathbf{Top}_k$. It follows that there exists a unique arrow $f^* : X^k \to Y$ such that $f = f^* \circ \alpha_X$ by the definition of reflective subcategory and the definition of adjoint in [1]. We want to prove that Y and X^k are homeomorphic.

Define $\alpha_X^* : Y \to X^k$ by

$$\alpha_X^*(x) = \begin{cases} \alpha_X(x), & x \in P; \\ \alpha_X(\top_1), & x = \top_2. \end{cases}$$

Claim 1: α_X^* is continuous.

Let $U \in \mathcal{O}(X^k)$. It suffices to prove that $(\alpha_X^*)^{-1}(U)$ is Scott open in Y. For any $x \in (\alpha_X^*)^{-1}(U)$, we consider two cases.

Case 1. $x \in P$. Then we have $x \in \alpha_X^{-1}(U)$. Since α_X is continuous, we have $\alpha_X^{-1}(U)$ is an upper set of P. It follows that $\alpha_X^{-1}(U) \cup \{\top_2\}$ is an upper set of L. Let D be a directed subset of L with $\sup D \in \alpha_X^{-1}(U) \cup \{\top_2\}$. If $\sup D \in \alpha_X^{-1}(U)$, then $\alpha_X(\sup D) = \sup(\alpha_X(D)) \in U$ because α_X is the homomorphism of the category $\mathbf{Top}_k$. Since $\alpha_X(D)$ is an irreducible subset of X^k and X^k is a k-bounded sober space, we have $\alpha_X(D) \cap U \neq \emptyset$, that is, $D \cap \alpha_X^{-1}(U) \neq \emptyset$. Else $\sup D = \{\top_2\}$, then $\top_2 \in D$ by the compactness of $\top_2$. Thus $D \cap (\alpha_X^{-1}(U) \cup \{\top_2\}) \neq \emptyset$, that is, $\alpha_X^{-1}(U) \cup \{\top_2\}$ is Scott open in L. Note that $\top_1 \in \alpha_X^{-1}(U)$ since $\alpha_X^{-1}(U) \in A(P)$. Then $\alpha_X^*(\top_2) = \alpha_X(\top_1) \in U$. This implies that $x \in \alpha_X^{-1}(U) \cup \{\top_2\} \subseteq (\alpha_X^*)^{-1}(U)$.

Case 2. $x = \top_2$. Then we have $x \in \{\top_2\} \subseteq (\alpha_X^*)^{-1}(U)$. Hence, α_X^* is continuous.

Claim 2: X^k is a retraction of Y.

For this, it suffices to prove that $\alpha_X^* \circ f^* = id_{X^k}$, where id_{X^k} denotes the identity map of X^k. From the above proof, we know $f = f^* \circ \alpha_X$. It follows that $\alpha_X = \alpha_X^* \circ f = \alpha_X^* \circ f^* \circ \alpha_X$ from the construct of α_X^*. Then $\alpha_X^* \circ f^* = id_{X^k}$ by the uniqueness of id_{X^k}.

Claim 3: $\alpha_X(\mathbb{J})$ is irreducible in X^k.

Let $\{U, V\} \subseteq \mathcal{O}(X^k)$, $U \cap \alpha_X(\mathbb{J}) \neq \emptyset$, $V \cap \alpha_X(\mathbb{J}) \neq \emptyset$. Then $\mathbb{J} \cap \alpha_X^{-1}(U) \neq \emptyset$ and $\mathbb{J} \cap \alpha_X^{-1}(V) \neq \emptyset$. Now we need to prove that $\alpha_X^{-1}(U)$ and $\alpha_X^{-1}(V)$ are both Scott open in P. It is obvious that $\alpha_X^{-1}(U)$ and $\alpha_X^{-1}(V)$ are two upper sets because α_X is continuous. Similar to the proof of **Claim 1**, we can get $\alpha_X^{-1}(U)$ and $\alpha_X^{-1}(V)$ are both Scott open in P. This implies that $\mathbb{J} \cap \alpha_X^{-1}(U) \cap \alpha_X^{-1}(V) \neq \emptyset$. So $\alpha_X(\mathbb{J}) \cap U \cap V \neq \emptyset$.

Claim 4: α_X is an injection.

Let $\{x, y\} \subseteq X$, $x \neq y$. Suppose $\alpha_X(x) = \alpha_X(y)$. Then $f^*(\alpha_X(x)) = f^*(\alpha_X(y))$. We have $x = f(x) = f(y) = y$ since $f = f^* \circ \alpha_X$, which contradicts $x \neq y$. Hence, $\alpha_X(x) \neq \alpha_X(y)$.

Claim 5: Y and X^k are homeomorphic.

We know that f^* is a topological embedding by Proposition 4.10.28 in [3] and **Claim 2**. So it suffices to prove that f^* is a surjection. Note that $\{\top_1\} \in \mathcal{O}(Y)$ and f^* is an injection. Then we have $(f^*)^{-1}(\{\top_1\}) = \{\alpha_X(\top_1)\}$ is an open subset of X^k. Since α_X is an injection, we have $\alpha_X(\top_1) \notin \alpha_X(\mathbb{J})$. Thus $\{\alpha_X(\top_1)\} \cap \alpha_X(\mathbb{J}) = \emptyset$. Hence, $\sup \alpha_X(\mathbb{J}) \neq \alpha_X(\top_1)$ by **Claim 3**. This implies that there exists $t \in X^k$ such that t is an upper bound of $\alpha_X(\mathbb{J})$ and $\alpha_X(\top_1) \nleq t$. Next, we want to prove that $t \notin \alpha_X(\mathbb{J})$. Suppose not, $t \in \alpha_X(\mathbb{J})$, then there exists $j \in \mathbb{J}$ such that $\alpha_X(j) = t$. It follows that $f^*(t) = f^*(\alpha_X(j)) = j$, which is an upper bound of $\mathbb{J}$. This contradicts that j is not an upper bound of $\mathbb{J}$. Since f^* is an injection, we have $f^*(t) = \top_2$. Thus f^* is a surjection.

Claim 6: **KBSob** is not reflective in $\mathbf{Top}_k$.

Let $Z = \Sigma 2$ be the Sierpinski space, f_0 be the constant mapping which maps x to 0 for all $x \in X$ and f_0^* the constant mapping which maps y to 0 for all $y \in Y$. Define $g : Y \to Z$ by

$$g(x) = \begin{cases} 0, & x \in P; \\ 1, & x = \top_2. \end{cases}$$

Note that $f_0 \in \mathrm{Mor}(\mathbf{Top}_k)$, $Z \in \mathrm{Ob}(\mathbf{KBSob})$ and $f_0 = f_0^* \circ f = g \circ f$, which contradicts the uniqueness of f_0^*. Hence, **KBSob** is not reflective in $\mathbf{Top}_k$. $\square$

4. *k*-Bounded well-filtered spaces

In this section, we introduce and study the *k-bounded well-filtered* spaces.

Definition 4.1. ([6]) Let X be a T_0 space. A non-empty subset A of X is called a KF-set if there exists $\mathfrak{K} \subseteq_{flt} Q(X)$ such that $cl(A)$ is a minimal closed set that intersects all members of $\mathfrak{K}$. The set of all closed KF-subsets of X is denoted by $\mathbf{KF}(X)$.

A KF-set is also named as Rudin set in [8].

Definition 4.2. A T_0 space (X, τ) is called *k-bounded well-filtered* if for any non-empty closed KF-subset A of X whose $\sup A$ exists, there is a unique point $x \in X$ such that $A = cl(\{x\})$.

Just like the fact that every sober space is well-filtered, we also have the following similar result. We omit the proof.

Lemma 4.3. *Every k-bounded sober space is k-bounded well-filtered.*

Proposition 4.4. *If X is a locally compact k-bounded well-filtered space, then it is k-bounded sober.*

Proof. Let $A \in \mathbf{IRR}(X)$. Then A is a closed KF-subset of X by Theorem 6.10 in [8]. If $\sup A$ exists, then $A = cl(\{\sup A\})$ since X is k-bounded well-filtered. $\square$

By the definition of k-bounded well-filtered spaces, it is clear that every well-filtered space is k-bounded well-filtered. The converse implication is not true as illustrated by the following example.

Example 4.5. Let $\mathbb{N}$ be the set of natural numbers with the usual order. Then $\Sigma\mathbb{N}$ is k-bounded well-filtered, but not well-filtered.

Proof. It is easy to see that $\mathbb{N}$ is an algebraic poset. Then $\Sigma\mathbb{N}$ is k-bounded sober by Theorem 5.2 in [5]. It follows that $\Sigma\mathbb{N}$ is k-bounded well-filtered by Lemma 4.3. Since $\mathbb{N}$ is not a dcpo, we have $\Sigma\mathbb{N}$ is not well-filtered. $\square$

In [10], Xi and Lawson proved that the Scott space of every complete lattice is well-filtered. Naturally, one wonders whether an L-dcpo with the Scott topology is k-bounded well-filtered. Next, we give a positive answer for the above problem.

Lemma 4.6. *Let X be a T_0 space and A a closed KF-subset of X. If $x \in X$ with $A \subseteq \downarrow x$, then A is a closed KF-subset of the subspace $\downarrow x$ of X.*

Proof. For any closed KF-subset A, there exists $\mathfrak{K} \subseteq_{flt} Q(X)$ such that $cl(A)$ is a minimal closed set that intersects all members of $\mathfrak{K}$. Note that $K \cap A$ is compact for any $K \in \mathfrak{K}$. If $A \subseteq \downarrow x$, then $(\uparrow(K \cap A) \cap \downarrow x)_{K \in \mathfrak{K}} \subseteq_{flt} Q(\downarrow x)$. It suffices to prove that A is a minimal closed set that intersects all members of

$\{\uparrow(K \cap A) \cap \downarrow x \mid K \in \mathfrak{K}\}$. Obviously, $A \cap (\uparrow(K \cap A) \cap \downarrow x) \neq \emptyset$. Let B be a closed subset of $\downarrow x$ and $B \subseteq A$, which intersects all members of $\{\uparrow(K \cap A) \cap \downarrow x \mid K \in \mathfrak{K}\}$, that is, $B \cap \uparrow(K \cap A) \neq \emptyset$. This implies that $B \cap K \cap A \neq \emptyset$ because B is a lower set in X. By the closedness of $\downarrow x$, we have B is a closed subset of X. It follows that $A = B$ by the minimality of A. Thus, A is a closed KF-subset of $\downarrow x$. □

Proposition 4.7. *Let X be a T_0 space. If every subspace $\downarrow x$ of X is well-filtered, then X is k-bounded well-filtered.*

Proof. Let A be a closed KF-subset of X with $\sup A$ existing. This implies that A is a closed KF-subset of $\downarrow \sup A$ by Lemma 4.6. Since $\downarrow \sup A$ is well-filtered, we have $A = cl(\{\sup A\})$. Hence, X is k-bounded well-filtered. □

Theorem 4.8. *For any L-dcpo L, ΣL is a k-bounded well-filtered space.*

Proof. By Proposition 4.7, it only needs to prove that $\downarrow x$ is a well-filtered space for all $x \in L$. Obviously, $\downarrow x$ is a complete lattice from the definition of L-dcpos. Since a complete lattice with the Scott topology is well-filtered, we have $\Sigma\downarrow x$ is well-filtered. Since $\sigma(\downarrow x)$ coincides with the relative topology from L, $\downarrow x$ is well-filtered. □

5. A closed set lattice whose Scott space is non-sober

Johnstone constructed the first dcpo whose Scott space is not sober. Soon after, Isbell constructed a complete lattice whose Scott space is not sober. Since Isbell's complete lattice is not distributive, Jung asked whether there is a distributive complete lattice whose Scott space is not sober. In [9], the authors proved that there is a special frame whose Scott space is not sober, thus answered Jung's problem. In [7], Xu and Zhao further asked whether there is a topological space X such that the lattice $\Gamma(X)$ of all closed sets of X has a non-sober Scott topology. In this section, we first answer Xu and Zhao's problem. Then we prove some further related results.

Lemma 5.1. *Let L be a complete lattice. Then L is a Scott retract of $(\Gamma(L), \subseteq)$, where $\Gamma(L)$ is the set of all Scott closed sets of L.*

Proof. Define $f : L \to \Gamma(L)$ by $f(x) = \downarrow x$ and $g : \Gamma(L) \to L$ by $g(A) = \sup A$ for any $A \in \Gamma(L)$.

Claim 1: f is Scott continuous.

Let D be a directed subset of L. Then $f(\sup D) = \downarrow \sup D = cl(\bigcup_{d \in D} \downarrow d) = \sup_{d \in D} \downarrow d = \sup_{d \in D} f(d)$.

Claim 2: g is Scott continuous.

Let $(A_i)_{i \in I}$ be a directed subset of $\Gamma(L)$. Then $g(\sup_{i \in I} A_i) = g(cl(\bigcup_{i \in I} A_i)) = \sup cl(\bigcup_{i \in I} A_i)$. We want to prove that $\sup cl(\bigcup_{i \in I} A_i) = \sup \bigcup_{i \in I} A_i$. It is obvious that $\sup cl(\bigcup_{i \in I} A_i) \geq \sup \bigcup_{i \in I} A_i$. Conversely, suppose not, $\sup cl(\bigcup_{i \in I} A_i) \not\leq \sup \bigcup_{i \in I} A_i$. It follows that there exists $x \in cl(\bigcup_{i \in I} A_i)$ such that $x \not\leq \sup \bigcup_{i \in I} A_i$. Thus, $x \in L \backslash \downarrow \sup \bigcup_{i \in I} A_i$. This implies that $(\bigcup_{i \in I} A_i) \cap (L \backslash \downarrow (\sup \bigcup_{i \in I} A_i) \neq \emptyset$, which contradicts $\bigcup_{i \in I} A_i \cap (L \backslash \downarrow (\sup \bigcup_{i \in I} A_i) = \emptyset$. Hence, $g(\sup_{i \in I} A_i) = \sup(\bigcup_{i \in I} A_i) = \sup_{i \in I} \sup A_i = \sup_{i \in I} g(A_i)$.

It remains to prove that $g \circ f = id_L$. For any $x \in L$, $g \circ f(x) = g(\downarrow x) = x$. The proof is completed. □

Corollary 5.2. *Let L be the non-sober complete lattice given by Isbell. Then $\Sigma\Gamma(L)$ is not sober.*

Proof. Suppose that $\Sigma\Gamma(L)$ is a sober space. By Lemma 5.1, L is a Scott retract of $\Sigma\Gamma(L)$. It follows that ΣL is sober by Lemma 2.2, which contradicts that ΣL is not sober. □

In general, we have the following properties about a complete lattice L.

Corollary 5.3. *Let L be a complete lattice. If f and g are the two mappings defined in the proof of Lemma 5.1, then f is the upper adjoint of g.*

Proof. Let $x \in L$, $A \in \Gamma(L)$. We only need to prove that $f(x) \geq A$ iff $x \geq g(A)$. Assume $f(x) \geq A$, that is $\downarrow x \supseteq A$. This implies that $x \geq \sup A = g(A)$. Conversely, if $x \geq g(A)$, then $x \geq \sup A$. Thus $\downarrow x \supseteq \downarrow \sup A \supseteq A$. Therefore, $f(x) \geq A$. $\square$

Theorem 5.4. *Let L be a poset and $f : L \to \Gamma(L)$ be the mapping defined by $f(x) = \downarrow x$. Then the following statements are equivalent:*

(1) *L is a complete lattice;*

(2) *f has a lower adjoint.*

Proof. (1) ⇒ (2) by Corollary 5.3.

(2) ⇒ (1) First, we want to prove that $\sup A$ exists for any $A \in \Gamma(L)$. Let d be the lower adjoint of f. Then $d(A) = \min f^{-1}(\uparrow A)$ by Theorem 0.3.2 in [2]. Note that $x \in f^{-1}(\uparrow A)$ iff $\downarrow x \supseteq A$ iff x is an upper bound of A. It follows that $d(A) = \min A^u$, where A^u denotes the set of all upper bound of A, that is, $d(A) = \sup A$. Second, suppose F is a finite subset of L. Then $d(\downarrow F) = \sup(\downarrow F) = \sup F$. Let D be a directed subset of L. Then $d(cl(D)) = \sup cl(D) = \sup D$ by the proof of Lemma 5.1. Obviously, $d(\emptyset)$ is the least element of L. Hence, L is a complete lattice. $\square$

Based on Corollary 5.2 one naturally asks the following problem.

Problem 5.5. If L is a sober complete lattice, must $\Sigma\Gamma(L)$ be sober?

At the moment, we do not have a complete answer yet. Here we give some partial answers.

Lemma 5.6. *If ΣL is locally compact, then $\Sigma\Gamma(L)$ is sober. In fact, in such case, $\Sigma\Gamma(L)$ is locally compact and $\sigma(\Gamma(L)) = \upsilon(\Gamma(L))$, where $\upsilon(\Gamma(L))$ denotes the upper topology of $\Gamma(L)$.*

Proof. Let $Q(L)$ denote the set of all Scott compact saturated subsets of L. Define $f : L \to \Gamma(L)$ by $f(x) = \downarrow x$ for any $x \in L$. Assume $\mathcal{B} = \{\Gamma(L) \backslash \uparrow f(K) \mid K \in Q(L)\}$. Then $\mathcal{B}$ is a basis of a topology of $\Gamma(L)$. We denote the topology induced from $\mathcal{B}$ by τ.

Claim 1: $(\Gamma(L), \subseteq, \upsilon(\Gamma(L)) \vee \tau)$ is a pospace.

Let $(A, B) \in (\Gamma(L) \times \Gamma(L)) \backslash \subseteq$. Then $A \nsubseteq B$. This implies that there exists one point $a \in A \cap (L \backslash B)$. Since ΣL is locally compact, we obtain that there exists $K \in Q(L)$ such that $a \in int(K) \subseteq K \subseteq L \backslash B$. It follows that $(A, B) \in \Diamond int(K) \times (\Gamma(L) \backslash \uparrow f(K))$. It remains to prove that $\Diamond int(K) \times (\Gamma(L) \backslash \uparrow f(K)) \subseteq (\Gamma(L) \times \Gamma(L)) \backslash \subseteq$. Suppose not, there exists $(C, D) \in \Diamond int(K) \times (\Gamma(L) \backslash \uparrow f(K))$ with $C \subseteq D$. Hence $C \in \Diamond int(K) \cap (\Gamma(L) \backslash \uparrow f(K))$. So $C \cap int(K) \neq \emptyset$. Pick $c \in C \cap int(K)$. Therefore, $f(c) \in f(K) \cap \downarrow C$, which contradicts $C \in \Gamma(L) \backslash \uparrow f(K)$.

By **Claim 1**, we know that $(\Gamma(L), \subseteq, \upsilon(\Gamma(L)) \vee \tau)$ is a pospace. Hence, $(\Gamma(L), \supseteq, \upsilon(\Gamma(L)) \vee \tau)$ and $(\Gamma(L), \supseteq, \sigma(\Gamma(L)) \vee \tau)$ are both pospaces. What is more, we can also show that $\sigma(\Gamma(L))$ and $\upsilon(\Gamma(L))$ are two separating dual topologies on (X, τ) (see Definition VI-6.17 in [2]).

Claim 2: $\sigma(\Gamma(L)) \vee \tau$ and $\upsilon(\Gamma(L)) \vee \tau$ are both compact.

It suffices to prove that $\sigma(\Gamma(L)) \vee \tau$ is compact. Because $\Gamma(L)$ is a complete lattice, $\Gamma(L)$ is Lawson compact. Thus $\Sigma\Gamma(L)$ is patch compact. This implies that $\sigma(\Gamma(L)) \vee \tau$ is compact since $\sigma(\Gamma(L)) \vee \tau$ is contained in the patch topology. By Theorem VI-6.18 in [2], we have $\sigma(\Gamma(L)) = \upsilon(\Gamma(L))$ and $\Sigma\Gamma(L)$ is a stable compact space. $\square$

Lemma 5.7. *For any poset L, if $\Sigma\Gamma(L)$ is locally compact then ΣL is core-compact.*

Proof. Let $x \in U \in \sigma(L)$. Then $\downarrow x \in \Diamond U \in \upsilon(\Gamma(L)) \subseteq \sigma(\Gamma(L))$. Thus there exists $\mathcal{K} \in Q(\Gamma(L))$ such that $\downarrow x \in int(\mathcal{K}) \subseteq \mathcal{K} \subseteq \Diamond U$ since $\Sigma\Gamma(L)$ is locally compact. Let $V = \{y \in L \mid y \in int(\mathcal{K})\}$. We want to prove that V is Scott open. Obviously, V is an upper set. For any directed subset D with $\sup D$ exists and $\sup D \in V$, we have $\downarrow \sup D = \sup_{d\in D} \downarrow d \in int(\mathcal{K})$. This implies that there exists $d \in D$ such that $\downarrow d \in int(\mathcal{K})$. Hence, $d \in V$. Note that $x \in V$. It suffices to prove that $V \ll U$. Let $(U_i)_{i\in I}$ be a directed subset of $\sigma(L)$ with $U \subseteq \bigcup_{i\in I} U_i$. Then we have $\Diamond U \subseteq \bigcup_{i\in I} \Diamond U_i$, that is, $\mathcal{K} \subseteq \bigcup_{i\in I} \Diamond U_i$. It follows that there exists $i \in I$ such that $\mathcal{K} \subseteq \Diamond U_i$. If $v \in V$, then $\downarrow v \in \mathcal{K} \subseteq \Diamond U_i$. Therefore, $v \in U_i$. So we have $x \in V \ll U$. □

From the above two lemmas, we deduce the following.

Theorem 5.8. *Let ΣL be well-filtered. Then ΣL is locally compact iff $\Sigma\Gamma(L)$ is locally compact.*

The following are some more relevant problems.

Problem 5.9. If ΣL is core-compact, must $\Sigma\Gamma(L)$ be locally compact?

Problem 5.10. If $\Sigma\Gamma(L)$ is locally compact, must ΣL be locally compact?

Problem 5.11. If $\Sigma\Gamma(L)$ is locally compact and $\sigma(\Gamma(L)) = \upsilon(\Gamma(L))$, must ΣL be locally compact?

By Lemma 5.1, a complete lattice is sober if its Scott closed set lattice is sober. This conclusion is not true for general dcpos as the following example shows.

Example 5.12. Let $\mathbb{J}$ be Johnstone's non-sober dcpo in [3]. Then $\Sigma\Gamma(\mathbb{J})$ is sober and $\mathbb{J}$ is a non-sober dcpo. To see this, let $\mathcal{A}$ be an irreducible closed subset of $\Gamma(\mathbb{J})$. Then $\bigcup\mathcal{A} \in \Gamma(\mathbb{J})$. We now distinguish two cases:

Case 1, $\bigcup\mathcal{A} \cap \max\mathbb{J}$ is infinite: It follows that $\bigcup\mathcal{A} = \mathbb{J}$ by the Scott-closedness of $\bigcup\mathcal{A}$. We only need to prove that $\mathbb{J} \in cl(\mathcal{A})$. For any $\mathcal{U} \in \sigma(\Gamma(\mathbb{J}))$, if $\mathbb{J} \in \mathcal{U}$, then $\mathbb{J} = \sup_{F\in Fin(\max\mathbb{J})} \downarrow F \in \mathcal{U}$, where $Fin(\max\mathbb{J})$ denotes the set of all finite subsets of $\max\mathbb{J}$. This implies that there exists $F \in Fin(\mathbb{J})$ such that $\downarrow F \in \mathcal{U}$ by the Scott openness of $\mathcal{U}$. Assume $E = \{m \in \mathbb{N} \mid (m, \infty) \in F\}$. Note that $\downarrow F = \sup_{n\in\mathbb{N}} \downarrow F_n$, $F_n = \{(m, n) \mid m \in E\}$. Hence, there exists $n \in \mathbb{N}$ such that $\downarrow F_n \in \mathcal{U}$. From the order of $\mathbb{J}$, we know there exists (s, t) such that it is an upper bound of F_n. Therefore, $\downarrow(s, t) \in \mathcal{U}$ because $\mathcal{U}$ is an upper set. Then we have $\downarrow(s, t) \in \mathcal{A} \cap \mathcal{U} \neq \emptyset$.

Case 2, $\bigcup\mathcal{A} \cap \max\mathbb{J}$ is finite: We want to prove that $\bigcup\mathcal{A}$ is a quasicontinuous domain. Assume $m_0 = \max E$. It is easy to prove that (s, t) is a compact element of $\bigcup\mathcal{A}$ for any $t \geq m_0 + 1$. Let $(m, n) \in \bigcup\mathcal{A}$. We now distinguish three cases:

Case 2.1, $m \in E, n = \infty$: Then $(m, \infty) = \sup_{t\geq(m_0+1)}(m, t)$. Note that $((m, t))_{t\geq(m_0+1)} \subseteq \downarrow(m, \infty) \cap K(\bigcup\mathcal{A})$ and is directed, where $K(\bigcup\mathcal{A})$ denotes the compact elements of $\bigcup\mathcal{A}$.

Case 2.2, $m \in E, n \neq \infty$: Let $G = \{s \in E \mid s \geq n\}$, $E_t = \{(s, t) \mid s \in G\} \cup \{(m, n)\}$. Then $E_t \ll (m, n)$ and $\uparrow(m, n) = \bigcap_{t\in\mathbb{N}} \uparrow E_t$.

Case 2.3, $m \notin E$: The proof is similar to that of Case 2.2.

Then we have $\Sigma\Gamma(\bigcup\mathcal{A})$ is sober. It remains to prove that $\sup\mathcal{A} = \bigcup\mathcal{A} \in \mathcal{A}$. Note that $\bigcup\mathcal{A}$ is a Scott closed subset of L. We obtain that the Scott topology of $\bigcup\mathcal{A}$ coincides with the relative topology of $\sigma(\mathbb{J})$. It follows that $\Gamma(\bigcup\mathcal{A})$ is a Scott closed subset of $\Gamma(L)$. Therefore, $\mathcal{A}$ is a irreducible closed subset of $\Sigma\Gamma(\bigcup\mathcal{A})$. Thus $\bigcup\mathcal{A} \in \mathcal{A}$ because $\Sigma\Gamma(\bigcup\mathcal{A})$ is sober.

Definition 5.13. [4] A dcpo L is called *strongly complete* if every irreducible subset of L has a supremum.

Theorem 5.14. *Let L be a poset. Then the following statements are equivalent:*

(1) *ΣL is sober;*

(2) *L is a strongly complete dcpo and $\sigma(\mathbf{IRR}(L))$ agrees with the lower Vietoris topology, where $\mathbf{IRR}(L)$ denotes the set of all irreducible Scott closed subsets of L.*

Proof. (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (1) Define $f : L \to \mathbf{IRR}(L)$ by $f(x) = \downarrow x$ and $g : \mathbf{IRR}(L) \to L$ by $g(A) = \sup A$ for any $A \in \mathbf{IRR}(L)$.

Claim 1: f is Scott continuous.

Let D be a directed subset of L. Then $f(\sup D) = \downarrow \sup D = cl(\bigcup_{d\in D} \downarrow d) = \sup_{d\in D} \downarrow d = \sup_{d\in D} f(d)$.

Claim 2: g is Scott continuous.

Let $(A_i)_{i\in I}$ be a directed subset of $\mathbf{IRR}(L)$. Then $g(\sup_{i\in I} A_i) = g(cl(\bigcup_{i\in I} A_i)) = \sup cl(\bigcup_{i\in I} A_i)$. From the proof of Lemma 5.1, we know that $\sup cl(\bigcup_{i\in I} A_i) = \sup(\bigcup_{i\in I} A_i)$. Hence, $g(\sup_{i\in I} A_i) = \sup(\bigcup_{i\in I} A_i) = \sup_{i\in I} \sup A_i = \sup_{i\in I} g(A_i)$.

Assume $x \in L$. Then $g \circ f(x) = g(\downarrow x) = x$, that is, $g \circ f = id_L$. Hence, L is a Scott retract of $\mathbf{IRR}(L)$. Since $\mathbf{IRR}(L)$ endowed with the lower Vietrois topology is sober and $\sigma(\mathbf{IRR}(L))$ agrees with the lower Vietoris topology, we have ΣL is sober. $\square$

The following two examples show that none of the condition (2) of the above Theorem can be dispensed with.

Example 5.15. Let $\mathbb{J}$ be Johnstone's Example in [3], $L = \mathbb{J} \cup \{\top\}$. We define an order $\leq$ on L as follows:

$x \leq y$ iff either

- $x, y \in \mathbb{J}$ and $x \leq y$ in $\mathbb{J}$; or
- $x \in L, y = \top$.

Then it is easy to prove that L is a non-sober and strongly complete dcpo. Note that $\{\mathbb{J}, L\} \in \sigma(\mathbf{IRR}(L))$, but $\{\mathbb{J}, L\}$ is not open in the lower Vietoris topology of $\mathbf{IRR}(L)$.

Example 5.16. Let $\mathcal{H}$ be the dcpo in [4]. From the paper [4], we know that $\sigma(\mathbf{IRR}(\mathcal{H}))$ coincides with the lower Vietoris topology and $\Sigma\mathcal{H}$ is not a strongly complete dcpo. It is easy to verify that $\Sigma\mathcal{H}$ is not sober.

In this paper, we gave the negative answers to two open problems on sobriety. We also proved some related results. The work here also led to some natural problems, in particular the Problem 5.5, whose solutions will enrich the theory of non-Hausdorff topology.

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