The Krull–Schmidt Theorem

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*Partially supported by Gruppo Nazionale Strutture Algebriche e Geometriche e loro Applicazioni of Istituto Nazionale di Alta Matematica, Italy, and by Ministero dell'Università e della Ricerca Scientifica e Tecnologica (progetto di ricerca di rilevante interesse nazionale "Nuove prospettive nella teoria degli anelli, dei moduli e dei gruppi abeliani"), Italy.

HANDBOOK OF ALGEBRA, VOL. 3
Edited by M. Hazewinkel

1. Introduction

In this chapter, we shall consider unital right modules over associative rings with identity $1 \neq 0$. If M is a module and $\{M_i \mid i \in I\}$, $\{N_j \mid j \in J\}$ are two families of submodules of M such that $M = \bigoplus_{i \in I} M_i = \bigoplus_{j \in J} N_j$, we say that these two direct sum decompositions of M are *isomorphic* if there is a one-to-one correspondence $\varphi : I \to J$ such that $M_i \cong N_{\varphi(i)}$ for every $i \in I$. A module is called *indecomposable* if it cannot be expressed as a direct sum of two of its non-zero submodules. The "classical" Krull–Schmidt theorem (Theorem 2.1) asserts that any two direct sum decompositions of a module of finite length into indecomposable summands are isomorphic.

The origin of the Krull–Schmidt theorem goes back to group theory. Joseph Henry Maclagan Wedderburn [W9] proved in 1909 that any two direct product decompositions of a finite group G into indecomposable factors $G = H_1 \times \cdots \times H_r = K_1 \times \cdots \times K_s$ are isomorphic. Later, Remak [R2] showed that the indecomposable factors are centrally isomorphic (that is, $r = s$ and there is an automorphism σ of G that is the identity modulo the center $Z(G)$ of G such that, after suitable relabeling of the indices if necessary, $\sigma(H_i) = K_i$ for every i). Subsequently, Krull [K4, Fundamentalsatz] and Schmidt [S1,S2] extended this result to "generalized Abelian groups" with the acc and the dcc on admissible subgroups. In 1950 Azumaya [A3] extended the theorem to the case of arbitrary (i.e. possibly infinite) direct sums of modules with local endomorphism rings. This extension of the Krull–Schmidt theorem is the celebrated Krull–Schmidt–Azumaya theorem (Theorem 2.2).

The problem considered in the Krull–Schmidt theorem, that is, asking whether the direct sum decomposition of a module into indecomposable summands is unique up to isomorphism, is very natural and is the topic of this chapter. After having proved the theorem for the class of modules of finite length, W. Krull [K6, last sentence of the paper] asked in 1932 whether the theorem holds also for the class of Artinian modules (for the answers to this question see Section 4.2). Similar problems were subsequently posed for a number of classes of modules. If $\mathcal{C}$ is a class of right modules over a ring R, we shall say that the *Krull–Schmidt theorem holds for* $\mathcal{C}$ if the class $\mathcal{C}$ is closed under direct summands, every module $M \in \mathcal{C}$ is a direct sum of indecomposable modules, and all direct sum decompositions of M into indecomposable direct summands are isomorphic. Obviously, it would be better to say that the Krull–Schmidt *property* holds for $\mathcal{C}$, but the expression "the Krull–Schmidt theorem holds for $\mathcal{C}$" is charming, stimulating and widespread. Thus the Krull–Schmidt theorem holds for the class of modules of finite length. For another example of class of modules for which the Krull–Schmidt theorem holds, consider the ring of integers $R = \mathbf{Z}$ and the class FG-$\mathbf{Z}$ of finitely generated Abelian groups. Every finitely generated Abelian group is a direct sum of copies of $\mathbf{Z}$ and $\mathbf{Z}/\mathbf{Z}p^n$, where p is a prime and n is a natural number (these are the only indecomposable finitely generated Abelian groups), and the Krull–Schmidt theorem holds for the class FG-$\mathbf{Z}$. For another non-trivial example, note that if Mod-R denotes the class of all right R-modules, then the Krull–Schmidt theorem holds for Mod-R if and only if R is right pure-semisimple (because if R is right pure-semisimple, then every right R-module is Σ-pure-injective, and Σ-pure-injective modules are direct sums of modules with local endomorphism rings [F2, Theorem 2.29]; conversely, if the Krull–Schmidt theorem holds for Mod-R, then every right R-module is a direct sum of indecomposable modules, hence R must be right pure-semisimple [Z4, Corollary 2]).

The outline of the chapter is the following. In Section 2 we give the basic concepts and definitions: we sketch two proofs of the Krull–Schmidt–Azumaya theorem, mention refinements of decompositions and the exchange property and study endomorphism rings of modules of finite length. Notice that the validity of the Krull–Schmidt theorem, at least for finite direct sum decompositions, is a property of the endomorphism ring. That is, if M_R is a module and $E = \mathrm{End}(M_R)$ is its endomorphism ring, then any two direct sum decompositions $M_R = M_1 \oplus \cdots \oplus M_t = M'_1 \oplus \cdots \oplus M'_s$ of M_R into finitely many indecomposable direct summands are isomorphic if and only if any two direct sum decompositions $E_E = e_1 E \oplus \cdots \oplus e_t E = e'_1 E \oplus \cdots \oplus e'_s E$ of the right E-module E_E into finitely many (projective cyclic) indecomposable direct summands are isomorphic. More generally, the category $\mathrm{add}(M_R)$ of all modules isomorphic to direct summands of direct sums M^n_R of finitely many copies of M_R and the category proj-E of all finitely generated projective right E-modules are equivalent. We recall the basic properties of semiperfect rings and semilocal rings, because endomorphism rings of modules of finite length are semiperfect rings, and semiperfect rings are semilocal.

In Section 3 we survey the validity of the Krull–Schmidt theorem for modules over commutative rings. Historically, in the setting of commutative rings, the first results obtained were the theorem about elementary divisors and Steinitz' theorem about the structure of finitely generated modules over a Dedekind domain. We consider finite rank torsion-free Abelian groups, introducing the results obtained by Jónsson [J1,J2,J3] and Walker [W1]. Finite rank torsion-free Abelian groups are *almost Krull–Schmidt* $\mathbf{Z}$-modules, in the sense that they have only finitely many direct sum decompositions up to isomorphism. Then we consider finitely generated modules over commutative rings. Here the notion of Henselian ring appears naturally because of the link between the Krull–Schmidt theorem and Henselian rings that was first pointed out by Swan and Evans [E]. In this area, the main results were obtained by Vámos [V3,V4], Siddoway [S3] and Levy [L3]. The case of commutative rings whose finitely generated modules are direct sums of cyclic submodules or of submodules generated by $\leqslant n$ elements is particularly interesting. Then we turn our attention to the case of torsion-free modules over valuation domains, another area in which Henselian rings appear naturally. In Section 4 we study a second class of almost Krull–Schmidt modules, the modules whose endomorphism ring is semilocal. In this setting the main results were obtained by Camps and Dicks [CD], Herbera and Shamsuddin [HS], Facchini, Herbera, Levy and Vámos [FHLV,FH3].

In Section 5 we consider biuniform modules in general and uniserial modules in particular. Warfield [W5] proved in 1975 that every finitely presented module over a serial ring is a finite direct sum of uniserial summands and asked whether such a finite direct sum decomposition is unique up to isomorphism, i.e. whether the Krull–Schmidt theorem holds for finitely presented modules over serial rings. The negative answer to Warfield's question was given by the author [F1] in 1996. Subsequently, part of the contents of [F1] was extended to direct sums of arbitrary (possibly infinite) families of uniserial modules by Nguyen Viet Dung and the author in [DF1], and results about direct summands of serial modules were obtained in [DF2]. The three articles [F1,DF1,DF2] deal with direct sum decompositions of serial modules, but most of the results they contain hold not only for serial modules, but also for modules that are direct sums of biuniform modules, as was observed independently by Nguyen Viet Dung, Dolors Herbera and Ladislav Bican [B3]. The most recent results about direct sum decompositions into uniserial modules were obtained by Puninski [P3,P4], who constructed wonderful examples.

In Section 6 we study projective modules over semilocal rings, because this is equivalent to studying direct sum decompositions of modules whose endomorphism ring is semilocal and these modules are almost Krull–Schmidt, as we have already said above. The isomorphism classes of finitely generated projective modules over a semilocal ring form a commutative monoid that is isomorphic to a full submonoid of the additive monoid $\mathbf{N}^n$. In this setting, the main results were obtained by the author and Herbera [FH1], R. Wiegand [W10] and Yakovlev [Y2]. In Section 7 we present some results about direct sums of modules whose endomorphism rings are homogeneous semilocal. Homogeneous semilocal rings are a class of rings that properly contains the class of local rings, and therefore it is natural to ask whether the Krull–Schmidt theorem holds for the modules that are direct sums of modules whose endomorphism rings are homogeneous semilocal; see [CF,BFRR, FH3]. In Section 8 we present some nice theorems due to Levy and Odenthal [LO1] about the Krull–Schmidt theorem for modules over a semiprime module-finite k-algebra, where k is a commutative, Noetherian ring of Krull dimension 1. Section 9 contains the theorem on torsion-free Krull–Schmidt for integral group rings by Hindman, Klingler and Odenthal [HKO].

This report is not exhaustive. We have not considered the validity of the Krull–Schmidt theorem for other algebraic structures (for the classical case with regard to groups see [R4, 3.3.8] or [R6], and for modular lattices of finite length and universal algebras see [C1]). We have only mentioned the case of additive categories in Appendix A. In this framework, we cite the nice introduction of the paper [WW1] for a detailed report on the history of the Krull–Schmidt theorem for additive categories.

Finally, Appendix B is devoted to the Goldie dimension of modular lattices, because the endomorphism ring of a finite direct sum of modules with local endomorphism rings is a semilocal ring and semilocal rings are characterized as the rings for which the dual lattice of the modular lattice of their right ideals has finite Goldie dimension. The *Goldie dimension* (or, *uniform dimension*, or *uniform rank*, or simply the *rank*) of a module was introduced by Alfred Goldie in 1960. Goldie, while studying the question of when it was possible to construct a classical right quotient ring that is semisimple Artinian, realized that a necessary condition was that the module R_R had finite Goldie dimension. Dually, Varadarajan [V5] realized in 1979 the importance of the Goldie dimension of the dual lattice $\mathcal{L}(M_R)^{\mathrm{op}}$ of the lattice $\mathcal{L}(M_R)$ of all submodules of a module M_R, the so-called *dual Goldie dimension* of M_R. Thus semilocal rings are exactly the rings with finite dual Goldie dimension. In our presentation we follow Grzeszczuk and Puczyłowski [GP], who introduced in 1984 the Goldie dimension for arbitrary modular lattices with 0 and 1.

The Krull–Schmidt theorem holds for a number of classes of modules, but apart from those classes in which the indecomposable modules have local endomorphism rings and therefore the Krull–Schmidt–Azumaya Theorem 2.2 applies directly, for the other classes of modules the proof of the validity of the Krull–Schmidt theorem depends strongly on properties of the class itself. In other words, a general theory is missing. The reason of this is, in our opinion, the following. The Krull–Schmidt theorem does not hold for arbitrary modules, that is, an arbitrary module usually has non-isomorphic direct sum decompositions. The validity of the Krull–Schmidt theorem for a class of modules is an exception, and when it holds, it must be examined separately, case by case, using the particular properties of the class in all the key points of the proof. And this must be the reason why a general theory cannot exist: there cannot be a general theory that includes all these exceptions.

I wish to thank Professor Lawrence Levy, to whom Sections 8 and 9 are almost entirely due. I am also indebted to Professors David Arnold, Lee Klingler, Peter Vámos and Paolo Zanardo for reading parts of the manuscript.

2. Basic concepts and definitions

2.1. *The Krull–Schmidt–Azumaya theorem*

The well-known classical Krull–Schmidt theorem asserts that

THEOREM 2.1. *If M is a module of finite length, then any two direct sum decompositions $M = M_1 \oplus M_2 \oplus \cdots \oplus M_t = N_1 \oplus N_2 \oplus \cdots \oplus N_s$ of M into indecomposable summands M_i, N_j are isomorphic.*

Subsequently this theorem was generalized, and its generalization is usually known as the Krull–Schmidt–Azumaya theorem (Theorem 2.2). For any ring R we shall denote the Jacobson radical of R by $J(R)$. A ring R is *local* if $R/J(R)$ is a division ring. Equivalently, a ring R is local if and only if the sum of two non-invertible elements of R is non-invertible. Therefore for us a local ring will not necessarily be either commutative or Noetherian.

THEOREM 2.2 (Krull–Schmidt–Azumaya theorem [A3]). *If a module $M = \bigoplus_{i \in I} M_i$ is a direct sum of modules M_i $(i \in I)$ and the endomorphism ring of each M_i is local, then any two direct sum decompositions of M into indecomposable direct summands are isomorphic.*

A concept strictly related to the Krull–Schmidt theorem and the notion of isomorphic decompositions is the concept of isomorphic refinements. Let M be a module over a ring R. Suppose that $\{M_i \mid i \in I\}$ and $\{N_j \mid j \in J\}$ are two families of submodules of M such that $M = \bigoplus_{i \in I} M_i = \bigoplus_{j \in J} N_j$. The second decomposition is a *refinement* of the first if there is a surjective mapping $\varphi : J \to I$ such that $N_j \subseteq M_{\varphi(j)}$ for every $j \in J$. Equivalently, the decomposition $M = \bigoplus_{j \in J} N_j$ is a *refinement* of the decomposition $M = \bigoplus_{i \in I} M_i$ if there is a surjective mapping $\varphi : J \to I$ such that $\bigoplus_{j \in \varphi^{-1}(i)} N_j = M_i$ for every $i \in I$. Here we shall not consider the property of having isomorphic refinements in detail. We just recall the following result due to Warfield:

THEOREM 2.3 [W3, Theorem 1]. *If a module $M = \bigoplus_{i \in I} M_i$ is a direct sum of modules M_i $(i \in I)$ and each M_i is countably generated and has a local endomorphism ring, then* *every direct sum decomposition of M has a refinement isomorphic to the decomposition $M = \bigoplus_{i \in I} M_i$.*

We recall two different proofs of Theorem 2.2. The first, that can be found in [AF2, Theorem 12.6], makes use of the notion of decomposition that complements (maximal) direct summands. If M is a module, all complements of a direct summand M' of M are isomorphic, because $M = M' \oplus M_1 = M' \oplus M_2$ implies $M_1 \cong M/M' \cong M_2$. A direct summand M' of M is a *maximal direct summand* of M if M' has an indecomposable complement in M. A direct sum decomposition $M = \bigoplus_{i \in I} M_i$ of a module M is said to *complement direct summands* if for every direct summand M' of M there is a subset $J \subseteq I$ such that $M = M' \oplus (\bigoplus_{i \in J} M_i)$, and is said to *complement maximal direct summands* if for every maximal direct summand M' of M there exists $J \subseteq I$ with $M = M' \oplus (\bigoplus_{i \in J} M_i)$. Right perfect rings can be characterized by the property that all projective right modules have a decomposition that complements direct summands [AF1, Theorem 6]. If $M = \bigoplus_{i \in I} M_i$ is a direct sum decomposition that complements direct summands, then all the modules M_i are indecomposable, and if a module M has a direct sum decomposition $M = \bigoplus_{i \in I} M_i$ that complements maximal direct summands and all the modules M_i are indecomposable, then any two direct sum decompositions of M into indecomposable direct summands are isomorphic [AF2, Theorem 12.4]. It is possible to prove that a decomposition $M = \bigoplus_{i \in I} M_i$ in which the modules M_i have local endomorphism rings complements maximal direct summands. Theorem 2.2 follows from this. For the details of this proof, see [AF2, §12].

A second proof of the Krull–Schmidt–Azumaya theorem, which can be found in [F2, Theorem 2.12], uses the so-called *exchange property*. Let $\aleph$ be a cardinal number. An R-module M is said to have the *$\aleph$-exchange property* if for any R-module G and any two direct sum decompositions

$$G = M' \oplus N = \bigoplus_{i \in I} M_i,$$

where $M' \cong M$ and the cardinality $|I|$ of I is $\leqslant \aleph$, then there are R-submodules N_i of M_i, $i \in I$, such that $G = M' \oplus (\bigoplus_{i \in I} N_i)$. A module has the *exchange property* if it has the $\aleph$-exchange property for every cardinal number $\aleph$, and has the *finite exchange property* if it has the n-exchange property for every finite cardinal n. A module with the 2-exchange property has the finite exchange property. If $\aleph$ is a cardinal number, a module $M = M_1 \oplus M_2$ has the $\aleph$-exchange property if and only if both M_1 and M_2 have the $\aleph$-exchange property. An indecomposable module M_R has the exchange property if and only if it has the finite exchange property, if and only if its endomorphism ring $\mathrm{End}(M_R)$ is local. It is possible to prove [CJ, 4.1] that if $\aleph$ is a cardinal, M is a module with the $\aleph$-exchange property, $M = \bigoplus_{i \in I} M_i = \bigoplus_{j \in J} N_j$ are two direct sum decompositions of M with I finite and $|J| \leqslant \aleph$, then the two direct sum decompositions of M have isomorphic refinements. Making use of these properties of modules with the exchange property, it is possible to show that for a module M that is a direct sum of modules with local endomorphism rings, the endomorphism ring of *any* indecomposable direct summand of M is local. From this the Krull–Schmidt–Azumaya Theorem 2.2 follows. For the details, see [F2, §§2.1–2.4].

2.2. *Endomorphism rings of modules of finite length*

Theorems about finite direct sum decompositions of a module M_R, like Theorem 2.1, are essentially theorems about the endomorphism ring of M_R because of the one-to-one correspondence between finite direct sum decompositions of M_R and complete sets of orthogonal idempotents in its endomorphism ring $\mathrm{End}(M_R)$. More generally, let M_R be a module over an arbitrary ring R and let $E = \mathrm{End}(M_R)$. All direct sum decompositions of M_R in the category Mod R are direct sum decompositions of M_R in the full subcategory $\mathrm{add}(M_R)$ of Mod R whose objects are the modules isomorphic to direct summands of direct sums M_R^n of finitely many copies of M_R. For any ring S, we shall denote by proj-S the full subcategory of Mod S whose objects are all finitely generated projective right S-modules. Thus proj-$S = \mathrm{add}(S_S)$.

THEOREM 2.4. *Let M_R be an R-module over a ring R and let $E = \mathrm{End}(M_R)$, so that ${}_EM_R$ is a bimodule. The categories* $\mathrm{add}(M_R)$ *and* proj-E *are equivalent via the functors* $\mathrm{Hom}_R(M_R, -) : \mathrm{add}(M_R) \to$ proj-E *and* $- \otimes_E M :$ proj-$E \to \mathrm{add}(M_R)$.

As category equivalences preserve direct sum decompositions, the study of the direct sum decompositions of a module N_R belonging to $\mathrm{add}(M_R)$ can be reduced to that of the corresponding projective E-module $\mathrm{Hom}_R({}_EM_R, N_R)$. Note that in this equivalence between $\mathrm{add}(M_R)$ and proj-E the module M_R correspond to the module E_E. Hence, studying the direct sum decompositions in the category $\mathrm{add}(M_R)$ is completely equivalent to studying the direct sum decompositions of finitely generated projective right modules over the ring $E = \mathrm{End}(M_R)$.

A ring S is *semiperfect* if it has a complete set $\{e_1, e_2, \dots, e_n\}$ of orthogonal idempotents for which every e_iSe_i is a local ring. Equivalently, a ring S is semiperfect if and only if $S/J(S)$ is semisimple Artinian and idempotents lift modulo $J(S)$, if and only if every finitely generated S-module has a projective cover [F2, Theorem 3.6]. The reason why semiperfect rings naturally appear in connection with the Krull–Schmidt theorem is explained by the next proposition.

PROPOSITION 2.5 [F2, Proposition 3.14]. *If M_R is a module and $E = \mathrm{End}(M_R)$ is its endomorphism ring, then M_R is a finite direct sum of R-modules with local endomorphism rings if and only if E_E is a finite direct sum of E-modules with local endomorphism rings, if and only if E is semiperfect.*

An immediate application of Theorem 2.3 to free modules over semiperfect rings yields the following result.

PROPOSITION 2.6. *The Krull–Schmidt theorem holds for the class of all projective right modules over a semiperfect ring.*

Theorem 2.1 considers modules of finite length. It is possible to prove that *the endomorphism ring $E = \mathrm{End}(M_R)$ of a module M_R of finite length is a semiprimary ring* (that is, $E/J(E)$ is semisimple Artinian and $J(E)$ is nilpotent) *with the acc and the dcc on* *right annihilators and on left annihilators.* More generally, *if M_R is an Artinian module, then* $\mathrm{End}(M_R)/J(\mathrm{End}(M_R))$ *is semisimple Artinian* [CD, Corollary 6], $\mathrm{End}(M_R)$ *satisfies the acc on left annihilators and every nil subring of* $\mathrm{End}(M_R)$ *is nilpotent* [F5, Theorem 1.5]. Here is a sketch of the proof of what we have stated for modules of finite length. A module M_R is a *Fitting module* if for every $f \in \mathrm{End}(M_R)$ there is an integer $n \geqslant 1$ such that $M = \ker(f^n) \oplus f^n(M)$. If M_R is a module of finite length, then M_R is a Fitting module (this is Fitting's lemma [F2, Lemma 2.20]). Hence, for every $f \in J(\mathrm{End}(M_R))$, there exists an n such that $M = \ker(f^n) \oplus f^n(M)$. In particular, f^n induces an automorphism on $f^n(M)$. Let α be the endomorphism of M that is equal to the inverse of this automorphism on $f^n(M)$ and is equal to zero on $\ker(f^n)$. Then $f^n\alpha$ is the idempotent endomorphism of M that is the identity on $f^n(M)$ and is equal to zero on $\ker(f^n)$. Since it belongs to $J(\mathrm{End}(M_R))$, it must be zero, i.e. $f^n(M) = 0$. Thus $J(\mathrm{End}(M_R))$ is nil, hence nilpotent [F5, Theorem 1.5]. In order to show that $\mathrm{End}(M_R)$ satisfies the acc and the dcc on left annihilators, let $\mathcal{LA}(\mathrm{End}(M_R))$ denote the set of all left annihilators of $\mathrm{End}(M_R)$ and $\mathcal{L}(M_R)$ the set of all submodules of M_R. The mapping $\varphi : \mathcal{LA}(\mathrm{End}(M_R)) \to \mathcal{L}(M_R)$ defined by $\varphi(I) = \bigcap_{f \in I} \ker(f)$ for every $I \in \mathcal{LA}(\mathrm{End}(M_R))$ is inclusion reversing. Similarly, if $\mathcal{L}(\mathrm{End}(M_R))$ denotes the set of all left ideals of $\mathrm{End}(M_R)$, the mapping $\psi : \mathcal{L}(M_R) \to \mathcal{L}(\mathrm{End}(M_R))$ defined by $\psi(N) = \{f \in \mathrm{End}(M_R) \mid f(N) = 0\}$ is inclusion reversing. If A is a subset of $\mathrm{End}(M_R)$, then its left annihilator $I = \{f \in \mathrm{End}(M_R) \mid fg = 0$ for every $g \in A\}$ is equal to $\psi(N)$ where $N = \sum_{g \in A} g(M)$. Trivially $N \subseteq \varphi(\psi(N))$, so that $I = \psi(N) \supseteq \psi(\varphi(\psi(N))) = \psi(\varphi(I))$. Conversely $I \subseteq \psi(\varphi(I))$, so that $I = \psi(\varphi(I))$. In particular, the mapping $\varphi : \mathcal{LA}(\mathrm{End}(M_R)) \to \mathcal{L}(M_R)$ is injective. As $\mathcal{L}(M_R)$ satisfies the acc and the dcc, it follows that $\mathcal{LA}(\mathrm{End}(M_R))$ satisfies the acc and the dcc, i.e. $\mathrm{End}(M_R)$ satisfies both the acc and the dcc on left annihilators. Finally, a ring satisfies the acc and the dcc on left annihilators if and only if it satisfies the acc and the dcc on right annihilators.

There exist examples of modules M_R of finite length whose endomorphism ring is a non-Noetherian commutative ring. For instance, let $K \subseteq F$ be an extension of fields of infinite degree $[F : K]$. Let $F^* = \mathrm{Hom}_K(F, K)$ be the dual of the K-vector space F, so that F^* is a K-vector space of infinite dimension and ${}_F F^*_K$ is an F-K-bimodule. Let R be the ring of all 3×3 matrices $\begin{pmatrix} a & b & \varphi \\ 0 & a & 0 \\ 0 & 0 & k \end{pmatrix}$ with $a, b \in F$, $\varphi \in F^*$ and $k \in K$ and let M be the set of all 1×3 row matrices (c, d, ℓ) with $c, d \in F$ and $\ell \in K$. Then M is a right module over R and it is easily verified that M_R is a cyclic module of length 3 whose endomorphism ring $\mathrm{End}(M_R)$ is isomorphic to the ring of all 2×2 matrices $\begin{pmatrix} k & a \\ 0 & k \end{pmatrix}$ with $k \in K$ and $a \in F$. This is a local non-Noetherian commutative ring.

2.3. *First applications of the Krull–Schmidt–Azumaya theorem*

The first example of trivial application of the Krull–Schmidt–Azumaya theorem concerns the class of semisimple modules. As the endomorphism ring of any simple module is a division ring by Schur's lemma, the Krull–Schmidt theorem holds for the class of semisimple modules.

A second natural application of the Krull–Schmidt–Azumaya Theorem 2.2 is that of deriving from it the classical Krull–Schmidt Theorem 2.1, which concerns modules of finite length. More generally, this can be done for Fitting modules. The endomorphism ring of an indecomposable Fitting module is local [F2, Lemma 2.21], so that the Krull–Schmidt–Azumaya Theorem 2.2 applies to any module M that is a direct sum of indecomposable Fitting modules. By Fitting's lemma [F2, Lemma 2.20], every module of finite length is a Fitting module, so that Theorem 2.1 follows from Theorem 2.2.

Our third immediate application of the Krull–Schmidt–Azumaya theorem is to the class of injective modules of finite Goldie dimension. For the definition of the Goldie dimension of a module, see Appendix B. Recall that every non-invertible element φ of the endomorphism ring $\mathrm{End}(M_R)$ of an injective indecomposable R-module M_R has non-zero kernel. It follows that [F2, Lemma 2.25]:

PROPOSITION 2.7. *An injective R-module M_R is indecomposable if and only if its endomorphism ring* $\mathrm{End}(M_R)$ *is local. Therefore the Krull–Schmidt theorem holds for the class of injective modules of finite Goldie dimension.*

Any Noetherian module N_R has finite Goldie dimension, so that the injective envelope $E(N_R)$ of any Noetherian module N_R has finite Goldie dimension. Thus the Krull–Schmidt theorem holds for the class of the injective envelopes $E(N_R)$ of Noetherian modules N_R.

THEOREM 2.8 (Matlis [M1], Papp [P1]). *A ring R is right Noetherian if and only if every injective right R-module is a direct sum of indecomposable injective R-modules.*

Therefore every injective right module over a right Noetherian ring R is a direct sum of modules whose endomorphism rings are local. Hence

COROLLARY 2.9. *If the ring R is right Noetherian, the Krull–Schmidt theorem holds for the class of injective right R-modules.*

2.4. *Semilocal rings*

For the details and the proofs of the results presented in this section, see [F2].

PROPOSITION 2.10. *The following conditions are equivalent for a ring R:*

(a) *The ring $R/J(R)$ is semisimple Artinian.*

(b) *The ring $R/J(R)$ is isomorphic to a finite direct product $\prod_{i=1}^{t} M_{n_i}(D_i)$ of rings $M_{n_i}(D_i)$ of $n_i \times n_i$ matrices over division rings D_i, $i = 1, 2, \dots, t$.*

(c) *The ring $R/J(R)$ is right Artinian.*

(d) *There exists a finite set $\{I_1, I_2, \dots, I_n\}$ of maximal right ideals of R such that $J(R) = I_1 \cap I_2 \cap \cdots \cap I_n$.*

(c*), (d*) *The left–right duals of* (c), (d).

A ring R is said to be *semilocal* if it satisfies the equivalent conditions of Proposition 2.10.

EXAMPLES 2.11.

(1) By condition (d), a commutative ring is semilocal if and only if it has only finitely many maximal ideals.
(2) By condition (c), every right (or left) Artinian ring is semilocal.
(3) Every local ring is semilocal.
(4) If R is semilocal, the ring $M_n(R)$ of $n \times n$ matrices over R is semilocal for every positive integer n. If $0 \neq e \in R$ is idempotent and R is semilocal, then eRe is semilocal.
(5) Let k be a commutative ring with identity. A *k-algebra* is a ring R that is a k-module and for which multiplication is k-bilinear. Equivalently, a k-algebra can be defined as a ring R with a ring homomorphism of k into the center of R. A k-algebra is said to be *module-finite* if it is finitely generated as a k-module. A module-finite algebra over a semilocal commutative ring is semilocal [F2, p. 7].
(6) Let k be either a semilocal commutative principal ideal domain or a valuation domain (that is, a commutative integral domain whose ideals are linearly ordered under inclusion). If R is a k-algebra which is torsion-free and of finite rank over k, then R is semilocal [W7, Theorems 5.2 and 5.4].
(7) Semiperfect rings, i.e. endomorphism rings of finite direct sums of modules with local endomorphism rings (Proposition 2.5), are semilocal rings.
(8) For examples of modules with semilocal endomorphism rings, see Examples 4.2.

A semilocal ring has only finitely many simple right modules up to isomorphism. We already know that in a semilocal ring the Jacobson radical is the intersection of finitely many maximal right ideals. We shall see in the next theorem that semilocal rings are exactly those of finite dual Goldie dimension (for the definition of dual Goldie dimension of a module, see Appendix B). In [FS2] it is proved that a semilocal ring has only finitely many indecomposable projective right modules up to isomorphism. Thus being semilocal is a finiteness condition on the ring.

THEOREM 2.12 [SV1, Corollary 1.14]. *A ring R is semilocal if and only if the right R-module R_R has finite dual Goldie dimension, if and only if the left R-module ${}_RR$ has finite dual Goldie dimension. If these equivalent conditions hold, then* $\operatorname{codim}(R_R) = \operatorname{codim}({}_RR) =$ *"Goldie dimension of the semisimple R-module $R/J(R)$".*

If $e \in R$ is idempotent, the dual Goldie dimension $\operatorname{codim}(eR)$ of the right R-module eR is equal to the dual Goldie dimension $\operatorname{codim}(eRe)$ of the (right or left) eRe-module eRe [SV1, Theorems 2.3 and 2.5]. As $\operatorname{codim}(R_R) = \operatorname{codim}(eR) + \operatorname{codim}((1-e)R)$, we get that

PROPOSITION 2.13. *For any idempotent e of a ring R*

$$\operatorname{codim}(R) = \operatorname{codim}(eRe) + \operatorname{codim}\big((1-e)R(1-e)\big).$$

In particular, R is semilocal if and only if eRe and $(1-e)R(1-e)$ are both semilocal.

3. Krull–Schmidt and modules over commutative rings

3.1. *Elementary divisors and Steinitz' theorem*

The origins of the Krull–Schmidt theorem with regard to modules over commutative rings go back to the well-known theorem about elementary divisors (modules over principal ideal domains) and Steinitz' theorem (modules over Dedekind domains). By a *principal ideal domain* we mean a commutative integral domain in which every ideal is principal.

THEOREM 3.1 (Elementary divisors). *Let M_R be a finitely generated module over a principal ideal domain R. Then there exist a natural number n and a chain $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n$ of proper ideals of R such that $M \cong \bigoplus_{k=1}^{n} R/I_k$. The number n and the ideals I_k, $k = 1, 2, \ldots, n$, are uniquely determined by M_R.*

Note that if R is a principal ideal domain and I is a non-zero proper ideal of R, then I is the product of a non-empty finite family of prime ideals of R, that is, $I = P_1^{m_1} P_2^{m_2} \ldots P_t^{m_t}$ with $t \geqslant 1$, $P_1, P_2, \ldots, P_t$ distinct prime ideals of R and $m_1, m_2, \ldots, m_t$ positive integers. Then $I = P_1^{m_1} \cap P_2^{m_2} \cap \cdots \cap P_t^{m_t}$, so that $R/I \cong R/P_1^{m_1} \oplus R/P_2^{m_2} \oplus \cdots \oplus R/P_t^{m_t}$ by the Chinese remainder theorem and $\mathrm{End}(R/P_j^{m_j}) \cong R/P_j^{m_j}$ is a local ring. Thus Theorem 3.1 – when the terms R/I are decomposed in this way – is a prototype theorem of Krull–Schmidt type.

The extension of Theorem 3.1 to Dedekind domains is the following:

THEOREM 3.2 (Steinitz, 1912). *Let M_R be a finitely generated module over a Dedekind domain R. Then*:

(a) *$M = T \oplus F$, where T is torsion and F is projective*;

(b) *$T \cong R/Q_1^{n_1} \oplus R/Q_2^{n_2} \oplus \cdots \oplus R/Q_t^{n_t}$ for a suitable integer $t \geqslant 0$, non-zero prime ideals $Q_1, Q_2, \ldots, Q_t$ and positive integers $n_1, n_2, \ldots, n_t$*;

(c) *$F \cong I_1 \oplus I_2 \oplus \cdots \oplus I_m$ for a suitable integer $m \geqslant 0$ and non-zero ideals $I_1, I_2, \ldots, I_m$ of R.*

Moreover, if $M = T' \oplus F'$, $T' \cong R/(Q_1')^{n_1'} \oplus R/(Q_2')^{n_2'} \oplus \cdots \oplus R/(Q_{t'}')^{n_{t'}'}$, $F' \cong I_1' \oplus I_2' \oplus \cdots \oplus I_{m'}'$ are direct sum decompositions with properties (a), (b) *and* (c), *then $T = T'$, $F \cong F'$, $t = t'$, $m = m'$, $I_1 I_2 \ldots I_m \cong I_1' I_2' \ldots I_{m'}'$, and there is a permutation σ of $\{1, 2, \ldots, t\}$ such that $Q_{\sigma(i)} = Q_i'$ and $n_{\sigma(i)} = n_i'$ for every $i = 1, 2, \ldots, t$.*

For a proof of the results of Section 3.1 see, for instance, [SV2].

Notice that every finitely generated module over a principal ideal domain is a direct sum of cyclic modules, and every finitely generated module over a Dedekind domain is a direct sum of modules generated by $\leqslant 2$ elements. As far as part (c) of Theorem 3.2 is concerned, note that two ideals I and J of an integral domain R with field of fractions K are isomorphic if and only if there exists a non-zero element α of K such that $I = \alpha J$. Here is a description of the torsion-free (= projective) part of the modules described in Theorem 3.2.

COROLLARY 3.3. *Let R be a Dedekind domain and let $I_1, I_2, \ldots, I_t, I'_1, I'_2, \ldots, I'_s$ be non-zero ideals of R. Then $I_1 \oplus I_2 \oplus \cdots \oplus I_t \cong I'_1 \oplus I'_2 \oplus \cdots \oplus I'_s$ if and only if $t = s$ and $I_1 I_2 \ldots I_t \cong I'_1 I'_2 \ldots I'_s$. In particular, $I_1 \oplus I_2 \oplus \cdots \oplus I_t \cong R^{t-1} \oplus I_1 I_2 \ldots I_t$.*

Every ideal of a Dedekind domain R is isomorphic to a direct summand of $R_R \oplus R_R$. It follows that the Krull–Schmidt theorem holds for the class of finitely generated projective modules over a Dedekind domain R if and only if R is a principal ideal domain.

For commutative rings R such that for any ideals $I_1, I_2, \ldots, I_t, I'_1, I'_2, \ldots, I'_s$ of R, if $I_1 \oplus I_2 \oplus \cdots \oplus I_t \cong I'_1 \oplus I'_2 \oplus \cdots \oplus I'_s$, then $t = s$ and after reindexing, $I_j \cong J_j$ for all j, see [GO2,GO3].

3.2. *Torsion-free Abelian groups*

In this section we consider torsion-free Abelian groups. We begin with the general case of finite rank torsion-free modules and finitely generated torsion-free modules over arbitrary commutative integral domains. For these classes the Krull–Schmidt theorem does not usually hold. Examples are easy to find. For instance, let I and J be two comaximal ideals of a commutative integral domain R, that is, two ideals of R with $I + J = R$. For instance, I and J could be two distinct maximal ideals of R. If I and J are not principal ideals, then the Krull–Schmidt theorem does not hold for the class of finite rank torsion-free R-modules (because the kernel of the canonical mapping $I \oplus J \to R$, $(i, j) \mapsto i + j$, is a direct summand). Similarly, if I and J are finitely generated ideals that are not principal, then the Krull–Schmidt theorem does not hold for the class of finitely generated torsion-free R-modules. As far as the case $R = \mathbf{Z}$ is concerned, the Krull–Schmidt theorem does not hold for the class of finite rank torsion-free Abelian groups [J1,J2], i.e. the direct sum decomposition is not unique up to isomorphism. Nevertheless, there are only finitely many direct sum decompositions up to isomorphism, as we shall see now.

A module M_R over an arbitrary ring R is said to be an *almost Krull–Schmidt module* [V4, p. 480] if it has only a finite number of direct sum decompositions up to isomorphism. Lady [L1] proved that:

LEMMA 3.4. *If R is a ring whose additive group is torsion-free of finite rank, then R_R is an almost Krull–Schmidt module.*

If E is the endomorphism ring of a finite rank torsion-free Abelian group, then the additive group of E is likewise a finite rank torsion-free Abelian group. Making use of Theorem 2.4 one obtains:

THEOREM 3.5 [L1]. *Every finite rank torsion-free Abelian group is an almost Krull–Schmidt $\mathbf{Z}$-module.*

For a characterization of the Dedekind domains of characteristic 0 over which every finite rank torsion-free module is almost Krull–Schmidt, see [FV1, Theorem 5.1]. We shall see in Lemma 4.3(c) that all modules with semilocal endomorphism rings are almost Krull–Schmidt modules.

Although the Krull–Schmidt theorem does not hold for finite rank torsion-free Abelian groups, it does hold in the category we shall describe now. We follow Walker [W1]. An Abelian group A is *bounded* if $nA = 0$ for some positive integer n. Let $\mathcal{B}$ be the class of all bounded Abelian groups. If A and B are two arbitrary Abelian groups, we say that A and B *belong to the same monogeny class* if there exist a monomorphism $A \to B$ and a monomorphism $B \to A$ (in this case we shall write $[A]_m = [B]_m$), and that they are *quasi-isomorphic* [J3] if there exist a subgroup S of A and a subgroup T of B with $A/S, B/T \in \mathcal{B}$ and $S \cong T$.

LEMMA 3.6 (Jónsson [J3], Warfield [W7, Lemma 5.1]). *If M is a finite rank torsion-free module over an integral domain R and $f : M \to M$ is a monomorphism, then there is an $r \in R$, $r \neq 0$, such that $rM \subseteq f(M)$.*

For $R = \mathbf{Z}$, that is, when M is an Abelian group, Lemma 3.6 states that every subgroup of M isomorphic to M has finite index. Two torsion-free Abelian groups A and B are quasi-isomorphic if and only if there exist a monomorphism $f : A \to B$ and a monomorphism $g : B \to A$ with $A/g(B), B/f(A) \in \mathcal{B}$; and two torsion-free Abelian groups of finite rank are quasi-isomorphic if and only if they belong to the same monogeny class.

Let $\mathcal{A}$ be the category of all Abelian groups. The class $\mathcal{B}$ is a *Serre class* of $\mathcal{A}$, that is, for every exact sequence $0 \to G' \to G \to G'' \to 0$ of Abelian groups, G belongs to $\mathcal{B}$ if and only if G' and G'' belong to $\mathcal{B}$. Therefore it is possible to construct the *quotient category* $\mathcal{A}/\mathcal{B}$ in the sense of Grothendieck [G2]. The objects of $\mathcal{A}/\mathcal{B}$ are all the objects of $\mathcal{A}$, that is, all Abelian groups, and $\mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(G, H) = \varinjlim \mathrm{Hom}_{\mathbf{Z}}(G', H/H')$, where the direct system consists of all subgroups G' of G and H' of H with $G/G' \in \mathcal{B}$ and $H' \in \mathcal{B}$. The category $\mathcal{A}/\mathcal{B}$ is Abelian, and we may look at the usual direct sum decompositions of the Abelian group G in the category $\mathcal{A}$ or the coproduct decompositions of G in the category $\mathcal{A}/\mathcal{B}$ (called *quasi-decompositions* of G).

There is an equivalent way to look at the category $\mathcal{A}/\mathcal{B}$. Let $\mathcal{A}_{\mathbf{Q}}$ be the category whose objects are all Abelian groups and whose morphism groups are defined by $\mathrm{Hom}_{\mathcal{A}_{\mathbf{Q}}}(G, H) = \mathbf{Q} \otimes \mathrm{Hom}_{\mathbf{Z}}(G, H)$ (Jónsson [J3]). Every element of $\mathbf{Q} \otimes \mathrm{Hom}_{\mathbf{Z}}(G, H)$ can be written in the form $1/n \otimes f$ for some positive integer n and some $f \in \mathrm{Hom}_{\mathbf{Z}}(G, H)$. The composition of morphisms in $\mathcal{A}_{\mathbf{Q}}$ is defined by $(1/n \otimes f) \circ (1/m \otimes g) = (1/nm \otimes fg)$. Then the categories $\mathcal{A}_{\mathbf{Q}}$ and $\mathcal{A}/\mathcal{B}$ are equivalent [W1].

For every $f \in \mathrm{Hom}_{\mathbf{Z}}(G, H)$, let $\overline{f}$ denote the element of $\mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(G, H)$ induced by f. There is a functor $J : \mathcal{A} \to \mathcal{A}/\mathcal{B}$ defined by $J(G) = G$ and $J(f) = \overline{f}$. The functor J has kernel $\mathcal{B}$ and is exact. If $f' \in \mathrm{Hom}_{\mathbf{Z}}(G', H/H')$ and $G/G', H' \in \mathcal{B}$, then f' induces a morphism $\overline{f'} \in \mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(G, H)$. One has that $\overline{f'}$ is an epimorphism if and only if $\operatorname{coker} f' \in \mathcal{B}$, and $\overline{f'}$ is a monomorphism if and only if $\ker f' \in \mathcal{B}$. As remarked by Walker [W1, pp. 147 and 154], two torsion-free Abelian groups are quasi-isomorphic if and only if they are isomorphic in the quotient category $\mathcal{A}/\mathcal{B}$. Hence the decomposition theory of torsion-free Abelian groups in $\mathcal{A}/\mathcal{B}$ is equivalent to the quasi-decomposition theory of torsion-free Abelian groups in $\mathcal{A}$. Notice that our definition of quasi-isomorphism applies to all Abelian groups, not just torsion-free ones, but quasi-isomorphism does not correspond to isomorphism in $\mathcal{A}/\mathcal{B}$ for arbitrary Abelian groups. The equivalent categories $\mathcal{A}/\mathcal{B}$ and $\mathcal{A}_{\mathbf{Q}}$ have enough projectives and injectives, but they do not have injective envelopes and infinite coproducts.

Let us restrict our attention to the full subcategory FRTF of $\mathcal{A}_{\mathbf{Q}}$ whose objects are all finite rank torsion-free Abelian groups. The category FRTF is an additive category and *idempotents split* in FRTF, that is, if G is an object of FRTF and $e \in \mathrm{End}_{\mathcal{A}_{\mathbf{Q}}}(G)$ is an idempotent, then there exist an object H of FRTF, $p \in \mathrm{Hom}_{\mathcal{A}_{\mathbf{Q}}}(G, H)$ and $q \in \mathrm{Hom}_{\mathcal{A}_{\mathbf{Q}}}(H, G)$ such that $qp = e$ and $pq = 1_H$ (see, for instance, [A1, p. 73]). As we have already said, two torsion-free Abelian groups of finite rank are isomorphic in the category FRTF if and only if the are quasi-isomorphic (in the category $\mathcal{A}$), if and only if they are in the same monogeny class. An object G of FRTF is a coproduct $G = G_1 \amalg \cdots \amalg G_t$ in FRTF if and only if there is a monomorphism $f : G_1 \oplus \cdots \oplus G_t \to G$ in $\mathcal{A}$ with $G/\mathrm{im}(f)$ finite, or, equivalently, if and only if $[G]_m = [G_1 \oplus \cdots \oplus G_t]_m$. An object G of FRTF is indecomposable in FRTF if and only if whenever $nG \subseteq H \oplus H' \subseteq G$ for some integer $n > 0$, then $H = 0$ or $H' = 0$. In this case one says that G is *strongly indecomposable*. A finite rank torsion-free Abelian group is strongly indecomposable if and only if its endomorphism ring $\mathbf{Q} \otimes \mathrm{End}_{\mathbf{Z}}(G)$ in $\mathcal{A}_{\mathbf{Q}}$ is local [R1].

The main result of B. Jónsson's paper [J3] is the Krull–Schmidt theorem for the category FRTF (it holds by Theorem A.1 in Appendix A) and its interpretation in the language of Abelian groups is the following:

THEOREM 3.7. *Let G be a finite rank torsion-free Abelian group. Then there exist a positive integer m and strongly indecomposable subgroups $G_1, \dots, G_t$ of G such that $mG \subseteq G_1 \oplus \cdots \oplus G_t \subseteq G$. Moreover, if k is a positive integer and $H_1, \dots, H_s$ are strongly indecomposable subgroups of G with $kG \subseteq H_1 \oplus \cdots \oplus H_s \subseteq G$, then $t = s$ and there is a permutations σ of $\{1, 2, \dots, t\}$ such that $[G_i]_m = [H_{\sigma(i)}]_m$ for every $i = 1, 2, \dots, t$.*

An extension of this theorem to Abelian groups of infinite rank is given in Fuchs and Viljoen [FV2]. For the proofs of the statements presented in this Section 3.2 and further results about torsion-free Abelian groups, see [A1].

Similar constructions and results are also possible for other classes of modules. For instance, it is possible to construct a category whose objects are Noetherian right modules over a ring but whose morphisms have been modified in such a way that indecomposable objects have local endomorphism ring. Hence the Krull–Schmidt theorem holds in this new category. See [B5].

Some most remarkable results obtained in the setting of torsion-free Abelian groups appear in the papers [C2,C3,C4]. Corner proved in [C2] that every reduced, torsion-free ring of rank $n \leqslant \aleph_0$ is isomorphic to the endomorphism ring of a reduced, torsion-free group of rank $2n$. This theorem exhibits the enormous amount of pathology which can be encountered in the structure of countable, torsion-free Abelian groups. For instance, applying this result, Corner [C3] showed that for any positive integer r there is a countable torsion-free Abelian group G such that G^m is isomorphic to G^n if and only if $m \equiv n \pmod r$. Also, he proved that there exists a countable, torsion-free group which has no non-zero indecomposable direct summand [C2]. Corner's results have been greatly generalized and have found several applications.

3.3. *Henselian rings*

A local commutative ring R with maximal ideal M is said to be *Henselian* if the following condition holds:

if $f, g_0, h_0 \in R[x]$ are monic polynomials such that
(1) $f \equiv g_0 h_0 \pmod{MR[x]}$ and
(2) $g_0 R[x] + h_0 R[x] + MR[x] = R[x]$,
then there exist monic polynomials $g, h \in R[x]$ such that $f = gh$, $g \equiv g_0 \pmod{MR[x]}$ and $h \equiv h_0 \pmod{MR[x]}$.

For example, complete local commutative rings are Henselian.

PROPOSITION 3.8 (Azumaya [A4]). *A local commutative ring R is Henselian if and only if every module-finite R-algebra (not necessarily commutative) is semiperfect.*

The next result was proved by Vámos [V3, Lemma 13] and Siddoway [S3] independently.

THEOREM 3.9. *Every finitely generated indecomposable module M_R over a Henselian local commutative ring R has a local endomorphism ring* $\mathrm{End}(M_R)$.

The proof is essentially the following. If M_R is a finitely generated indecomposable module over a Henselian ring R, then $\mathrm{End}(M_R)$ is a module-finite R-algebra, hence a semiperfect ring by Proposition 3.8. Thus M_R is a direct sum of modules with local endomorphism rings (Proposition 2.5). As M_R is indecomposable, its endomorphism ring $\mathrm{End}(M_R)$ must be local.

For a ring R, we denote by FG-R the class of all finitely generated right R-modules. From the Krull–Schmidt–Azumaya Theorem 2.2 and Theorem 3.9 we immediately obtain:

COROLLARY 3.10. *If R is a Henselian local commutative ring, the Krull–Schmidt theorem holds for the class* FG-R.

The sketch of proof we have given for Theorem 3.9 also shows that if R is any module-finite algebra over a Henselian local commutative ring, then every finitely generated indecomposable R-module has a local endomorphism ring, so that the Krull–Schmidt theorem holds for the class FG-R.

From Example 4.2(5) and Lemma 4.3(c) it will follow that finitely generated modules over semilocal commutative rings (more generally, finitely generated modules over module-finite k-algebras, where k is a semilocal commutative ring [W7, Lemma 2.3]) are almost Krull–Schmidt modules.

3.4. *Finitely generated modules over valuation rings*

A *valuation ring* is a commutative ring whose ideals are linearly ordered under inclusion. A *valuation domain* is a valuation ring that is an integral domain. Vámos proved that the Krull–Schmidt theorem does not necessarily hold for the class FG-R of finitely generated modules over a valuation ring R making use of the next theorem. In order to prove it, Vámos used the fact that the Krull–Schmidt theorem does not hold for the class of torsion-free modules of finite rank over a discrete valuation domain of rank one, a class of modules that will be considered in Theorem 3.21. Recall that a subgroup H of a totally ordered Abelian group G is said to be *convex* if $0 \leqslant g \leqslant h$, $g \in G$ and $h \in H$ imply $g \in H$. If R is a valuation domain and G is its value group, then R is said to be a *discrete* valuation domain if whenever $H \subset H'$ are convex subgroups of G and there is no convex subgroup of G properly between H and H', then H'/H is order isomorphic to $\mathbf{Z}$. A discrete valuation domain R is *of rank n* if its value group G has a chain $0 = H_0 \subseteq H_1 \subset \cdots \subset H_n = G$ of convex subgroups with H_i/H_{i-1} order isomorphic to $\mathbf{Z}$ for all $i = 1, 2, \ldots, n$. Thus a valuation domain R is a discrete valuation domain of rank one if and only if its value group G is order isomorphic to $\mathbf{Z}$.

THEOREM 3.11 [V3, Theorem 20]. *Let R be a valuation ring (not necessarily a domain) with a prime ideal P and an element $a \in P$ such that $V = R/P$ is a non-Henselian discrete valuation domain of rank one, $Pa \neq 0$ and P is the radical of Pa. Then the Krull–Schmidt theorem does not hold for the class* FG-R.

A commutative ring R is said to be *almost Henselian* [V3] if every proper homomorphic image of R is a local Henselian ring. Let R be an almost Henselian valuation ring. Then the endomorphism ring of every indecomposable finitely generated R-module is local. In particular, the Krull–Schmidt theorem holds for FG-R [V3, Theorem 19].

COROLLARY 3.12 [V3, Corollary 21]. *Let R be a discrete valuation ring of finite rank. The Krull–Schmidt theorem holds for the class* FG-R *if and only if R is almost Henselian.*

If R is a valuation domain and $\mathrm{FG}_{\oplus 2}$-R is the class of all modules that are direct sums of finitely many modules generated by $\leqslant 2$ elements, then the Krull–Schmidt theorem holds for $\mathrm{FG}_{\oplus 2}$-R [SZ]. For a commutative ring R, let FG_n-R denote the class of all R-modules that can be generated by $\leqslant n$ elements. If a module over a valuation domain is generated by $\leqslant 4$ elements and is not indecomposable, then it decomposes as the direct sum of a cyclic module and a module generated by $\leqslant 3$ elements or as the direct sum of two modules generated by $\leqslant 2$ elements. In both cases the direct sum decomposition as a direct sum of indecomposables is unique [SZ], so that the Krull–Schmidt theorem holds for the class FG_4-R for every valuation domain R. Zanardo [Z3] showed that there exist valuation domains R such that the Krull–Schmidt theorem does not hold for the class FG_6-R. It is not known whether the Krull–Schmidt theorem holds for the class FG_5-R for any valuation domain R.

3.5. *Finitely generated modules over subrings of* $\mathbf{Z}^n$

A nice set of examples showing a number of ways in which the Krull–Schmidt uniqueness fails for the class FG-R of Noetherian modules over suitable subrings R of the direct product $\mathbf{Z}^n$ was discovered by Levy [L3] (these rings are all commutative Noetherian rings of Krull dimension one). Levy showed that:

(1) (Non-uniqueness of the number of indecomposable direct summands) For every $m \geqslant 3$ there is a Noetherian module M over a suitable subring R of $\mathbf{Z}^{m-1}$ that has a direct sum decomposition $M = M_{1,i} \oplus M_{2,i} \oplus \cdots \oplus M_{i,i}$ into i indecomposable direct summands $M_{1,i}, M_{2,i}, \ldots, M_{i,i}$ for every $i = 2, 3, \ldots, m$.

(2) (Failure of the cancellation property) There exist Noetherian modules M, N, N' over suitable subrings R of $\mathbf{Z}^n$ such that $M \oplus N \cong M \oplus N'$, but N is not isomorphic to N'. We shall see in Lemma 4.3(a) that the cancellation property holds for modules whose endomorphism ring is semilocal.

(3) (Failure of the n-th root property) There exist Noetherian modules M, N over a suitable subring R of $\mathbf{Z}^m$ such that $M^n \cong N^n$ (for suitable $n > 1$), but M is not isomorphic to N. We shall see in Lemma 4.3(b) that the n-th root property holds for modules whose endomorphism ring is semilocal.

(4) (Failure of the interchange of localizations property) There exist four pairwise non-isomorphic indecomposable Noetherian modules H, K, M, N over a suitable subring R of $\mathbf{Z}^n$ and two maximal ideals P and Q of R such that $H \oplus K \cong M \oplus N$, $H_P \cong M_P \not\cong K_P \cong N_P$ and $H_Q \cong N_Q \not\cong K_Q \cong M_Q$.

In (1), (4), (5), and (7) the symbol $\mathbf{Z}$ denotes the ring of integers, but it can be any integral domain with at least four maximal ideals. In (2) and (3) also, the symbol $\mathbf{Z}$ can denote the ring of integers, but it can be any integral domain with at least four maximal ideals that satisfies an additional non-triviality condition involving non-liftability of units modulo maximal ideals to units of $\mathbf{Z}$. If $\mathbf{Z}$ denotes the polynomial ring $F[x]$, where F is any field which is not an algebraic extension of a finite field, then there are subrings R of the direct product $\mathbf{Z}^n$ that satisfy all of (1)–(7).

(5) (Indecomposability is not a local property) There exists an indecomposable Noetherian R-module M such that M_P is not indecomposable, for some subring R of $\mathbf{Z}^n$ and some maximal ideal P of R.

(6) (Projective modules of rank 1) There exists a projective R-module M of rank 1 such that $M^n \not\cong R^n$ for every positive n, R a suitable subring of $\mathbf{Z}^2$. Here $\mathbf{Z}$ can be $F[x]$, F a field that is not an algebraic extension of a finite field, but $\mathbf{Z}$ cannot be the ring of the integers.

(7) (Isomorphism is not a local property) There exist two non-isomorphic Noetherian modules M and N such that $M_P \cong N_P$ for every maximal ideal P of a subring R of $\mathbf{Z}^n$.

We note that, in example (1) above, the integer m must be selected before the ring R and the R-module M are formed. If one does not insist that M be Noetherian, then this is not necessary, as shown by the following example of Osofsky which Levy obtains, as a corollary of his methods.

(8) (B. Osofsky [O]) There exists a (non-Noetherian!) module M_R that is not the direct sum of infinitely many submodules, but that is the direct sum of m indecomposable modules for every $m \geqslant 2$. In particular, M_R is not an almost Krull–Schmidt module. (Here the

ring R is an ultraproduct of subrings of $\mathbf{Z}$. The M Levy obtains is a ring, considered as a module over itself. Like every ring with identity, M cannot be written as the direct sum of an infinite number of non-zero submodules.)

Levy's examples can be used, via Theorem 2.4, to reconstruct some of the classical examples of the failure of the Krull–Schmidt theorem for torsion-free Abelian groups of finite rank mentioned in Section 3.2; see [A2].

3.6. *Commutative rings whose finitely generated modules are direct sums of cyclic submodules or of n-generated submodules*

A commutative ring R is called an FG(n)-ring if every finitely generated R-module is a direct sum of submodules generated by $\leqslant n$ elements. Thus FG(1)-rings are exactly the commutative rings over which every finitely generated module is a direct sum of cyclics. For instance, principal ideal domains are FG(1)-rings (Theorem 3.1) and Dedekind domains are FG(2)-rings (Theorem 3.2). FG(n)-rings are also called *rings of bounded module type*. The characterization of FG(1)-rings, the solution of the so-called Kaplansky's problem, was found in 1976 (for a nice presentation of this topic, the history of the problem and the proofs of the results presented in this section, see [B4,V2] or [WW2]). In order to present the characterization of FG(1)-rings, we need a number of definitions. Recall that a commutative ring R is *maximal* (or *linearly compact*) if whenever $\{I_\alpha \mid \alpha \in A\}$ is a family of ideals of R, $\{x_\alpha \mid \alpha \in A\}$ is a family of elements of R and the family of cosets $\{x_\alpha + I_\alpha \mid \alpha \in A\}$ has the finite intersection property (that is, the intersection of every finite subfamily is non-empty), then $\bigcap_{\alpha\in A} x_\alpha + I_\alpha \neq \emptyset$. A commutative ring is *almost maximal* if R/I is a maximal ring for all non-zero ideals I of R, and is *locally almost maximal* if R_M is an almost maximal ring for all maximal ideals M of R.

If R and S are valuation rings, $R \subseteq S$ and M is the maximal ideal of R, then S is an *immediate extension* of R if the embedding $R \to S$ induces an isomorphism between the value groups of R and S and an isomorphism $R/M \to S/MS$. A valuation domain R is maximal if and only if it is has no proper immediate extension. Any valuation domain R has an immediate extension $\widetilde{R}$ that is a maximal valuation domain ([K5]; see also [W2, p. 717]). The R-module $\widetilde{R}$ is the pure-injective envelope of the R-module R, so that $\widetilde{R}$ is unique (up to isomorphism) as an R-module, though, [K1], it is not unique as an R-algebra. A valuation ring that is not a domain is maximal if and only if it is almost maximal [G1]. A valuation domain is maximal if and only if it is almost maximal and complete in the valuation topology. The completion of a valuation domain R in the valuation topology will be denoted by $\widehat{R}$. If $K = K(R)$, $\widetilde{K} = K(\widetilde{R})$ are the fields of fractions of R, $\widetilde{R}$, respectively, then $\widetilde{K}$ is the completion of K as a valuated field and the degree $[\widetilde{K} : K]$ is the torsion-free rank of the R-module $\widetilde{R}$. A valuation domain R is almost maximal if and only if $\widehat{R} = \widetilde{R}$.

A commutative integral domain R is *h-local* if every non-zero element of R belongs to only finitely many maximal ideals of R and every non-zero prime ideal of R is contained in only one maximal ideal. A *Bezout ring* is a commutative ring in which every finitely generated ideal is principal. A right module M is said to be *uniserial* if its lattice of submodules is linearly ordered by set inclusion, that is, if for any submodules A and B of M either $A \subseteq B$ or $B \subseteq A$. We are ready to define torch rings. A *torch ring* is a commutative ring R

that satisfies all the following conditions: (1) the ring R is not local; (2) the ring R has a unique minimal prime ideal P and P is a non-zero uniserial R-module; (3) the ring R/P is h-local; (4) R is a locally almost maximal Bezout ring. Torch rings were first considered by Shores and R. Wiegand in [SW], and their name was suggested by Vámos in [V1]. The name refers to the ideal lattice of a torch ring R: all ideals of R either contain P or belong to the linearly ordered set of the ideals contained in P.

THEOREM 3.13 (W. Brandal, T. Shores, P. Vámos, R. Wiegand and S. Wiegand). *A commutative ring is an* FG(1)*-ring if and only if it is a finite direct product of rings of the following three types*:
(a) *maximal valuation rings*;
(b) *almost maximal Bezout domains*;
(c) *torch rings.*

THEOREM 3.14 [B4, Theorem 9.2]. *If R is an* FG(1)*-ring, the Krull–Schmidt theorem holds for the class* FG-R.

Another form of uniqueness of direct sum decompositions is given by the so-called canonical form decompositions. A *canonical form decomposition* of a module M over a commutative ring R is a decomposition of the form $M \cong R/I_1 \oplus R/I_2 \oplus \cdots \oplus R/I_t$ where $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_t$ are proper ideals of R. The following result, proved by Kaplansky, partially extends Theorem 3.1.

THEOREM 3.15 [K3, Theorem 9.3]. *If a module M over a commutative ring R has a canonical form decomposition, then the canonical form decomposition is unique up to isomorphism. More generally, if R is a ring in which every one-sided ideal is two-sided, $R/I_1 \oplus R/I_2 \oplus \cdots \oplus R/I_t \cong R/I'_1 \oplus R/I'_2 \oplus \cdots \oplus R/I'_s$ and $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_t$, $I'_1 \subseteq I'_2 \subseteq \cdots \subseteq I'_s$ are proper ideals of R, then $t = s$ and $I_k = I'_k$ for every $k = 1, 2, \ldots, t$.*

THEOREM 3.16 [B4, Theorem 9.5]. *Every finitely generated module over an* FG(1)*-ring has a canonical form decomposition.*

We now turn our attention to FG(n)-rings for arbitrary n. If R is an FG(n)-ring, then: (1) the localization R_M is an FG(n) valuation ring for every maximal ideal M of R (Warfield, [W4, Theorem 2]); (2) R has only finitely many minimal prime ideals (Midgarten and S. Wiegand, [MW, Theorem 1.8]); (3) R_M/P_M is complete for all maximal ideals M and prime ideals $P \subseteq M$ that are not minimal primes (Zanardo, [Z2, Theorem 4]).

THEOREM 3.17 (Warfield, [W4, Corollary 4.1]). *Let R be a commutative Noetherian* FG(n)*-ring for some $n \geqslant 1$. Then R is a finite product of Dedekind domains and valuation rings. In particular, R is an* FG(2)*-ring.*

THEOREM 3.18 (Vámos, [V3]). *If R is a valuation domain that is an* FG(n)*-ring for some $n \geqslant 1$ and is an algebra over the field $\mathbf{Q}$ of rational numbers, then R is an almost maximal valuation domain, i.e. an* FG(1)*-ring.*

More generally, Vámos conjectured that every FG(n) local commutative ring has to be almost maximal (as we have already said above, it is necessarily a valuation ring by [W4, Theorem 2]). Couchot proved in [C5,C6] that if every FG(n) valuation domain of Krull dimension 1 is almost maximal, then Vámos' conjecture is true for every local commutative ring.

As a corollary of Theorem 3.9, Vámos [V3, Theorem 19] obtains:

COROLLARY 3.19. *If R is an* FG(n) *valuation domain for some $n \geqslant 1$, the Krull–Schmidt theorem holds for the class* FG-R.

3.7. *Torsion-free modules over valuation domains*

In this section, R will always be a commutative integral domain, usually a valuation domain, and FRTF-R will denote the class of all finite rank torsion-free R-modules.

THEOREM 3.20 ([S3, Theorem 9], [V3, Lemma 14]). *The endomorphism ring* End(M_R) *of a finite rank torsion-free indecomposable module M_R over a Henselian valuation domain R is local. Hence the Krull–Schmidt theorem holds for the class* FRTF-R *whenever R is a Henselian valuation domain.*

Historically, the first example that shows that the Krull–Schmidt theorem does not hold for FRTF-$\mathbf{Z}_p$, where $\mathbf{Z}_p$ is the discrete valuation domain of the integers localized at a prime p, is due to Butler and goes back to the sixties (see [A1, Example 2.15] and [W6, p. 461]). The following theorem, due to Vámos [V3, Theorem 17], shows why $\mathbf{Z}_p$ can be used to construct examples of torsion-free modules for which Krull–Schmidt does not hold. The "if" part of the theorem was also proved by Lady.

THEOREM 3.21. *Let R be a discrete rank one valuation domain. The Krull–Schmidt theorem holds for the class* FRTF-R *if and only if R is Henselian.*

Theorem 3.21 does not hold for arbitrary valuation domains: Vámos gave examples of non-Henselian valuation domains R such that the Krull–Schmidt theorem holds for FRTF-R, and proved in [V3, Theorems 5 and 10 and last note of the paper] that

THEOREM 3.22. *If R is a non-Henselian valuation domain and the maximal immediate extension $\widetilde{R}$ of R belongs to* FRTF-R, *then the Krull–Schmidt theorem holds for the class* FRTF-R.

Notice that the maximal immediate extension $\widetilde{R}$ of a valuation domain R is always a torsion-free R-module. Hence it belongs to FRTF-R if and only if it has finite rank. May and Zanardo, extending previous results of Goldsmith and May [GM, Theorem 2], proved in [MZ, Theorem 3] that

THEOREM 3.23. *If a valuation domain R contains a prime ideal P such that R/P is not Henselian and the completion $\widehat{R/P}$ in the valuation topology of R/P is an R/P-module of torsion-free rank $\geqslant 6$, then the Krull–Schmidt theorem does not hold for* FRTF-R.

For instance, Theorem 3.23 applies to the valuation domain $R = \mathbf{Z}_p + x\mathbf{Q}[[x]]$, which is a complete valuation domain, with P the prime ideal $x\mathbf{Q}[[x]]$ of R, because the completion $\widehat{\mathbf{Z}}_p$ of $\mathbf{Z}_p$ in the valuation topology has infinite torsion-free rank as a $\mathbf{Z}_p$-module.

The following is a corollary of Theorem 3.21.

COROLLARY 3.24 [V3, Corollary 18]. *Let R be a Dedekind domain. The Krull–Schmidt theorem holds for the class* FRTF-R *if and only if R is a Henselian valuation domain.*

For a commutative integral domain R let $\mathrm{fr}(R)$ denote the supremum of the ranks of all indecomposable finite rank torsion-free R-modules. One can prove [AD] that if R is a discrete valuation domain, then $\mathrm{fr}(R) = 1, 2, 3$ or ∞.

THEOREM 3.25 ([K2, Theorem 12], [M2, Theorem 65]). *For a valuation domain R, $\mathrm{fr}(R) = 1$ if and only if R is a maximal valuation domain.*

More generally, the integrally closed domains with $\mathrm{fr}(R) = 1$ were determined by Matlis [M2].

THEOREM 3.26 [V3]. *Let n be a positive integer and let R be a valuation domain such that every torsion-free R-module is a direct sum of modules of rank $\leqslant n$. Then the Krull–Schmidt theorem holds for the class of torsion-free R-modules. If, moreover, R is an algebra over the field $\mathbf{Q}$ of rational numbers, then every torsion-free R-module is a direct sum of modules of rank $\leqslant 2$.*

4. Modules with a semilocal endomorphism ring

4.1. *Examples and properties of modules whose endomorphism ring is semilocal*

The aim of this section is to study modules whose endomorphism ring is semilocal. Further details and the proofs can be found in [F2, Chapter 4]. By Theorem 2.4, if a module M_R has a semilocal endomorphism ring E, the study of the direct sum decompositions of M_R reduces to the study of the idempotents of the semilocal ring E. Theorem 4.1 will allow us to give examples of modules with a semilocal endomorphism ring. For a module M_R set $\delta(M_R) = \mathrm{codim}(\mathrm{End}(M_R))$. Thus $\delta(M_R)$ is a non-negative integer if and only if $\mathrm{End}(M_R)$ is semilocal (Theorem 2.12); otherwise, $\delta(M_R) = \infty$. From Proposition 2.13 one immediately obtains that $\delta(M_R \oplus M'_R) = \delta(M_R) + \delta(M'_R)$ for any M_R, M'_R. Immediate consequences of this formula and the definition of the dimension δ are that: $\delta(M_R) = 0$ if and only if $M_R = 0$; $\delta(M_R^n) = n\delta(M_R)$; $\delta(M_R) = 1$ if and only if $\mathrm{End}(M_R)$ is local. Notice that under the conditions of the Krull–Schmidt theorem, that is, if $M_R = M_1 \oplus \cdots \oplus M_t$ is the direct sum of t modules M_i and all the endomorphism rings $\mathrm{End}(M_i)$ are local, then $\delta(M_R) = t$.

THEOREM 4.1 (D. Herbera and A. Shamsuddin [HS]). *Let M_R be a module over a ring R. Then the following statements hold*:

(a) $\delta(M_R) \leqslant \dim(M_R) + \operatorname{codim}(M_R)$.

(b) *If every injective endomorphism of M_R is bijective, then* $\delta(M_R) \leqslant \dim(M_R)$ (Camps and Dicks [CD, Theorem 5]).

(c) *If every surjective endomorphism of M_R is bijective, then* $\delta(M_R) \leqslant \operatorname{codim}(M_R)$.

EXAMPLES 4.2.

(1) Artinian modules have semilocal endomorphism rings [CD]. In fact, every injective endomorphism of an Artinian module is bijective, so that Theorem 4.1(b) can be applied. More generally, modules that are linearly compact in the discrete topology have semilocal endomorphism rings [HS].

(2) Noetherian modules of finite dual Goldie dimension have semilocal endomorphism rings. This is the dual of the previous Example 4.2(1). In particular, finitely generated right modules over semilocal right Noetherian rings have semilocal endomorphism rings.

(3) Modules of finite Goldie dimension and finite dual Goldie dimension have semilocal endomorphism rings by Theorem 4.1(a).

(4) Let R be either a semilocal commutative principal ideal domain or a valuation domain. If M_R is a finite rank torsion-free R-module, then $\operatorname{End}(M_R)$ is semilocal by Example 2.11(6).

(5) Let k be a semilocal commutative ring and R a module-finite k-algebra. The endomorphism ring of any finitely generated R-module is a semilocal ring [W7, Lemma 2.3].

(6) For further examples of modules whose endomorphism ring is semilocal, see [HS] and [F2, §4.3].

If M_R is a module and its endomorphism ring $E = \operatorname{End}(M_R)$ is semilocal, then every direct sum decomposition of M_R has finitely many direct summands, that is, if $M_R = \bigoplus_{\lambda \in \Lambda} M_\lambda$ is a decomposition of M_R as a direct sum of non-zero modules M_λ and $E = \operatorname{End}(M_R)$ is semilocal, then the index set Λ must be necessarily finite. In the next lemma we have collected some further properties of modules whose endomorphism ring is semilocal.

LEMMA 4.3. *Let M_R, N_R, N'_R be modules over an arbitrary ring R and suppose* $\operatorname{End}(M_R)$ *is semilocal. The following properties hold*:

(a) (Cancellation property) *If $M \oplus N \cong M \oplus N'$, then $N \cong N'$.*

(b) (n-th root property) *If $M^n \cong N^n$ for some positive integer n, then $M \cong N$.*

(c) *The module M_R is an almost Krull–Schmidt module.*

Part (a) of Lemma 4.3 follows from results of Bass [B1], who proved that semilocal rings have stable range 1, and Evans [E], who showed that modules whose endomorphism ring has stable range 1 cancel from direct sums. We shall not insist on the cancellation property and the n-th root property here. For a nice survey on the n-th root property, see [L2].

Notice that, by (c), not only are all direct sum decompositions of a module with semilocal endomorphism ring finite, but also there are just a finite number of such decompositions.

We conclude this section with a further result about modules M_R with a semilocal endomorphism ring, that is, modules M_R with $\delta(M_R) < \infty$.

THEOREM 4.4 [FH3, Theorem 2.3]. *Let M_R be a module over a ring R with $\delta(M_R) = n < \infty$. Let $N_1, N_2, \dots, N_m$ be R-modules such that M_R is isomorphic to a direct summand of $\bigoplus_{i=1}^m N_i$. Then there exists a subset α of $\{1, 2, \dots, m\}$ of cardinality $\leqslant n$ such that M_R is isomorphic to a direct summand of $\bigoplus_{j\in\alpha} N_j$.*

4.2. *Krull–Schmidt fails for Artinian modules*

Every Artinian right module decomposes as a finite direct sum of indecomposable modules, but the Krull–Schmidt theorem does not hold for the class of Artinian modules, as the following two examples that appear in [FHLV] show.

EXAMPLE 4.5 (*Non-uniqueness of the number of indecomposable summands*). Let $n \geqslant 2$ be an integer. There exists an Artinian module M over a suitable ring R that has a direct sum decomposition

$$M = M_{1,i} \oplus M_{2,i} \oplus \cdots \oplus M_{i,i}$$

into i indecomposable direct summands $M_{1,i}, M_{2,i}, \dots, M_{i,i}$ for every $i = 2, 3, \dots, n$.

EXAMPLE 4.6 (*Simple failure of Krull–Schmidt for Artinian modules*). There exist four indecomposable, pairwise non-isomorphic, Artinian modules M_1, M_2, M_3, M_4 over a suitable ring R such that $M_1 \oplus M_2 \cong M_3 \oplus M_4$.

These two examples were constructed showing that every module-finite algebra over a semilocal commutative Noetherian ring can be realized as the endomorphism ring of a suitable Artinian right module over a suitable ring [FHLV]. Further nice examples of Artinian modules with particular direct sum decompositions were later constructed by Yakovlev [Y1] and Pimenov–Yakovlev [PY]. Making use of the techniques of Pimenov–Yakovlev, Ringel showed in [R3] that the Krull–Schmidt theorem does not necessarily hold for the class of Artinian R-modules even when the base ring R is *local*. This answered a question posed in [F2, Problem 7, p. 268]. For instance, Ringel constructed five pairwise non-isomorphic indecomposable Artinian modules M_1, M_2, M_3, M_4, M_5 over a local ring R such that $M_1 \oplus M_2 \cong M_3 \oplus M_4 \oplus M_5$.

Notice that Warfield showed in 1969 that:

PROPOSITION 4.7 [W3, Proposition 5]. *If a ring R is either right Noetherian or commutative, then the Krull–Schmidt theorem holds for the class of all Artinian right R-modules.*

5. Biuniform modules

5.1. *Generalities*

A module A_R is called *uniform* if it is non-zero and the intersection of any two non-zero submodules of A_R is non-zero. Thus "uniform" means "of Goldie dimension one". Dually, a module A_R is *couniform* (or *hollow*) if it is non-zero and the sum of any two proper submodules of A_R is a proper submodule of A_R (equivalently, if its dual Goldie dimension is one). A module is *biuniform* if it is uniform and couniform. For example, non-zero uniserial modules (recall that a right module M is said to be *uniserial* if its lattice of submodules is linearly ordered by set inclusion) are biuniform. Sections 5.3 and 5.4 will be completely devoted to uniserial modules and direct sum decompositions of direct sums of uniserial modules. If A_R is a biuniform module over an arbitrary ring R, then its endomorphism ring $E = \mathrm{End}(A_R)$ has at most two maximal right ideals, the subset I of E whose elements are all the endomorphisms of A_R that are not injective, and the subset K of E whose elements are all the endomorphisms of A_R that are not surjective [F2, Theorem 9.1]. These two right ideals I and K are two-sided completely prime ideals of E, and every proper right ideal of E and every proper left ideal of E is contained either in I or in K. Moreover, if the ideals I and K are comparable, then E is a local ring and $I \cup K$ is its maximal ideal, whereas if I and K are not comparable, then $J(E) = I \cap K$ and $E/J(E)$ is canonically isomorphic to the direct product of the two division rings E/I and E/K.

5.2. *Weak Krull–Schmidt theorem for biuniform modules*

Two modules A and B *belong to the same monogeny class* if there exist a monomorphism $A \to B$ and a monomorphism $B \to A$. In this case, we write $[A]_m = [B]_m$. Similarly, A and B *belong to the same epigeny class* if there exist an epimorphism $A \to B$ and an epimorphism $B \to A$, and in this case we write $[A]_e = [B]_e$. Clearly, belonging to the same monogeny class and belonging to the same epigeny class are two equivalence relations.

PROPOSITION 5.1 ([F1, Proposition 1.6], [F2, Proposition 9.3]). *Two biuniform modules A and B are isomorphic if and only if $[A]_m = [B]_m$ and $[A]_e = [B]_e$.*

THEOREM 5.2 ([DF1, Theorem 3.1], [F2, Theorem 9.11]). *If $\{A_i \mid i \in I\}$ and $\{B_j \mid j \in J\}$ are two families of biuniform modules over a ring R and there exist two bijections $\sigma, \tau : I \to J$ such that $[A_i]_m = [B_{\sigma(i)}]_m$ and $[A_i]_e = [B_{\tau(i)}]_e$ for every $i \in I$, then $\bigoplus_{i\in I} A_i \cong \bigoplus_{j\in J} B_j$.*

Only a half of the implication in Theorem 5.2 can be reversed, as the next theorem and Example 5.11 show.

THEOREM 5.3 ([DF1, Theorem 3.3], [F2, Theorem 9.12]). *Let R be a ring and let $\{A_i \mid i \in I\}$, $\{B_j \mid j \in J\}$ be two families of biuniform right R-modules such that $\bigoplus_{i\in I} A_i \cong \bigoplus_{j\in J} B_j$. Then there exists a bijection $\sigma : I \to J$ such that $[A_i]_m = [B_{\sigma(i)}]_m$ for every $i \in I$.*

The next result describes when two direct sums of biuniform modules are isomorphic in the finite case.

THEOREM 5.4 (Weak Krull–Schmidt theorem for biuniform modules, [F1, Theorem 1.9]). *Let $A_1, \dots, A_t, B_1, \dots, B_s$ be biuniform modules over a ring R. Then $A_1 \oplus \cdots \oplus A_t \cong B_1 \oplus \cdots \oplus B_s$ if and only if $t = s$ and there are two permutations σ, τ of $\{1, 2, \dots, t\}$ such that $[A_i]_m = [B_{\sigma(i)}]_m$ and $[A_i]_e = [B_{\tau(i)}]_e$ for every $i = 1, 2, \dots, t$.*

We shall see in Example 5.7 that Theorem 5.4 cannot be improved, in the sense that, over suitable rings R, the Krull–Schmidt theorem does not hold for the class of the R-modules that are finite direct sums of biuniform modules.

In this setting we mention a result due to L. Diracca. Part (a) was first proved by Zanardo for modules over commutative rings [Z1, Theorem 2].

THEOREM 5.5. *Let R be an arbitrary ring.*

(a) *If $A_1, \dots, A_t, B_1, \dots, B_s$ are uniform R-modules and $[A_1 \oplus \cdots \oplus A_t]_m = [B_1 \oplus \cdots \oplus B_s]_m$, then $t = s$ and there is a permutation σ of $\{1, 2, \dots, t\}$ such that $[A_i]_m = [B_{\sigma(i)}]_m$ for every $i = 1, 2, \dots, t$.*

(b) *Dually, if $A_1, \dots, A_t, B_1, \dots, B_s$ are couniform R-modules and $[A_1 \oplus \cdots \oplus A_t]_e = [B_1 \oplus \cdots \oplus B_s]_e$, then $t = s$ and there is a permutation τ of $\{1, 2, \dots, t\}$ such that $[A_i]_e = [B_{\tau(i)}]_e$ for every $i = 1, 2, \dots, t$.*

5.3. *Uniserial modules*

A module is *serial* if it is a direct sum of uniserial submodules. In particular, serial modules of finite Goldie dimension are exactly direct sums of finitely many uniserial modules, and their endomorphism ring is semilocal (Theorem 4.1(a)). For instance, the Prüfer groups $\mathbf{Z}(p^\infty)$ are uniserial modules over the ring $\mathbf{Z}$ of integers, so that every torsion injective Abelian group is a serial $\mathbf{Z}$-module. Finite Abelian groups are serial $\mathbf{Z}$-modules. Vector spaces are serial modules. More generally, semisimple modules are serial.

A *chain ring* is a ring R such that both R_R and ${}_RR$ are uniserial modules. A ring R is *serial* if both R_R and ${}_RR$ are serial modules. Hence chain rings are serial rings, and valuation rings are exactly commutative chain rings. Semisimple Artinian rings and rings of $n \times n$ upper triangular matrices with entries in a field are serial rings.

Let R be a serial ring. Then there exists a complete set of orthogonal idempotents $\{e_1, e_2, \dots, e_n\}$ of R such that both e_iR and Re_i are uniserial modules. Moreover, all the rings e_iRe_i are chain rings. Every indecomposable projective right R-module is isomorphic to e_iR for some $i = 1, 2, \dots, n$ (although e_iR and e_jR may be isomorphic for $i \neq j$). Since serial rings are semiperfect, the Krull–Schmidt theorem holds for the class of all projective modules over a serial ring (Proposition 2.6). In particular, every projective module over a serial ring is a direct sum of indecomposables. This fact plus the fact that every indecomposable projective module is isomorphic to an e_iR imply that every projective module over a serial ring is serial. Moreover:

THEOREM 5.6 [W5]. *Every finitely presented module over a serial ring is serial.*

The next example shows that the Krull–Schmidt theorem does not hold for the class of serial modules.

EXAMPLE 5.7 [F1]. Let $n \geqslant 2$ be an integer. There exist n^2 pairwise non-isomorphic finitely presented uniserial modules $U_{i,j}$ ($i, j = 1, 2, \dots, n$) over a suitable serial ring R satisfying the following properties:

(a) for every $i, j, k, \ell = 1, 2, \dots, n$, $[U_{i,j}]_m = [U_{k,\ell}]_m$ if and only if $i = k$;

(b) for every $i, j, k, \ell = 1, 2, \dots, n$, $[U_{i,j}]_e = [U_{k,\ell}]_e$ if and only if $j = \ell$.

Thus the n^2 modules $U_{i,j}$ ($i, j = 1, 2, \dots, n$) can be represented as the entries of a square $n \times n$ matrix in such a way that two modules $U_{i,j}$ belong to the same monogeny class if and only if they are on the same row, and they belong to the same epigeny class if and only if they are on the same column. If σ and τ are any two permutations of $\{1, 2, \dots, n\}$, then $U_{1,1} \oplus U_{2,2} \oplus \cdots \oplus U_{n,n} \cong U_{\sigma(1),\tau(1)} \oplus U_{\sigma(2),\tau(2)} \oplus \cdots \oplus U_{\sigma(n),\tau(n)}$ by Theorem 5.4, and these two decompositions have the property that σ and τ are exactly the two permutations that mix the monogeny classes and the epigeny classes, respectively (compare with Theorem 5.4).

By Theorem 5.4 and Example 5.7 the Krull–Schmidt theorem does not hold for the class of finitely generated (respectively, finitely presented) uniserial modules over a serial ring.

In opposition to the general case of finitely presented modules over *serial* rings, the Krull–Schmidt theorem holds for finitely presented modules over *chain* rings, a result due to Puninski (see [F2, Theorem 9.19]):

THEOREM 5.8 (Puninski). *The Krull–Schmidt theorem holds for the class of finitely presented modules over chain rings.*

A module M_R over a ring R is said to be *small* if for every family $\{N_i \mid i \in I\}$ of R-modules and any homomorphism $\varphi : M_R \to \bigoplus_{i \in I} N_i$, there exists a finite subset F of I such that $\varphi(M) \subseteq \bigoplus_{i \in F} N_i$. For instance, finitely generated modules are small. A proof due to Fuchs and Salce shows that every uniserial module that is not countably generated is small (cf. [DF1, Lemma 4.2] and [FS1, Lemma 24].) An R-module M_R is called *quasi-small* if for any family $\{N_i \mid i \in I\}$ of R-modules such that M_R is isomorphic to a direct summand of $\bigoplus_{i \in I} N_i$, there exists a finite subset F of I such that M_R is isomorphic to a direct summand of $\bigoplus_{i \in F} N_i$. Small modules, modules with local endomorphism rings and Artinian modules are quasi-small ([DF1, p. 112] and [FH3, Corollary 3.3]).

THEOREM 5.9 [DF1, Theorem 4.7]. *Let $\{U_i \mid i \in I\}$ and $\{V_j \mid j \in J\}$ be two families of non-zero uniserial modules over an arbitrary ring R and suppose that $\bigoplus_{i \in I} U_i \cong \bigoplus_{j \in J} V_j$. Set $I' = \{i \in I \mid U_i$ is quasi-small$\}$ and $J' = \{j \in J \mid V_j$ is quasi-small$\}$. Then there is a bijection $\tau : I' \to J'$ such that $[U_i]_e = [V_{\tau(i)}]_e$ for every $i \in I'$.*

COROLLARY 5.10. *The following conditions on a ring R are equivalent:*

(a) *All uniserial right R-modules are quasi-small.*

(b) *For two arbitrary families $\{U_i \mid i \in I\}$ and $\{V_j \mid j \in J\}$ of uniserial right R-modules, $\bigoplus_{i \in I} U_i \cong \bigoplus_{j \in J} V_j$ if and only if there exist two bijections $\sigma : I \to J$ and $\tau : I \to J$ such that $[U_i]_m = [V_{\tau(i)}]_m$ and $[U_i]_e = [V_{\tau(i)}]_e$ for every $i \in I$.*

EXAMPLE 5.11. An interesting example discovered by Puninski [P3] shows that there exist rings R for which the equivalent conditions of Corollary 5.10 do not hold, that is, over which there exist uniserial modules that are not quasi-small. This solves a problem posed in [DF1, p. 111] (see also [F2, Problem 15 on p. 269]). Recall that a chain ring R is called a *nearly simple* chain ring if it has exactly three two-sided ideals, which must be necessarily the ideals $0 \subset J(R) \subset R$. For an example of a nearly simple chain domain, see [BBT, §6.5]. If R is a nearly simple chain domain and a, b are non-zero non-invertible elements of R, then the right R-modules R/aR and R/bR are always isomorphic [P3]. Hence, let a be a non-zero non-invertible element of R and $U_R = R/aR$. Every finitely presented right module over the nearly simple chain domain R is isomorphic to $R_R^n \oplus U^m$ for two uniquely determined non-negative integers n and m (Theorems 5.6 and 5.8). Puninski showed that there exists a suitable countably generated uniserial module V_R such that $V_R \oplus U_R^{(\aleph_0)} \cong U_R^{(\aleph_0)}$, where $U_R^{(\aleph_0)}$ denotes the direct sum of countably many copies of U_R. Since U_R is cyclic and V_R is not, the epigeny classes of U_R and V_R are different, so that in the two direct sum decompositions $V_R \oplus U_R^{(\aleph_0)} \cong U_R^{(\aleph_0)}$ there cannot be a one-to-one correspondence between the epigeny classes.

A suitable generalization of Theorem 5.4 to arbitrary families of uniserial modules is not known yet.

5.4. *Direct summands of serial modules*

Until now we have studied direct sum decompositions of a serial module into uniserial modules and the uniqueness of these decompositions up to isomorphism. Now a natural question is: Does every direct sum decomposition of a serial module M_R refine to a direct sum decomposition into uniserial direct summands? Equivalently, if N_R is a direct summand of a serial module M_R, is N_R also a serial module? Here are some partial answers to this question.

If M_R is a direct sum of uniserial modules each of which has a local endomorphism ring, then N_R is serial [F2, Corollary 2.54]. Thus the answer to our question is "yes" in this case. This happens, for instance, for any serial R-module when the base ring R is either commutative or right Noetherian.

The answer is "no", in general, for a serial module M_R of infinite Goldie dimension over an arbitrary ring R, as is shown by Puninski in [P4]. Puninski considers uniserial modules over prime coherent nearly simple chain rings that are not domains. Over such a ring R, he constructs a pure-projective module that is not a direct sum of indecomposable modules. This shows that not every direct summand of a serial module over a chain ring is serial, solving Problems 10 and 11 of [F2, pp. 268, 269]. An example of a prime coherent nearly simple chain ring R that is not a domain can be found in [D].

It is not known yet whether every direct summand of a serial module *of finite Goldie dimension* is serial [F2, Problem 9 on p. 268]. The best result in this direction is the following theorem, concerning the case of a direct sum of finitely many isomorphic uniserial modules.

THEOREM 5.12 [DF2, Theorem 2.7]. *If R is an arbitrary ring, U is a uniserial right R-module and $n \geqslant 0$ is an integer, then every direct summand of U^n is isomorphic to U^m for some $m \leqslant n$.*

Some generalizations of Theorem 5.12 can be found in [DF2].

6. Projective modules over semilocal rings

For the proofs of the results presented in this section, see [FH1] and [FH2].

A module A_S whose endomorphism ring is semilocal is an almost Krull–Schmidt module, that is, has only finitely many direct sum decompositions up to isomorphism (Lemma 4.3(c)). It is therefore natural to try to describe the direct sum decompositions of A_S. More generally, we shall consider the direct sum decompositions of the objects in the category $\mathrm{add}(A_S)$. By Theorem 2.4, this is equivalent to describing finitely generated projective modules over the semilocal ring $R = \mathrm{End}(A_S)$.

The set of isomorphism classes of finitely generated projective right modules over a ring R can be given the structure of a commutative monoid in the following way. For every right R-module P_R, let $\langle P_R \rangle$ denote the isomorphism class of P_R, that is, the class of all right R-modules isomorphic to P_R. Let $V(R)$ be the additive commutative monoid whose elements are the isomorphism classes $\langle P_R \rangle$ of all finitely generated projective right R-modules P_R, and whose addition is defined by $\langle P_R \rangle + \langle Q_R \rangle = \langle P_R \oplus Q_R \rangle$.

For example, if R is a semisimple Artinian ring, each finitely generated R-module is a finite direct sum of simple modules. It follows that $V(R)$ is the free commutative monoid having the isomorphism classes of simple R-modules as a free set of generators, that is, $V(R) \cong \mathbf{N}^n$, where n is the number of simple R-modules up to isomorphism.

If for every ring morphism $\varphi : R \to S$ we denote by $V(\varphi) : V(R) \to V(S)$ the monoid morphism defined by $V(\varphi)(\langle P_R \rangle) = \langle P \otimes_R S \rangle$, V becomes a functor from the category of associative rings with identity to the category of commutative monoids.

Notice that $V(R)$ describes both right projective R-modules and left projective R-modules, because the contravariant functor $\mathrm{Hom}_R(-, R)$ induces a duality between the full subcategory of $\mathrm{Mod}\, R$ whose objects are all finitely generated projective right R-modules and the full subcategory of R-Mod whose objects are all finitely generated projective left R-modules, so that if we use finitely generated projective left modules instead of finitely generated projective right modules in the construction of $V(R)$, we obtain a monoid canonically isomorphic to $V(R)$.

Two projective modules P_R and Q_R are *stably isomorphic* if $P_R \oplus R_R^n \cong Q_R \oplus R_R^n$ for some integer $n \geqslant 0$. We shall denote by $[P_R]$ the *stable isomorphism class* of P_R, that is, the class of all modules Q_R stably isomorphic to P_R. For a ring R, the *Grothendieck group* of R is the Abelian group

$$K_0(R) = \big\{[P_R] - [Q_R] \mid P_R, Q_R \text{ finitely generated projective right } R\text{-modules}\big\}.$$

Here two elements $[P_R] - [Q_R]$ and $[P'_R] - [Q'_R]$ are equal if and only if $P_R \oplus Q'_R$ and $P'_R \oplus Q_R$ are stably isomorphic. The addition is defined by $([P_R] - [Q_R]) + ([P'_R] -$

$[Q'_R]) = [P_R \oplus P'_R] - [Q_R \oplus Q'_R]$. The assignment K_0 also turns out to be a functor from the category of associative rings with identity to the category of Abelian groups. The canonical monoid morphism $V(R) \to K_0(R)$ defined by $\langle P_R \rangle \mapsto [P_R]$ yields a natural transformation of functors $V \to K_0$.

For the rest of this section we suppose R is semilocal. In this case, finitely generated projective R-modules cancel from direct sums (Example 2.11(4) and Lemma 4.3(a)), that is, stable isomorphism coincides with isomorphism, i.e. $[P_R] = \langle P_R \rangle$ for every P_R. Thus the morphism $V(R) \to K_0(R)$ is an embedding, so that we may suppose $V(R) \subseteq K_0(R)$ for R semilocal. The canonical projection $\pi : R \to R/J(R)$ induces a monoid embedding $V(\pi) : V(R) \to V(R/J(R))$, and, as we have already remarked, the semisimplicity of $R/J(R)$ implies that $V(R/J(R)) \cong \mathbf{N}^n$, where n is the number of simple $R/J(R)$-modules up to isomorphism. Thus $V(R)$ is isomorphic to a submonoid of $\mathbf{N}^n$. Similarly, the canonical projection $\pi : R \to R/J(R)$ induces an embedding of Abelian groups $K_0(\pi) : K_0(R) \to K_0(R/J(R))$ and $K_0(R/J(R)) \cong \mathbf{Z}^n$, so that $K_0(R)$ is isomorphic to a subgroup of $\mathbf{Z}^n$. There is a commutative diagram

$$\begin{array}{ccccc} V(R) & \xrightarrow{V(\pi)} & V(R/J(R)) & \cong & \mathbf{N}^n \\ \downarrow & & \downarrow & & \downarrow \\ K_0(R) & \xrightarrow{K_0(\pi)} & K_0(R/J(R)) & \cong & \mathbf{Z}^n \end{array}$$

in which all the arrows are embeddings. Hence the monoid $V(R)$ is isomorphic to its image in $\mathbf{Z}^n$, and the image of $V(R)$ in $\mathbf{Z}^n$ is the intersection of $\mathbf{N}^n$ with the image of the Abelian group $K_0(R)$ in $\mathbf{Z}^n$. In particular, $V(R)$ is isomorphic to the intersection of $\mathbf{N}^n$ with a subgroup of $\mathbf{Z}^n$ [FH1]. A submonoid M of the additive monoid $\mathbf{N}^n$ is a *full submonoid* if there exists a subgroup G of $\mathbf{Z}^n$ such that $M = G \cap \mathbf{N}^n$. Thus $V(R)$ is isomorphic to a full submonoid of $\mathbf{N}^n$ for every semilocal ring R. In this isomorphism, the element $\langle R_R \rangle$ of $V(R)$ corresponds to an n-tuple $u = (d_1, \ldots, d_n)$ of positive integers. Conversely:

THEOREM 6.1. *Let M be a full submonoid of $\mathbf{N}^n$ and let $T : M \to \mathbf{N}^n$ be the embedding. Assume that $u \in M$ is such that $T(u) = (d_1, \ldots, d_n)$ with $d_1, \ldots, d_n > 0$. Then there exist a semilocal ring R and two monoid isomorphisms $g : M \to V(R)$ and $h : \mathbf{N}^n \to V(R/J(R))$ such that if $\pi : R \to R/J(R)$ denotes the canonical projection, then the diagram of commutative monoids and monoid homomorphisms*

$$\begin{array}{ccc} M & \xrightarrow{T} & \mathbf{N}^n \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ V(R) & \xrightarrow{V(\pi)} & V(R/J(R)) \end{array} \qquad (1)$$

commutes and $g(u) = \langle R_R \rangle$.

Moreover, R can be

(a) *either a hereditary k-algebra over an arbitrary field k* [FH1, Theorem 6.1],

(b) *or the endomorphism ring of a finitely generated reflexive module M_S over a commutative Noetherian local unique factorization domain S of Krull dimension* 2 [W10, Theorem 4.1],

(c) *or the endomorphism ring of a finite rank torsion-free module over the local ring* $\mathbf{Z}_{(p)}$ *(the ring of integers* $\mathbf{Z}$ *localized at its prime ideal* (p) *for an arbitrary prime* p*)* [Y2, Theorem].

For (b), recall that if M is a module over a commutative ring S and $M^* = \mathrm{Hom}_S(M, S)$ denotes the dual module, then M is *reflexive* if the canonical mapping $M \to M^{**}$ is an isomorphism.

Theorem 6.1 describes all possible direct sum decompositions of Artinian modules, of finitely generated modules over commutative Noetherian semilocal rings, and of finite rank torsion-free $\mathbf{Z}_{(p)}$-modules. For some examples that show how the method expounded in this section allows us to solve the problem of the existence of modules with particular direct sum decompositions whenever we are considering finitely generated modules over commutative Noetherian semilocal rings, or Artinian modules over arbitrary rings, or projective modules over semilocal rings, etc., see the examples at the end of Section 2 of [F4].

THEOREM 6.2 [F3, Theorem 4.4]. *The following conditions are equivalent for a semilocal ring* R:

(a) *The Krull–Schmidt theorem holds for the class* proj-R *of all finitely generated projective right* R*-modules.*

(b) $V(R) \cong \mathbf{N}^t$ *for some* $t \geqslant 1$.

(c) *There exists a direct sum decomposition* $R_R = I_1 \oplus \cdots \oplus I_m$ *such that if* $\{I_1, \ldots, I_t\}$ *is a complete irredundant set of representatives of the isomorphism classes of* $I_1, \ldots, I_m$, *then every finitely generated projective right* R*-module is isomorphic to a direct sum of copies of* $I_1, \ldots, I_t$ *in a unique way.*

Notice that the condition "there exists a direct sum decomposition $R_R = I_1 \oplus \cdots \oplus I_m$ such that every finitely generated projective right R-module is isomorphic to a direct sum of the I_j's" is strictly weaker than condition (c); see [F3].

7. Homogeneous semilocal rings, modules whose endomorphism ring is homogeneous semilocal

In this section we shall present some results that appear in [CF] and [BFRR]. A ring R is a *homogeneous semilocal* ring if $R/J(R)$ is simple Artinian, that is, if $R/J(R)$ is isomorphic to the ring $M_n(D)$ of $n \times n$ matrices with entries in some division ring D for some positive integer n. Equivalently, a ring R is homogeneous semilocal if and only if there exists a finite set $\{I_1, I_2, \ldots, I_n\}$ of maximal right ideals of R such that $J(R) = I_1 \cap I_2 \cap \cdots \cap I_n$ and $R/I_j \cong R/I_k$ for every $j, k = 1, 2, \ldots, n$.

EXAMPLES 7.1.

(1) Every local ring is homogeneous semilocal. More generally, if R is a local ring and t is a non-negative integer, the ring $M_t(R)$ of $t \times t$ matrices with entries in R is a homogeneous semilocal ring.

(2) If a ring R is commutative, then R is homogeneous semilocal if and only if it is local.
(3) Homomorphic images of homogeneous semilocal rings are homogeneous semilocal rings.
(4) If R is a right Noetherian ring and P is a right localizable prime ideal of R, the localization of R with respect to P is a homogeneous semilocal ring.
(5) For further examples of homogeneous semilocal rings, see [CF, §§4 and 5].

A number of properties of local rings extend to homogeneous semilocal rings [CF, BFRR], and it is natural to ask whether the Krull–Schmidt theorem, which holds for direct sums of modules whose endomorphism rings are local, holds also for direct sums of modules whose endomorphism rings are homogeneous semilocal. The next theorem considers this case.

THEOREM 7.2 (Krull–Schmidt theorem for direct sums of modules with homogeneous semilocal endomorphism rings, [BFRR]). *Let $\overline{M}_R = M_1 \oplus \cdots \oplus M_t = N_1 \oplus \cdots \oplus N_s$ be two direct sum decompositions of a module $\overline{M}_R$ over a ring R. If all the R-modules M_i and N_j are indecomposable and all their endomorphism rings* $\mathrm{End}(M_i)$ *and* $\mathrm{End}(N_j)$ *are homogeneous semilocal, then the two direct sum decompositions are isomorphic.*

Notice that in order to get the uniqueness of the decompositions, it does not suffice that in the decompositions $\overline{M}_R = M_1 \oplus \cdots \oplus M_t = N_1 \oplus \cdots \oplus N_s$ only *one* decomposition $M_1 \oplus \cdots \oplus M_t$ is into indecomposable direct summands M_i whose endomorphism rings $\mathrm{End}(M_i)$ are homogeneous semilocal. In fact, for any pair of integers $t > 1$ and $s > 1$, there exists a module $\overline{M}_R$ over a suitable ring R with the property that $\overline{M}_R$ has two non-isomorphic direct sum decompositions into indecomposable direct summands $\overline{M}_R = M_1 \oplus \cdots \oplus M_t \cong N^s$ and such that the endomorphism rings $\mathrm{End}(M_i)$ are homogeneous semilocal rings for every $i = 1, \ldots, t$ ([FH2, Lemma 5.5] and [BFRR, Example 3.4]).

We conclude this section with a generalization of Theorem 7.2 to arbitrary, possibly infinite, direct sums. For other generalizations of the theorem to infinite direct sums, for instance to indecomposable Artinian direct summands, see [FH3].

THEOREM 7.3 [FH3, Corollary 3.5]. *Let $\overline{M}_R = \bigoplus_{i \in I} M_i = \bigoplus_{j \in J} N_j$ be two direct sum decompositions of a module $\overline{M}_R$ into indecomposable direct summands such that all the endomorphism rings* $\mathrm{End}(M_i)$ *and* $\mathrm{End}(N_j)$ *are homogeneous semilocal. Suppose that each module M_i and each module N_j is either small, or quasi-small and countably generated. Then the two direct sum decompositions of $\overline{M}_R$ are isomorphic.*

8. Module-finite algebras in Krull dimension 1

In this section we shall present some results obtained recently by Levy and Odenthal [LO1]. Recall that a ring is *semiprime* if it has no non-zero nilpotent ideal. In Sections 8 and 9, the symbol R will denote a semiprime module-finite k-algebra, where k is a commutative Noetherian ring of Krull dimension 1. Without loss of generality, we may assume that k is contained in the center $Z(R)$, and hence k has no non-zero nilpotent elements, because any such element would generate a nilpotent ideal of R. Thus R can be any of the orders studied in integral representation theory. Without loss of generality, we shall assume that R is indecomposable as a ring; and we shall assume that R is not Artinian, since the Krull–Schmidt theorem holds for finitely generated modules over Artinian rings (Theorem 2.1).

Let Q denote the finite set of minimal prime ideals of k. Since k has no nilpotent elements, $k - \bigcup Q$ is the set of regular elements of k, that is, elements that are not zero-divisors. Let k_Q and R_Q denote the localizations of k and R respectively formed by inverting all of the elements of $k - \bigcup Q$. The ring k_Q is Artinian (because all prime ideals are maximal), has no nilpotent element, and therefore it is a direct product of fields.

Since k_Q is an Artinian ring and R_Q is module-finite over k_Q, R_Q is also an Artinian ring. Moreover, since R is semiprime, the localization R_Q is a semisimple Artinian ring. In fact, R is a *k-order in the semisimple Artinian ring R_Q* in the sense that the natural map $R \to R_Q$ is one-to-one. (The simple proof of this last fact uses the hypotheses that R is indecomposable and not Artinian. See the introduction to [LO1] for details.)

In this situation, Levy and Odenthal define a *normalization* R' of R (in R_Q) to be a maximal element of the family of subrings of R_Q that contain R and are integral over k. In the classical situation, where R is contained in a maximal k-order in R_Q, the normalization R' is easily seen to be such a maximal order. And when R is commutative, R' becomes a normalization of R in the sense of commutative ring theory; that is, the integral closure in the total quotient ring. In all cases, R' turns out to be a direct product of classical maximal orders over Dedekind domains in simple Artinian rings, and thus provides a link between Levy and Odenthal's theory and classical integral representation theory [LO2, §4].

Two finitely generated R-modules A, B *are in the same genus* if $A_M \cong B_M$ (as R_M-modules) for every maximal ideal M of k. This property is independent of the particular ring k over which R is a module-finite algebra.

We shall denote by FG-R the class of all finitely generated right R-modules, and by FGTF-R the class of all finitely generated torsion-free right R-modules, that is, the A_R for which the natural map $A \to A \otimes_k Q$ is injective. In the language of integral representation theory (when R is a classical k-order) every finitely generated torsion-free right R-module is an R-lattice. The Krull–Schmidt theorem holds for FG-R and FGTF-R only in the presence of extreme restrictions on the structure of R, as the next two results show.

THEOREM 8.1 [LO1, Theorem 1.1]. *The Krull–Schmidt theorem holds for* FG-R *if and only if the following three conditions hold.*

(a) *Every genus of finitely generated right R-modules consists of a single isomorphism class.*

(b) *Either $R = R'$ or exactly one maximal ideal N of $Z(R)$ is* singular *with respect to R; that is, $R_N \neq R'_N$ (localizations of R, R').*

(c) *If this singular N occurs, then primitive idempotents of R'_N remain primitive modulo $J(R'_N)$.*

Condition (c) is equivalent to the statement that the Krull–Schmidt theorem holds for FG-R_N, as one can see by applying the theorem to the k_N-algebra R_N, and noting that $Z(R_N) = Z(R)_N$.

THEOREM 8.2 [LO1, Theorem 1.3]. *The Krull–Schmidt theorem holds for* FGTF-R *if and only if the following conditions are satisfied.*

(a) *Every genus of finitely generated torsion-free R-modules consists of a single isomorphism class.*
(b) *Either $R = R'$ or exactly one maximal ideal N of $Z(R)$ is singular with respect to R (i.e. $R_N \neq R'_N$).*
(c) *If this singular N occurs, then either condition* (c1) *or* (c2) *holds*:
    (c1) *Primitive idempotents of R'_N remain primitive modulo $J(R'_N)$; that is, the Krull–Schmidt theorem holds for* FG-R_N,
    (c2) *R_N is a member of a very small class of exceptional rings. [See Remark* (2) *below.*]

REMARKS.

(1) [on condition (a)] It is possible for every genus of torsion-free R-modules to consist of a single isomorphism class without the same being true for genera of modules that are not torsion-free. An example is given in [KL]. In this example the Krull–Schmidt theorem holds for FGTF-R and fails for FG-R; in fact, direct-sum cancellation fails for modules that are not torsion-free.
(2) [on condition (c2)] It is interesting that these exceptional rings are "classical" in the sense that R'_N is module-finite over R_N. Moreover, R'_N is a prime ring, and has at most two maximal 2-sided ideals. See [LO1] for full details.

We remark that, when k is semilocal, the notions of genus and isomorphism class coincide. This has a number of interesting consequences. The most obvious of these is:

THEOREM 8.3. *Suppose that k is semilocal. Then the Krull–Schmidt theorem holds for the class of finitely generated right R-modules if and only if it holds for the class of finitely generated left R-modules. The same is true for the classes of finitely generated torsion-free right and left R-modules.*

It seems to be unknown whether the "semilocal" hypothesis can be dropped, because there seems to be no criterion that decides whether every genus consists of a single isomorphism class.

An often-asked question is whether indecomposable modules have local endomorphism rings whenever the Krull–Schmidt theorem holds for FG-R. This is never asked in print, of course, since the ring of integers gives an immediate counterexample; and the 6-localization of this ring is even a semilocal counterexample. These examples seem to have discouraged looking at this question seriously. In fact, a more subtle version of the question has a positive answer, after one ignores the trouble caused by maximal orders:

COROLLARY 8.4 [LO1, Corollary 2.11]. *Suppose that the Krull–Schmidt theorem holds for* FG-R, *and $R \neq R'$. Let N be the singular maximal ideal of $Z(R)$ with respect to R.*

Then the endomorphism ring of the N-localization of every indecomposable R-module is a local ring.

The analogous result for the class FGTF-R fails, because of the exceptional rings in Theorem 8.2. See [LO1, Remark 2.12(iv)].

A well-known theorem of A. Heller (implicit in the proof of [H, 2.5]) states that if k is a discrete valuation ring of characteristic zero and R is contained in a maximal order, then condition (a) of the following corollary implies that the Krull–Schmidt theorem holds for FG-R. The characteristic 0 hypothesis was removed in [CR, (30.18)], but only for torsion-free R-modules. The following corollary removes these special hypotheses, and condition (b) provides a simple explanation of condition (a) in terms of the structure of the ring R'.

COROLLARY 8.5 [LO1, Corollary 2.5]. *The following statements are equivalent, if k is semilocal.*

(a) *The $J(k)$-induced completion $\hat{k} \otimes_k V$ of every simple R_Q-module V is an indecomposable $\widehat{R}_Q$-module.*

(b) *Primitive idempotents of R' remain primitive modulo $J(R')$.*

When the conditions hold, the Krull–Schmidt theorem holds for FG-R *and $Z(R)$ is a direct product of local rings.*

Again the analogous result for the class FGTF-R fails, because of the exceptional rings [LO1, Remarks 2.12].

9. Integral group rings

It may be surprising that the Krull–Schmidt theorem holds for FGTF-$\mathbf{Z}G$ (and hence for FG-$\mathbf{Z}G$) for only finitely many integral group rings $\mathbf{Z}G$ of finite groups G. In more detail, Hindman, Klingler, and Odenthal [HKO] – building upon earlier work of Swan, Wiegand, and others – prove:

THEOREM 9.1. *If G is not dihedral of order* 16 *(the only undetermined case), then the Krull–Schmidt theorem holds for* FGTF-$\mathbf{Z}G$ *exactly when G is one of the following groups:*

(a) *Cyclic of prime order $p \leqslant 19$.*

(b) *Cyclic of order* 1, 4, 8, *or* 9.

(c) *Klein* 4-*group.*

(d) *Dihedral of order* 8.

This is the main result of [HKO, 1.6], except that their condition (a) can be simplified to our present statement, in view of the following two facts. (1) Let $m \not\equiv 2 \pmod 4$, and let ω_m be a primitive m-th root of unity. Then the class number of $\mathbf{Q}(\omega_m)$ is 1 if and only if m is one of the following: 1, 3, 4, 5, 7, 8, 9, 11, 12, 13, 15, 16, 17, 19, 20, 21, 24, 25, 27, 28, 32, 33, 35, 36, 40, 44, 45, 48, 60, 84. See, e.g. [W8, Theorem 11.1] (only the case $m = p$ is needed); and (2) the Krull–Schmidt theorem holds for FGTF-$\mathbf{Z}G$ when G is cyclic of order 2. See the last paragraph of [HKO, p. 3247] for this.

It seems to be unknown for which of these groups G the Krull–Schmidt theorem holds for FG-$\mathbf{Z}G$. The critical property is whether every genus of finitely generated R-modules – not just the torsion-free ones – consists of a single isomorphism class. (See Theorems 8.1 and 8.2.)

The Krull–Schmidt theorem holds for the finitely generated modules over the rings of Theorem 9.1(a). This follows from the following two facts: (1) $\mathrm{Pic}(R)$ is trivial (has order 1) since the Krull–Schmidt theorem holds for FGTF-R, because Theorem 8.2 states that every genus of torsion-free modules consists of a single isomorphism class, and (2) these rings are, by [L5], special cases of the Dedekind-like rings studied in [L4]; and for the Dedekind-like rings studied in [L4], every genus class group is a homomorphic image of $\mathrm{Pic}(R)$.

Appendix A. The Krull–Schmidt theorem for additive categories

In this appendix we mention a generalization of the Krull–Schmidt theorem to additive categories. An additive category $\mathcal{A}$ with kernels is said to satisfy a *weak Grothendieck condition* if for every index set I and every non-zero subobject A of $\coprod_{i\in I} C_i$, there exists a finite subset F of I such that $A \cap \coprod_{i\in F} C_i \neq 0$. A *small* object is an object S such that every morphism into a coproduct $f: S \to \coprod_{i\in I} C_i$ factors though $\coprod_{i\in F} C_i$ for some finite subset F of I. An object M is *finitely approximable* if for any object L and morphism $f: L \to M$, f is an isomorphism if and only if (1) f is monic, and (2) for any small object S and any morphism $g: S \to M$ there is a morphism $h: S \to L$ such that $g = fh$.

The Krull–Schmidt–Azumaya theorem generalizes to additive categories as follows.

THEOREM A.1 [B2,WW1,AHR]. *Let $\mathcal{A}$ be an additive category in which idempotent endomorphisms have kernels. Let C_i, $i \in I$, be objects of $\mathcal{A}$ with local endomorphism rings. If I is infinite, suppose that $\mathcal{A}$ has kernels and infinite coproducts and that either* (1) *the C_i are finitely approximable or* (2) *the category $\mathcal{A}$ satisfies a weak Grothendieck condition. Then any two coproduct decompositions of $\coprod_{i\in I} C_i$ into indecomposables are isomorphic.*

The finite case of this theorem is due to Bass [B2], and the infinite case to Walker and Warfield [WW1] (see [AHR]). For the details, see [WW1] and [AHR]. For an analogue of Theorem A.1 in the case where the endomorphism rings of the C_i are principal ideal domains, see [AHR].

In this setting we must mention *spectral* categories, that is, Abelian categories in which every object is injective. In these categories the theory of decompositions is particularly good [GO1,GB,P2,R5].

Appendix B. Goldie dimension of modular lattices

In this chapter we have heavily used the notion of dual Goldie dimension of a module. Since the knowledge of this notion is not so widespread among algebraists, we recall its definition in this appendix. The correct framework for the notions of Goldie dimension and dual Goldie dimension is that of modular lattices with a smallest element 0 and a greatest element 1. We shall briefly review how it is possible to define the Goldie dimension in this setting.

Let L be a modular lattice with a smallest element 0 and a greatest element 1. A finite subset $\{a_i \mid i \in I\}$ of $L \setminus \{0\}$ is said to be *join-independent* if $a_i \wedge (\bigvee_{j \neq i} a_j) = 0$ for every $i \in I$. An arbitrary subset A of $L \setminus \{0\}$ is *join-independent* if all finite subsets of A are join-independent. An element a of L is *uniform* if $a \neq 0$ and for every $x, y \in L$, $x \leqslant a$, $y \leqslant a$ and $x \wedge y = 0$ imply $x = 0$ or $y = 0$. An element $a \in L$ is *essential* in L if $a \wedge x \neq 0$ for every non-zero element $x \in L$. If a, b are elements of L, $a \leqslant b$ and a is essential in the interval $[0, b] = \{x \in L \mid x \leqslant b\}$, then a is said to be *essential* in b.

THEOREM B.1 [GP]. *Let L be a modular lattice with 0 and 1. The following conditions are equivalent*:

(a) *L does not contain infinite join-independent subsets.*

(b) *There exists a finite join-independent subset $\{a_1, a_2, \ldots, a_n\}$ of L with $a_1 \vee a_2 \vee \cdots \vee a_n$ essential in L and $a_1, a_2, \ldots, a_n$ uniform elements.*

(c) *The cardinality of all join-independent subsets of L is $\leqslant m$ for some non-negative integer m.*

(d) *For every ascending chain $a_0 \leqslant a_1 \leqslant a_2 \leqslant \cdots$ of elements of L there exists an index $i \geqslant 0$ with a_i essential in a_j for every $j \geqslant i$.*

If these equivalent conditions hold and $\{a_1, a_2, \ldots, a_n\}$ is a finite join-independent subset of L with $a_1 \vee a_2 \vee \cdots \vee a_n$ essential in L and $a_1, a_2, \ldots, a_n$ uniform elements, then any other join-independent subset of L has cardinality $\leqslant n$.

For a modular lattice L satisfying the equivalent conditions of the theorem, the integer n of the statement, that is, the greatest element of the set of the cardinalities of all join-independent subsets of $L \setminus \{0\}$, is called the *Goldie dimension* of the lattice L, denoted $\dim L$. If L does not satisfy the equivalent conditions of Theorem B.1, that is, if L contains an infinite join-independent subset, the lattice L is said to have *infinite Goldie dimension*.

The *Goldie dimension* $\dim(M_R)$ of a module M_R is the Goldie dimension of its lattice $\mathcal{L}(M_R)$ of submodules. As the dual (= opposite) L^{op} of any modular lattice L is a modular lattice, the *dual Goldie dimension* $\operatorname{codim}(L) = \dim(L^{\mathrm{op}})$ of a modular lattice L is always defined [GP] (it is possibly infinite). In particular, the *dual Goldie dimension* $\operatorname{codim}(M_R)$ of a module M_R is defined as the Goldie dimension of the dual lattice $\mathcal{L}(M_R)^{\mathrm{op}}$. Thus a module M has finite dual Goldie dimension $\operatorname{codim}(M_R) = n$ if and only if there is a coindependent set $\{N_1, N_2, \ldots, N_n\}$ of proper submodules of M with $N_1 \cap N_2 \cap \cdots \cap N_n$ superfluous in M and M/N_i couniform for $i = 1, 2, \ldots, n$.

An injective module has finite Goldie dimension n if and only if it is the direct sum of n indecomposable modules. More generally, let $E(M_R)$ denote the injective envelope of a module M_R. If $E(M_R)$ is the direct sum of n indecomposable modules $E_1, E_2, \ldots, E_n$, the number n depends only on M_R by Theorem 2.2 and Proposition 2.7 and coincides with the Goldie dimension of M_R. Otherwise, that is, if $E(M_R)$ is not a direct sum of finitely many indecomposable modules, then M_R has infinite Goldie dimension.

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