Sheaf Cohomology
Lecture 2: Leray Theorem

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Let $\mathcal{F}$ be a sheaf over a complex manifold M and $\mathfrak{U} = \{U_\alpha, \alpha \in I\}$ and $\mathfrak{V} = \{V_\beta, \beta \in J\}$ two open covers of M.

Definition 1. $\mathfrak{V}$ is called a *refinement* of $\mathfrak{U}$ is for every $\beta \in J$ there exists $\alpha \in I$ such that $V_\beta \subset U_\alpha$.

Definition 2. The *n^{th} Čech cohomology group of $\mathcal{F}$ on M* is the direct limit of the sheaf cohomology groups $H^n(\mathfrak{U}; \mathcal{F})$ taken over refinements of the cover

$$\check{H}^n(M; \mathcal{F}) := \varinjlim_{\mathfrak{U}} H^n(\mathfrak{U}; \mathcal{F}) \ .$$

Definition 3. The cover $\mathfrak{U}$ is called *acyclic* for $\mathcal{F}$ if $H^q(U_{i_1} \cap \cdots \cap U_{i_p}) = 0$ for $q > 0$ and any $i_1, \ldots, i_p$.

Theorem 4 (Leray Theorem)**.** *If the covering $\mathfrak{U}$ is acyclic for $\mathcal{F}$, then*

$$\check{H}^n(M; \mathcal{F}) = H^n(\mathfrak{U}; \mathcal{F}) \ .$$

I will discuss the case of $n = 1$, and the other cases are **homework**. The statement of Leray theorem for $n = 1$ takes a simpler form:

Theorem 5 (First order Leray covers)**.** *Suppose $H^1(U_i; \mathcal{F}) = 0$ for all $i \in I$, then*

$$\check{H}^1(M; \mathcal{F}) = H^1(\mathfrak{U}; \mathcal{F}) \ .$$

If the cover $\mathfrak{V}$ is finer than the cover $\mathfrak{U}$ of M, then there exists a map $\tau\colon J \to I$ such that $V_\beta \subset U_{\tau(\beta)}$. Then τ induces a map

$$\tau^{\mathfrak{U}}_{\mathfrak{V}}\colon Z^1(\mathfrak{U}, \mathcal{F}) \to Z^1(\mathfrak{V}, \mathcal{F})$$

$$(f_{ij}) \mapsto (g_{kl})$$

where $g_{kl} := f_{\tau(k)\tau(l)}|V_k \cap V_l$ for all $k, l \in J$. This map takes coboundaries to coboundaries, so it induces a homomorphism of cohomology groups

$$\tau^{\mathfrak{U}}_{\mathfrak{V}}\colon H^1(\mathfrak{U}, \mathcal{F}) \to H^1(\mathfrak{V}, \mathcal{F}).$$

This induced map in cohomology does not depend on the choice of refinement (exercise), and it is injective.

Injectivity: Suppose $(f_{ij}) \in Z^1(\mathfrak{U}, \mathcal{F})$ is a cocycle mapped to a coboundary in $Z^1(\mathfrak{V}, \mathcal{F})$ we need to show that (f_{ij}) itself is a coboundary. Write the image $f_{\tau(k)\tau(l)} = g_k - g_l$ on $V_k \cap V_l$ where $g_k \in \mathcal{F}(V_k)$. Then over the triple intersection $U_i \cap V_k \cap V_l$ we have

$$g_k - g_l = f_{\tau(k)\tau(l)} = f_{i\tau(k)} + f_{\tau(l)i} = f_{i\tau(l)} - f_{i\tau(k)},$$

which gives

$$f_{i\tau(k)} + g_k = f_{i\tau(l)} + g_l.$$

Now, applying the second axiom of sheaves to the family of open sets $(U_i \cap V_k)_{k \in K}$ we obtain $h_i \in \mathcal{F}(U_i)$ such that

$$h_i = f_{i\tau(k)} + g_k \text{ on } U_i \cap V_k.$$

With elements h_i found in this way, over $U_i \cap U_j \cap V_k$ we have

$$f_{ij} = f_{i\tau(k)} + f_{\tau(k)j} = f_{i\tau(k)} + g_k - f_{j\tau(k)} - g_k = h_i - h_j.$$

Since k is arbitrary, applying the first axiom of sheaves, we see that this equation is valid over $U_i \cap U_j$, so (f_{ij}) is itself a coboundary and injectivity follows.

Direct limit: If one has three open coverings such that $\mathfrak{W}$ is a refinement of $\mathfrak{V}$ and $\mathfrak{V}$ is a refinement of $\mathfrak{U}$ then

$$\tau^{\mathfrak{V}}_{\mathfrak{W}} \circ \tau^{\mathfrak{U}}_{\mathfrak{V}} = \tau^{\mathfrak{U}}_{\mathfrak{W}}$$

and this allows us to define the direct limit

$$\check{H}^1(M; \mathcal{F}) := \varinjlim_{\mathfrak{U}} H^1(\mathfrak{U}; \mathcal{F}) = \bigsqcup H^1(\mathfrak{U}; \mathcal{F}) / \sim \ ,$$

where two classes $\xi, \eta \in H^1(\mathfrak{U}; \mathcal{F})$ are equivalent if they coincide in some refinement, that is, if there exist $\mathfrak{W}$ such that $\tau^{\mathfrak{U}}_{\mathfrak{W}}(\xi) = \tau^{\mathfrak{V}}_{\mathfrak{W}}(\eta)$. Now, is follows from the definition of the inductive limit that the injectivity of the maps $\tau^{\mathfrak{U}}_{\mathfrak{V}}$ imply injectivity of the map

$$\tau^U \colon H^1(\mathfrak{U}; \mathcal{F}) \to \check{H}^1(M; \mathcal{F}) \ .$$

Remark. In particular it follows that $H^1(M; \mathcal{F}) = 0$ precisely if $H^1(\mathfrak{U}; \mathcal{F}) = 0$ for every open cover $\mathfrak{U}$ of M.

Proof of theorem 5. Now, assuming $H^1(U_i; \mathcal{F}) = 0$ for every i, we want to prove that the map $\tau^U \colon H^1(\mathfrak{U}; \mathcal{F}) \to \check{H}^1(M; \mathcal{F})$ is an isomorphism. Since injectivity was proven above, it suffices to show that for every refinement $\mathfrak{V}$ of $\mathfrak{U}$ the map $\tau^{\mathfrak{U}}_{\mathfrak{V}} \colon H^1(\mathfrak{U}; \mathcal{F}) \to H^1(\mathfrak{V}, \mathcal{F})$ is surjective. Given any cocycle $(f_{\alpha\beta}) \in H^1(\mathfrak{V}; \mathcal{F})$ to show surjectivity we must prove that there exists a cocycle $(F_{ij}) \in H^1(\mathfrak{U}; \mathcal{F})$ such that the cocycle

$$(F_{\tau(\alpha)\tau(\beta)}) - (f_{\alpha\beta}) \sim 0$$

is cohomologous to zero relative to the covering $\mathfrak{V}$. The family $(U_i \cap V_\alpha)_{\alpha \in J}$ is an open cover of U_i which we denote by $U_i \cap \mathfrak{V}$. The assumption implies that $H^1(U_i \cap \mathfrak{V}; \mathcal{F}) = 0$ (see remark above) so there exists $g_{i\alpha} \in U_i \cap V_\alpha$ such that

$$f_{\alpha\beta} = g_{i\alpha} - g_{i\beta} \text{ on } U_i \cap V_\alpha \cap V_\beta.$$

On the intersetion $U_i \cap U_j \cap V_\alpha \cap V_\beta$ we have

$$g_{j\alpha} - g_{j\beta} = g_{i\alpha} - g_{i\beta} \Rightarrow g_{j\alpha} - g_{i\alpha} = g_{j\beta} - g_{i\beta}$$

so by the second axiom of sheaves there exists elements F_{ij} on $U_i \cap U_j$ such that

$$F_{ij} = g_{j\alpha} - g_{i\alpha} \text{ on } U_i \cap U_j \cap V_\alpha.$$

Then $(F_{ij}) \in Z^1(\mathfrak{U}, \mathcal{F})$. Setting

$$h_\alpha := g_{\tau(\alpha)\alpha}|V_\alpha \in \mathcal{F}(V_\alpha),$$

on $V_\alpha \cap V_\beta$ we have

$$F_{\tau(\alpha)\tau(\beta)} - f_{\alpha\beta} = (g_{\tau(\beta)\alpha} - g_{\tau(\alpha)\alpha}) - (g_{\tau(\beta)\alpha} - g_{\tau(\beta)\beta}) = g_{\tau(\beta)\beta} - g_{\tau(\alpha)\alpha} = h_\beta - h_\alpha.$$

Hence $(F_{\tau(\alpha)\tau(\beta)}) - (f_{\alpha\beta}) \sim 0$. $\square$

Corollary 6. *The open cover of $\mathbb{P}^1$ presented in Lecture 1, namely $\mathfrak{U} = \{U, V\}$ with $U = \{z\} \cong \mathbb{C} \cong \{\xi\} = V$ satifying $\xi = z^{-1}$ on $U \cap V = \mathbb{C} - \{0\}$ calculates Čech cohomology of $\mathbb{P}^1$ with $\mathcal{O}(n)$-coefficients. Consequently,*

$$H^1\big(\mathbb{P}^1; \mathcal{O}(n)\big) = H^1\big(\mathfrak{U}; \mathcal{O}(n)\big) = \mathbb{C}^{\max\{0,-n-1\}}\ .$$

Note that in particular it follows that two line bundles $\mathcal{O}(-n)$ and $\mathcal{O}(-m)$ with $n \neq m$ are not isomorphic. Calculating H^0 as in Lecture 1, it then follows that for any pair of integers $n \neq m$ the bundles on $\mathbb{P}^1$ satisfy $\mathcal{O}(-n) \ncong \mathcal{O}(-m)$. In contrast, we mention here a very general result for non-compact Riemann surfaces:

Theorem 7 (see [F, 30.4])**.** *Every holomorphic bundle on a non-compact Riemann surface is holomorphically trivial*

We observe that such a result is not true in the algebraic category.

References

[F] Forster, O. *Lectures on Riemann Surfaces*, GTM 81 Springer–Verlag (1980)

[GH] Griffiths, P. and Harris, J. *Principles of Algebraic Geometry*, John Wiley & Sons (1978)