Harvard Undergraduate Topology Seminar

Lecture 2

The Serre Spectral Sequence: Construction and first examples

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1 Homology with Local Coefficients

With ordinary singular homology, we look at linear combinations of simplices with coefficients in some abelian group A. For the Serre spectral sequence, it's important that we can generalize this slightly: rather than always work with the group A, we can allow local variations of our group, using *local coefficients*.

Recall the *fundamental groupoid* $\Pi(X)$ of a topological space X is the homotopy category of the category whose objects are points of x, and $\mathrm{Mor}(x, y) = \mathrm{Paths}_X(x, y)$. More explicitly, its objects are the points $x \in X$, and a morphism $x \to y$ is given by a path from x to y, modulo homotopy. Composition, as usual, is given by concatenation.

Definition 1.1. A *local system* is a functor $\mathcal{L} : \Pi(X) \to \mathsf{Ab}$, where Ab is the category of abelian groups.

We will let $\mathsf{LocSys}(X)$ denote the category of local systems on X, whose morphisms are natural transformations.

Example 1.2. When X is path connected, and $x \in X$, we have an equivalence of categories $\mathsf{LocSys}(X) \simeq \mathrm{Mod}_{\mathbb{Z}[\pi_1(X,x)]}$. We can see this quite explicitly: viewing $\pi_1(X, x)$ as a one-object groupoid, we see that an object of $\mathrm{Mod}_{\mathbb{Z}[\pi_1(X,x)]}$ contains the same data as a functor $\mathcal{F} : \pi_1(X, x) \to \mathsf{Ab}$. Moreover, we have the obvious inclusion of categories $\pi_1(X, x) \to \Pi(X)$, and since X is path connected, this is an equivalence of categories. Thus a functor $\pi_1(X, x) \to \mathsf{Ab}$ induces a unqiue functor $\Pi(X) \to \mathsf{Ab}$, or in other words, a local system.

For any map of topological spaces $X \xrightarrow{f} Y$, and any local system $\mathcal{L} : \Pi(Y) \to \mathsf{Ab}$, we can "pull back" the local system to X, by setting $f * \mathcal{L} : \Pi(X) \to \mathsf{Ab}$ be given by $\mathcal{L} \circ f$ on objects, and similarly for paths. This construction is functorial, in the sense that it makes the assignment $X \mapsto \mathsf{LocSys}(X)$ into a functor $\mathsf{Top}^{\mathrm{op}} \to \mathsf{Cat}$.

In this way, given any simplex $\Delta^n \xrightarrow{f} X$ in X, given a local system $\mathcal{L}$ on X, we can associate the abelian group $A_f = \lim f^*\mathcal{L}$. Via the inclusions $\iota_i : \Delta^{n-1} \to \Delta^n$ of the faces, we get induced maps $d_i : A_f \to A_{f\circ\iota_i}$.

This construction probably feels quite familiar—we're essentially just mimicing singular homology. Indeed, we can set

$$C_n(X, \mathcal{L}) := \bigoplus_{f:\Delta^n \to X} \mathcal{L}_f.$$

We can define the usual boundary operator $\partial = \sum_i (-1)^i d_i$. It's easy to see $\partial^2 = 0$, and thus we can take the homology of the complex.

Example 1.3. If $\mathcal{L}$ is a trivial local system (i.e. a constant functor at an abelian group A, taking all morphisms to the identity map id_A), we just get the usual singular singular homology with coefficients in A.

2 Interlude on homological algebra, or: everyone's favorite magic black box

Last week, Reuben discussed how, given an exact couple

$$\begin{tikzcd} X \arrow[rr, "i"] & & X \arrow[dl, "j"] \\ & A \arrow[ul, "k"] & \end{tikzcd}$$

we can arrive at a spectral sequence $E^n_{p,q}$. Here, we'll discuss briefly how to use this to get a spectral sequence from a filtration. Given a filtration

$$Z_0 \subset Z_1 \subset \ldots \subset Z_n \subset \ldots$$

of a space Z, with $\operatorname{colim} Z_n \cong Z$, we can set

$$\begin{aligned} X_{p,q} &:= H_{p+q}(Z_p; \mathbb{Z}) \\ A_{p,q} &:= H_{p+q}(Z_p, Z_{p-1}; \mathbb{Z}). \end{aligned}$$

We have a map $X \xrightarrow{i} X$ given by the inclusion $Z_p \to Z_{p+1}$, of degree $(1, -1)$. Similarly, there is a natural map $X \xrightarrow{j} A$ induced by the inclusion $(X_p, \emptyset) \to (X_p, X_{p-1})$, of degree $(0, 0)$, and we'll define a map $A \xrightarrow{k} X$ as the natural map $H_{p+q}(X_p, X_{p-1}) \to H_{p+q-1}(X_{p-1})$ from the snake lemma, and is of degree $(-1, 0)$.

We should certainly check that this sequence is exact. Following Reuben's construction, we get a spectral sequence with

$$E^1_{p,q} = H_{p+q}(X_p, X_{p-1}; \mathbb{Z}),$$

and $d^1 = j \circ k$.

3 The Serre Spectral Sequence

Given a fiber bundle $F \to E \to B$, or more generally, a Serre fibration $E \to B$, we'd like to find a way to describe the homology of E based on that of F and B (or in the case of a Serre fibration, based on $\{E_b\}$ and B).

Definition 3.1. Recall, a *Serre fibration* is a map $E \to B$ with the homotopy lifting property with respect to spheres. That is, for any topological space X and maps

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\{0\}\times X"'] & E \arrow[d] \\ X \times [0,1] \arrow[r] \arrow[ur, dashed, "\exists !"] & B \end{tikzcd}$$

the dashed line exists.

Example 3.2. Given a Serre fibration $E \xrightarrow{\pi} B$, we can define a local system by setting

$$\mathcal{L} : \Pi(B) \to \mathsf{Ab},$$

given by $b \mapsto H_1(\pi^{-1}(b))$.

Theorem 3.3. *Given a Serre fibration $E \xrightarrow{\pi} B$, there's a spectral sequence $E^r_{p,q}$ converging to $G_pH_{p+q}(E)$ with*

$$E^2_{p,q} = H_p(B; \{H_q(\pi^{-1}(b))\}_{b\in B}).$$

Proof. We'll sketch the argument when B is a CW-complex. Let B^m denote the m-skeleton of B. Then we can define a filtration on E by setting $E_m := \pi^{-1}(B^m)$. Of course, it's clear $\operatorname{colim} E_m \cong E$. Then we can apply the results of Section 2 to get a spectral sequence with

$$E^1_{p,q} = H_{p+q}(E_p, E_{p-1}).$$

Given any simplex $\Delta^p \xrightarrow{f} B$ of B, we can form the pullback

$$\begin{tikzcd} E_f \arrow[r] \arrow[d] & E \arrow[d, "\pi"] \\ \Delta^p \arrow[r, "f"] & B. \end{tikzcd}$$

Of course, we get a map $(E_f, E_{\partial f}) \to (E_p, E_{p-1})$, and this induces an isomorphism

$$\bigoplus_f H_{p+q}(E_f, E_{\partial f}) \to H_{p+q}(\coprod E_f, \coprod E_{\partial f}) = E^1_{p,q}.$$

Consider the diagram

$$\begin{tikzcd} E_f \arrow[r, "\mathrm{id}\times\pi"] \arrow[d, "\pi"] & E_f \times \Delta^p \arrow[d, "\pi"] \\ \Delta^p \arrow[r, "\mathrm{id}"] & \Delta^p. \end{tikzcd}$$

The vertical maps are Serre fibrations, and the horizontal maps are weak homotopy equivalences. Taking fibers and the long exact sequence of homotopy groups, the Five Lemma tells us the fibers are (weakly) homotopic. Similarly on the horizontal arrow, we get a weak homotopy equivalence $E_{\partial f} \to E_f \times \partial\Delta^p$.

Note that via the Künneth formula, we have

$$H_q(E_f) \cong H_q(E_f) \otimes H_p(\Delta^p, \Delta^{p-1}) \cong H_{p+q}(E_f \times \Delta^p, E_f \times \partial\Delta^p).$$

Throwing this into our definitions of $E^1_{p,q}$, we have

$$E^1_{p,q} \cong \bigoplus_f H_q(E_f) \cong C_p(B, \{H_q(E_f)\}),$$

where the last isomorphism is simply the definition of chains on B with local coefficient system $\{H_q(E_f)\}$. We're now just left to figure out what the differential looks like, and compute the E^2 page. Perhaps as we'd expect, it turns out that the differential map d^1 is given by taking $H_q(E_f)$ to $(-1)^i H_q(E_{f_i})$, where f_i denotes the i^{th} face of f. But this is just the ordinary differential of the chains $C_p(B, \{H_q(E_f)\})$. Thus we arrive at

$$E^2_{p,q} \cong H_p(B, \{H_q(E_f)\}).$$

□

4 Examples

By far the nicest case to physically work examples with is a fiber bundle $F \to E \to B$, where the action of $\pi_1(B)$ on F is trivial. (Or, even more "simple," when B is simply connected.)

4.1 The Loop Space of the Sphere

Reuben did this last week.

4.2 Trivial fibrations

Suppose you remember the Serre Spectral Sequence, but not the Künneth formula. Then we can quite quickly recover the formula (though this doesn't work as an actual derivation, since we used the Künneth formula in the proof of the Serre Spectral Sequence) by considering $B \times F \to F$. Then we have

$$E^2(p,q) = H_p(B; H_q(F)),$$

and all differentials at this page vanish.

In this certain sense, we can interpret higher differentials as measuring how much our Serre fibration deviates from being weakly homotopic to the trivial fibration.

4.3 The Hopf Fibration

We have a fiber bundle $S^1 \to S^3 \to S^2$. This isn't all that interesting, since we know the homology of all the participants in the spectral sequence, but we can compute it if we'd like.

4.4 SU(3)

The quotient $SU(3) \to SU(3)/SU(2) \cong S^5$ is a fiber bundle, with fiber $SU(2) \cong S^3$. Then we have

$$E^2_{p,q} = H_p(S^5) \otimes H_q(S^3).$$

The differentials all vanish, hence $H_*(SU(3))$ is $\mathbb{Z}$ in degrees $0, 3, 5, 8$, and vanishes elsewhere.

4.5 Homotopy Groups of Spheres

Let $K(\mathbb{Z}, n)$ be the n^{th} Eilenberg-MacLane space, and convert $S^n \to K(\mathbb{Z}, n)$ into a fiber bundle. The fiber X is n-connected, with $\pi_i(X) \cong \pi_i(S^n)$ for $i > n$ by the long exact sequence of a fibration. We can thereby modify the inclusion $X \to S^n$ into a different fibration

$$K(\mathbb{Z}, n) \to X \to S^n.$$

The E^2 page of the Serre spectral sequence for this fibration when n is odd over $\mathbb{Q}$ is

$$\begin{tikzcd}
\cdots & & \\
3n-3 & \mathbb{Q}a^3 & \mathbb{Q}a^3x \\
2n-2 & \mathbb{Q}a^2 & \mathbb{Q}a^2x \arrow[ul] \\
n-1 & \mathbb{Q}a & \mathbb{Q}ax \arrow[ul] \\
0 & \mathbb{Q} & \mathbb{Q}x \arrow[ul] \\
 & 0 & n
\end{tikzcd}$$

Note X is n-connected, so we must have the first differential is an isomorphism. It follows that all of the differentials are isomorphisms, hence the rational homology $\widetilde{H}_i(X, \mathbb{Q}) = 0$ for all i, hence $\pi_i(X)$ is torsion for all i, which implies the same is true of S^n for $i > n$.

References

For the most part, I'm following these notes, by Galatius. I also referred to this for the discussion of homotopy groups of spheres.