

SPARSE DOWKER NERVES

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ABSTRACT. We propose sparse versions of filtered simplicial complexes used to compute persistent homology of point clouds and of networks. In particular we extend a slight variation of the Sparse Čech Complex of Cavanna, Jahanseir and Sheehy [7] from point clouds in Cartesian space to point clouds in arbitrary metric spaces. Along the way we formulate interleaving in terms of strict 2-categories, and we introduce the concept of Dowker dissimilarities that can be considered as a common generalization of metric spaces and networks.

1. INTRODUCTION

This paper is the result of an attempt to obtain the interleaving guarantee for the sparse Čech complex of Cavanna, Jahanseir and Sheehy [7] without using the Nerve Theorem. The rationale for this was to generalize the result to arbitrary metric spaces. We have not been able to show that the constructions of [21] or [7] are interleaved with the Čech complex in arbitrary metric spaces. However, changing the construction slightly, we obtain a sub-complex of the Čech complex that is interleaved in a similar way. When applied to point clouds in $\mathbb{R}^d$ with a convex metric this sub-complex is homotopic to the construction of [7].

The search for a more general version of the sparse Čech complex led us to study both different versions of filtered covers and extended metrics. We discovered that these concepts are instances of filtered relations given by functions of the form

$$\Lambda: L \times W \rightarrow [0, \infty]$$

from the product of two sets L and W to the interval $[0, \infty]$. Given $t \in [0, \infty]$, the relation Λ_t at filtration level t is

$$\Lambda_t = \{(l, w) \in L \times W \mid \Lambda(l, w) < t\}.$$

[14] observed that a relation $R \subseteq L \times W$ gives a cover $(R(l))_{l \in L}$ of the set

$$R_W = \{w \in W \mid \text{there exists } l \in L \text{ with } (l, w) \in R\}$$

with

$$R(l) = \{w \in W \mid (l, w) \in R\}.$$

The *Dowker complex* of the relation R is the Borsuk Nerve of this cover. The *Dowker Homology Duality Theorem* [14, Theorem 1] states that the Dowker complexes of R and the transposed relation

$$R^t = \{(w, l) \mid (l, w) \in R\} \subseteq W \times L$$

have isomorphic homology. In [12] Chowdhury and Mémoli have sharpened the Dowker Homology Duality Theorem to a Dowker Homotopy Duality Theorem stating that the Dowker complexes of R and R^t are homotopy equivalent after geometric realization. That result is a central ingredient in this paper.

In honor of Dowker we name functions $\Lambda: L \times W \rightarrow [0, \infty]$ *Dowker dissimilarities*. Forming the Dowker complexes of the relations Λ_t for $t \in [0, \infty]$ we obtain a filtered simplicial complex, the *Dowker Nerve* $N\Lambda$ of Λ , with $N\Lambda_t$ equal to the Dowker complex of Λ_t .

The main result of our work is Theorem 11.5 on sparsification of Dowker nerves. Here we formulate it in the context of a finite set P contained in a metric space (M, d) . Let $p_0, \dots, p_n$ be a farthest point sampling of P with insertion radii $\lambda_0, \dots, \lambda_n$. That is, $p_0 \in P$ is arbitrary, $\lambda_0 = \infty$ and for each $0 < k \leq n$, the point $p_k \in P$ is of maximal distance to $p_0, \dots, p_{k-1}$, and this distance is λ_k . Let $\varepsilon > 0$ and let $\Lambda: P \times M \rightarrow [0, \infty]$ be the Dowker dissimilarity given by the metric d , that is, $\Lambda(p, w) = d(p, w)$. Then the Dowker Nerve $N\Lambda$ is equal to the relative Čech complex $\check{C}(P, M)$ of P in M consisting of all balls in M centered at points in P . Let $[n] = \{0, \dots, n\}$ and let $\varphi: [n] \rightarrow [n]$ be a function with $\varphi(0) = 0$ and $\varphi(k) < k$ and

$$d(p_k, p_{\varphi(k)}) + (\varepsilon + 1)\lambda_k/\varepsilon \leq (\varepsilon + 1)\lambda_{\varphi(k)}/\varepsilon$$

for $k = 1, \dots, n$. The *Sparse Dowker Nerve* of Λ is the filtered subcomplex $N(\Lambda, \varphi, (\varepsilon + 1)\lambda/\varepsilon)$ of $N\Lambda$ with $N(\Lambda, \varphi, (\varepsilon + 1)\lambda/\varepsilon)_t$ consisting of subsets $\sigma \subseteq P$ such that there exists $w \in M$ with

$$d(p_k, w) < \min\{t, (\varepsilon + 1)\lambda_k/\varepsilon, (\varepsilon + 1)\lambda_{\varphi(k)}/\varepsilon\}$$

for every $k, l \in [n]$.

Theorem 1.1. *The Sparse Dowker Nerve $N(\Lambda, \varphi, (\varepsilon + 1)\lambda/\varepsilon)$ is multiplicatively $(1, 1 + \varepsilon)$ -interleaved with the relative Čech complex $\check{C}(P, M)$ of P in M .*

Explicitly, there are maps $f_t: N\Lambda_t \rightarrow N(\Lambda, \varphi, (\varepsilon + 1)\lambda/\varepsilon)_{(1+\varepsilon)t}$ so that if $g_t: N(\Lambda, \varphi, (\varepsilon + 1)\lambda/\varepsilon)_t \rightarrow N\Lambda_t$ are the inclusion maps, then $f_t g_t$ and $g_{(1+\varepsilon)t} f_t$ are homotopic to the inclusion of the space of level

t into the space of level $(1 + \varepsilon)t$ of the filtered simplicial complexes $N(\Lambda, \varphi, (\varepsilon + 1)\lambda/\varepsilon)$ and $N\Lambda$ respectively. For $M = \mathbb{R}^d$ the Sparse Dowker Nerve is a closely related to the Sparse Čech Complex of [7]. We have implemented both constructions made them available at GitHub [2]. It turned out that the two constructions are of similar size. We will leave it for further work to implement a Sparse Dowker Nerve version of the Witness Complex.

Chazal et al. observed in [8] that witness complexes and Čech complexes are both instances of Dowker dissimilarities. The weighted Čech complex in [4, Definition 5.1] is also an instance of a Dowker complex. Also the filtered clique complex of a finite weighted undirected simple graph (G, w) is an instance of a Dowker nerve: let $\mathcal{P}(G)$ be the set of subsets of G and define

$$\Lambda: G \times \mathcal{P}(G) \rightarrow [0, \infty], \quad (v, V) \mapsto \begin{cases} \text{diam}(V) & \text{if } v \in V \\ \infty & \text{otherwise,} \end{cases}$$

where $\text{diam}(V) = \max_{v, v' \in V} w(v, v')$. Then the Dowker Nerve of Λ is equal to the filtered clique complex of G .

For disjoint sets L and W a Dowker dissimilarity $\Lambda: L \times W \rightarrow [0, \infty]$ is the same thing as a weighted simple bipartite graph. On the other hand, a Dowker dissimilarity of the form $\Lambda: X \times X \rightarrow [0, \infty]$ is the same thing as a weighted directed graph with no multiple edges. In [12] Dowker dissimilarities of this form are called weighted networks, and their Dowker nerves are studied thoroughly under the name Dowker complexes. In particular they show that the persistent homology of the Dowker Nerve of a network is sensitive to the direction its edges. For example, for the networks A and B in Figure 1, with self-loops of weight 0, the Dowker Nerve of network A is contractible while the Dowker Nerve of network B is homotopic to a circle at all filtration levels. Chowdhury and Mémoli also formulate a stability result for homology of Dowker nerves [12]. We formulate interleaving of Dowker dissimilarities in such a way that their network distance is bounded below by our interleaving distance. Together with functoriality for interleaving distance and the Algebraic Stability Theorem [11] this implies the stability result of [12]. In the context of metric spaces, this Stability Theorem is contained in [8].

Imposing conditions on a Dowker dissimilarity of the form

$$\Lambda: X \times X \rightarrow [0, \infty]$$

we arrive at concepts of independent interest. Most importantly, (X, Λ) is a metric space if and only if Λ satisfies

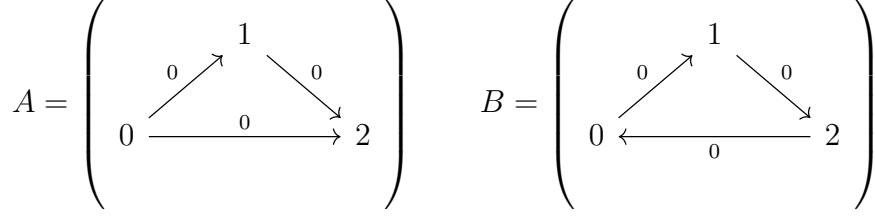


FIGURE 1. The Dowker Nerve of network A is contractible while the Dowker Nerve of network B is homotopic to a circle.

Finiteness: $\Lambda(x, y) < \infty$ for all $x, y \in X$

Triangle inequality: $\Lambda(x, z) \leq \Lambda(x, y) + \Lambda(y, z)$ for $x, y, z \in X$.

Identity of indiscernibles: $d(x, y) = 0$ if and only if $x = y$

Symmetry: $d(x, y) = d(y, x)$ for all $x, y \in X$

Removing some of the above conditions on Λ leads to various generalizations of metric spaces. In particular the situation where Λ only is required to satisfy the triangle inequality has been studied by Lawvere [18]. He noticed that $[0, \infty]$ is a closed symmetric monoidal category and that when the triangle inequality holds, then Λ gives X the structure of a category enriched over $[0, \infty]$.

Guided by the Functorial Dowker Theorem we have chosen to work with interleavings in the homotopy category instead of on the level of homology groups. We leave it for further investigation to decide if the Functorial Dowker Theorem can be extended to homotopy interleavings in the sense of Blumberg and Lesnick [1].

We extend the usual notion of interleaving between $[0, \infty]$ -filtered objects in two ways. Firstly, we consider interleavings in 2-categories. We were led to do this because Dowker dissimilarities form a 2-category, and the proof of the Stability Theorem is streamlined by working in this generality. Secondly, following [3] we allow interleaving with respect to order preserving functions of the form $\alpha: [0, \infty] \rightarrow [0, \infty]$ satisfying $t \leq \alpha(t)$ for all t . In this context additive interleaving corresponds to functions of the form $\alpha(t) = t + a$ and multiplicative interleaving corresponds to functions of the form $\alpha(t) = ct$.

After setting terminology and notation, the proof of our main result, Theorem 11.5, is a quite simple application of the functorial Dowker Theorem. It consists of two parts. First we truncate the Dowker dissimilarity associated to a metric by replacing certain distances by infinity and show that the truncated Dowker dissimilarity is interleaved with

the original Dowker dissimilarity. At that point we use the functorial Dowker Theorem. Second we give conditions that allow us to sparsify the Dowker Nerve of the truncated Dowker dissimilarity without changing the filtered homotopy type.

The paper is organized as follows: In Section 2 we present the homotopy category of simplicial complexes. In Section 3 we recollect basic terminology about 2-categories. The main motivation for going to this level of generality is that interleaving distance in the 2-category Dow of Dowker dissimilarities defined in 7.7 generalizes network distance from [12]. Section 4 introduces interleavings in 2-categories. In Section 5 we introduce the 2-category of sets and relations. Section 6 uses the Dowker Nerve construction to define a 2-category with relations as objects. In Section 7 we define the 2-category of Dowker dissimilarities and introduce the concept of a triangle relation used as a substitute for the triangle equation for metric spaces. In Section 8 we relate interleaving distance of Dowker dissimilarities to Gromov–Hausdorff distance of metric spaces. Section 9 shows that, under certain conditions, when some of the values $\Lambda(l, w)$ in a Dowker dissimilarity are set to infinity the homotopy type of the Dowker Nerve is only changed up to a certain interleaving. This is the first step in our proof of Theorem 11.5. In Section 10 we give a criterion ensuring that a certain sub-complex is homotopy equivalent to the Dowker Nerve of a Dowker dissimilarity. Finally in Section 11 we combine the results of sections 9 and 10 to obtain Theorem 11.5. We also show how Theorem 1.1 is a consequence of Theorem 11.5 and how the Sparse Čech complex [7] fits into this context.

2. THE HOMOTOPY CATEGORY OF SIMPLICIAL COMPLEXES

Recall that a simplicial complex $K = (V, K)$ consists of a vertex set V and a set K of finite subsets of V with the property that if σ is a member of K , then every subset of σ is a member of K . Given a subset $V' \subseteq V$ and a simplicial complex $K = (V, K)$, we write $K_{V'}$ for the simplicial complex $K_{V'} = (V', K_{V'})$ consisting of subsets of V' of the form $\sigma \cap V'$ for $\sigma \in K$. The *geometric realization* of a simplicial complex $K = (V, K)$ is the space $|K|$ consisting of all functions $f: V \rightarrow \mathbb{R}$ satisfying:

- (1) The support $\{v \in V \mid f(v) \neq 0\}$ of f is a member of K
- (2) $\sum_{v \in V} f(v) = 1$.

If V is finite, then $|K|$ is given the subspace topology of the Euclidean space $\mathbb{R}^V$. Otherwise $U \subseteq |K|$ is open if and only if for every finite $V' \subseteq V$, the set $U \cap |K_{V'}|$ is open in $|K_{V'}|$.

A *simplicial map* $f: K \rightarrow L$ of simplicial complexes $K = (V, K)$ and $L = (W, L)$ consists of a function $f: V \rightarrow W$ such that

$$f(\sigma) = \{f(v) \mid v \in \sigma\}$$

is in L for every $\sigma \in K$. Observe that a simplicial map $f: K \rightarrow L$ induces a continuous map $|f|: |K| \rightarrow |L|$ of geometric realizations and that this promotes the geometric realization to a functor $|\cdot|: \mathbf{Cx} \rightarrow \mathbf{Top}$ from the category $\mathbf{Cx}$ of simplicial complexes and simplicial maps to the category $\mathbf{Top}$ of topological spaces and continuous maps.

Definition 2.1. The *homotopy category* $\mathbf{hCx}$ of simplicial complexes has the class of simplicial complexes as objects. Given simplicial complexes K and L , the morphism set $\mathbf{hCx}(K, L)$ is the set of homotopy classes of continuous maps from the geometric realization of K to the geometric realization of L . Composition in $\mathbf{hCx}$ is given by composition of functions representing homotopy classes.

We remark in passing that the homotopy category of simplicial complexes is equivalent to the weak homotopy category of topological spaces.

3. BACKGROUND ON 2-CATEGORIES

The material in this section is standard. We have taken it from [19]. Recall that a 2-category $\mathcal{C}$ consists of

- (1) A class of objects $A, B, \dots$,
- (2) For all objects A, B a category $\mathcal{C}(A, B)$. The objects of $\mathcal{C}(A, B)$ are the morphisms in $\mathcal{C}$ and the morphisms $\alpha: f \Rightarrow g$ of $\mathcal{C}(A, B)$ are the 2-cells in $\mathcal{C}$.
- (3) For every object A of $\mathcal{C}$ there is an identity morphism $\text{id}_A: A \rightarrow A$ and an identity 2-cell $\text{id}_{\text{id}_A}: \text{id}_A \Rightarrow \text{id}_A$.
- (4) For all objects A, B and C of $\mathcal{C}$ there is a functor

$$\begin{aligned} \mathcal{C}(A, B) \times \mathcal{C}(B, C) &\rightarrow \mathcal{C}(A, C) \\ (f, g) &\mapsto g \cdot f \end{aligned}$$

which is associative and admits the identity morphisms and identity 2-cells of $\mathcal{C}$ as identities.

Definition 3.1. Given 2-categories $\mathcal{C}$ and $\mathcal{D}$, a *functor* $F: \mathcal{C} \rightarrow \mathcal{D}$ consists of

- (1) Function $F: \text{ob } \mathcal{C} \rightarrow \text{ob } \mathcal{D}$
- (2) Functors $F: \mathcal{C}(A, B) \rightarrow \mathcal{D}(FA, FB)$

such that $F(\text{id}_A) = \text{id}_{FA}$ and $Fg \circ Ff = F(g \circ f)$ for A an object of $\mathcal{C}$ and $f: A \rightarrow B$ and $g: B \rightarrow C$ morphisms of $\mathcal{C}$.

Definition 3.2. Given two functors $F, G: \mathcal{C} \rightarrow \mathcal{D}$ of 2-categories, a *transformation* $\alpha: F \rightarrow G$ consists of

- (1) A morphism $\alpha_A: FA \rightarrow GA$ for every $A \in \text{ob } \mathcal{C}$
- (2) A 2-cell $\alpha_f: Gf \circ \alpha_A \rightarrow \alpha_B \circ Ff$ for every morphism $f: A \rightarrow B$ in $\mathcal{C}$.

This structure is subject the axioms given by commutativity of the following two diagrams:

$$\begin{array}{ccc}
 & Gg \circ \alpha_B \circ Ff & \\
 \text{id}_{Gg} \circ \alpha_f \nearrow & & \searrow \alpha_g \circ \text{id}_{Ff} \\
 Gg \circ Gf \circ \alpha_A & \xrightarrow{\alpha_{g \cdot f}} & \alpha_C \circ Fg \circ Ff
 \end{array}$$

$$\begin{array}{ccc}
 & \sigma_A & \\
 \text{id} \nearrow & & \searrow \text{id} \\
 G(\text{id}_A) \circ \alpha_A & \xrightarrow{\alpha_{\text{id}_A}} & \sigma_A \circ F(\text{id}_A).
 \end{array}$$

Definition 3.3. Given two functors $F, G: \mathcal{C} \rightarrow \mathcal{D}$ of 2-categories, and transformations $\alpha, \beta: F \rightarrow G$, a *modification* $M: \alpha \rightarrow \beta$ consists of a 2-cell

$$M_A: \alpha_A \rightarrow \beta_A$$

for every object A of $\mathcal{C}$ such that for every morphism $f: A \rightarrow B$ of $\mathcal{C}$ the following diagram commutes:

$$\begin{array}{ccc}
 Gf \circ \alpha_A & \xrightarrow{\text{id}_{Gf} \circ M_A} & Gf \circ \beta_A \\
 \downarrow \alpha_f & & \downarrow \beta_f \\
 \alpha_B \circ Ff & \xrightarrow{M_B \circ \text{id}_{Ff}} & \beta_B \circ Ff.
 \end{array}$$

Definition 3.4. Given 2-categories $\mathcal{C}$ and $\mathcal{D}$, the *functor 2-category* $[\mathcal{C}, \mathcal{D}]$ is the 2-category with functors $F: \mathcal{C} \rightarrow \mathcal{D}$ as objects, transformations of such functors as morphisms and with 2-cells given by modifications.

Given a category $\mathcal{C}$ we will consider it as a 2-category with only identity 2-cells. Thus, if $\mathcal{C}$ is a category and $\mathcal{D}$ is a 2-category we have defined the functor 2-categories $[\mathcal{C}, \mathcal{D}]$ and $[\mathcal{D}, \mathcal{C}]$.

Definition 3.5. The opposite of a 2-category $\mathcal{C}$ is the 2-category $\mathcal{C}^{\text{op}}$ with the same objects as $\mathcal{C}$, with

$$\mathcal{C}^{\text{op}}(A, B) = \mathcal{C}(B, A)$$

and with composition obtained from composition in $\mathcal{C}$.

4. INTERLEAVINGS

We write $[0, \infty]$ for the extended set of non-negative real numbers and consider it as a partially ordered set. We also consider $[0, \infty]$ as a category with object set $[0, \infty]$ and with a unique morphism $s \rightarrow t$ if and only if $s \leq t$.

Definition 4.1. Let $\mathcal{C}$ be a 2-category. The *category of filtered objects* in $\mathcal{C}$ is the functor 2-category $[[0, \infty], \mathcal{C}]$. A *filtered object* in $\mathcal{C}$ is an object $C: [0, \infty] \rightarrow \mathcal{C}$ of $[[0, \infty], \mathcal{C}]$, that is, C is a functor from $[0, \infty]$ to $\mathcal{C}$. A *morphism* $f: C \rightarrow C'$ of filtered objects in $\mathcal{C}$ is a transformation.

Definition 4.2. Let $\mathcal{C}$ be a 2-category and let $\alpha: [0, \infty] \rightarrow [0, \infty]$ be a functor under the identity, that is, order preserving function satisfying $t \leq \alpha(t)$ for all $t \in [0, \infty]$.

- (1) The pull-back functor $\alpha^*: [[0, \infty], \mathcal{C}] \rightarrow [[0, \infty], \mathcal{C}]$ is the functor taking a filtered object $C: [0, \infty] \rightarrow \mathcal{C}$ in $\mathcal{C}$ to the filtered object $\alpha^*C = C \circ \alpha$.
- (2) The *unit of the functor* $\alpha^*: [[0, \infty], \mathcal{C}] \rightarrow [[0, \infty], \mathcal{C}]$ is the natural transformation $\alpha_*: \text{id} \rightarrow \alpha^*$ defined by

$$\alpha_{*C}(t) = C(t \leq \alpha(t)): C(t) \rightarrow \alpha^*(C)(t).$$

Definition 4.3. Let C and C' be filtered objects in a 2-category $\mathcal{C}$ and let $\alpha, \alpha': [0, \infty] \rightarrow [0, \infty]$ be functors under the identity.

- (1) An (α, α') -interleaving between C and C' is a pair (F, F') of morphisms $F: C \rightarrow \alpha^*C'$ and $F': C' \rightarrow \alpha'^*C$ in $[[0, \infty], \mathcal{C}]$ such that there exist 2-cells

$$(\alpha' \circ \alpha)_* \rightarrow (\alpha^*F') \circ F \quad \text{and} \quad (\alpha \circ \alpha')_* \rightarrow (\alpha'^*F) \circ F'.$$

- (2) We say that C and C' are (α, α') -interleaved if there exists an (α, α') -interleaving between C and C' .

The following results appear in [3, Proposition 2.2.11 and Proposition 2.2.13].

Lemma 4.4 (Functoriality). *Let C and C' be filtered objects in a 2-category $\mathcal{C}$, let $\alpha, \alpha': [0, \infty] \rightarrow [0, \infty]$ be functors under the identity and let $H: \mathcal{C} \rightarrow \mathcal{D}$ be a functor of 2-categories. If C and C' are (α, α') -interleaved, then the filtered objects HC and HC' in $\mathcal{D}$ are (α, α') -interleaved.*

Lemma 4.5 (Triangle inequality). *Let C , C' and C'' be filtered objects in a 2-category $\mathcal{C}$. If C and C' are (α, α') -interleaved and C' and C'' are (β, β') -interleaved, then C and C'' are $(\beta\alpha, \alpha'\beta')$ -interleaved.*

5. RELATIONS

Definition 5.1. Let X and Y be sets. A *relation* $R: X \rightrightarrows Y$ is a subset $R \subseteq X \times Y$.

Definition 5.2. We define a partial order on the set of relations between X and Y by set inclusion. That is, for relations $R: X \rightrightarrows Y$ and $R': X \rightrightarrows Y$, we have $R \leq R'$ if and only if R is contained in the subset of R' of $X \times Y$.

Definition 5.3. Given two relations $R: X \rightrightarrows Y$ and $S: Y \rightrightarrows Z$, their composition

$$S \circ R: X \rightrightarrows Z$$

is

$$S \circ R = \{(x, z) \in X \times Z \mid \exists y \in Y : (x, y) \in R \text{ and } (y, z) \in S\}.$$

Definition 5.4. The 2-category $\mathcal{S}$ of sets and relations has as objects the class of sets and as morphisms the class of relations. The 2-cells are given by the inclusion partial order on the class of relations. Composition of morphisms is composition of relations and composition of 2-cells is given by composition of inclusions. The identity morphism on the set X is the diagonal

$$\Delta_X = \{(x, x) \mid x \in X\}.$$

The identity 2-cell on a relation R is the identity inclusion $R \leq R$.

Definition 5.5. The transposition functor $T: \mathcal{S} \rightarrow \mathcal{S}^{\text{op}}$ is defined by $T(X) = X$,

$$T(R) = R^t = \{(y, x) \mid (x, y) \in R\}$$

and $T(i) = i^t$, where $i^t: R^t \rightarrow S^t$ takes (y, x) to (z, w) when $(w, z) = i(x, y)$.

Definition 5.6. A *correspondence* $C: X \rightrightarrows Y$ is a relation such that:

- (1) for every $x \in X$ there exists $y \in Y$ so that $(x, y) \in C$ and
- (2) for every $y \in Y$ there exists $x \in X$ so that $(x, y) \in C$.

Lemma 5.7. A relation $C: X \rightrightarrows Y$ is a correspondence if and only if there exists a relation $D: Y \rightrightarrows X$ so that $\Delta_X \leq D \circ C$ and $\Delta_Y \leq C \circ D$.

Proof. By definition of a correspondence, for every $x \in X$, there exists $y \in Y$ so that $(x, y) \in C$. This means that $\Delta_X \subseteq C^t \circ C$, where

$$C^t \circ C = \{(x, z) \in X \times X \mid \exists y \in Y : (x, y) \in C \text{ and } (y, z) \in C^t\}.$$

Reversing the roles of C and C^t we get the inclusion $\Delta_Y \subseteq C \circ C^t$. Conversely, if C and D are relations with $\Delta_Y \subseteq C \circ D$, then for every

$y \in Y$, the element (y, y) is contained in $C \circ D$. This means that there exists $x \in X$ so that $(x, y) \in C$, and $(y, x) \in D$. In particular, for every $y \in Y$, there exists $x \in X$ so that $(x, y) \in C$. Reversing the roles of C and D we get that for every $x \in X$ there exists $y \in Y$ so that $(x, y) \in C$. $\square$

6. THE CATEGORY OF RELATIONS

We start by recalling Dowker's definition of the nerve of a relation. (Called the complex K in [14, Section 1].)

Definition 6.1. Let $R \subseteq X \times Y$ be a relation. The *nerve* of R is the simplicial complex

$$NR = \{ \text{finite } \sigma \subseteq X \mid \exists y \in Y \text{ with } (x, y) \in R \text{ for all } x \in \sigma \}.$$

Example 6.2. Let X be a space, and let Y be a cover of X . In particular every element $y \in Y$ is a subset of X . Let R be the relation $R \subseteq X \times Y$ consisting of pairs (x, y) with $x \in y$. A direct inspection reveals that the nerve of R is equal to the Borsuk Nerve of the cover Y .

Definition 6.3. The 2-category $\mathcal{R}$ of relations has as objects the class of relations. A morphism $C: R \rightarrow R'$ in $\mathcal{R}$ between relations $R \subseteq X \times Y$ and $R' \subseteq X' \times Y'$ consists of a relation $C \subseteq X \times X'$ such that for every $\sigma \in NR$, the set

$$(NC)(\sigma) = \{x' \in X' \mid \text{there exists } x \in \sigma \text{ with } (x, x') \in C\}$$

is an element $(NC)(\sigma) \in NR'$ of the nerve of R' . In particular $(NC)(\sigma)$ is finite and non-empty. The class of 2-cells in $\mathcal{R}$ is the class of inclusions $R \subseteq S$ for $R, S \subseteq X \times Y$. Composition in $\mathcal{R}$ is given by composition of relations.

Lemma 6.4. Let $C_1, C_2: R \rightarrow R'$ be morphisms in $\mathcal{R}$. If there exists a 2-cell $\alpha: C_1 \rightarrow C_2$, then the simplicial maps NC_1 and NC_2 are contiguous. In particular, their geometric realizations are homotopic maps.

Proof. Let $\sigma \in NR$. Since $C_1 \subseteq C_2$, we have an inclusion

$$(NC_1)(\sigma) \subseteq (NC_2)(\sigma),$$

and thus $(NC_2)(\sigma) \in NR'$ implies

$$(NC_1)(\sigma) \cup (NC_2)(\sigma) = (NC_2)(\sigma) \in NR'.$$

This shows that NC_1 and NC_2 are contiguous. For the statement about contiguous maps having homotopic realizations see [22, Lemma 2, p. 130]. $\square$

Definition 6.5. The *nerve functor* $N: \mathcal{R} \rightarrow \mathbf{hCx}$ is the functor taking a relation R to its nerve NR and taking a morphism $C: R \rightarrow R'$ in $\mathcal{R}$ to the morphism $|NC|: |NR| \rightarrow |NR'|$ in $\mathbf{hCx}$.

Let us emphasize that if $\alpha: C_1 \rightarrow C_2$ is a 2-cell in $\mathcal{R}$, then $|NC_1| = |NC_2|$ in $\mathbf{hCx}$.

7. FILTERED RELATIONS AND DOWKER DISSIMILARITIES

Definition 7.1. A *filtered relation* is a functor from $[0, \infty]$ to $\mathcal{R}$. We define the 2-category of filtered relations to be the 2-category $[[0, \infty], \mathcal{R}]$ of functors from $[0, \infty]$ to $\mathcal{R}$.

Definition 7.2. The *filtered nerve functor* is the functor

$$N: [[0, \infty], \mathcal{R}] \rightarrow [[0, \infty], \mathbf{hTop}]$$

from the 2-category of filtered relations to the category of homotopy filtered spaces taking $X: [0, \infty] \rightarrow \mathcal{R}$ to the composition

$$[0, \infty] \xrightarrow{X} \mathcal{R} \xrightarrow{N} \mathbf{hTop}.$$

From Lemma 4.4 we get:

Corollary 7.3. *If R and R' are (α, α') -interleaved filtered relations, then NR and NR' are (α, α') -interleaved filtered simplicial complexes.*

Definition 7.4. A *Dowker dissimilarity* Λ consists of two sets L and W and a function $\Lambda: L \times W \rightarrow [0, \infty]$. Given $t \in [0, \infty]$, we let

$$\Lambda_t = \{(l, w) \in L \times W \mid \Lambda(l, w) < t\}$$

considered as an object of the category $\mathcal{R}$ of relations, and given $s \leq t$ in $[0, \infty]$ we let

$$\Lambda_{s \leq t} = \Delta_L$$

considered as a morphism $\Lambda_{s \leq t}: \Lambda_s \rightarrow \Lambda_t$ in $\mathcal{R}$.

Definition 7.5. The *filtered relation associated to a Dowker dissimilarity* $\Lambda: L \times W \rightarrow [0, \infty]$ is the functor

$$\Lambda: [0, \infty] \rightarrow \mathcal{R}$$

taking $t \in [0, \infty]$ to the relation Λ_t and taking $s \leq t$ in $[0, \infty]$ to the morphism $\Lambda_{s \leq t}$ in $\mathcal{R}$.

Definition 7.6. Let $\Lambda: L \times W \rightarrow [0, \infty]$ and $\Lambda': L' \times W' \rightarrow [0, \infty]$ be Dowker dissimilarities. A morphism $C: \Lambda \rightarrow \Lambda'$ of filtered relations is a *morphism of Dowker dissimilarities* if there exists a relation $C \subseteq L \times L'$ so that $C_t = C: \Lambda_t \rightarrow \Lambda'_t$ for every $t \in [0, \infty]$.

Definition 7.7. The 2-category *Dow* of *Dowker dissimilarities* is the 2-category with Dowker dissimilarities as objects and morphisms of Dowker dissimilarities as morphisms. Given morphisms $C_1, C_2: \Lambda \rightarrow \Lambda'$ of Dowker dissimilarities, we define the set of 2-cells $\alpha: C_1 \rightarrow C_2$ in *Dow* by letting $\text{Dow}(C_1, C_2) = [[0, \infty], \mathcal{R}](C_1, C_2)$.

Definition 7.8. Let $\Lambda: L \times W \rightarrow [0, \infty]$ be a Dowker dissimilarity. The *Dowker Nerve* $N\Lambda$ of Λ is the filtered nerve of the underlying filtered relation.

Note that the Dowker Nerve is filtered by inclusion of sub-complexes, that is, if $s \leq t$, then $N\Lambda_{s \leq t}: N\Lambda_s \rightarrow N\Lambda_t$ is an inclusion of simplicial complexes.

Definition 7.9. The *cover radius* of a Dowker dissimilarity

$$\Lambda: L \times W \rightarrow [0, \infty]$$

is

$$\rho_\Lambda = \sup_{w \in W} \inf_{l \in L} \Lambda(l, w).$$

Definition 7.10. Let $\Lambda: L \times W \rightarrow [0, \infty]$ be a Dowker dissimilarity. Given $l \in L$ and $t > 0$, the Λ -ball of radius t centered at l is

$$B_\Lambda(l, t) = \{w \in W \mid \Lambda(l, w) < t\}.$$

Example 7.11. Let (M, d) be a metric space and L and W be subsets of M . Then the restriction $\Lambda: L \times W \rightarrow [0, \infty]$ of d to $L \times W$ is a Dowker dissimilarity. Its cover radius $\rho_\Lambda = \sup_{w \in W} \inf_{l \in L} d(l, w)$ is the directed Hausdorff distance from W to L . The Dowker Nerve of Λ is the composite

$$[0, \infty] \xrightarrow{\Lambda} \mathcal{R} \xrightarrow{N} \text{Cx}$$

taking $t \in [0, \infty]$ to

$$\{\text{finite } \sigma \subseteq L \mid \text{there exists } w \in W \text{ with } d(l, w) < t \text{ for all } l \in \sigma\}.$$

If $L = W = M$, then the Λ -ball of radius t centered at l is the usual open ball in M of radius t centered at l and the Dowker Nerve of Λ is equal to the Čech complex $\check{C}(M)$.

Lemma 7.12. Let $\Lambda: L \times W \rightarrow [0, \infty]$ be a Dowker dissimilarity. Given $t > 0$, the nerve $N\Lambda_t$ is isomorphic to the Borsuk Nerve of the cover of the set

$$\bigcup_{l \in L} B_\Lambda(l, t)$$

by balls $B_\Lambda(l, s)$ of radius $s \leq t$ centered at points in L .

Corollary 7.3 gives:

Corollary 7.13. *If $\Lambda: L \times W \rightarrow [0, \infty]$ and $\Lambda': L' \times W' \rightarrow [0, \infty]$ are (α, α') -interleaved Dowker dissimilarities, then $N\Lambda$ and $N\Lambda'$ are (α, α') -interleaved filtered simplicial complexes.*

Definition 7.14. Let $\Lambda: L \times W \rightarrow [0, \infty]$ be a Dowker dissimilarity. The *Rips complex* of Λ is the filtered simplicial complex $R\Lambda$ defined by

$$(R\Lambda)(t) = \{\text{finite } \sigma \subseteq L \mid \text{every } \tau \subseteq \sigma \text{ with } |\tau| \leq 2 \text{ is in } (N\Lambda)(t)\}.$$

Corollary 7.15. *If $\Lambda: L \times W \rightarrow [0, \infty]$ and $\Lambda': L' \times W' \rightarrow [0, \infty]$ are (α, α') -interleaved Dowker dissimilarities, then $R\Lambda$ and $R\Lambda'$ are (α, α') -interleaved filtered simplicial complexes.*

Proof. Use Corollary 7.13 and the fact that the Rips complex depends functorially on the one skeleton of the Dowker Nerve. $\square$

The following definition is an instance of the generalized inverse in [16].

Definition 7.16. Let $\alpha: [0, \infty] \rightarrow [0, \infty]$ be order preserving with

$$\lim_{t \rightarrow \infty} \alpha(t) = \infty.$$

The generalized inverse function $\alpha^\leftarrow: [0, \infty] \rightarrow [0, \infty]$ is the order preserving function

$$\alpha^\leftarrow(s) = \inf\{t \in [0, \infty] \mid \alpha(t) \geq s\}.$$

Lemma 7.17. *Given a Dowker dissimilarity $\Lambda: L \times W \rightarrow [0, \infty]$ and an order preserving function $\alpha: [0, \infty] \rightarrow [0, \infty]$, the filtered relation associated to the Dowker dissimilarity Λ given as the composite function*

$$L \times W \xrightarrow{\Lambda} [0, \infty] \xrightarrow{\alpha^\leftarrow} [0, \infty],$$

is equal to $\alpha^ \Lambda$.*

Definition 7.18. A *triangle relation* for a Dowker dissimilarity

$$\Lambda: L \times W \rightarrow [0, \infty]$$

is a relation $T \subseteq L \times W$ with the following properties:

- (1) For every $w \in W$ there exists $l \in L$ so that $(l, w) \in T$.
- (2) For all $(l, w) \in T$ and $(l', w') \in L \times W$, the triangle inequality

$$\Lambda(l', w') \leq \Lambda(l', w) + \Lambda(l, w') + \Lambda(l, w)$$

holds.

Remark 7.19.

- (1) If $\Lambda_M: M \times M \rightarrow [0, \infty]$ satisfies the triangle inequality

$$\Lambda_M(x, z) \leq \Lambda_M(x, y) + \Lambda_M(y, z)$$

for all $x, y, z \in Z$, then every relation $T \subseteq M \times M$ satisfies part (2) of Definition 7.18. Moreover, if L and W are subsets of M and $\Lambda: L \times W \rightarrow M$ is the restriction of Λ_M to $L \times W$, then every relation $T \subseteq L \times W$ satisfies part (2) of Definition 7.18.

- (2) Given a Dowker dissimilarity $\Lambda: L \times W \rightarrow [0, \infty]$ so that the set $\Lambda(L \times \{w\})$ has a least upper bound for every $w \in W$, there exists a triangle relation T for Λ consisting of the pairs (l, w) satisfying $\Lambda(l', w) \leq \Lambda(l, w)$ for all $l' \in L$.

8. STABILITY AND INTERLEAVING DISTANCE

The functoriality of interleaving implies that all functorial constructions are stable with respect to interleaving. In this section we relate interleaving distance of Dowker dissimilarities to Gromov–Hausdorff distance of [15, 17] and to the network distance defined in [12]

Definition 8.1. Let C and C' be filtered objects in a 2-category $\mathcal{C}$.

- (1) Given $a, a' \in [0, \infty]$ we say that the filtered objects C and C' are *additively (a, a') -interleaved* if they are (α, α') -interleaved for the functions $\alpha(t) = a + t$ and $\alpha'(t) = a' + t$.
- (2) Let

$$A(C, C') = \{a \in [0, \infty] \mid C \text{ and } C' \text{ are additively } (a, a)\text{-interleaved}\}.$$

The *interleaving distance* of C and C' is

$$d_{\text{int}}(C, C') = \begin{cases} \inf A(C, C') & \text{if } A(C, C') \neq \emptyset \\ \infty & \text{otherwise.} \end{cases}$$

Definition 8.2. A *non-negatively weighted network* is a pair (X, ω_X) of a set X and a weight function $\omega_X: X \times X \rightarrow [0, \infty)$.

Definition 8.3. Let $\omega_X: X \times X \rightarrow [0, \infty)$ and $\omega_{X'}: X' \times X' \rightarrow [0, \infty)$ be non-negatively weighted networks and let $C \subseteq X \times X'$. The *distortion* of C is

$$\text{dis}(C) = \sup_{(x, x'), (y, y') \in C} |\omega_X(x, y) - \omega_{X'}(x', y')|.$$

Recall from 5.6 that $C \subseteq X \times X'$ is a correspondence if the projections of C on both X and X' are surjective.

Definition 8.4. Let $\omega_X: X \times X \rightarrow [0, \infty)$ and $\omega_{X'}: X' \times X' \rightarrow [0, \infty)$ be non-negatively weighted networks and let $\mathcal{R}$ be the set of correspondences $C \subseteq X \times X'$. The *network distance* between X and X' is

$$d_{\mathcal{N}}(X, X') = \frac{1}{2} \inf_{C \in \mathcal{R}} \text{dis}(C).$$

The Stability Theorem [12, Proposition 15] for networks is a consequence of functoriality of interleaving distance, the Algebraic Stability Theorem for bottleneck distance [10, Theorem 4.4] and the following result:

Proposition 8.5. *Let $\omega_X: X \times X \rightarrow [0, \infty)$ and $\omega_{X'}: X' \times X' \rightarrow [0, \infty)$ be networks, and write*

$$\Lambda: X \times X \rightarrow [0, \infty] \quad \text{and} \quad \Lambda': X' \times X' \rightarrow [0, \infty]$$

for the corresponding Dowker dissimilarities with $\Lambda(x, y) = \omega_X(x, y)$ and $\Lambda'(x', y') = \omega_{X'}(x', y')$. Then

$$d_{\text{int}}(\Lambda, \Lambda') \leq 2 d_{\mathcal{N}}(X, X').$$

Proof. We have to show that $d_{\text{int}}(\Lambda, \Lambda') \leq \text{dis}(C)$ for every correspondence $C \subseteq X \times X'$. So let $C \subseteq X \times X'$ be a correspondence and let $a > \text{dis}(C)$. By definition of $\text{dis}(C)$, for all (l, l') and (w, w') in C we have

$$|\omega_X(l, w) - \omega_{X'}(l', w')| < a.$$

Defining $\alpha: [0, \infty] \rightarrow [0, \infty]$ by $\alpha(t) = t + a$, by symmetry, it suffices to show that C defines a morphism

$$C: \Lambda \rightarrow \alpha^* \Lambda'.$$

That is, we have to show that if $\sigma \in \Lambda_t$, then $(NC)(\sigma) \in \Lambda'_{\alpha t}$. So suppose that $w \in X$ satisfies $\Lambda(l, w) < t$ for all $l \in \sigma$. Since C is a correspondence we can pick $w' \in X'$ so that $(w, w') \in C$. By definition of NC , for every $l' \in (NC)(\sigma)$, there exists $l \in \sigma$ so that $(l, l') \in C$. By definition of distortion distance this gives

$$\Lambda'(l', w') = \omega_{X'}(l', w') < a + \omega_X(l, w) = a + \Lambda(l, w) < a + t = \alpha t.$$

We conclude that $\sigma \in N\Lambda_t$ implies $(NC)(\sigma) \in N\Lambda'_{\alpha t}$ as desired. $\square$

The Stability Theorem [9, Theorem 5.2] for metric spaces is a consequence of functoriality of interleaving distance, the Algebraic Stability Theorem for bottleneck distance [10, Theorem 4.4] and the following result:

Corollary 8.6. *Let (M, d) and (M', d') be metric spaces, and write*

$$\Lambda: M \times M \rightarrow [0, \infty] \quad \text{and} \quad \Lambda': M' \times M' \rightarrow [0, \infty]$$

for the corresponding Dowker dissimilarities with $\Lambda(p, q) = d(p, q)$ and $\Lambda'(p', q') = d'(p', q')$. Then

$$d_{\text{int}}(\Lambda, \Lambda') \leq 2d_{GH}(M, M').$$

Proof. By [5, Theorem 7.3.25] the Gromov–Hausdorff distance of the metric spaces (M, d) and (M', d') agrees with their network distance when we consider them as non-negatively weighted networks. That is, $d_{GH}(M, M') = d_{\mathcal{N}}(M, M')$. The result now follows from Proposition 8.5. $\square$

9. TRUNCATED DOWKER DISSIMILARITIES

Definition 9.1. Let $\Lambda: L \times W \rightarrow [0, \infty]$ be a Dowker dissimilarity, let $T \subseteq L \times W$ be a triangle relation for Λ and let $\beta: [0, \infty] \rightarrow [0, \infty]$ be an order preserving function. A *T -insertion function for Λ of resolution at most β* is a function $\lambda: W \rightarrow [0, \infty]$ with the property that for every $t \in [0, \infty]$ and for every $(l, w) \in T$ there exists $w_0 \in W$ so that

$$\Lambda(l, w_0) \leq \beta(t) < \lambda(w_0).$$

Example 9.2. Recall the Dowker dissimilarity $\Lambda: L \times W \rightarrow [0, \infty]$ from Example 7.11 for two subsets L and W of a metric space (M, d) and let β be any order preserving function with $\beta(t) > \rho_\Lambda$. Then for every $T \subseteq L \times W$ the function $\lambda \equiv \infty$ is a T -insertion function for Λ of resolution at most β .

In the following definition we use the generalized inverse from Definition 7.16.

Definition 9.3. Let $\Lambda: L \times W \rightarrow [0, \infty]$ be a Dowker dissimilarity with a triangle relation T and a T -insertion function λ of resolution at most β for an order preserving $\beta: [0, \infty] \rightarrow [0, \infty]$ with

$$\lim_{t \rightarrow \infty} \beta(t) = \infty.$$

Given an order preserving function $\alpha: [0, \infty] \rightarrow [0, \infty]$, satisfying that $\alpha(t) \geq t + \beta(t)$ for all t , the (λ, α, β) -truncation of Λ is the Dowker dissimilarity $\Lambda^{(\lambda, \alpha, \beta)}: L \times W \rightarrow [0, \infty]$ defined by

$$\Lambda^{(\lambda, \alpha, \beta)}(l, w) = \begin{cases} \Lambda(l, w) & \text{if } \Lambda(l, w) \leq \alpha\beta^{\leftarrow}\lambda(w) \\ \infty & \text{otherwise.} \end{cases}$$

Lemma 9.4. *Let $\Lambda: L \times W \rightarrow [0, \infty]$ be a Dowker dissimilarity with a triangle relation T and a T -insertion function λ of resolution at most $\beta: [0, \infty] \rightarrow [0, \infty]$. If $\alpha: [0, \infty] \rightarrow [0, \infty]$ is an order preserving function satisfying*

$$\alpha(t) \geq t + \beta(t) + \sup \Lambda(T)$$

for all $t \in [0, \infty]$, then Δ_L is a morphism $\Delta_L: \Lambda \rightarrow \alpha^ \Lambda^{(\lambda, \alpha, \beta)}$ of Dowker dissimilarities.*

Proof. Let $t \in [0, \infty]$ and $\sigma \in N\Lambda_t$. We need to show $\sigma \in N\Lambda_{\alpha t}^{(\lambda, \alpha, \beta)}$. Pick $w \in W$ with $\Lambda(l, w) < t$ for all $l \in \sigma$. Since T is a triangle relation we can pick $l_0 \in L$ so that $(l_0, w) \in T$. Since λ is a T -insertion function of resolution at most β we can pick $w_0 \in W$ so that

$$\Lambda(l_0, w_0) \leq \beta(t) < \lambda w_0.$$

The triangle inequality for T now gives

$$\Lambda(l, w_0) \leq \Lambda(l_0, w_0) + \Lambda(l_0, w) + \Lambda(l, w).$$

We have picked l_0, w and w_0 so that $\Lambda(l_0, w) \leq \sup \Lambda(T)$ and also $\Lambda(l_0, w_0) \leq \beta t$. If $l \in \sigma$, then $\Lambda(l, w) < t$, and thus

$$\Lambda(l, w_0) < \beta t + \sup \Lambda(T) + t = \alpha t.$$

From part (5) in [16, Proposition 1] the inequality $\beta(t) < \lambda(w_0)$ gives $t \leq \beta^{\leftarrow} \lambda(w_0)$. Since α is order preserving we get $\Lambda(l, w_0) < \alpha \beta^{\leftarrow} \lambda w_0$. We conclude that $\sigma \in N\Lambda_{\alpha t}^{(\lambda, \alpha, \beta)}$. $\square$

Proposition 9.5. *Let $\Lambda: L \times W \rightarrow [0, \infty]$ be a Dowker dissimilarity with an insertion function $\lambda: W \rightarrow [0, \infty]$ of resolution at most $\beta: [0, \infty] \rightarrow [0, \infty]$ and a triangle relation $T \subseteq L \times W$. If an order preserving function $\alpha: [0, \infty] \rightarrow [0, \infty]$ satisfies*

$$\alpha(t) \geq t + \beta(t) + \sup \Lambda(T)$$

for all $t \in [0, \infty]$, then the Dowker dissimilarities Λ and $\Lambda^{(\lambda, \alpha, \beta)}$ are (α, id) -interleaved.

Proof. By Lemma 9.4, the relation Δ_L gives a morphism

$$\Delta_L: \Lambda \rightarrow \alpha^* \Lambda^{(\lambda, \alpha, \beta)}$$

of Dowker dissimilarities. Since $\Lambda(l, w) \leq \Lambda^{(\lambda, \alpha, \beta)}(l, w)$ for all $(l, w) \in L \times W$, the relation Δ_L also gives a morphism $\Delta_L: \Lambda^{(\lambda, \alpha, \beta)} \rightarrow \Lambda$ of Dowker dissimilarities. $\square$

10. SPARSE DOWKER NERVES

Definition 10.1. Let $\Lambda: L \times W \rightarrow [0, \infty]$ be a Dowker dissimilarity and let $\varphi: L \rightarrow L$ and $\lambda: L \rightarrow [0, \infty]$ be functions. Given $\sigma \in N\Lambda_\infty$, the *radius* of σ is

$$r(\sigma) = \inf\{t \mid \sigma \in N\Lambda_t\}.$$

The *sparse* (φ, λ) -nerve of Λ is the filtered simplicial complex $N(\Lambda, \varphi, \lambda)$ defined by

$$N(\Lambda, \varphi, \lambda)(t) = \{\sigma \in N\Lambda_t \mid r(\sigma) \leq \lambda(\varphi(l)) \text{ for all } l \in \sigma\}.$$

Proposition 10.2. Let $\Lambda: L \times W \rightarrow [0, \infty]$ be a Dowker dissimilarity and let $\varphi: L \rightarrow L$ and $\lambda: L \rightarrow [0, \infty]$ be functions. Suppose there exists $l_0 \in L$ and an integer $N \geq 0$ so that for all $l \in L$ and all $t \in [0, \infty]$:

- (1) $\varphi^N(l) = l_0$.
- (2) $B_\Lambda(l, \lambda(l)) \subseteq B_\Lambda(\varphi(l), \lambda(\varphi(l)))$.
- (3) $B_\Lambda(l, t) = B_\Lambda(l, \lambda(l))$ if $\lambda(l) \leq t$.
- (4) $\lambda(\varphi(l)) \geq \lambda(l)$.

Then for every $t \in [0, \infty]$ the inclusion of $N(\Lambda, \varphi, \lambda)(t)$ in $(N\Lambda)(t)$ is a homotopy equivalence.

Proof. Assumptions (1), (3) and (4) together imply that $N\Lambda_t = N\Lambda_{\lambda(l_0)}$ and $N(\Lambda, \varphi, \lambda)(t) = N(\Lambda, \varphi, \lambda)(\lambda(l_0))$ for $t \geq \lambda(l_0)$. Thus it suffices to prove the claim for $t \leq \lambda(l_0)$. In this situation we will show that the inclusions of

$$N_t(\Lambda, \varphi, \lambda) = \{\sigma \in N\Lambda_t \mid t \leq \lambda(\varphi(l)) \text{ for all } l \in \sigma\}$$

in both $N(\Lambda, \varphi, \lambda)(t)$ and in $N\Lambda_t$ are deformation retracts. For this it suffices to find a map $f: N\Lambda_t \rightarrow N\Lambda_t$ with the following three properties: firstly both f and its restriction $f: N(\Lambda, \varphi, \lambda)(t) \rightarrow N(\Lambda, \varphi, \lambda)(t)$ are contiguous to the identity. Secondly we have $f(\sigma) = \sigma$ for every $\sigma \in N_t(\Lambda, \varphi, \lambda)$, and thirdly $f(\sigma) \in N_t(\Lambda, \varphi, \lambda)$ for every $\sigma \in N\Lambda_t$.

For $t \leq \lambda(l_0)$ we use assumption (1) to define a function $f: L \rightarrow L$ by

$$f(l) = \varphi^m(l) \text{ for } m \geq 0 \text{ minimal with } \lambda(\varphi^{m+1}(l)) \geq t.$$

Given $\sigma \in N\Lambda_t$ we let $f(\sigma) = \{f(l) \mid l \in \sigma\}$. By construction, if $\sigma \in N_t(\Lambda, \varphi, \lambda)$, then $f(\sigma) = \sigma$. On the other hand, by construction, $\lambda(\varphi(f(l))) \geq t$ for all $l \in \sigma$ so $f(\sigma) \in N_t(\Lambda, \varphi, \lambda)$.

Note that if $\lambda(\varphi(l)) < t$, then assumption (2) gives

$$B_\Lambda(l, \lambda(l)) \subseteq B_\Lambda(\varphi(l), \lambda(\varphi(l))) \subseteq B_\Lambda(\varphi(l), t),$$

and together with assumptions (3) and (4) we get

$$B_\Lambda(l, t) \subseteq B_\Lambda(\varphi(l), t).$$

On the other hand, if $\lambda(\varphi(l)) \geq t$, then $f(l) = l$. It follows by induction that $B_\Lambda(l, t) \subseteq B_\Lambda(f(l), t)$ for every $l \in L$. This implies that the map $f: L \rightarrow L$ induces simplicial maps $f: N\Lambda_t \rightarrow N\Lambda_t$ and $f: N(\Lambda, \varphi, \lambda)(t) \rightarrow N(\Lambda, \varphi, \lambda)(t)$ which are contiguous to the respective identity maps. $\square$

11. DOWKER DISSIMILARITIES ON FINITE ORDINALS

In this section give a sparse approximation to the Dowker Nerve for Dowker dissimilarities of the form

$$\Lambda: L \times [n] \rightarrow [0, \infty],$$

where $[n] = \{0 < 1 < \dots < n\}$.

Definition 11.1. Let $n \geq 0$ be a natural number, let

$$\Lambda: L \times [n] \rightarrow [0, \infty]$$

be a Dowker dissimilarity and let $T \subseteq L \times [n]$ be a triangle relation for Λ .

(1) The *domain* of T is the set

$$D(T) = \{l \in L \mid \text{there exists } k \in [n] \text{ with } (l, k) \in T\}.$$

(2) The *insertion radius* of $k \in [n]$ with respect to Λ and T is

$$\lambda_{\Lambda, T}(k) = \begin{cases} \infty & \text{if } k = 0 \\ \sup_{l \in D(T)} \inf_{i \in [k-1]} \Lambda(l, i) & \text{if } k > 0. \end{cases}$$

Recall the definition of the cover radius ρ_Λ of a Dowker dissimilarity in Definition 7.9 and the definition of T -insertion functions for Λ in 9.1.

Lemma 11.2. Let $\Lambda: L \times [n] \rightarrow [0, \infty]$ be a Dowker dissimilarity and let $\beta: [0, \infty] \rightarrow [0, \infty]$ be an order preserving function with $\beta(t) \geq \rho_\Lambda$ for all $t \in [0, \infty]$. The insertion radius $\lambda_{\Lambda, T}: [n] \rightarrow [0, \infty]$ with respect to Λ and T is a T -insertion function for Λ of resolution at most β .

Proof. Given $t \in [0, \infty]$ and $l \in L$, let $i \in [n]$ be minimal under the condition that $\Lambda(l, i) \leq \beta t$. Then, by definition of $\lambda_{\Lambda, T}$, we have $\lambda_{\Lambda, T}(i) > \beta t$. $\square$

Definition 11.3. Let $\Lambda: L \times [n] \rightarrow [0, \infty]$ be a Dowker dissimilarity, let $T \subseteq L \times [n]$ be a triangle relation for Λ and let $\beta: [0, \infty] \rightarrow [0, \infty]$ be an order preserving function with $\beta(t) \geq \rho_\Lambda$ for all $t \in [0, \infty]$. Suppose that $\lim_{t \rightarrow \infty} \beta(t) = \infty$ and let $\alpha: [0, \infty] \rightarrow [0, \infty]$ be the function

$$\alpha(t) = t + \beta(t) + \sup(\Lambda(T))$$

and let $\lambda: [n] \rightarrow [n]$ be the function

$$\lambda(k) = \alpha\beta^{\leftarrow}\lambda_{\Lambda,T}(k)$$

The *parent function* $\varphi: [n] \rightarrow [n]$ is defined by letting $\varphi(0) = 0$ and $\varphi(k) = \max\{i \in [k-1] \mid B_{\Lambda}(k, \lambda(k)) \subseteq B_{\Lambda}(i, \lambda(i)) \text{ and } \lambda(k) \leq \lambda(i)\}$.

The following result is about sparsification of truncated Dowker dissimilarities. We remind that the truncated Dowker dissimilarity $\Lambda^{(\lambda_{\Lambda}, \alpha, \beta)}$ comes from Definition 9.3.

Theorem 11.4. *Suppose, in the situation of Definition 11.3, that $B_{\Lambda^t}(0, \infty) = L$. If we let $\Gamma = (\Lambda^{(\lambda_{\Lambda}, \alpha, \beta)})^t$, then the Dowker Nerve $N\Lambda$ of Λ is (α, id) -interleaved with the filtered simplicial complex $N(\Gamma, \varphi, \lambda)$.*

Proof. We first check that Proposition 10.2 applies to the Dowker dissimilarity $\Gamma: [n] \times L \rightarrow [0, \infty]$ and the functions $\varphi: [n] \rightarrow [n]$ and $\lambda: [n] \rightarrow [0, \infty]$. By construction $\varphi(0) = 0$ and $\varphi(k) < k$ for $k > 0$, so $\varphi^n(k) = 0$ for every $k \in [n]$. Thus condition (1) of 10.2 holds for φ . By construction of φ the assumption that $B_{\Lambda^t}(0, \infty) = L$ implies conditions (2) and (4) of 10.2. Condition (3) of 10.2 holds by construction of $\Lambda^{(\lambda_{\Lambda}, \alpha, \beta)}$. We conclude that by Proposition 10.2 the filtered simplicial complexes $N(\Gamma, \varphi, \lambda)$ and $N\Gamma$ are homotopy equivalent. The functorial Dowker theorem [12, Corollary 20] implies that the filtered simplicial complexes $N\Gamma$ and $N(\Lambda^{(\lambda_{\Lambda}, \alpha, \beta)})$ are homotopy equivalent. By Lemma 11.2 the function $\lambda_{\Lambda}: [n] \rightarrow [0, \infty]$ is an insertion function for Λ , so by Proposition 9.5 the filtered simplicial complexes $N\Lambda$ and $N(\Lambda^{(\lambda_{\Lambda}, \alpha, \beta)})$ are (α, id) -interleaved. $\square$

Theorem 11.5. *Let*

$$\Lambda: L \times [n] \rightarrow [0, \infty]$$

be a Dowker dissimilarity with $B_{\Lambda^t}(0, \infty) = L$. Let T be a triangle relation for Λ and let $\beta: [0, \infty] \rightarrow [0, \infty]$ be an order preserving function with $\lim_{t \rightarrow \infty} \beta(t) = \infty$. Let $\alpha: [0, \infty] \rightarrow [0, \infty]$ be the function

$$\alpha(t) = t + \beta(t) + \sup(\Lambda(T))$$

and let $\lambda: [n] \rightarrow [n]$ be the function

$$\lambda(k) = \alpha\beta^{\leftarrow}\lambda_{\Lambda,T}(k).$$

Let $\varphi: [n] \rightarrow [n]$ be the parent function defined by letting $\varphi(0) = 0$ and $\varphi(k) = \max\{i \in [k-1] \mid B_{\Lambda}(k, \lambda(k)) \subseteq B_{\Lambda}(i, \lambda(i)) \text{ and } \lambda(k) \leq \lambda(i)\}$.

If we let $\Gamma = (\Lambda^{(\lambda_{\Lambda}, \alpha, \beta)})^t$, then the Dowker Nerve $N\Lambda^t$ of Λ^t is (α, id) -interleaved with the filtered simplicial complex $N(\Gamma, \varphi, \lambda)$.

Proof. By Theorem 11.4 we have that $N\Lambda$ and $N(\Gamma, \varphi, \lambda)$ are (α, id) -interleaved. Now use the functorial Dowker Theorem to get that the filtered simplicial complexes $N\Lambda$ and $N\Lambda^t$ are homotopy equivalent. $\square$

As a special case of Theorem 11.5 we get the following result:

Corollary 11.6. *In the situation of Theorem 11.5, let $c > 1$, let $\beta: [0, \infty] \rightarrow [0, \infty]$ be the function*

$$\beta(t) = \max((c-1)t, \rho_\Lambda)$$

and let $\alpha: [0, \infty] \rightarrow [0, \infty]$ be the function

$$\alpha(t) = t + \beta(t) + \sup(\Lambda(T)).$$

The Dowker Nerve $N\Lambda$ of Λ is (α, id) -interleaved with the filtered simplicial complex

$$N((\Lambda^{(\lambda_\Lambda, \alpha, \beta)})^t, \varphi, \lambda).$$

Specializing even further, we get obtain a variation of the Sparse Čech complex of [7]:

Corollary 11.7. *Let (M, d) be a metric space, let $L \subseteq M$ be a compact subset, let P be a finite subset of M and let $[n] \xrightarrow{p} P$ be a bijection. Let $\Lambda: M \times [n] \rightarrow [0, \infty]$ be the function*

$$\Lambda(x, k) = d(x, p_k),$$

where we write $p_k = p(k)$. Let $T \subseteq M \times [n]$ be the triangle relation for Λ consisting of the pairs (l, k) such that $d(l, p_k) \leq d(l', p_k)$ for every $l' \in L$. Let $c > 1$, let $\beta: [0, \infty] \rightarrow [0, \infty]$ be the function

$$\beta(t) = \max((c-1)t, \rho_\Lambda)$$

and let $\alpha: [0, \infty] \rightarrow [0, \infty]$ be the function

$$\alpha(t) = t + \beta(t) + \sup(\Lambda(T)).$$

For φ and λ as in Definition 11.3, the Dowker Nerve $N\Lambda^t$ of Λ^t is (α, id) -interleaved with the filtered simplicial complex

$$N((\Lambda^{(\lambda_\Lambda, \alpha, \beta)})^t, \varphi, \lambda)$$

and $N\Lambda^t$ is additively $(2d_{GH}(L, P), 2d_{GH}(L, P))$ -interleaved with the relative Čech complex $\check{C}(L, M)$ consisting of all balls in M with centers in L .

Proof. Corollary 11.6 gives that $N\Lambda$ is (α, id) -interleaved with

$$N((\Lambda^{(\lambda_\Lambda, \alpha, \beta)})^t, \varphi, \lambda).$$

For second statement note that the stability 8.5 implies that the Dowker dissimilarities $d: M \times P \rightarrow [0, \infty]$ and $d: M \times L \rightarrow [0, \infty]$ are additively $(2d_{GH}(L, P), 2d_{GH}(L, P))$ -interleaved. Now use that $N\Lambda$ is isomorphic to the Dowker Nerve of $d: M \times P \rightarrow [0, \infty]$, and that the Dowker Nerve of $d: M \times L \rightarrow [0, \infty]$ is the relative Čech complex $\check{C}(L, M)$. $\square$

Finally, we relate the Sparse Dowker Nerve to the Sparse Čech complex of [7]:

Proposition 11.8. *Let d be a convex metric on $\mathbb{R}^d$ and let P be a finite subset of $\mathbb{R}^d$ together with a greedy order $[n] \xrightarrow{p} P$. Let the function $\Lambda: \mathbb{R}^d \times [n] \rightarrow [0, \infty]$ be given by*

$$\Lambda(x, k) = d(x, p_k),$$

where we write $p_k = p(k)$. Let $\varepsilon > 0$ and let $\alpha, \beta: [0, \infty] \rightarrow [0, \infty]$ be the functions $\beta(t) = \varepsilon t$ and $\alpha(t) = (1 + \varepsilon)t$. In the notation of Definition 11.1, let $T = P \times [n]$ and let $\lambda = \lambda_{\Lambda, T}(1 + \varepsilon)^2/\varepsilon$. Then the filtered simplicial complex

$$N((\Lambda^{(\lambda, \alpha, \beta)})^t, \text{id}, \lambda)(t)$$

is isomorphic to the filtered simplicial complex $\{\bigcup_{s < t} S^s\}_{t \geq 0}$ obtained from the sparse Čech complex $\{S^t\}_{t \geq 0}$ constructed in [7, Section 4].

Proof. A subset $\sigma \subseteq [n]$ is in

$$N((\Lambda^{(\lambda, \alpha, \beta)})^t)$$

if and only if there exists $w \in \mathbb{R}^d$ so that for all $l \in \sigma$ we have

$$d(p_l, w) < t \quad \text{and} \quad d(p_l, w) \leq \lambda_{\Lambda, T}(l)(1 + \varepsilon)/\varepsilon.$$

Moreover

$$\sigma \in N(((\Lambda^t)^{(\lambda, \alpha, \beta)})^t, \text{id}, \lambda)(t)$$

if and only if there exists $x \in \mathbb{R}^d$ so that for all $k, l \in \sigma$ we have $d(p_k, x) < t$ and

$$d(p_k, x) \leq \lambda_{\Lambda, T}(k)(1 + \varepsilon)/\varepsilon \quad \text{and} \quad d(p_k, x) \leq \lambda_{\Lambda, T}(l)(1 + \varepsilon)^2/\varepsilon.$$

On the other hand, $\sigma \in S^t$ if and only if there exists $s \leq t$ and $w \in \mathbb{R}^d$ so that $w \in b_l(s)$ for all $l \in \sigma$. By the definition of $b_l(s)$ defined in [7, Section 3]. This is the case if and only if $s \leq t$ and

$$s \leq \lambda_{\Lambda, T}(l)(1 + \varepsilon)^2/\varepsilon \quad \text{and} \quad d(p_l, w) \leq \min(s, \lambda_{\Lambda, T}(l)(1 + \varepsilon)/\varepsilon)$$

for every $l \in \sigma$. We conclude that $\sigma \in S^t$ if and only if there exists $w \in \mathbb{R}^d$ satisfying $d(p_l, w) \leq t$ and

$$d(p_l, w) \leq \lambda_{\Lambda, T}(l)(1 + \varepsilon)/\varepsilon \quad \text{and} \quad d(p_k, w) \leq \lambda_{\Lambda, T}(l)(1 + \varepsilon)^2/\varepsilon.$$

for all $k, l \in \sigma$. $\square$

We have not performed any complexity analysis of Sparse Dowker Nerves. Instead we have made proof-of-concept implementations of slight variations of both the Sparse Čech Complex of [7] described in Proposition 11.8 and the Sparse Dowker Nerve described in Corollary 11.7. These implementations come with the same interleaving guarantees, but for practical reasons concerning the miniball algorithm we consider complexes that are slightly bigger than the ones described above. We have tested these implementations the following data: The optical patch data sets called $X(300, 30)$ and $X(15, 30)$ in [6], 6,040 points from the cyclo-octane conformation space as analyzed in [23] the Clifford data set consisting of 2,000 points on a curve on a torus considered in [20, Chapter 5] and the double torus from [13]. Computing the Sparse Čech complexes and the Sparse Dowker Nerves on these data sets with the same interleaving constant c the resulting simplicial complexes are almost of the same size, with the size of the Sparse Dowker Nerve slightly smaller than the size of the Sparse Čech Complex. Our implementations, the data sets mentioned above and the scripts used to run compute persistent homology is available [2]

12. CONCLUSION

We have generalized the Sparse Čech construction of [7] to arbitrary metric spaces and to a large class of Dowker dissimilarities. The abstract context of Dowker dissimilarities is well suited for sparse nerve constructions. The concepts of filtered relations and strict 2-categories enable us to easily formulate and prove basic stability results. An implementation of the Sparse Dowker Nerve most similar to the Sparse Čech complex is available at GitHub [2]. This implementation is not practical for analysis of high dimensional data. The current bottleneck is the construction of a clique complex. In further work we will improve this construction and we will make Sparse Dowker Nerve versions of the Witness Complex.

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