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THE GEOMETRIC REALIZATION OF A SEMI-SIMPLICIAL COMPLEX

BY JOHN MILNOR

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Corresponding to each (complete) semi-simplicial complex K , a topological space $|K|$ will be defined. This construction will be different from that used by Giever [4] and Hu [5] in that the degeneracy operations of K are used. This difference is important when dealing with product complexes.

If K and K' are countable it is shown that $|K \times K'|$ is canonically homeomorphic to $|K| \times |K'|$. It follows that if K is a countable group complex then $|K|$ is a topological group. In particular $|K(\pi, n)|$ is an abelian topological group.

In the last section it is shown that the space $|K|$ has the correct singular homology and homotopy groups.

The terminology for semi-simplicial complexes will follow John Moore [7]. In particular the face and degeneracy maps of K will be denoted by $\partial_i: K_n \rightarrow K_{n-1}$ and $s_i: K_n \rightarrow K_{n+1}$ respectively.

1. The definition

As standard n -simplex Δ_n take the set of all $(n+2)$ -tuples $(t_0, \dots, t_{n+1})$ satisfying $0 = t_0 \leq t_1 \leq \dots \leq t_{n+1} = 1$. The face and degeneracy maps

$$\partial_i: \Delta_{n-1} \rightarrow \Delta_n$$

and $s_i: \Delta_{n+1} \rightarrow \Delta_n$ are defined by

$$\partial_i(t_0, \dots, t_n) = (t_0, \dots, t_i, t_i, \dots, t_n)$$

$$s_i(t_0, \dots, t_{n+2}) = (t_0, \dots, t_i, t_{i+2}, \dots, t_{n+2}).$$

Let $K = \bigcup_{i \geq 0} K_i$ be a semi-simplicial complex. Giving K the discrete topology, form the topological sum

$$\bar{K} = (K_0 \times \Delta_0) + (K_1 \times \Delta_1) + \dots + (K_n \times \Delta_n) + \dots.$$

Thus $\bar{K}$ is a disjoint union of open sets $k_i \times \Delta_i$. An equivalence relation in $\bar{K}$ is generated by the relations

$$(\partial_i k_n, \delta_{n-1}) \sim (k_n, \partial_i \delta_{n-1})$$

$$(s_i k_n, \delta_{n+1}) \sim (k_n, s_i \delta_{n+1}),$$

for each $k_n \in K_n$, $\delta_{n \pm 1} \in \Delta_{n \pm 1}$ and for $i = 0, 1, \dots, n$. The identification space $|K| = \bar{K}/(\sim)$ will be called the *geometric realization* of K . The equivalence class of (k_n, δ_n) will be denoted by $|k_n, \delta_n|$. (The equivalence class $|k_0, \delta_0|$ may be abbreviated by $|k_0|$.)

THEOREM 1. $|K|$ is a CW -complex having one n -cell corresponding to each non-degenerate n -simplex of K .

For the definition of CW -complex see Whitehead [8].

LEMMA 1. Every simplex $k_n \in K_n$ can be expressed in one and only one way as $k_n = s_{j_p} \cdots s_{j_1} k_{n-p}$ where k_{n-p} is non-degenerate and $0 \leq j_1 < \cdots < j_p < n$. The indices j_α which occur are precisely those j for which $k_n \in s_j K_{n-1}$.

The proof is not difficult. (See [3] 8.3). Similarly we have:

LEMMA 2. Every $\delta_n \in \Delta_n$ can be written in exactly one way as $\delta_n = \partial_{i_q} \cdots \partial_{i_1} \delta_{n-q}$ where δ_{n-q} is an interior point (that is the coordinates t_i of δ_{n-q} satisfy $t_0 < t_1 < \cdots < t_{n-q+1}$) and $0 \leq i_1 < \cdots < i_q \leq n$.

By a non-degenerate point of $\bar{K}$ will be meant a point (k_n, δ_n) with k_n non-degenerate and δ_n interior.

LEMMA 3. Each $(k_n, \delta_n) \in \bar{K}$ is equivalent to a unique non-degenerate point.

Define the map $\lambda: \bar{K} \rightarrow \bar{K}$ as follows. Given k_n choose $j_1, \cdots, j_p, k_{n-p}$ as in Lemma 1 and set

$$\lambda(k_n, \delta_n) = (k_{n-p}, s_{j_1} \cdots s_{j_p} \delta_n).$$

Define the discontinuous function $\rho: \bar{K} \rightarrow \bar{K}$ by choosing $i_1 \cdots i_q, \delta_{n-q}$ as in Lemma 2 and setting

$$\rho(k_n, \delta_n) = (\partial_{i_1} \cdots \partial_{i_q} k_n, \delta_{n-q}).$$

Now the composition $\lambda\rho: \bar{K} \rightarrow \bar{K}$ carries each point into an equivalent, non-degenerate point. It can be verified that if $x \sim x'$ then $\lambda\rho(x) = \lambda\rho(x')$; which proves Lemma 3.

Take as n -cells of $|K|$ the images of the non-degenerate simplexes of $\bar{K}$. By Lemma 3 the interiors of these cells partition $|K|$. Since the remaining conditions for a CW -complex are easily verified, this proves Theorem 1.

LEMMA 4. A semi-simplicial map $f: K \rightarrow K'$ induces a continuous map $|K| \rightarrow |K'|$.

In fact the map $|f|$ defined by $|k_n, \delta_n| \rightarrow |f(k_n), \delta_n|$ is clearly well defined and continuous.

As an example of the geometric realization, let C be an ordered simplicial complex with space $|C|$. (See [2] pp. 56 and 67). From C we can define a semi-simplicial complex K , where K_n is the set of all $(n+1)$ -tuples $(a_0, \cdots, a_n)$ of vertices of C which (1) all lie in a common simplex, and (2) satisfy $a_0 \leq a_1 \leq \cdots \leq a_n$. The operations ∂_i, s_i are defined in the usual way.

ASSERTION. The space $|C|$ is homeomorphic to the geometric realization $|K|$. In fact the point $|(a_0, \cdots, a_n); (t_0, \cdots, t_{n+1})|$ of $|K|$ corresponds to the point of $|C|$ whose a^{th} barycentric coordinate, a being a vertex of C , is the sum, over all i for which $a_i = a$, of $t_{i+1} - t_i$. The proof is easily given.

2. Product complexes

Let $K \times K'$ be the cartesian product of two semi-simplicial complexes (that is $(K \times K')_n = K_n \times K'_n$). The projection maps $\rho: K \times K' \rightarrow K$ and $\rho': K \times K' \rightarrow K'$ induce maps $|\rho|$ and $|\rho'|$ of the geometric realizations. A map

$$\eta: |K \times K'| \rightarrow |K| \times |K'|$$

is defined by $\eta = |\rho| \times |\rho'|$.

THEOREM 2. η is a one-one map of $|K \times K'|$ onto $|K| \times |K'|$. If either (a) K and K' are countable, or (b) one of the two CW -complexes $|K|$, $|K'|$ is locally finite; then η is a homeomorphism.

The restrictions (a) or (b) are necessary in order to prove that $|K| \times |K'|$ is a CW -complex. (For the proof in case (b) see [8] p. 227 and for case (a) see [6] 2.1.)

PROOF (Compare [2] p. 68). If x'' is a point of $|K \times K'|$ with non-degenerate representative $(k_n \times k'_n, \delta_n)$ we will first determine the non-degenerate representative of $|\rho|(x'') = |k_n, \delta_n|$. Since δ_n is an interior point of Δ_n , this representative has the form

$$(k_{n-p}, s_{i_1} \cdots s_{i_p} \delta_n) \quad \text{where} \quad k_n = s_{i_p} \cdots s_{i_1} k_{n-p}$$

(see proof of Lemma 3). Similarly $|\rho'|(x'')$ is represented by

$$(k'_{n-q}, s_{j_1} \cdots s_{j_q} \delta_n)$$

where $k'_n = s_{j_q} \cdots s_{j_1} k'_{n-q}$. The indices i_α and j_β must be distinct; for if $i_\alpha = j_\beta$ for some α, β then $k_n \times k'_n$ would be an element of $s_{i_\alpha}(K_{n-1} \times K'_{n-1})$.

However the point x'' can be completely determined by its image.

$$|k_{n-p}, s_{i_1} \cdots s_{i_p} \delta_n| \times |k'_{n-q}, s_{j_1} \cdots s_{j_q} \delta_n|.$$

In fact given any pair $(x, x') \in |K| \times |K'|$ define $\bar{\eta}(x, x') \in |K \times K'|$ as follows. Let (k_a, δ_a) and (k'_b, δ'_b) be the non-degenerate representatives: where $\delta_a = (t_0, \cdots, t_{a+1})$, $\delta'_b = (u_0, \cdots, u_{b+1})$. Let $0 = w_0 < \cdots < w_{n+1} = 1$ be the distinct numbers t_i and u_j arranged in order. Set $\delta''_n = (w_0, \cdots, w_{n+1})$. Then if $\mu_1 < \cdots < \mu_{n-a}$ are those integers $\mu = 0, 1, \cdots, n-1$ such that $w_{\mu+1}$ is not one of the t_i , we have $\delta_a = s_{\mu_1} \cdots s_{\mu_{n-a}} \delta''_n$. Similarly $\delta'_b = s_{\nu_1} \cdots s_{\nu_{n-b}} \delta''_n$ where the sets $\{\mu_i\}$ and $\{\nu_j\}$ are disjoint. Now define

$$\bar{\eta}(x, x') = |(s_{\mu_{n-a}} \cdots s_{\mu_1} k_a) \times (s_{\nu_{n-b}} \cdots s_{\nu_1} k'_b), \delta''_n|.$$

Clearly

$$\begin{aligned} |\rho| \bar{\eta}(x, x') &= |s_{\mu_{n-a}} \cdots s_{\mu_1} k_a, \delta''_n| = |k_a, s_{\mu_1} \cdots s_{\mu_{n-a}} \delta''_n| \\ &= |k_a, \delta_a| = x \end{aligned}$$

and $|\rho'| \bar{\eta}(x, x') = x'$, which proves that $\eta \bar{\eta}$ is the identity map of $|K| \times |K'|$. On the other hand, taking x'' as above we have

$$\begin{aligned} \bar{\eta} \eta(x'') &= \bar{\eta}(|k_{n-p}, s_{i_1} \cdots s_{i_p} \delta_n|, |k'_{n-q}, s_{j_1} \cdots s_{j_q} \delta_n|) \\ &= |(s_{i_p} \cdots s_{i_1} k_{n-p}) \times (s_{j_q} \cdots s_{j_1} k'_{n-q}), \delta_n| = x''. \end{aligned}$$

To complete the proof it is only necessary to show that $\bar{\eta}$ is continuous. However it is easily verified that $\bar{\eta}$ is continuous on each product cell of $|K| \times |K'|$. Since we know that this product is a CW -complex, this completes the proof.

An important special case is the following. Let I denote the semi-simplicial complex consisting of a 1-simplex and its faces and degeneracies.

COROLLARY. *A semi-simplicial homotopy $h: K \times I \rightarrow K'$ induces an ordinary homotopy $|K| \times [0, 1] \rightarrow |K'|$.*

In fact the interval $[0, 1]$ may be identified with $|I|$. The homotopy is now given by the composition

$$|K| \times |I| \xrightarrow{\eta} |K \times I| \xrightarrow{|h|} |K'|.$$

3. Product operations

Now let K be a countable complex. Any semi-simplicial map $p: K \times K \rightarrow K$ induces by Lemma 4 and Theorem 2 a continuous product

$$|p| \eta: |K| \times |K| \rightarrow |K|.$$

If there is an element e_0 in K_0 such that $s_0^n e_0$ is a two-sided identity in K_n for each n , then it follows that $|e_0|$ is a two-sided identity in $|K|$; so that $|K|$ is an H -space. If the product operation p is associative or commutative then it is easily verified that $|p| \eta$ is associative or commutative. Hence we have the following.

THEOREM 3. *If K is a countable group complex (countable abelian group complex), then $|K|$ is a topological group (abelian topological group).*

Let $K(\pi, n)$ denote the Eilenberg MacLane semi-simplicial complex (see [1]). Since $K(\pi, n)$ is an abelian group complex we have:

COROLLARY. *If π is a countable abelian group, then for $n \geq 0$ the geometric realization $|K(\pi, n)|$ is an abelian topological group.*

It will be shown in the next section that $|K(\pi, n)|$ actually is a space with one non-vanishing homotopy group.

The above construction can also be applied to other algebraic operations. For example a pairing $K \times K' \rightarrow K''$ between countable group complexes induces a pairing between their realizations. If K is a countable semi-simplicial complex of Λ -modules, where Λ is a discrete ring, then $|K|$ is a topological Λ -module.

4. The topology of $|K|$

For any space X let $S(X)$ be the total singular complex. For any semi-simplicial complex K a one-one semi-simplicial map $i: K \rightarrow S(|K|)$ is defined by

$$i(k_n)(\delta_n) = |k_n, \delta_n|.$$

Let $H_*(K)$ denote homology with integer coefficients.

LEMMA 5. *The inclusion $K \rightarrow S(|K|)$ induces an isomorphism $H_*(K) \approx H_*(S|K|)$ of homology groups.*

By the n -skeleton $K^{(n)}$ of K is meant the subcomplex consisting of all K_i , $i \leq n$ and their degeneracies. Thus $|K^{(n)}|$ is just the n -skeleton of $|K|$ considered as a CW -complex. The sequence of subcomplexes

$$K^{(0)} \subset K^{(1)} \subset \dots$$

gives rise to a spectral sequence $\{E_{pq}^r\}$; where E^∞ is the graded group corresponding to $H_*(K)$ under the induced filtration; and

$$E_{pq}^1 = H_{p+q}(K^{(p)} \bmod K^{(p-1)}).$$

It is easily verified that $E_{pq}^1 = 0$ for $q \neq 0$, and that E_{p0}^1 is the free abelian group generated by the non-degenerate p -simplexes of K . From the first assertion it follows that $E_{p0}^2 = E_{p0}^\infty = H_p(K)$.

On the other hand the sequence

$$S(|K^{(0)}|) \subset S(|K^{(1)}|) \subset \dots$$

gives rise to a spectral sequence $\{\bar{E}_{pq}^r\}$ where $\bar{E}^\infty$ is the graded group corresponding to $H_*(S(|K|))$. Since it is easily verified that the induced map $E_{pq}^1 \rightarrow \bar{E}_{pq}^1$ is an isomorphism, it follows that the rest of the spectral sequence is also mapped isomorphically; which completes the proof.

Now suppose that K satisfies the Kan extension condition, so that $\pi_1(K, k_0)$ can be defined.

LEMMA 6. *If K is a Kan complex then the inclusion i induces an isomorphism of $\pi_1(K, k_0)$ onto $\pi_1(S(|K|), i(k_0)) = \pi_1(|K|, |k_0|)$.*

Let K' be the Eilenberg subcomplex consisting of those simplices of K whose vertices are all at k_0 . Then $\pi_1(K, k_0)$ can be considered as a group with one generator for each element of K'_1 and one relation for each element of K'_2 .

The space $|K'|$ is a CW -complex with one vertex. For such a space the group π_1 is known to have one generator for each edge and one relation for each face. Comparing these two descriptions it follows easily that the homomorphism $\pi_1(K) = \pi_1(K') \rightarrow \pi_1(|K'|)$ is an isomorphism.

We may assume that K is connected. Then it is known (see [7] Chapter I, appendix C) that the inclusion map $K' \rightarrow K$ is a semi-simplicial homotopy equivalence. By the corollary to Theorem 2 this proves that the inclusion $|K'| \rightarrow |K|$ is a homotopy equivalence; which completes the proof of Lemma 6.

REMARK 1. From Lemmas 5 and 6 it can be proved, using a relative Hurewicz theorem, that the homomorphisms

$$\pi_n(K, k_0) \rightarrow \pi_n(|K|, |k_0|)$$

are isomorphisms for all n . (The proof of the relative Hurewicz theorem given in [9] §3 carries over to the semi-simplicial case without essential change, making use of [7] Chapter I, appendices A and C. This theorem is applied to the pair $(S(|\tilde{K}|), \tilde{K})$ where $\tilde{K}$ denotes the universal covering complex of K .)

REMARK 2. The space $|K(\pi, n)|$ has n^{th} homotopy group π , and other homotopy groups trivial. This clearly follows from the preceding remark. Alternatively the proof given by Hu [5] may be used without essential change.

Now let X be any topological space. There is a canonical map

$$j: |S(X)| \rightarrow X$$

defined by $j(|k_n, \delta_n|) = k_n(\delta_n)$.

THEOREM 4. *The map $j: |S(X)| \rightarrow X$ induces isomorphisms of the singular homology and homotopy groups.*

(This result is essentially due to Giever [4]).

The map j induces a semi-simplicial map $j_*: S(|S(X)|) \rightarrow S(X)$. A map i in the opposite direction was defined at the beginning of this section. The composition $j_*i: S(X) \rightarrow S(X)$ is the identity map. Together with Lemma 5 this implies that j induces isomorphisms of the singular homology groups of $|S(X)|$ onto those of X . Together with Remark 1 it implies that j induces isomorphisms of the homotopy groups of $|S(X)|$ onto those of X . This completes the proof.

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