On the Vietoris–Rips Complexes and a Cohomology Theory for Metric Spaces

Jean-Claude Hausmann

1 Introduction

Let X be a pseudo-metric space with pseudo-distance d (i.e. d satisfies the axioms of a distance except that $d(x,y)=0$ does not imply that $x=y$). Let ε be a positive real number. We denote by X_ε the abstract simplicial complex defined as follows: the vertices of X_ε are the points of X and a q-simplex of X_ε is a subset $\{x_0,\dots,x_q\}$ of X such that $\mathrm{Diam}\,(\{x_0,\dots,x_q\})<\varepsilon$. Recall that if $P\subset X$, the diameter $\mathrm{Diam}\,P$ of P is the least upper bound of $d(x,y)$ for all $x,y\in P$.

Examples.

1. Let $X:=\{e^{\frac{2ik\pi}{n}}\mid k=0,1,\dots,n\}$, the set of n^{th} roots of 1 with the geodesic distance on the circle S^1. If $\varepsilon\le 2\pi/n$, then X_ε is a discrete set with n points. If $2\pi/n<\varepsilon\le 4\pi/n$ and $n\ge 4$, X_ε is a polygon with n sides. One checks easily that if $4\pi/n<\varepsilon\le 6\pi/n$ and $n\ge 6$ then X_ε is a combinatorial surface (with boundary if $n\ge 7$). By computing the Euler characteristic and the number of connected components of the boundary, one sees that this surface is

   - the sphere S^2 (an octaedron) if $n=6$
   - the band $S^1\times[0,1]$ if $n\ge 8$ is even and
   - the Möbius band if $n\ge 7$ is odd.

   The first case generalizes as follows: if $n=2m$ and $(m-1)\pi/m<\varepsilon<\pi$ then X_ε is a triangulation of the sphere S^{m-1}.

2. Let X be the set of vertices of a cube of edge 1 in $\mathbb{R}^3$. If $\sqrt{2}<\varepsilon\le\sqrt{3}$, then X_ε is a triangulation of the sphere S^3.

The complex X_ε was once called the *Vietoris complex* ([Le], p. 271). Since its re-introduction by E. Rips for studying hyperbolic groups, it has been popularized under the name of *Rips complex* ([Gr], § 1.7 and 2.2 or [G–H], p. 68).

This paper is devoted to the study of X_ε, principally when ε is small (in contrast with the framework of Rips). In § 2, we establish a few general properties of the complexes X_ε, the most interesting being that $|X_\varepsilon|$ is homotopy equivalent to $|\widehat{X}_\varepsilon|$ if $\widehat{X}$ is the metric completion or X. In § 3, we prove theorems concerning the ε-complex of a Riemannian manifold (or of a geodesic space). The main consequence of these results is that for M a compact Riemannian manifold and ε small enough, the geometric realization of M_ε is homotopy equivalent to M.

In § 4, we introduce the *metric cohomology* $\mathcal{H}^*(X;\mathcal{R})$ of a pseudo-metric space X in a coefficient ring $\mathcal{R}$. The group $\mathcal{H}^*(X;\mathcal{R})$ is defined as the inductive limit of the simplicial cohomology $H^*(X_\varepsilon;\mathcal{R})$ when $\varepsilon \to 0$. We show that this produces a cohomology theory for the category of pseudo-metric spaces and uniformly continuous maps (with a "metric version" of the excision axiom and Mayer–Vietoris sequences). In § 5, a short direct proof is given that if X is a compact metric space, then $\mathcal{H}^*(X;\mathcal{R})$ is canonically isomorphic to the Cech cohomology $\check{H}^*(X;\mathcal{R})$. We finish by other properties of the metric cohomology which are quite different than those of previously known cohomology theories: for instance, the metric cohomology of X is isomorphic to that of its metric completion. Also, a contractible metric space might not be acyclic for the metric cohomology.

The corresponding construction for homology ($\mathcal{H}_*(X)$ being the projective limit of $= H_*(X_\varepsilon)$) was the one originally considered by L. Vietoris [Vi] (for compact metric spaces) as one of the first attempt to define homology groups for spaces which were not triangulated. Thus, "Vietoris cohomology" could have been an appropriate denomination for our cohomology. We think however that "metric cohomology" is less confusing, since the definition of Vietoris homology has been subsequently modified in the literature ([Be] and [Do]) in such a way to become the homology counterpart of the Alexander–Spanier cohomology (which is not isomorphic to the metric cohomology; see § 5).

In general, little is known about the ε-complex of a metric space, even for simple spaces like the spheres. Interesting questions seem to arise from the results and techniques of this paper (see, for instance, § 3). We would be happy if these techniques may be of some use in problems involving metrics or pseudo-metrics (approximation by discrete or finite subsets, controlled topology, fractals, image analysis, etc).

I was helped by M. San Millan who worked out some details of § 4 and 5 in her "mémoire de diplôme". I wish also to thank P.Y. Gaillard, J. Roe and H. Munkholm for useful remarks.

2 Some properties of X_ε

(2.1) Lemma. *Let K be a finite pseudo-metric space. Let $\varepsilon > 0$. Then there exists $\delta > 0$ such that* id_K *induces a simplicial isomorphism from* $K_{\varepsilon-2\delta} \xrightarrow{\simeq} K_\varepsilon$.

Proof. Let

$$m = \max\{d(x,y) \mid x,y \in K \text{ and } d(x,y) < \varepsilon\}.$$

As K is finite, there exist $\delta > 0$ such that $m < \varepsilon - 2\delta$. The simplicial complexes $K_{\varepsilon-2\delta}$ and K_ε have thus the same 1- skeleton. But for any pseudo-metric space X, the complex X_ε is determined by its 1- skeleton ($\{x_0, \dots, x_q\}$ is a q-simplex of X_ε if and only if $\{x_i, x_j\}_{i<j}$ are 1-simplexes of X_ε). □

Definition. Let X be a pseudo-metric spaces and A a subspace of X. A continuous map $F : X \times [0,1] \longrightarrow X$ satisfying

a) $F(x,1) = x$, $F(x,0) \in A$, $F(a,t) = a$ if $a \in A$

b) $d(F(x,t'), F(y,t')) \leq d(F(x,t), F(y,t))$ whenever $t' \leq t$

is called a *crushing* from X onto A. We call a pseudo-metric space X *crushable* if there is a crushing from X onto one of its point.

Remarks. 1) A crushing is a retraction by deformation and crushable spaces are contractible. The converse is not true, as will be seen in § 3.

2) Let V be a real normed vector space. Then the map $(x,t) \mapsto t \cdot x$ is a crushing of V onto $\{0\}$. Trees (metric or real, see Chapter 2 of [G–H] for definitions) are also crushable.

In the statement below, if P is a simplicial complex, we denote as usual by $|P|$ its geometric realization.

(2.2) Proposition. *Let X be a pseudo-metric space admitting a crushing onto a subspace $A \subset X$. Then, for any $\varepsilon > 0$, the map $|A_\varepsilon| \longrightarrow |X_\varepsilon|$ induced by the inclusion is a homotopy equivalence.*

Proof. Let $g : (|K|, |L|) \longrightarrow (|X_\varepsilon|, |A_\varepsilon|)$ be a continuous map, where K is a finite simplicial complex and L a subcomplex of K. To prove Proposition (2.2), it is enough to show that such a map g is homotopic (rel $|L|$) to a map sending $|K|$ into $|A_\varepsilon|$.

By simplicial approximation we may, after replacing K and L by one of their barycentric subdivisions, suppose that $g = |h|$ for a simplicial map $h : (K, L) \longrightarrow (X_\varepsilon, A_\varepsilon)$. The sets $\overline{K} := h(K^0)$ and $\overline{L} := h(L^0)$ are finite subsets of X and A and h factors through the pair $(\overline{K}_\varepsilon, \overline{L}_\varepsilon)$. By (2.1), there is $\delta > 0$ such that $(\overline{K}_{\varepsilon-2\delta}, \overline{L}_{\varepsilon-2\delta}) = (\overline{K}_\varepsilon, \overline{L}_\varepsilon)$. Therefore, one has a factorization

$$(K, L) \xrightarrow{\overline{h}} (\overline{K}_{\varepsilon-2\delta}, \overline{L}_{\varepsilon-2\delta}) \xrightarrow{j} (X_\varepsilon, A_\varepsilon)$$

Let $F : X \times [0,1] \longrightarrow X$ be a crushing of X onto A. Chose an integer p so that for all $x \in \overline{K} - \overline{L}$ and all $0 \leq k < p$ one has

$$d(F(x, \frac{k}{p}), F(x, \frac{k+1}{p}) < \delta$$

The map $x \mapsto F(x, \frac{k+1}{p})$ induces then a simplicial map

$$f^k : \overline{K}_{\varepsilon-2\delta} \longrightarrow X_\varepsilon$$

such that $f^p|\overline{L}_{\varepsilon-2\delta} = j$ and $f^0(\overline{L}_{\varepsilon-2\delta}) \subset A_\varepsilon$. One has

$$\begin{aligned} &\mathrm{Diam}\,(\{x_0,\dots,x_q\}) < \varepsilon - 2\delta \\ &\qquad\qquad \Longrightarrow \mathrm{Diam}\,(\{f^k(x_0),\dots,f^k(x_q),f^{k+1}(x_0),\dots,f^{k+1}(x_q)\}) < \varepsilon \end{aligned}$$

which shows f^k and f^{k+1} are contiguous (see [Sp], Chapter 3 § 5). Therefore, $f^p = j$ and f^0 are in the same contiguity class (rel $\overline{L}_{\varepsilon-2\delta}$). This implies that the induces maps $|j|$ and $|f^0|$ are homotopic (rel $|L_{\varepsilon-2\delta}|$) ([Sp], Lemma 2 p. 130). By composing with $\overline{h}$, this proves that $g = |h|$ is homotopic (rel $|L|$) to $|\overline{h}|\circ|f^0|$ which sends $|K|$ into $|A_\varepsilon|$. □

(2.3) Corollary. *Let X be a crushable pseudo-metric space. Then, for any $\varepsilon > 0$, the space $|X_\varepsilon|$ is contractible.*

(2.4) Proposition. *Let X be a pseudo-metric space and Y a dense subset of X. Then, for any $\varepsilon > 0$, the map $|Y_\varepsilon| \longrightarrow |X_\varepsilon|$ induced by the inclusion is a homotopy equivalence.*

Proof. As in the proof of (2.2), let us consider a continuous map $g\colon(|K|,|L|) \longrightarrow (|X_\varepsilon|,|Y_\varepsilon|)$, where K is a finite simplicial complex and L a subcomplex of K and try to prove that g is homotopic (rel $|L|$) to a map sending $|K|$ into $|Y_\varepsilon|$. As in the proof of (2.2), we reduce to the case where g is induced by a simplicial map $h : (K,L) \longrightarrow (X_\varepsilon, Y_\varepsilon)$ with a factorization

$$(K,L) \xrightarrow{\overline{h}} (\overline{K}_{\varepsilon-2\delta}, \overline{L}_{\varepsilon-2\delta}) \xrightarrow{j} (X_\varepsilon, Y_\varepsilon)$$

where $\overline{K} := h(K^0)$ and $\overline{L} := h(L^0)$ are finite subsets of X and Y and $0 < 2\delta < \varepsilon$. For each $u \in \overline{K}$ chose $u' \in Y$ such that $d(u,u') < \delta$ and $u' = u$ if $u \in Y$. The correspondance $u \mapsto u'$ gives a simplicial map $j' : \overline{K}_{\varepsilon-2\delta} \longrightarrow Y_\varepsilon \subset X_\varepsilon$ which is contiguous (rel $\overline{L}_{\varepsilon-2\delta}$) to j. Therefore, $g = |h| = |\overline{j}|\circ|\overline{h}|$ is homotopic (rel $|L|$) to the map $|\overline{j}|\circ|\overline{h}|$ which sends $|K|$ into $|Y|_\varepsilon$. □

(2.5) Corollary. *Let $j : X \longrightarrow \widehat{X}$ be the inclusion of a metric space into a metric complection $\widehat{X}$ of X. Then, for any $\varepsilon > 0$, the map $|j| : |X_\varepsilon| \longrightarrow |\widehat{X}_\varepsilon|$ is a homotopy equivalence.*

3 The ε-complex of a Riemannian manifold

Let M be a Riemannian manifold. We consider M as equipped with the distance $d(x,y) = \inf\{\mathrm{length}(\gamma)\}$, where γ runs among all smooth curves joining x to y. Define $r(M) \in \mathbb{R}_{\geq 0}$ as the least upper bound of the set of real numbers r satisfying the following three conditions:

a) for all $x, y \in M$ such that $d(x,y) < 2r$ there is a unique shortest geodesic joining x to y. Its length is $d(x,y)$.

b) let x, y, z, u with $d(x, y) < r$, $d(u, x) < r$, $d(u, y) < r$ and z be a point on the shortest geodesic joining x to y. Then $d(u, z) \leq \max\{d(u, x), d(u, y)\}$.

c) if γ and γ' are arc-length parametrized geodesics such that $\gamma(0) = \gamma'(0)$ and if $0 \leq s, s' < r$ and $0 \leq t \leq 1$, then $d(\gamma(ts), \gamma'(ts')) \leq d(\gamma(s), \gamma'(s'))$.

Remarks. 1) It is well known that $r(M) > 0$ if M admits a strictly positive injectivity radius and an upper bound on its sectional curvature. In particular, a compact Riemannian manifold M has $r(M) > 0$. For example, the sphere S^n of radius 1 has $r(S^n) = \pi/2$.

2) Conditions a), b) and c) above make sense for a geodesic space (see [G–H], p. 16) and all the theorems of this section which are stated for Riemannian manifolds with $r(M) > 0$ are true for a geodesic space X with $r(X) > 0$.

Chose a total ordering on the points of M. From now on, whenever we describe a finite subset of M by $\{x_0, \ldots, x_q\}$, we suppose that $x_0 < x_1 < \ldots < x_q$. Suppose that $r(M) > 0$ and let $\varepsilon < r(M)$. we shall associate to each q-simplex $\sigma := \{x_0, \ldots, x_q\}$ of M_ε a singular q-simplex $T_\sigma : \Delta^q \longrightarrow M$. Here, Δ^q denotes the standard euclidean q-simplex:

$$\Delta^q := \left\{ \sum_{i=0}^{q} t_i e_i \mid t_i \in [0,1] \text{ and } \sum t_i = 1 \right\} \tag{3.1}$$

where $e_0, \ldots, e_q$ are the standard basis vectors of $\mathbb{R}^{q+1}$. The map T_σ is defined inductively as follows: set $T(e_0) = x_0$. Suppose that $T_\sigma(z)$ is defined for $y = \sum_{i=0}^{p-1} \tau_i e_i$. Let $z := \sum_{i=0}^{p} t_i e_i$. If $t_p = 1$, we pose $T_\sigma(z) = x_p$. Otherwise, let

$$x := T_\sigma \left(\frac{1}{1 - t_p} \sum_{i=0}^{p-1} t_i e_i \right)$$

We define $T_\sigma(z)$ as the point on the shortest geodesic joining x to x_p with $d(x, T_\sigma(z)) = t_p \cdot d(x, x_p)$ (the shortest geodesic exists by Condition b) above). To sum up, T_σ is defined inductively on Δ^p for $p \leq q$ as the "geodesic join" of $T_\sigma(\Delta^{p-1})$ with x_p.

If σ' is a face of σ of dimension p we form the euclidean sub-p-simplex Δ' of Δ^q formed by the points $\sum_{i=0}^{q} t_i e_i \in \Delta^q$ with $t_i = 0$ if $x_i \notin \sigma'$. One checks by induction on $\dim \sigma'$ that

$$T_{\sigma'} = T_\sigma | \Delta' \tag{3.2}$$

(3.3) Proposition. *Let M be a Riemannian manifold with $r(M) > 0$. Let $0 < \varepsilon' \leq \varepsilon \leq r(M)$. Then the inclusion $M_{\varepsilon'} \subset M_\varepsilon$ induces an isomorphism on homology.*

Proof. Let $\sigma = \{x_0, \ldots, x_q\}$ be a simplex of M_ε and let $I_\sigma = \text{Image}(T_\sigma)$. If σ' is a face of σ then $I_{\sigma'} \subset I_\sigma$ by (3.2) and thus $(I_{\sigma'})_\delta$ is a subcomplex of

$(I_\sigma)_\delta$ for all $\delta > 0$. On the other hand, $(I_\sigma)_\delta$ is acyclic for all $\delta > 0$. Indeed, I_σ is contained in the geodesic ball $B := B(x_q, r(M))$. By Condition c) of the definition of $r(M)$ the obvious deformation retraction of B on x_q along the geodesics satisfies the hypothesis of (0.179) and I_σ is invariant under this deformation. These considerations show that for $0 < \delta' \le \delta \le r(M)$, the correspondence

$$\sigma \mapsto (I_\sigma)_{\delta'} \quad (\sigma \in M_\delta)$$

is an acyclic carrier $\Phi_{\delta'\delta}$ from M_δ to $M_{\delta'}$ (see [Mu], § 13).

We now use the theorem of acyclic carrier ([Mu], Theorem 13.3). It implies that there exists an augmentation preserving chain map $\nu : C_*(M_\varepsilon) \longrightarrow C_*(M_{\varepsilon'})$ which is carried by $\Phi_{\varepsilon'\varepsilon}$. Let μ denote the inclusion from $M_{\varepsilon'}$ into M_ε. Then, $\Phi\varepsilon'\varepsilon'$ is an acyclic carrier for both $\nu \circ \mu_*$ and the identity of $C_*(M\varepsilon')$. By theorem of acyclic carrier again, these two maps are chain homotopic and thus $\nu \circ \mu_*$ induces the identity on $H_*(M_{\varepsilon'})$. The same argument shows that $\mu_* \circ \nu$ induces the identity on $H_*(M_\varepsilon)$ (using the acyclic carrier $\Phi_{\varepsilon\varepsilon}$). □

We will now compare the simplicial homology of M_ε with the singular homology of M. Formula (3.2) shows that the correspondence $\sigma \mapsto T_\sigma$ gives rise to a chain map

$$T_\sharp : C_*(M_\varepsilon) \longrightarrow SC_*(M)$$

where $SC_*(M)$ denotes the singular chain complex of M.

(3.4) Proposition. *If $0 < \varepsilon \le r(M)$ the chain map T_* induces an isomorphism on homology.*

Proof. We shall need a few accessory chain complexes. For $\delta > 0$, denote by $SC_*(M;\delta)$ the sub-chain complexe of $SC_*(M)$ based on singular simplexes τ such that there exists $x \in M$ with Image(τ) is contained in the ball $B(x,\delta)$. Recall that the inclusion $SC_*(M;\delta) \longrightarrow SC_*(M)$ induces an isomorphism on homology ([Mu], Theorem 31.5).

We shall also use the ordered chain complex $C_*^o(M_\delta)$: the group $C_q^o(M_\delta)$ is free abelian on $(q+1)$-uples $(x_0,\dots,x_q)$ such that $\{x_0\}\cup\dots\cup\{x_q\}$ is a simplex of M_δ. On can see $C_*(M_\delta)$ as a sub-chain complex of $C_*^o(M_\delta)$ by associating to a q-simplex $\{x_0,\dots,x_q\}$ of M_ε (with our convention that $x_0 < x_1 < \dots < x_q$ for the well ordering on M) the $(q+1)$-uple $(x_0,\dots,x_q)$. It is also classical that this inclusion is a homology isomorphism ([Mu], Theorem 13.6). Observe that the construction $\sigma \mapsto T_\sigma$ does not required that the vertices of σ are all distincts. One can then define T_σ for a basis element of $C_q^o(M_\delta)$ (if $\delta \le r(M)$) and $t^\sharp$ thus extends to a chain map $T^\sharp = T^{\sharp,\delta} : C_*^o(M_\delta) \longrightarrow SC_*(M;\delta)$ (the image of $T_{(x_0,\dots,x_q)}$ being contained in $B(x_q,\delta)$). If $0 < \varepsilon \le r(M)$, one then has a commutative diagram

$$\begin{array}{ccc} C^o_*(M_{\varepsilon/2}) & \xrightarrow{T^{\sharp,\varepsilon/2}} & SC_*(M;\varepsilon/2) \\ \downarrow & & \downarrow \\ C^o_*(M_\varepsilon) & \xrightarrow{T^{\sharp,\varepsilon}} & SC_*(M;\varepsilon) \end{array}$$

Let $\tau : \Delta^q \longrightarrow M$ be a singular simplex whose image is contained in some ball of radius $\varepsilon/2$. The $(q+1)$-uple $(\tau(e_0), \ldots, \tau(e_q))$ is element of $C^o_q(M_\varepsilon)$. This correspondence gives rise to a chain map

$$R^\varepsilon : SC_*(M;\varepsilon/2) \longrightarrow C^o_*(M_\varepsilon)$$

The composition $R^\varepsilon \circ T^{\sharp,\varepsilon/2}$ is equal to the inclusion $C^o_*(M_{\varepsilon/2}) \subset C^o_*(M_\varepsilon)$ which induces a homology isomorphsim by (3.3). Let us now understand the composition $T^{\sharp,\varepsilon} \circ R^\varepsilon$. Let $\tau : \Delta^q \longrightarrow M$ be a singular simplex such that $\tau(\Delta^q) \subset B(y,\varepsilon/2)$ for some $y \in M$. Therefore, $\tau' := T^{\sharp,\varepsilon} \circ R^\varepsilon(\tau)$ also satisfies $\tau'(\Delta^q) \subset B(y,\varepsilon/2)$ by Condition b). Hence, τ and τ' are cannonically homotopic (following, for each $s \in \Delta^q$, the shortest geodesic joining $\tau(s)$ to $\tau'(s)$). As in the proof of the homotopy axiom for singular homology ([Mu], § 30), these homotopies provide a chain homotopy between $T^{\sharp,\varepsilon} \circ R^\varepsilon$ and the inclusion $SC_*(M;\varepsilon/2) \subset SC_*(M;\varepsilon)$. As said before, this inclusion is known to induce a homology isomorphism. Therefore, $T^{\sharp,\varepsilon} \circ R^\varepsilon$ induces an isomorphism on homology.

We have shown that both $R^\varepsilon \circ T^{\sharp,\varepsilon/2}$ and $T^{\sharp,\varepsilon} \circ R^\varepsilon$ induce homology isomorphisms. Therefore, $T^{\sharp,\varepsilon}$ induces a homology isomorphism. □

We are now in position to state our final theorem. By (3.2) the correspondence $\sigma \mapsto T_\sigma$ gives rise to a continuous map

$$T : |M_\varepsilon| \longrightarrow M$$

where $|M_\varepsilon|$ denotes the geometric realization of M_ε.

(3.5) Theorem. *Let M be a Riemannian manifold with $r(M) > 0$. Let $0 < \varepsilon \leq r(M)$. Then the map $T : |M_\varepsilon| \longrightarrow M$ is a homotopy equivalence.*

Proof. The proof goes in several steps.

(3.6) We first show that T induces an isomorphism on the fundamental groups. Let $\gamma : [0,1] \longrightarrow M$ represent an element of $\pi_1(M)$. Let $p \in \mathbb{N}$ such that $1/p$ is a Lebesgue number for the covering $\{\gamma^{-1}(B(x,\varepsilon/2)\}_{x\in M}$. The path $\gamma|[k/p,(k+1)/p]$ is then canonically homotopic to a parameterisation of the shortest geodesic joining $\gamma(k/p)$ to $\gamma((k+1)/p)$. This shows that γ is homotopic to a composition $\gamma'$of geodesics in balls of radius $\varepsilon/2$. Such a path γ' represents the image by T of an element of $\pi_1(|M_\varepsilon|)$, the later being identifiend with the edge-path group of the simplicial complex M_ε ([Sp], pp. 134–138). This proves that $\pi_1 T : \pi_1(|M_\varepsilon|) \longrightarrow \pi_1(M)$ is surjective. A similar argument using, as usual, a small triangulation of $[0,1] \times [0,1]$ shows the injectivity of $\pi_1 T$.

(3.7) Let $\tilde{M}$ be the universal covering of M endowed with the Riemannian metric so that the covering projection is a local isometry. We leave to the reader to check that $r(\tilde{M}) \geq r(M)$. By Proposition (3.4), the map $\widehat{T} : |(\tilde{M})_\varepsilon| \longrightarrow \tilde{M}$ is a homology isomorphism. By (3.6), $\pi_1((\tilde{M})_\varepsilon) \simeq \pi_1(\tilde{M}) \simeq \{1\}$. Therefore, $\widehat{T}$ is a homotopy equivalence.

(3.8) Let $p_\varepsilon : \widetilde{|M_\varepsilon|} \longrightarrow M_\varepsilon$ be the universal covering projection over M_ε. By (3.6), the following commutative diagram

$$\begin{array}{ccc} \widetilde{|M_\varepsilon|} & \xrightarrow{\tilde{T}} & \tilde{M} \\ {\scriptstyle p_\varepsilon}\downarrow & & \downarrow{\scriptstyle p} \\ |M_\varepsilon| & \xrightarrow{T} & M \end{array} \tag{3.9}$$

is a pull-back diagram. Therefore, the map $\widehat{T} : |(\tilde{M})_\varepsilon| \longrightarrow \tilde{M}$ factors as $\widehat{T} = \tilde{T} \circ \check{T}$ where

$$|(\tilde{M})_\varepsilon| \xrightarrow{\check{T}} \widetilde{|M_\varepsilon|} \xrightarrow{\tilde{T}} \tilde{M}$$

(all these maps are unique once choice of base points have been made). We shall see in (3.10) that the map $\check{T}$ is a homeomorphism. Since the map $\widehat{T}$ is a homotopy equivalence by (3.7), the maps $\tilde{T}$ (and then T) will be homotopy equivalences and Theorem (3.5) will be proven.

(3.10) Let L be the simplicial complex defined as follows:

- the vertex set $L^{(0)}$ of L is $L^{(0)} := \{p_\varepsilon^{-1}(v) \mid v$ is a vertex of M_ε
- $\overline{\sigma} := \{\overline{v}_0, \ldots, \overline{v}_q\} \subset L^{(0)}$ is a q-simplex of L if there is a q-simplex $\sigma = \{v_0, \ldots, v_q\}$ of M_ε and a lifting $f_{\overline{\sigma}} : |\sigma| \longrightarrow \widetilde{|M_\varepsilon|}$ such that $f_{\overline{\sigma}}(v_i) = \overline{v}_i$. It is known that the collection $\{f_{\overline{\sigma}} \mid \overline{\sigma} \in L\}$ produces a homeomorphism $f : |L| \xrightarrow{\simeq} \widetilde{|M_\varepsilon|}$ ([Sp], pp. 144–145; in other words, (L, f) is a triangulation of $\widetilde{|M_\varepsilon|}$).

Using the pull-back diagram (3.9), one gets a bijection $\tilde{M} \xrightarrow{\simeq} L^{(0)}$. As $(\tilde{M})_\varepsilon^{(0)} = \tilde{M}$ one obtains a bijection $\check{t} : (\tilde{M})_\varepsilon^{(0)} \xrightarrow{\simeq} L^{(0)}$. It is easy to check that $\check{t}$ extends to a simplicial isomorphism $\check{t} : (\tilde{M})_\varepsilon \xrightarrow{\simeq} L$ and $|\check{t}| = \check{T}$. Therefore, $\check{T}$ is a homeomorphism. As said in (0.51), this finishes the proof of Theorem (3.5). $\square$

Problems.

(3.11) Given a compact Riemannian manifold M, does there exists a finite set $F \subset M$ and an $0 < \varepsilon < r(M)$ such that the restriction of the map T to $|F_\varepsilon|$ gives a homotpopy equivalence $|F_\varepsilon| \xrightarrow{\sim} M$?

Suppose that F is a subset of M which is δ-dense with $2\delta < r(M)$. Then one can choose a (possibly non-continuous) map $h : M \longrightarrow F$ such that $d(x, h(x)) < \delta$ for all $x \in M$. Let $0 < \eta < r(M) - 2\delta$ and let $\varepsilon = 2\delta + \eta$. Then, the composed simplicial map

$$M_\eta \longrightarrow F_\varepsilon \longrightarrow M_\varepsilon$$

is contiguous to the inclusion $M_\eta \subset M_\varepsilon$. By (3.5), this shows that $|F_\varepsilon| \xrightarrow{\sim} M$ is a domination.

(3.12) How does M_ε change when ε grows bigger ? I expected M_ε to become more and more connected till $\varepsilon = \operatorname{Diam} M$ where M_ε is contractible (this is suggested by the examples of the introduction).

4 The metric cohomology

Let $\mathcal{R}$ be a commutative ring. Let X be a pseudo-metric space. One denotes by $H^*(X_\varepsilon; \mathcal{R})$ the simplicial cohomology with coefficient in $\mathcal{R}$ of the ε-complex X_ε of X. One defines

$$\mathcal{H}^*(X; \mathcal{R}) := \lim_{\varepsilon \to 0} H^*(X_\varepsilon; \mathcal{R})$$

Here, one uses the fact that if $\varepsilon' \leq \varepsilon$, then $X_{\varepsilon'}$ is a subcomplex of X_ε, giving rise to a homomorphism $H^*(X_\varepsilon; \mathcal{R}) \longrightarrow H^*(X_{\varepsilon'}; \mathcal{R})$. The graded structure of $H^*(X; \mathcal{R})$ passes to the inductive limit and makes $\mathcal{H}^*(X; \mathcal{R})$ a graded ring.

In the same way, let us consider a *pair of pseudo-metric spaces* (X, Y). This means that Y is a subspace of X endowed with the induced pseudo-distance. Then Y_ε is a subcomplex of X_ε for all $\varepsilon > 0$. The simplicial cohomology of the pair $(X_\varepsilon, Y_\varepsilon)$ is then defined and one sets

$$\mathcal{H}^*(X, Y; \mathcal{R}) := \lim_{\varepsilon \to 0} H^*(X_\varepsilon, Y_\varepsilon; \mathcal{R})$$

We shall consider the following category $\mathcal{M}$:

- an object of $\mathcal{M}$ is a pair (X, Y) of pseudo-metric spaces.
- a morphism from f from (X, Y) to (X', Y') is a uniformly continuous map $f : X \longrightarrow X'$ such that $f(Y) \subset Y'$.
- If X is a pseudo-metric space, the space $X \times [0, 1]$ is understood to be endowed with the product pseudo-distance $d((x, t), (x', t')) = d(x, x') + |t - t'|$

The category $\mathcal{M}$ is an admissible category for a homology theory ([Mu], p. 146). This section is devoted to the proof of

(4.1) Theorem. *The correspondance $(X,Y) \mapsto \mathcal{H}^*(X,Y;\mathcal{R})$ satisfies the Eilenberg–Steenrod axioms of an ordinary cohomology theory for the category $\mathcal{M}$, with a "metric version" of the excision axiom (see (0,2) below).*

Proof. Let first establish the functoriality. Let $f : (X,Y) \longrightarrow (X',Y')$ be a uniformly continuous map. For $\varepsilon > 0$, there is $\delta > 0$ such that $d(f(u),f(v)) < \varepsilon$ if $d(u,v) < \delta$. Therfore one has a simplicial map $f_{\delta,\varepsilon} : (X_\delta, Y_\delta) \longrightarrow (X'_\varepsilon, Y'_\varepsilon)$. If $\varepsilon' \leq \varepsilon$, one can find $\delta' \leq \delta$ so that the following diagram

$$\begin{array}{ccc} (X_{\delta'}, Y_{\delta'}) & \xrightarrow{f_{\delta',\varepsilon'}} & (X'_{\varepsilon'}, Y'_{\varepsilon'}) \\ \downarrow & & \downarrow \\ (X_\delta, Y_\delta) & \xrightarrow{f_{\delta,\varepsilon}} & (X'_\varepsilon, Y'_\varepsilon) \end{array}$$

commutes. The induced homomorphisms $f^*_{\delta,\varepsilon} : H^*(X'_\varepsilon, Y'_\varepsilon; \mathcal{R}) \longrightarrow H^*(X_\delta, Y_\delta; \mathcal{R})$ give, passing to the inductive limit, a well defined homomorphism

$$f^* : \mathcal{H}^*(X',Y';\mathcal{R}) \longrightarrow \mathcal{H}^*(X,Y;\mathcal{R})$$

The functorial properties of f^* are easy to check.

Now, for each $\varepsilon > 0$ one has the long exact sequence of the pair $(X_\varepsilon, Y_\varepsilon)$. The exactness being preserved by passing to inductive limits, one gets the long exact sequence

$$\cdots \longrightarrow \mathcal{H}^{i+1}(X,Y) \xrightarrow{\delta} \mathcal{H}^i(Y) \longrightarrow \mathcal{H}^i(X) \longrightarrow \mathcal{H}^i(X,Y) \longrightarrow \cdots$$

One easily checks that this exact sequence is functorial.

The dimension axiom is obvious since $\{\mathrm{pt}\}_\varepsilon$ is a point for all $\varepsilon > 0$.

For the homotopy axiom, let

$$F : X \times [0,1] \longrightarrow Z$$

be a uniformly continuous map. Let $\varepsilon > 0$. Choose $\delta > 0$ so that $d(F((x,t)), F((x',t'))) < \varepsilon/3$ when $d((x,t),(x',t')) < \delta$. Chose an integer m such that $1/m < \delta/2$. Define $f_{k/m} : X \longrightarrow Z$ by $f_{k/m}(x) := F(x, k/m)$. With these choices, $f_{k/m}$ induces a simplicial map $f_{k/m} : X_{\delta/2} \longrightarrow Z_\varepsilon$ and if $\{x_0, \dots, x_q\}$ is a q-simplex of $X_{\delta/2}$ then

$$\{f_{\frac{k}{m}}(x_0), \dots, f_{\frac{k}{m}}(x_q), f_{\frac{k+1}{m}}(x_0), \dots, f_{\frac{k+1}{m}}(x_q)\}$$

span a simplex of Z_ε. The simplicial maps $f_{k/m}$ and $f_{(k+1)/m}$ are thus contiguous and then induce the same homomorphism from $H^*(Z_\varepsilon;\mathcal{R}) \longrightarrow H^*(X\delta/2;\mathcal{R})$. This argument can be made for each ε in a coherent enough way and this proves the equality $f_0^* = f_1^*$ as homomorphisms from $\mathcal{H}^*(Z;\mathcal{R}) \longrightarrow \mathcal{H}^*(X;\mathcal{R})$. The proof for pairs of spaces is similar.

We turn now our attention to the excision axiom or rather its metric version. Its precise statement is as follows

(4.2) Proposition (Metric Excision Axiom). *Let (X,Y) be a pair of pseudo-metric spaces. Let $U \subset Y$ such that $d(U, X-Y) > 0$. Then the homomorphism $\mathcal{H}^*(X,Y;\mathcal{R}) \longrightarrow \mathcal{H}^*(X-U, Y-U;\mathcal{R})$ induced by inclusion is an isomorphism.*

Proof. Let $\mu := d(U, X-Y)$. As soon as $\varepsilon < \mu$, one has

$$X_\varepsilon = (X-U)_\varepsilon \cup Y_\varepsilon \quad , \quad (X-U)_\varepsilon \cap Y_\varepsilon = (Y-U)_\varepsilon$$

and therefore the homomorphism

$$H^*(X_\varepsilon, Y_\varepsilon;\mathcal{R}) \longrightarrow H^*((X-U)\varepsilon, (Y-U)\varepsilon;\mathcal{R})$$

induced by inclusion is an isomorphism. □

Remark. The usual condition for the excision axiom is $\overline{U} \subset \overset{\circ}{Y}$ which is equivalent to $\overline{U} \cap \overline{X-Y} = \emptyset$. The hypothesis for the metric version is equivalent to $d(\overline{U}, \overline{X-Y}) > 0$. The classical version is false for the theory $\mathcal{H}^*$ as seen by the following example: take $X = \mathbb{R} - \{0\}$ and $U = Y = \mathbb{R}_{<0}$ (an open and closed subset of X). Then $\mathcal{H}^0(X,Y;\mathcal{R}) = 0$ while $\mathcal{H}^0(X-U, Y-U;\mathcal{R}) = \mathcal{H}^0(\mathbb{R}_{<0}, \emptyset;\mathcal{R}) \simeq \mathcal{R}$.

We finish by a Mayer–Vietoris sequence for the metric cohomology. As for the excision property, a slight modification is necessary whit respect to classical statemnts. Let $\mathcal{U}$ be an open covering of the pseudo-metric space X.

(4.3) Proposition. *Let $X = U \cup V$ an open covering of X admitting a Lebesgue number. Then, one has the Mayer–Vietoris exact sequence*

$$\begin{aligned} \cdots \longrightarrow \mathcal{H}^k(X;\mathcal{R}) \longrightarrow \mathcal{H}^k(U;\mathcal{R}) \oplus \mathcal{H}^k(V;\mathcal{R}) \longrightarrow \mathcal{H}^k(U \cap V;\mathcal{R}) \\ \longrightarrow \mathcal{H}^{k+1}(X;\mathcal{R}) \longrightarrow \cdots \end{aligned}$$

Proof. If the covering $X = U \cup V$ admits a Lebesgue number, then if ε is small enough, one has

$$X_\varepsilon = U_\varepsilon \cup V_\varepsilon \quad , \quad U_\varepsilon \cap V_\varepsilon = (U \cap V)_\varepsilon$$

Thus one has the Mayer–Vietoris for simplicial cohomology which, by passing to the inductive limit, gives the corresponding exact for the metric cohomology. □

5 Properties of the metric cohomology

Between compact space, uniform continuity is equivalent to continuity and the metric excision axiom (4.2) is equivalent to the usual excision property. Therefore $\mathcal{H}^*$ can be seen as a ordinary cohomology theory for the admissible category of pairs of compact metric (or even metrizable) spaces.

(5.1) Theorem. *For the category of compact metric pairs, the metric cohomology $\mathcal{H}^*$ and the Cech cohomology $\check{H}^*$ in a ring $\mathcal{R}$ are canonically isomorphic cohomology theories.*

Proof. Let X be a compact metric space. We denote by $\mathcal{N}_\varepsilon(X)$ the nerve of $\mathcal{B}_\varepsilon(X) := \{B(x,\varepsilon)\}_{x\in X}$, the covering of X by all balls of radius ε. We label the vertices of $\mathcal{N}_\varepsilon(X)$ by the points of X, identifying x with $B(x,\varepsilon)$.

If $\sigma = \{x_0,\dots,x_q\}$ is a q-simplex of X_ε, then $\sigma \subset B(x_0,\varepsilon)\cap\ldots\cap B(x_q,\varepsilon)$ which shows that σ is a q-simplex of $\mathcal{N}_\varepsilon(X)$. This provides us a simplicial map $g_\varepsilon : X_\varepsilon \longrightarrow \mathcal{N}_\varepsilon(X)$. On the other hand, if $\tau = \{y_0,\dots,y_q\}$ is a q-simplex of $\mathcal{N}_\varepsilon(X)$, then $d(x_i,x_j) < 2\varepsilon$ and thus τ is a q-simplex of $X_{2\varepsilon}$. This gives rise to a simplicial map $h_\varepsilon : \mathcal{N}_\varepsilon(X) \longrightarrow X_{2\varepsilon}$. It is clear that the composition $h_\varepsilon\circ g_\varepsilon$ is equal to the inclusion $X_\varepsilon \subset X_{2\varepsilon}$ and that $g_{2\varepsilon}\circ h_\varepsilon$ is the canonical simplicial injection $\mathcal{N}_\varepsilon(X) \longrightarrow \mathcal{N}_{2\varepsilon}(X)$. Passing to the inductive limits, one gets isomorphisms

$$g^* : \lim_{\varepsilon\to 0} H^*(\mathcal{N}_\varepsilon(X);\mathcal{R}) \xrightarrow{\sim} \mathcal{H}^*(X;\mathcal{R})\ ,$$
$$h^* : \mathcal{H}^*(X;\mathcal{R}) \xrightarrow{\sim} \lim_{\varepsilon\to 0} H^*(\mathcal{N}_\varepsilon(X);\mathcal{R})$$

which are inverses one of another. As X is compact, the coverings $\{B(x,\varepsilon)\}_{x\in X}$ form, by the Lebesgue number, a cofinal system in the projective system of all the coverings. Therefore, one has an isomorphism

$$\lim_{\varepsilon\to 0} H^*(\mathcal{N}_\varepsilon(X);\mathcal{R}) \xrightarrow{\sim} \check{H}^*(X;\mathcal{R})$$

and thus $\mathcal{H}^*(X;\mathcal{R})$ is isomorphic to $\check{H}^*(X;\mathcal{R})$. The same kind of argument holds for pairs of compact spaces and one checks easily that one gets this way an isomorphism od cohomology theories. □

(5.2) Remark. It is known, but not easy to prove in general, that the Cech cohomology is isomorphic to the Alexander–Spanier cohomology $\overline{H}^*$. (see, for instance, [Do], Theorem 2). This shows, using Theorem (5.1) that the metric cohomology $\mathcal{H}^*$ and the Alexander–Spanier cohomology are isomorphic for compact metric spaces. In addition, explicit isomorphisms (without choices) are easily obtainable in both directions: let X be a compact metric space. Recall that one of the definition of $\overline{H}^*(X;\mathcal{R})$ is the inductive limit of $H^*(X(\mathcal{U});\mathcal{R})$ where $\mathcal{U}$ is an open covering of X and $X(\mathcal{U})$ is the abstract simplicial complex for which $\sigma = \{x_0, \ldots, x_q\} \subset X$ is a q-simplex if and only if there exists $U \in \mathcal{U}$ such that $\sigma \subset U$. As X is compact, the inductive limit can be taken over the coverings $\mathcal{B}_\varepsilon := \{B(x,\varepsilon)\}_{x\in X}$ ([Sp], § 6.5). Now, if $\sigma = \{x_0, \ldots, x_q\} \in X_\varepsilon$ then σ is also a q-simplex of $X(\mathcal{B}_\varepsilon)$ since, for instance, $\sigma \subset B(x_0, \varepsilon)$. This gives a simplicial map $\alpha_\varepsilon : X_\varepsilon \longrightarrow X(\mathcal{B}_\varepsilon)$. In the other direction, if τ is a q-simplex of $X(\mathcal{B}_\varepsilon)$, then τ is a q-simplex of $X_{2\varepsilon}$ which provides us a simplicial map $\beta_\varepsilon : X(\mathcal{B}_\varepsilon) \longrightarrow X_{2\varepsilon}$. Passing to the inductive limits, these maps provides, for *compact* metric spaces, isomorphisms between the metric cohomology and the Alexander–Spanier cohomology theories. For general metric spaces, the two cohomologies are not isomorphic. For instance, $\overline{H}^0(\mathbb{R} - \{0\}; \mathcal{R}) \simeq \mathcal{R} \oplus \mathcal{R}$ while $\mathcal{H}^0(\mathbb{R} - \{0\}; \mathcal{R}) \simeq \mathcal{R}$ by (5.5) below.

Even for complete spaces, the metric cohomology might be different from the Cech comomology. For exemple, the space $X := \{(x,y) \in \mathbb{R}^2 \mid (xy)^2 = 1\}$. Then X has the same Cech cohomology as four points while it has the metric cohomology of a circle.

The next results about the metric cohomology are direct consequences of (2.2), (2.3) and (2.5).

(5.3) Proposition. *Let X be a pseudo-metric space admitting a crushing onto a subspace $A \subset X$. Then $\mathcal{H}^*(X, A; \mathcal{R}) = 0$.*

(5.4) Corollary. *Let X be a crushable pseudo-metric space. Then, $\mathcal{H}^*(X; \mathcal{R}) \simeq \mathcal{H}^*(\mathrm{pt}; \mathcal{R})$.*

(5.5) Theorem. *Let $j : X \longrightarrow \widehat{X}$ be the inclusion of a metric space into a metric complection $\widehat{X}$ of X. Then $j_* : \mathcal{H}^*(\widehat{X}; \mathcal{R}) \longrightarrow \mathcal{H}^*(X; \mathcal{R})$ is an isomorphism.*

(5.6) Example. Let $X = S^n - \{\mathrm{pt}\}$. By the above results, one has

$$\mathcal{H}^*(X; \mathbb{Z}) \xrightarrow{\simeq} \mathcal{H}^*(S^n; \mathbb{Z}) \xrightarrow{\simeq} \check{H}^*(S^n; \mathbb{Z})$$

Therefore, though that X is contractible, $\mathcal{H}^*(X; \mathbb{Z}) \not\simeq \mathcal{H}^*(\mathrm{pt}; \mathbb{Z})$. This shows that X is not crushable.

References

[Be] Begle, E.G., The Vietoris mapping theorem for bicompact spaces, *Annals of Math.* **51** (1950), 534–543.

[Do] Dowker, C.H., Homology groups of relations, *Annals of Math.* **56** (1952), 84–95.

[G-H] Ghys. E. – De la Harpe, P., *Sur les groupes hyperboliques d'après M. Gromov*, Birkhäuser, 1990.

[Gr] Gromov, M., Hyperbolic groups, in *Essays in group theory*, S.M. Gersten (ed.), MSRI Publ. **8** (1987), 75–263.

[Le] Lefschetz, S., Algebraic Topology, *AMS Coll. Publ.* **27** (1942).

[Mu] Munkres, J. *Elements of algebraic topology*, Addison–Wesley, 1984.

[Sp] Spanier, E., *Algebraic Topology*, McGraw–Hill, 1966.

[Vi] Vietoris, L., Über die höheren Zusammenhang kompakter Räume und eine Klasse von zusammenhangstreuen Abbildungen, *Math. Annalen* **97** (1927), 454–472.