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Stable representations of quivers

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Abstract

Let Q be a finite quiver without oriented cycles and let kQ the path algebra of Q over an algebraically closed field k. We investigate stable finite dimensional representations of Q. That is for a fixed dimension vector d and a fixed weight θ we consider θ-stable representations of Q with dimension vector d. If we wish to compare also representations with different dimension vectors, then it is more convenient to consider a slope μ instead of a weight θ. In particular, we apply the results of Harder–Narasimhan on natural filtrations associated to any fixed slope μ to the category of representations of Q. Further we introduce the wall system for weights with respect to a fixed dimension vector d and consider several examples. © 2001 Published by Elsevier Science B.V.

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1. Introduction

Let Q be a finite quiver without oriented cycles. Fix k an algebraically closed field. Finite dimensional representations of Q over k can be considered as finite dimensional left kQ-modules, where kQ is the path algebra of Q. We denote by $\operatorname{mod} Q$ the category of all finite dimensional representations of the quiver Q.

Fix a dimension vector d and a weight θ with respect to d (see Section 2 for a definition). The set of θ-*stable* representations of dimension vector d forms an algebraic

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variety in a natural way: the so-called moduli space (see [9]). Thus, θ-stable representations are of particular interest for geometric questions in representation theory. Beside these geometric properties, θ-stable representations also have other remarkable properties, which we study in this work.

First we introduce a different stability notion as follows: take two $\mathbb{Z}$-linear functions θ and κ on the Grothendieck group $K_0(Q)$ of the category $\operatorname{mod} Q$ (with values in $\mathbb{R}$ or $\mathbb{Z}$, in the latter case we say θ is integral). Assume, moreover, that κ is positive on $\operatorname{mod} Q$, that is $\kappa(M) \geqslant 0$ for all objects in $\operatorname{mod} Q$ and if $\kappa(M) = 0$ then $M = 0$. For any object $M \neq 0$ we define the *slope* of M as $\mu(M) := \theta(M)/\kappa(M)$. In a similar way as for weights we define a stability notion for the slope μ (see Section 2). The purpose of this definition is to get a stability notion for all objects (except the zero object) and not only for those of a fixed dimension vector d.

Objects which are stable for a certain slope were considered in algebraic geometry for a long time. In particular, stable vector bundles or stable sheaves play an important role. The aim of this note is to collect the basic properties of stable representations, some of them known, however never applied to our particular category $\operatorname{mod} Q$. Since we have two equivalent stability notions, the weight stability and the slope stability, we propose the following rule to use them. Whenever we work with representations of a fixed dimension vector d, then it is more convenient to use weights, and whenever we wish to define stability for representations of different dimension vectors we use slopes. Consequently, we use slopes in Section 2 and weights in Sections 3 and 4.

The basic properties of stable objects developed in [5] are valid with some slight changes also for stable representations of quivers. This is the starting point of our investigation in Section 2. We also note that all results in Section 2 are also true for any abelian finite length category with Grothendieck group of finite rank; in particular, also for the category of finite dimensional modules over any finite dimensional algebra. For a different approach to stable objects in abelian categories we also refer to [13].

As already mentioned, we collect the basic properties of stable representations in Section 2. This includes the Harder–Narasimhan filtration (Theorem 2.5), basic properties of the category of μ-semistable representations of fixed slope (Proposition 2.7), and the functorial behavior of the Harder–Narasimhan filtration (Theorem 2.8). In Section 3 we study the variation of the set of θ-stable representations: this leads naturally to the notion of a wall and the wall system. Associated to each weight which lies on exactly one wall we obtain a flip, that is the corresponding variation of the moduli space. We study basic properties of flips in Section 4. The last section is devoted to some examples: walls for tame hereditary algebras, stable representations of the generalized Kronecker quiver and examples of inner walls. See also [8].

Basic notation. In this note Q is a finite quiver without oriented cycles. We work over a ground field k, which is algebraically closed except in Section 2. We denote by $\subset$ strict inclusion. All categories, representations and algebraic varieties are always over k. We denote by $\operatorname{mod} Q$ the category of finite dimensional representations of Q. By θ and κ we denote $\mathbb{Z}$-linear functions on the Grothendieck group $K_0(Q)$ of $\operatorname{mod} Q$, where the latter function is a positive linear function, that is $\kappa(M) > 0$ for all $M \neq 0$. Moreover, μ denotes a slope. Further, we denote by τ the Auslander–Reiten translation in $\operatorname{mod} Q$ and $\langle -, - \rangle$ is the Euler form in $K_0(Q)$ defined by $\langle M, N \rangle =$

$\dim_k \mathrm{Hom}(M,N) - \dim_k \mathrm{Ext}^1(M,N)$. We shall identify functions on $K_0(Q)$ with its induced function on $\mathrm{mod}\, Q$, so we write e.g. $\theta(M)$ instead of θ applied to the class of M in $K_0(Q)$. Whenever we work with algebraic varieties we use the Zariski topology, whereas we use the standard topology in any $\mathbb{R}$-vector space, e.g. in the space of weights with respect to a dimension vector.

2. First properties of stable representations

We start this section with a definition of stable and semistable representations with respect to a weight and a slope, respectively.

Definition. An element d of the Grothendieck group $K_0(Q)$ is called a *dimension vector*. A *weight* with respect to a dimension vector d is a $\mathbb{Z}$-linear function on the Grothendieck group of $\mathrm{mod}\, Q$, which vanishes on the dimension vector d. The weight $\theta(M)$ of a representation M is just its value on the class of M in the Grothendieck group. We denote the set of all weights with respect to d by $\mathbb{H}(d)$. A *slope* μ is the quotient of a $\mathbb{Z}$-linear function θ by a positive $\mathbb{Z}$-linear function κ. Thus $\mu(M) = \theta(M)/\kappa(M)$, where the value of κ is positive on each non-zero representation.

Note that the weight of the zero representation is well-defined and zero, while the slope of the zero representation is *not* defined. Thus, we always assume M to be non-zero, whenever we write $\mu(M)$.

Definition. Let M be a representation of dimension vector d and θ be a weight with respect to d. The representation M is θ-*stable* if for all proper non-zero subrepresentation N of M we have $\theta(N) < 0$. It is said to be θ-*semistable* if for each subrepresentation N we have $\theta(N) \leqslant 0$.

A representation M is μ-*stable* (μ-*semistable*, respectively) for a certain slope μ if for each non-zero proper subrepresentation N of M we have

$$\mu(N) < \mu(M) \quad (\mu(N) \leqslant \mu(M), \text{ respectively}).$$

Two slopes μ and μ' are said to be *equivalent* (we write $\mu \sim \mu'$) if the set of μ-stable representations coincides with the set of μ'-stable representations, and the set of μ-semistable representations coincides with the set of μ'-semistable representations. For a fixed dimension vector d and θ, θ' in $\mathbb{H}(d)$, we say θ is d-*equivalent* to θ' if the set of θ-stable representations of dimension vector d coincides with the set of θ'-stable representations of dimension vector d, and the set of θ-semistable representations of dimension vector d coincides with the set of θ'-semistable representations of dimension vector d.

A representation M is *stable* if there exists a slope μ, so that M is μ-stable.

Lemma 2.1. *Let* $0 \to L \to M \to N \to 0$ *be an exact sequence of representations and let* μ *be a slope. Then the following conditions are equivalent*:

(1) $\mu(L) \leqslant \mu(M)$,

(2) $\mu(L) \leqslant \mu(N)$, *and*
(3) $\mu(M) \leqslant \mu(N)$.

Proof. Write μ in the form $\mu = (\theta + c\kappa)/\kappa = \theta/\kappa + c$ for a certain constant c and a new weight θ such that $\theta(M) = 0$. Then the lemma is true for μ precisely when it is true for θ/κ. Since κ is positive, and $\theta(M) = 0$, it remains to show that $\theta(L) \geqslant 0$ if and only if $\theta(N) \leqslant 0$. But this follows from the linearity of θ. □

The property in the lemma above is also called *seesaw property* in [13]. Observe that the lemma implies that a representation M is stable with respect to some slope precisely in case it is stable with respect to some weight. Moreover, a representation is μ-semistable precisely if for all μ-stable subrepresentations N we have $\mu(N) \leqslant \mu(M)$.

Lemma 2.2. *Let μ be a slope. Then each representation M has a unique subrepresentation N which satisfies the following two conditions*:
(1) *$\mu(N)$ is maximal for all subrepresentations of M*, *and*
(2) *N is maximal among all subrepresentations of maximal slope.*
Moreover, *the representation N is μ-semistable.*

The unique subrepresentation of Lemma 2.2 is called scss (strongly contradicting semistability) in [5] (see also Proposition 1.3.4 therein). Note that the scss subrepresentation of a representation M equals with M precisely when M is semistable.

Proof. Since the category of all representations is noetherian, the existence of a representation N with (1) and (2) is obvious. It remains to prove the uniqueness: Let N_1 and N_2 be two non-isomorphic representations which satisfy (1) and (2) of the lemma. We consider the exact sequence

$$0 \to N_1 \cap N_2 \to N_1 \oplus N_2 \to N_1 + N_2 \to 0.$$

By Condition (1) we obtain $\mu(N_1 \cap N_2) \leqslant \mu(N_1) = \mu(N_2)$. Thus $\mu(N_1 + N_2) \geqslant \mu(N_1 \oplus N_2) = \mu(N_1) = \mu(N_2)$ by Lemma 2.1. By our Assumption (1) we have $\mu(N_1 + N_2) = \mu(N_1) = \mu(N_2)$. Hence $N_1 = N_1 + N_2 = N_2$. The last assertion is obvious by Condition (1), indeed, each subrepresentation of maximal slope is μ-semistable. □

Lemma 2.3. *Let M be not μ-semistable and let $N \neq 0$ be a subrepresentation of M satisfying*
(1) *N and M/N are μ-semistable*, *and*
(2) $\mu(N) > \mu(M/N)$.
Then N is scss *in M.*

Proof. Let L' be a subrepresentation of M with maximal slope $\mu(L')$. Then we have an exact sequence

$$0 \to L' \cap N \to L' \to L'/(L' \cap N) \simeq (L' + N)/N \to 0,$$

where $\mu(L' \cap N) \leqslant \mu(N)$ and $\mu((L'+N)/N) \leqslant \mu(M/N)$ by (1). Consequently, by Lemma 2.1, $\mu(L') \leqslant \mu((L'+N)/N) < \mu(N)$ in case $L' \not\subset N$. This contradiction shows that $L' \subset N$. Thus N is scss. □

Lemma 2.4. *Let $0 = M_0 \subset M_1 \subset \cdots \subset M_{r-1} \subset M_r = M$ be a filtration of M, so that M_i/M_{i-1} is μ-semistable. Then the following two conditions are equivalent*:
(1) M_i/M_{i-1} *is* scss *in M/M_{i-1}, and*
(2) $\mu(M_i/M_{i-1}) > \mu(M_{i+1}/M_i)$ *for all* $i = 1, \ldots, r-1$.

Proof. First we show (1) implies (2). We consider the exact sequence

$$0 \to M_i/M_{i-1} \to M/M_{i-1} \to M/M_i \to 0.$$

Since M_i/M_{i-1} is scss in M/M_{i-1} we obtain $\mu(M_i/M_{i-1}) > \mu(M_{i+1}/M_{i-1}) > \mu(M_{i+1}/M_i)$. The last inequality follows from Lemma 2.1.

The converse implication is slightly more complicated. We use downward induction on r. First we show that M_{r-1}/M_{r-2} is scss in M/M_{r-2}. Consider the exact sequence

$$0 \to M_{r-1}/M_{r-2} \to M/M_{r-2} \to M/M_{r-1} \to 0.$$

By Condition (2) we have $\mu(M_{r-1}/M_{r-2}) > \mu(M/M_{r-1})$. Thus we can apply Lemma 2.3.

Next we establish the induction step.

Let $N \subset M/M_i$. We have to show $\mu(N) \leqslant \mu(M_{i+1}/M_i)$, and equality holds only for $N \subseteq M_{i+1}/M_i$. First assume $N \subset M_{i+1}/M_i$. Then the assertion is clear, since M_{i+1}/M_i is semistable. So we assume N to be not contained in M_{i+1}/M_i. Moreover, we can assume N is a subrepresentation of maximal slope, so it is itself semistable. We denote the composition of maps $N \subset M/M_i \to M/M_{i+1}$ by f. By assumption f is nontrivial. Then we have an exact sequence

$$0 \to \mathrm{Ker}(f) \to N \to \mathrm{Im}(f) \to 0.$$

Since N is semistable we obtain

$$\mu(\mathrm{Ker}(f)) \leqslant \mu(N) \leqslant \mu(\mathrm{Im}(f)) \leqslant \mu(M_{i+2}/M_{i+1}),$$

where the last inequality follows because M_{i+2}/M_{i+1} is scss in M/M_{i+1} by induction hypothesis. Now we can apply Assumption (2) and obtain the claim. □

Theorem 2.5 (Harder–Narasimhan filtration [5]). *Let M be a representation and let μ be a slope. Then there exists a unique filtration $0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_{r-1} \subset M_r = M$ such that*
(1) $\mu(M_i/M_{i-1}) > \mu(M_{i+1}/M_i)$ for $i = 1, \ldots, r-1$, *and*
(2) *the representations M_i/M_{i+1} for $i = 1, \ldots, r-1$ are μ-semistable.*

Proof. First we define the representation M_1 as the unique scss subrepresentation of M. Next we describe the induction step: Assume we already have defined representations $M_1, \ldots, M_i$ which satisfy the conditions of the theorem. Then we consider the unique maximal subrepresentation N_{i+1} in M/M_i with maximal slope according to Lemma 2.2.

It is μ-semistable, and we define M_{i+1} to be the pre-image of N_{i+1} in M. By Lemma 2.1 the slope of M_{i+1}/M_i is strictly smaller than the slope of M_i and by construction $M_{i+1}/M_i = N_{i+1}$ is μ-semistable. Hence we obtain a filtration of M with the desired conditions.

The uniqueness of the filtration follows from Lemma 2.4 using induction. □

Lemma 2.6. *Let M and N be both μ-semistable representations and let $f: M \to N$ be a non-trivial homomorphism. Then $\mu(M) \leqslant \mu(N)$. In particular, each homomorphism $f: M \to N$ between μ-semistable representations with $\mu(M) > \mu(N)$ is the zero map. Moreover, if M and N are μ-stable and $\mu(M) \geqslant \mu(N)$, then either f is the zero map or an isomorphism.*

Proof. Let $f: M \to N$ be a non-zero homomorphism of μ-semistable representations. We consider the two exact sequences

$$0 \to \mathrm{Ker}(f) \to M \to \mathrm{Im}(f) \to 0, \quad \text{and}$$

$$0 \to \mathrm{Im}(f) \to N \to \mathrm{Coker}(f) \to 0.$$

Because N is μ-semistable we obtain $\mu(\mathrm{Im}(f)) \leqslant \mu(N)$. Assume $\mu(M) > \mu(N)$, then $\mu(\mathrm{Im}(f)) < \mu(M)$. Thus by Lemma 2.1, $\mu(\mathrm{Ker}(f)) > \mu(M)$, which contradicts the semistability of M. This proves the first claim, and the second claim is just another formulation of the first one. The last claim is also clear. □

We denote by $\mathscr{S}^\mu$ the set of all μ-semistable representations and by $\mathscr{S}^\mu_\lambda$ the set of all μ-semistable representations with fixed slope λ. We define the zero representation to be an element of $\mathscr{S}^\mu_\lambda$. Compare with [9, 3.1].

Proposition 2.7. *The category of μ-semistable representations $\mathscr{S}^\mu_\lambda$ with fixed slope λ is an abelian subcategory of the category of representations. The simple objects in the category $\mathscr{S}^\mu_\lambda$ are the μ-stable representations of slope λ. In particular, each μ-semistable representation in $\mathscr{S}^\mu_\lambda$ has a Jordan–Hölder filtration with μ-stable quotients in $\mathscr{S}^\mu_\lambda$. Moreover, if k is algebraically closed, then each μ-stable representation has endomorphism ring k.*

Proof. First we show that $\mathscr{S}^\mu_\lambda$ has kernels and cokernels. Assume $f: M \to N$ is a homomorphism of μ-semistable representations, both of slope λ. Proceeding as in the lemma above, we get that $\mu(\mathrm{Ker}\, f) = \mu(M) = \mu(N) = \mu(\mathrm{Coker}\, f) = \mu(\mathrm{Im}\, f)$ and therefore the kernel, the image, and the cokernel of f are all μ-semistable of slope λ. Moreover each extension of representations in $\mathscr{S}^\mu_\lambda$ is in $\mathscr{S}^\mu_\lambda$ by Lemma 2.1. Also by the lemma before, a representation in $\mathscr{S}^\mu_\lambda$ has only trivial subrepresentations in $\mathscr{S}^\mu_\lambda$, precisely if it is μ-stable. Moreover, if the field k is algebraically closed, then the lemma above shows that the only endomorphisms of a stable representation are the homotheties.

Finally, a module $M \in S^\mu_\lambda$ is artinian in an abelian category. Therefore M accepts a filtration whose composition factors are simple objects, this is the desired Jordan–Hölder filtration. □

Next we introduce for a given slope μ and each representation M a family of representations $\{M_\lambda\}$ for $\lambda \in \mathbb{R}$. Let M_i be the Harder–Narasimhan filtration. Then we define $M_\lambda = M_i$ for $\mu(M_i/M_{i-1}) \geqslant \lambda > \mu(M_{i+1}/M_i)$. The sequence $(\mu(M_1), \dots, \mu(M_i/M_{i-1}), \dots, \mu(M/M_{r-1}))$ is also called the *Harder–Narasimhan sequence*.

Theorem 2.8. *Let $f: M \to N$ be a homomorphism of representations and μ be a slope. Then f preserves the Harder–Narasimhan filtration, that is $f(M_\lambda) \subset N_\lambda$.*

Proof. We prove the claim by induction on the sum of the number of terms in the Harder–Narasimhan filtration of M and N. Assume both M and N have only one term in its Harder–Narasimhan filtration. Then the claim follows from Lemma 2.6, because both are μ-semistable.

Let $\mu(M_1) > \mu(N_1)$. Then we show $f(M_1) = 0$: assume $f(M_1) \neq 0$ and let j minimal with $f(M_1) \subseteq N_j$. Then the restriction $f: M_1 \to N_j/N_{j-1}$ is not the zero map, which contradicts the semistability by Lemma 2.6.

Next assume $\mu(M_1) \leqslant \mu(N_1)$. Let j be the unique maximal index with $\mu(M_1) \leqslant \mu(N_j)$. Consider the induced map $\bar{f}: M \to N/N_j$, by the previous argument $\bar{f}(M_1) = 0$. Thus $f(M_1) \subseteq N_j$ and f induces a map $f': M/M_1 \to N/N_j$. We can apply the induction hypothesis to f'. Since $(M/M_i)_\lambda = M_\lambda/M_i$ for $\lambda \leqslant \mu(M_i)$ we obtain the result also for f. $\square$

We remark that the functorial properties of Harder–Narasimhan filtrations were first observed by Faltings [3].

3. The wall system

In this part we define the system of walls associated to the equivalence defined in the beginning of Section 2 for weights with respect to a fixed dimension vector d. We restrict to weights in this part, however it is certainly possible to consider similar notions for slopes. Further we need some facts about generic subrepresentations (see [14] for further properties) and define special subrepresentations.

Definition. We fix a dimension vector d. Then we can define the *wall system* with respect to this dimension vector. It consists of a finite set of hyperplanes $\{W_i\}_{i \in I}$ in $\mathbb{H}(d)$, where I is a finite index set. By definition the wall system is the minimal set of hyperplanes in $\mathbb{H}(d)$ with the following property: whenever two weights θ and θ' in $\mathbb{H}(d)$ lie on the same side of each of these hyperplanes W_i, then they are equivalent. The existence of such a wall system follows from the lemma below.

Moreover, we define *outer walls* and *inner walls*: Each wall defines two open half spaces in $\mathbb{H}$. A wall is an outer wall, if for one open half space *each* weight θ in this half space admits no θ-semistable representation of dimension vector d. It is an inner wall if in both open half spaces there exist weights which admit semistable representations of dimension vector d.

Examples. The wall system with respect to a thin dimension vector is known [6]. It has also a very useful description in terms of the flow polytope of Q [1]. Moreover, the wall system is known for a tame hereditary algebra (Section 5) and for the subspace quiver with dimension vector related to one-dimensional subspaces [7]. We also refer to [2]: here one finds a description of the wall system for rank-two parabolic vector bundles on a projective line. It is closely related to rank-two vector bundles on a weighted projective line in the sense of Geigle and Lenzing [4]. For general properties of the wall system for vector bundles we also refer to [15].

Remark. Let θ be a weight not on a wall of the wall system of $\mathbb{H}(d)$. Then there is a neighborhood U of θ such that U does not intersect any wall. Therefore for any weight θ' in U, the set of θ-semistable representations and the set of θ'-semistable representations coincide.

Let d be a fixed dimension vector and d' a non-trivial proper subdimension vector, which is *not* an element of the line through d. Then we define a hyperplane

$$W(d') := \{\theta \in \mathbb{H}(d) \mid \theta(d') = 0\}.$$

Without loss of generality we can assume that d' is *indivisible*, that is the greatest common divisor of the numbers d_q for q a vertex of Q is one. Note that $W(d') = W(d - d')$.

Lemma 3.1. *Wall systems in $\mathbb{H}(d)$ exist. If W is a wall in $\mathbb{H}(d)$, then $W = W(d')$ for a certain indivisible subdimension vector d' of d.*

Proof. This follows immediately from the definition: if θ and θ' are not equivalent, then there exists a dimension vector $d' \subset d$ with $\mathrm{sign}(\theta(d')) \neq \mathrm{sign}(\theta'(d'))$ (here sign takes values in $\{+, -, 0\}$). If θ and θ' are in a sufficiently small open disk, then no different d'' with $\mathrm{sign}(\theta(d'')) \neq \mathrm{sign}(\theta'(d''))$ exists. The existence of wall systems follows by compacity.

For a wall W in $\mathbb{H}(d)$, choose $d' \subset d$ such that $\mathrm{sign}(\theta(d')) \neq \mathrm{sign}(\theta(d'))$ for θ and θ' in different sides of $\mathbb{H}(d)$. Clearly $W = W(d')$. $\square$

We recall that a subdimension vector d' of d is said to be a *generic subdimension* of d if each representation of dimension vector d has a subrepresentation of dimension vector d'.

Lemma 3.2. *Let W be an outer wall. Then $W = W(d')$ for a generic subdimension vector d' of d. Moreover, the hyperplane $W(d')$ for d' a generic subrepresentation of d is not an inner wall.*

Proof. Let $W = W(d')$ be an outer wall for a certain dimension vector $d' \subset d$. Observe that the representations accepting a subrepresentation of dimension vector d' form a constructible set containing an open set. Moreover, if θ-semistable representations exist then they form an open dense subset. Thus, each representation has a subrepresentation of dimension vector d'. Consequently d' is a generic subrepresentation. Conversely, consider d' a generic subdimension of d. Then each θ-semistable representation has a subrepresentation of dimension d'. If $W(d')$ were an inner wall, then there exists $\theta \in \mathbb{H}(d)$ with $\theta(d') > 0$ and a θ-stable representation M of weight θ with a subrepresentation N of dimension d'. This yields a contradiction. $\square$

Note that Lemma 3.2 does not give a precise characterization of outer walls. It is not true, that each generic subrepresentation d' defines an outer wall $W(d')$.

It is much harder to characterize inner walls. For a given dimension vector even the existence of inner walls is an open problem. We only can show some special cases in Section 5.

Definition. A dimension vector $d' \subset d$ is called a *special subdimension vector* if there exist stable representations M and N of dimension vector d, such that M contains a subrepresentation of dimension vector d', but N does not contain such a subrepresentation.

Clearly, there exist subdimension vectors which are neither generic nor special. However, certain special subdimension vectors correspond to inner walls.

Lemma 3.3. *Let d be Schurian. An inner wall W with respect to d is of the form $W(d')$ for a certain special subdimension vector d'.*

Proof. First note that stable representations of dimension vector d exist, since d is Schurian (see [9, 4.4]). By Lemma 3.1 we know, that W is of the form $W(d')$ for a certain subdimension vector d' of d. Since W is an inner wall, d' is not a generic subdimension vector. In particular, there exist stable representations which do not have a subrepresentation of dimension vector d'. Since W is a wall, there exists at least one stable representation of dimension vector d, with a subrepresentation of dimension vector d'. Thus d' is a special subdimension vector. $\square$

Remark. Note that the generic subdimension vectors and the Schur roots were determined by Schofield [14]. It is desirable to obtain also a description of the special subdimension vectors. This is known only for a few particular cases (see the examples at the end of this work).

Observe that the Lemmata 3.2 to 3.4 may be reformulated in more algebraic geometric terms. Indeed, let $\mathscr{R}(Q,d)$ be the variety of kQ-representations of vector dimension d, that is, $\mathscr{R}(Q,d) = \prod_{i\to j} k^{d(i)\times d(j)}$ where the product runs over all the arrows in Q. The isomorphy of representations in $\mathscr{R}(Q,d)$ given by the action of the linear group $\prod_i G\ell_{d(i)}(k)$. Given a wall W we define $X(W)$ as the variety of those modules M in $\mathscr{R}(Q,d)$ such that there is a submodule N of M of dimension d' with $W = W(d')$. We get the following characterizations:

Lemma 3.4. *Let W be a wall. Then $X(W)$ is a non-empty closed variety satisfying*:
(a) *$X(W) = \mathscr{R}(Q,d)$ if and only if W is an outer wall.*
(b) *$X(W) \subsetneq \mathscr{R}(Q,d)$ if and only if W is an inner wall.*

4. Moduli spaces and flips

Recall from [9] the existence of the moduli space $\mathscr{M}^\theta(Q,d)$ of θ-semistable representations of dimension vector d. It is of dimension $1-\langle d,d\rangle$ for some weight θ precisely when the dimension vector is Schurian (see [14] for properties of Schurian representations). Moreover, it is smooth for d indivisible and θ sufficiently general, that is θ does not lie on a wall with respect to d. In this case, the tth symmetric power of $\mathscr{M}^\theta(Q,d)$ is a closed subvariety of $\mathscr{M}^\theta(Q,t\cdot d)$. Indeed, let d be indivisible and Schurian. Let θ be a weight such that $\mathscr{M}^\theta(Q,d)$ has dimension $1-\langle d,d\rangle$ and every θ-semistable representation of dimension d is θ-stable (see [9, 5.4]). Let $M_1,\dots,M_t$ be points in $\mathscr{M}^\theta(Q,d)$, that is M_i is a θ-semistable for all $i=1,\dots,t$. Then $\oplus_i^t M_i$ is also θ-semistable. Thus, it is a point in $\mathscr{M}^\theta(Q,t\cdot d)$. This defines a morphism ϕ from the tth symmetric power of $\mathscr{M}^\theta(Q,d)$ to $\mathscr{M}^\theta(Q,t\cdot d)$. Since $\mathscr{M}^\theta(Q,d)$ is projective, its symmetric power is also projective, and its image is closed. It remains to show, that the morphism ϕ is an embedding. First note that it is injective since every M_i is θ-stable. Then we use the universality property of a moduli space and obtain that ϕ induces an isomorphism between the tth symmetric power of $\mathscr{M}^\theta(Q,d)$ and the closed subvariety of $\mathscr{M}^\theta(Q,t\cdot d)$ consisting of all representations of the form $\oplus_i^t M_i$ for M_i θ-semistable of dimension vector d.

For a fixed dimension vector d, moduli spaces for the various weights θ are related via some operations that we shall now consider. Let θ be fixed weight. We consider the wall system in a small neighborhood of θ. In general, if θ does not lie on a wall, then all moduli spaces are the same in a sufficiently small neighborhood of θ. If θ lies on exactly one wall W, then we have at most three different moduli spaces $\mathscr{M}^+$, $\mathscr{M}^-$, and $\mathscr{M}^0$, which correspond to the three classes of weights lying in one of the open half spaces, lying in the other open half space, and lying on the wall W. Note that these varieties $\mathscr{M}^+$, $\mathscr{M}^-$, and $\mathscr{M}^0$ might be isomorphic (e.g. in the tame case), however they are viewed as different if they parameterize different sets of stable, respectively, semistable representations.

Again these moduli spaces $\mathscr{M}^+$, $\mathscr{M}^-$, and $\mathscr{M}^0$ are defined in a sufficiently small neighborhood of $\theta\in W$. The situation becomes more complicated, if θ lies on more than one wall. However, we always have the following result.

Following [9], we say that two θ-semistable representations are *S-equivalent* if they have the same composition factors in the category of θ-semistable representations (see Proposition 2.7).

Theorem 4.1. *We consider a fixed dimension vector d and walls with respect to d.*

Assume θ is a fixed weight and θ' is any weight in a sufficiently small neighborhood of θ. Then there exists a morphism of algebraic varieties

$$\mathscr{M}^{\theta'}(Q,d)\xrightarrow{\pi_{\theta',\theta}}\mathscr{M}^\theta(Q,d),$$

which sends an S-equivalence class $[M]$ of θ'-semistable representations to the S-equivalence class of θ-semistable representations containing $[M]$. This morphism satisfies the following properties:

(1) *The morphism $\pi_{\theta',\theta}$ is a projective morphism.*

(2) *The morphism* $\pi_{\theta',\theta}$ *is surjective or* $\mathcal{M}^{\theta'}(Q,d)$ *is empty.*
(3) *Assume* θ *does not lie on any wall, then* $\pi_{\theta',\theta}$ *is an isomorphism, in fact it is equality.*
(4) *Assume* θ *does not lie on outer walls, then* $\pi_{\theta',\theta}$ *is a birational projective morphism.*
(5) *For any weight* θ'' *in a sufficiently small neighborhood of* θ' *the morphism* $\pi_{\theta'',\theta}$ *also exists and we have* $\pi_{\theta',\theta}\pi_{\theta'',\theta'} = \pi_{\theta'',\theta}$.

Definition. Let θ be a weight which lies on exactly one inner wall W, and let θ^+ and θ^- be weights in the opposite open half spaces of W which are sufficiently nearby θ, that is no other wall contains one of θ^+ or θ^- or is between them. Then we call the diagram

$$\begin{tikzcd} & \mathcal{M}^+ \times_{\mathcal{M}^0} \mathcal{M}^- \arrow[dl, "p^+"'] \arrow[dr, "p^-"] & \\ \mathcal{M}^+ := \mathcal{M}^{\theta+}(Q,d) \arrow[dr, "\pi^+"'] & & \mathcal{M}^- := \mathcal{M}^{\theta-}(Q,d) \arrow[dl, "\pi^-"] \\ & \mathcal{M}^0 := \mathcal{M}^{\theta}(Q,d) & \end{tikzcd}$$

the *flip associated to* θ *in* W. Note that these morphisms exist by the theorem above. Sometimes we consider only a part of this diagram, e.g. in [6] we only consider the lower part.

Proof (of the Theorem 4.1).

Recall from the first remark in Section 3 that the set of θ-semistable representations contains the set of θ'-semistable representations for each θ' in a sufficiently small neighborhood of θ. Also note, that the moduli space $\mathcal{M}^{\theta}(Q,d)$ is a good quotient of the subvariety of θ-semistable representations $\mathcal{R}^{\theta}(Q,d)$ contained in the space of all representations $\mathcal{R}(Q,d)$ of Q with dimension vector d. In particular, it is a categorical quotient. Consequently we have an injective morphism $\mathcal{R}^{\theta'}(Q,d) \subset \mathcal{R}^{\theta}(Q,d)$. This morphism is equivariant. Thus it factors through the quotients and we obtain a morphism

$$\mathcal{M}^{\theta'}(Q,d) \xrightarrow{\pi_{\theta',\theta}} \mathcal{M}^{\theta}(Q,d).$$

Note that this morphism, in general, is not injective. Next we prove the properties (1)–(5).

First note that $\pi_{\theta',\theta}$ is always projective, since $\mathcal{M}^{\theta'}(Q,d)$ and $\mathcal{M}^{\theta}(Q,d)$ are projective for quivers Q without oriented cycles (see [9]). This in particular proves (1).

Assertion (3) is obvious by definition of a moduli space: the set of θ'-semistable representations and the set of θ-semistable representations coincides, and this implies that the corresponding moduli spaces are the same. Also (5) is clear by definition.

We proceed with (2). We assume $\mathcal{M}^{\theta'}(Q,d)$ is nonempty. Then the set of θ'-semistable representations is open and dense in the representation space $\mathcal{R}(Q,d)$. Let M be a θ-semistable representation considered as a point in the representation space $\mathcal{R}(Q,d)$. Choose a line L through M which intersects the open dense subset of θ'-semistable representations. Consider its image, after restricting to a sufficiently small open neighborhood, in $\mathcal{M}^{\theta}(Q,d)$ and its unique continuation in $\mathcal{M}^{\theta'}(Q,d)$; the latter exists, since $\mathscr{M}^{\theta'}(Q,d)$ is projective. Consequently, M has a preimage and $\pi_{\theta',\theta}$ is surjective.

It remains to prove (4). Let M and N be two θ'-semistable representations which represent different points in $\mathscr{M}^{\theta'}(Q,d)$, that is they are not S-equivalent with respect to θ', so that $\pi_{\theta',\theta}(M) = \pi_{\theta',\theta}(N)$, that is they are S-equivalent with respect to θ. Then both have a subrepresention of dimension vector d' with $\theta(d')=0$. Consider all representations which have a subrepresentation of dimension vector d' as a subrepresentation. This is a closed subvariety in $\mathscr{R}(Q,d)$, according to [14]. Let X be the closed subvariety of those representations which have a subrepresentation of some dimension vector d' with $\theta'(d')\neq 0$ and $\theta(d')=0$. If X contains all representations, then θ lies on an outer wall, since some d' is a generic subrepresentation. Otherwise we have an open dense subset U in the representation space consisting of representations for which S-equivalence with respect to θ and θ' coincides. Consequently the moduli spaces $\mathscr{M}^{\theta'}(Q,d)$ and $\mathscr{M}^{\theta}(Q,d)$ contain open dense subvarieties which are isomorphic. □

5. Examples

In this section, we investigate certain well-known examples, which were never considered using the theory above. Before we start with tame hereditary algebras we prove a useful lemma. Recall that a representation M is exceptional if $\mathrm{Ext}^1(M,M)=0$ and $\mathrm{End}(M)=k$. The bilinear form $\langle -,-\rangle$ defined for modules, may be extended to the Grothendieck group $K_0(kQ)$.

Lemma 5.1. *Let M be an exceptional representation of Q, then M is stable for the weight θ_M* defined by $\theta_M(N):=\langle M,N\rangle - \langle N,M\rangle$.

Proof. Let N be a proper non-zero subrepresentation of M and consider the exact sequence

$$0\to N\to M\to L\to 0.$$

If we apply $\mathrm{Hom}(M,-)$ and $\mathrm{Hom}(-,M)$ to this sequence, then we obtain

$$0\to \mathrm{Hom}(M,N)=0\to \mathrm{Hom}(M,M)=k\to \mathrm{Hom}(M,L)\to \mathrm{Ext}(M,N)\to$$
$$\to \mathrm{Ext}(M,M)=0,$$

and

$$0\to \mathrm{Hom}(L,M)=0\to \mathrm{Hom}(M,M)=k\to \mathrm{Hom}(N,M)\to \mathrm{Ext}(L,M)\to$$
$$\to \mathrm{Ext}(M,M)=0,$$

since $\mathrm{Hom}(M,M)=k\to \mathrm{Hom}(M,L)$ and $\mathrm{Hom}(M,M)=k\to \mathrm{Hom}(N,M)$ are injective. Consequently $\langle M,N\rangle\leqslant 0$ and $\langle N,M\rangle\geqslant 1$ and M is θ_M stable. □

5.1. Tame hereditary algebras

The classification of these algebras and its category of representations is well-known: the underlying graph of the algebra is an extended Dynkin graph and for properties of the category of representations we refer to [11]. First we note that we have a distinguished linear function rk on the Grothendieck group: it vanishes on all regular representations, it takes negative value on each indecomposable preprojective representation and it takes positive value on each indecomposable preinjective representation. To make it unique we claim that its image is just $\mathbb{Z}$. Up to a scalar it is known as the rank function or the defect (see e.g. [11]). Note our sign convention to get a non-trivial weight for the imaginary Schurian root δ. We choose δ to be indivisible. Thus we obtain $\operatorname{rk} M = -\langle M, \delta\rangle$, where $\langle -,-\rangle$ is the Euler form in $\operatorname{mod} Q$. Also we have to choose a positive linear function κ.

In the next proposition we consider the moduli space $\mathscr{M}^{\mu}(Q,d)$ of μ-stable representations in $\mathscr{R}(Q,d)$ modulo the action of the linear group $\prod_i G\ell_{d(i)}(k)$ (see remark after Lemma 3.3). We observe that we can choose an arbitrary κ, without changing the moduli spaces for the regular representations. Thus we just take the dimension during this example.

Proposition 5.2. (1) *Let θ be any weight with respect to δ in a sufficiently small neighborhood of* rk *and d a root. Then the moduli space $\mathscr{M}^{\mu}(Q,d)$, for $\mu = \theta/\kappa$ is a point or empty for d not a multiple of δ. If d is a multiple of the imaginary root, say $d = n\delta$, then $\mathscr{M}^{\mu}(Q,d)$ is isomorphic to the n-dimensional projective space. Moreover, the statements above are independent of the choice of κ, we can replace the dimension by any positive linear function. For a fixed tube there is at most one representation of dimension vector δ which is θ-stable. If, in addition, θ does not lie on an inner wall, then there is precisely one representation of dimension vector δ in this tube which is θ-stable.*

(2) *The stable representations are exactly the indecomposable preprojective representations, the indecomposable preinjective representations and those indecomposable regular representations with dimension vector less or equal to the imaginary root. That is a representation is stable precisely when it has trivial endomorphism ring.*

(3) *We consider walls with respect to δ. Then each outer wall is of the form $W(d)$ for some preprojective dimension vector d, where $d < \delta$ and each inner wall is of the form $W(d')$, where d' is a regular dimension vector smaller than δ, that is d' is regular Schurian. Moreover, for each d' as before, $W(d')$ is an inner wall.*

Proof. First note that a root with non-zero defect is either preinjective or preprojective. In both cases an indecomposable representation M with this dimension vector is exceptional. Thus the moduli space is a point or empty. Let M be regular of dimension vector δ in a homogeneous tube. Then M is μ-stable, since it has only preprojective subrepresentations. Thus let M be in a rank-r tube of dimension vector δ. Denote its chain of regular subrepresentations by $M_1 \subset M_2 \subset \cdots \subset M_r = M$. Since the classes of $M_1, \ldots M_{r-1}$ are linearly independent in the Grothendieck group we can choose a weight θ with $\theta(M_i) < 0$ for $i = 1, \ldots, r-1$. Thus M is stable with respect to this weight θ. However one can check that then all other representations M' in the same tube and with dimension vector δ are not μ-stable. Since this is an easy exercise using the Auslander–Reiten exact sequences in a tube, we omit the details. In particular, in a sufficiently small neighborhood of rk we have only inner walls. Again we fix a tube. Consider θ not on an inner wall, then we have a surjective morphism $\mathscr{M}^{\theta}(Q,\delta) \xrightarrow{\pi} \mathscr{M}^{\mathrm{rk}/\kappa}(Q,\delta)$. Since the first moduli space consists only of stable representations, there exists a θ-stable representation M which maps to the S-equivalence class of the representations of dimension vector δ in the fixed tube: note that all representations of dimension vector δ in one tube are S-equivalent with respect to rk. Consequently M is in this fixed tube.

Thus we have shown statement (1) for all indecomposable representations with dimension vector not a multiple of the imaginary root.

It is well-known, that the moduli space $\mathscr{M}^{\mu}(Q,\delta)$ must be a projective line: this can be easily seen for the Kronecker quiver directly. For the other tame quivers it follows from the fact that they have a rational, projective moduli space of dimension one. Thus it is a projective line. It remains to show $\mathscr{M}^{\mu}(Q,n\delta) \simeq \mathbb{P}^n$. First note, that each representation of dimension vector $n\delta$ is non-stable for $n > 1$. Moreover, it is S-equivalent to the direct sum $N_1 \oplus \cdots \oplus N_n$ of n μ-semistable representations of dimension vector δ. By the remark in the first paragraph of Section 4, we know that $\mathscr{M}^{\mu}(Q,n\delta)$ is isomorphic to the nth symmetric power of $\mathbb{P}^1$. This is known to be the n-dimensional projective space.

We proceed with (2): using the arguments before we see that for each regular representation M of dimension vector smaller than δ we can choose a weight in a sufficiently small neighborhood of rk, so that M is θ-stable; just solve the system of inequalities $\theta(N) < 0$ for each regular subrepresentation of M. If the dimension vector of M is larger than δ then M has a non-trivial endomorphism ring, thus it is not stable by Proposition 2.7.

Moreover, exceptional representations are always stable by Lemma 5.1.

The arguments above for subrepresentations of regular representations of dimension vector δ show that inner walls are of the form $W(d')$ for some regular subdimension vector d' of δ, by Lemma 5.4. Since for such a d' there exists precisely one indecomposable regular representation of dimension vector δ containing the unique indecomposable regular representation of dimension vector d', the hypersurface $W(d')$ is an inner wall. Similar with outer walls: each regular representation of dimension vector δ contains any indecomposable preprojective representation of dimension vector d'', where d'' is smaller than δ. Consequently any outer wall is of the form $W(d'')$. □

5.2. Generalized Kronecker quivers

Let Q be the quiver with two vertices $\{0,1\}$ and t arrows starting in 0 and ending in 1.

$$\begin{tikzcd} 0 \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\vdots"] & 1 \end{tikzcd}$$

Since $t = 1$ is representation finite and $t = 2$ was already considered above we assume $t > 2$, so Q is a wild quiver. First we fix a positive linear function κ on the

Grothendieck group. The space of all weights $\mathbb{S}_\kappa$ of the form θ/κ is two dimensional. There exist exactly three classes of weights under the natural action of $\mathbb{R}^+ \times \mathbb{R}$ via $\mu \mapsto a\mu + b$; note that this action preserves the equivalence class. Assume $\mu = (\theta_0, \theta_1)/(\kappa_0, \kappa_1)$, then the equivalence classes are given by the sign of $\theta_0/\kappa_0 - \theta_1/\kappa_1$ (for details see the proof of the proposition below). If the sign is zero, we obtain the constant slope. If the sign is negative then only the simple representations are μ-stable. In both cases the set of μ-stable representations consists of the simple ones. Thus, these weights form two equivalence classes, which we call the trivial classes. Consequently there exists one other equivalence class which is characterized by the property $0 < \theta_0/\kappa_0 - \theta_1/\kappa_1$, called the nontrivial class.

From now on we consider a fixed weight μ in the non-trivial equivalence class.

Proposition 5.3. (1) *A representation is stable precisely if it is μ-stable.*

(2) *A representation of dimension vector $d = (d_0, d_1)$ is stable if it does not contain a proper nontrivial subrepresentation of dimension vector (e_0, e_1) with $e_0/e_1 \geqslant d_0/d_1$.*

(3) *There exist no inner walls.*

(4) *Each indecomposable preprojective and each indecomposable preinjective representation is stable.*

(5) *Let P be preprojective, let R be regular and let I be preinjective. Then $\mu(P) < \mu(R) < \mu(I)$.*

(6) *An indecomposable non-projective representation M is stable if and only if its Auslander–Reiten translate τM is also stable.*

(7) *Let M be an indecomposable non-projective representation. Then $\mu(\tau M) > \mu(M)$ precisely when M is regular.*

(8) *Let M be a stable regular representation, then it is quasi-simple.*

Proof. First we compare $\mu(d_0, d_1) = (\theta_0 d_0 + \theta_1 d_1)/(\kappa_0 d_0 + \kappa_1 d_1)$ and $\mu(e_0, e_1) = (\theta_0 e_0 + \theta_1 e_1)/(\kappa_0 e_0 + \kappa_1 e_1)$. We have $\mu(d) > \mu(e)$ precisely when

$$\theta_1 \kappa_0 (d_0 e_1 - e_0 d_1) < \theta_0 \kappa_1 (d_0 e_1 - d_1 e_0).$$

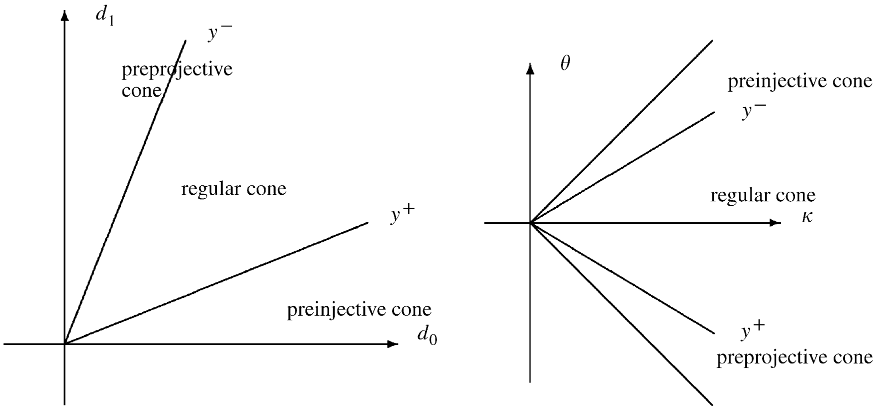
Since by assumption $\theta_1 \kappa_0 < \theta_0 \kappa_1$ we obtain $(d_0 e_1 - e_0 d_1) > 0$ if and only if $\mu(d) > \mu(e)$, where the first inequality is equivalent to $e_0/e_1 - d_0/d_1 < 0$. This proves (2).

Assertion (1) also follows from the arguments above and as a consequence we get (3).

Next we show, that all preprojective and preinjective representations are stable, that is they are μ-stable. Moreover, it is sufficient to prove this claim for the preprojective representations, the other statement is dual. Note that a proper subrepresentation N of a preprojective representation P is preprojective, and of smaller dimension vector. Further the simple projective representation $P(1)$ is stable and calculation of the subrepresentations of $P(0)$ shows that the other indecomposable projective representation $P(0)$ is also stable. In particular, $\mu(P(1)) < \mu(P(0))$. Using the Auslander–Reiten sequences in the preprojective component and Lemma 2.1 we obtain $\mu(\tau^{-l}P(1)) < \mu(\tau^{-l}P(0)) < \mu(\tau^{-l-1}P(1))$. Consequently all preprojective representations are μ-stable.

Since we know that all non-trivial slopes are equivalent we can choose a particular one: so we take $\theta_0, \kappa_0, \kappa_1 = 1$ and $\theta_1 = -1$.

Let ρ be the largest eigenvalue of the Coxeter transformation and y^+ an eigenvector of ρ and y^- an eigenvector of $1/\rho$ (see [10,12] for further properties). Then an indecomposable representation M of dimension vector (d_0, d_1) is preprojective if $d_1/d_0 > y_1^-/y_0^-$ and it is preinjective if $d_0/d_1 > y_0^+/y_1^+$. Then a direct calculation shows (5) and (7). This can be seen also directly from the following figure. The first one uses coordinates (d_0, d_1) and indicates the preprojective cone, the preinjective cone, and the regular cone. If we apply a coordinate transformation corresponding to our fixed slope μ then we obtain the figure on the right-hand side, so that the slope of a representation M is exactly the usual slope of the line through $(\theta(M), \kappa(M))$.



Using (3) claim (6) is obvious for preprojective and preinjective representations M. Thus it remains to show the claim for M regular. By (5) it is sufficient to consider a regular subrepresentation N of M. The Auslander–Reiten translation is an equivalence on the subcategory of regular representations. Thus $\tau N \subset \tau M$ precisely when $N \subset M$. Since the Coxeter transformation Φ is a linear map with eigenvectors y^- and y^+ and corresponding positive eigenvalues, it preserves the regular cone and $\mu(\Phi(d')) < \mu(\Phi(d))$ precisely when $\mu(d') < \mu(d)$ for any regular dimension vector d' and d. From the latter claim (6) follows.

It remains to prove (8). Let M be quasi-simple, then we have an Auslander–Reiten sequence

$$0 \to \tau M \to N \to M \to 0$$

with indecomposabe middle term. Thus $\mu(M) < \mu(N) < \mu(\tau(M))$ by (7). Then we use induction on the height of an Auslander–Reiten sequence in a regular component to show for any Auslander–Reiten sequence

$$0 \to \tau M \to N_1 \oplus N_2 \to M \to 0$$

with N_1 and N_2 indecomposable and M not quasi-simple

$$\mu(\tau M) > \mu(N_1), \quad \mu(N_2) > \mu(M).$$

Since one of the representations N_1, N_2 is a subrepresentation of M, the representation M is not stable. $\square$

5.3. *Inner walls*

In this part we consider some particular examples. The first examples we need in the proof of Lemma 5.4.

Example 1. We consider the quiver Q with three vertices $\{1,2,3\}$ and arrows $\alpha, \beta : 1 \to 2$ and $\gamma : 2 \to 3$.

We consider the dimension vector $(2,3,1)$ and construct an inner wall W for this dimension vector. With notation in Section 3 we get $W = W(1,1,1)$. Let θ be a weight with $\theta_2, \theta_3 < 0$ and $\theta_1 + 2\theta_2 > 0$. For example we can take $\theta = (3,-1,-3)$. We claim that any representation M, whose restriction to the full subquiver with vertices 1 and 2 is preprojective for this quiver, is θ-stable. Moreover, all other representations are *not* stable. Consequently the moduli space is isomorphic to the projective plane, since we can only choose the linear map corresponding to γ.

Next we choose a different weight θ', again $\theta'_2, \theta'_3 < 0$ and $\theta'_1 + \theta'_2 + \theta'_3 < 0$. Since θ' is a weight the latter condition is equivalent to $\theta'_1 + 2\theta'_2 > 0$, that is to the reversed inequality for θ. E.g. we can take $\theta = (2,-1,-1)$. There exists a one-parameter family of representations in $\mathscr{M}^{\theta}(Q,d)$ as follows. Choose a one-dimensional subspace N_1 in M_1 and consider its two-dimensional image N_2 under α and β in M_2. Then we choose γ so that $\gamma(N_2) = 0$. This way we see that there exists a one-parameter family of representations in $\mathscr{M}^{\theta}(Q,d)$ which have a subrepresentation of dimension vector $(1,2,0)$. These representations are not θ'-stable. Consequently $W(1,2,0) = W(1,1,1)$ is an inner wall. Other calculations show that this is the only inner wall and the outer walls are $W(0,0,1)$ and $W(0,1,0)$.

Example 2. Let Q be a one-point extension of the generalized Kronecker quiver with at least 3 arrows by a single arrow. Thus Q has 3 vertices $\{0,1,2\}$ and t arrows from 0 to 1, and one arrow from 2 to 3.

We consider the dimension vector $(2,3,1)$ and wish to construct an inner wall for this dimension vector. First, we consider the restriction of a representation M of Q to the Kronecker quiver. Thus this restriction has dimension vector $(2,3)$. The dimension of the corresponding moduli space $\mathscr{M}(2,3)$ is $6t - 12$, since the Euler quadratic from has value $13 - 6t$.

Lemma 5.4. *Let Q be a representation infinite quiver, with more than two vertices. Then there exists an inner wall for a certain dimension vector d.*

Proof. Let Q be a tame quiver. Then inner walls exist for the imaginary root, except for the Kronecker quiver. Any representation infinite quiver with three vertices contains either a tame quiver with more than two vertices or a one-point extension of the Kronecker quiver. Thus it remains to show the existence of inner walls for the minimal one-point extensions of the Kronecker quiver. We show this in the example above, for a particular orientation of such a quiver. Using reflection functors one gets inner walls for the other orientations. □

Lemma 5.5. *The moduli space $\mathcal{M}(2,3,1)$ has dimension $6t-10$, and it is for a certain slope μ just a $\mathbb{P}^2$ bundle over $\mathcal{M}(2,3)$. Moreover, for this slope, the restriction of a μ-stable representation to the Kronecker quiver does not contain a subrepresentation of dimension vector $(1,1)$.*

Proof. Let $\theta=(3,-2,0)$. A representation M of Q of dimension vector $(2,3,1)$ is θ-stable precisely if it has trivial endomorphism ring and it does not contain a subrepresentation of dimension vector $(1,1,1)$. This, in particular, implies that the restriction of M to the Kronecker quiver does not contain a subrepresentation of dimension vector $(1,1)$, thus it is stable (see example: Generalized Kronecker quiver). Vice versa let N be a stable representation of the Kronecker quiver and choose a surjective map $\beta: N_1 \to k$, then the representation (N,β,k) over Q is θ-stable. Thus the fiber of the restriction map $\mathcal{M}^\theta(2,3,1) \to \mathcal{M}(2,3)$ is a projective plane. □

Next we construct a stable representation M over Q, whose restriction to the Kronecker quiver contains $(1,1)$ as a direct summand. Consequently $(1,1,1)$ is a special subrepresentation and we obtain an inner wall.

Let N_1 be an indecomposable representation over the Kronecker quiver of dimension vector $(1,1)$ and N_2 be an indecomposable representation of the Kronecker quiver of dimension vector $(1,2)$, so that $\mathrm{Hom}(N_1,N_2)=\mathrm{Hom}(N_2,N_1)=0$. Let M be a representation of Q, with trivial endomorphism ring, whose restriction to the Kronecker quiver is $N_1 \oplus N_2$; e.g. M is the following representation (for $t=3$):

$$M(\alpha_1)=\begin{pmatrix}0&0\\1&0\\0&\lambda\end{pmatrix},\quad M(\alpha_2)=\begin{pmatrix}1&0\\0&0\\0&0\end{pmatrix},\quad M(\alpha_3)=\begin{pmatrix}0&0\\0&0\\0&1\end{pmatrix},\quad \text{and}$$

$$M(\beta)=(1\quad 0\quad 1).$$

Example 3. Let Q be the quiver with three vertices $\{1,2,3\}$ and arrows α_1, α_2 from 1 to 2, and arrows β_1, β_2 from 2 to 3:

$$\bullet \rightrightarrows \bullet \rightrightarrows \bullet$$

We consider the dimension vector $(1,2,1)$ and show, that there exists an inner wall for this dimension vector. Consider the following representation:

$$M(\alpha_1) = (1 \quad 0), \quad M(\alpha_2) = (1 \quad \lambda), \quad M(\beta_1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad M(\beta_2) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The restriction of the representation M to the left Kronecker quiver is a direct sum of a regular and a simple representation, and the restriction to the right-hand side Kronecker quiver is an injective representation. The endomorphism ring of this representation is trivial and it is stable. It contains a special subrepresentation of dimension vector $(1,1,1)$. Consequently there exists an inner wall.

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