1 Whitehead's theorem.

Statement: If $f : X \to Y$ is a map of CW complexes inducing isomorphisms on all homotopy groups, then f is a homotopy equivalence. Moreover, if f is the inclusion of a subcomplex X in Y, then there is a deformation retract of Y onto X.

For future reference, we make the following definition:

Definition: $f : X \to Y$ is a Weak Homotopy Equivalence (WHE) if it induces isomorphisms on all homotopy groups π_n.

Notice that a homotopy equivalence is a weak homotopy equivalence.

Using the definition of Weak Homotopic Equivalence, we paraphrase the statement of Whitehead's theorem as:

If $f : X \to Y$ is a weak homotopy equivalences on CW complexes then f is a homotopy equivalence.

In order to prove Whitehead's theorem, we will first recall the homotopy extension property and state and prove the Compression lemma.

Homotopy Extension Property (HEP):

Given a pair (X, A) and maps $F_0 : X \to Y$, a homotopy $f_t : A \to Y$ such that $f_0 = F_0|_A$, we say that (X, A) has (HEP) if there is a homotopy $F_t : X \to Y$ extending f_t and F_0. In other words, (X, A) has homotopy extension property if any map $X \times \{0\} \cup A \times I \to Y$ extends to a map $X \times I \to Y$.

Question: Does the pair $([0, 1], \{\frac{1}{n}\}_{n\in\mathbb{N}})$ have the homotopic extension property?

(Answer: No.)

Compression Lemma: If (X, A) is a CW pair and (Y, B) is a pair with $B \neq \emptyset$ so that for each n for which $X \backslash A$ has n-cells, $\pi_n(Y, B, b_0) = 0$ for all $b_0 \in B$, then any map $f : (X, A) \to (Y, B)$ is homotopic to $f' : X \to B$ fixing A. (i.e. $f'|_A = f|_A$).

To prove the compression lemma, we will first prove the following proposition:

Proposition: Any CW pair has the homotopy extension property. In fact, for every CW pair (X, A), there is a deformation retract $r : X \times I \to X \times \{0\} \cup A \times I$ and we can then define $X \times I \to Y$ by $X \times I \xrightarrow{r} X \times \{0\} \cup A \times I \to Y$.

Proof: We have that $D^n \times I \xrightarrow{r} D^n \times \{0\} \cup S^{n-1} \times I$ (where r is a deformation retraction). For every n, looking at the n-skeleton X_n and considering the pair $(X_n, A_n \cup X_{n-1})$, we get that $X_n \times I = [X_n \times \{0\} \cup (A_n \cup X_{n-1}) \times I] \cup D^n \times I$ where the cylinders $D^n \times I$ corresponding to n-cells D^n in $X \backslash A$ are glued along $D^n \times \{0\} \cup S^{n-1} \times I$ to the pieces $[X_n \times \{0\} \cup (A_n \cup X_{n-1}) \times I]$.

By deforming these cylinders $D^n \times I$ we get a deformation retraction $r_n : X_n \times I \to X_n \times \{0\} \cup (A_n \cup X_{n-1}) \times I$. Concatenating these deformation retractions by performing r_n over $[1 - \frac{1}{2^{n-1}}, 1 - \frac{1}{2^n}]$, we get a deformation retraction of $X \times I$ onto $X \times \{0\} \cup A \times I$. Continuity follows since CW complexes have the weak topology with respect to their skeleta, so a map is continuous iff its restriction to each skeleton is continuous.

□

Proof of the Compression Lemma: We will prove this by induction on n.

Assume $f(X_{k-1} \cup A) \subseteq B$. Let e^k be a k-cell in $X \backslash A$. Look at its characteristic map $\alpha : (D^k, S^k) \to (X_k, X_{k-1} \cup A)$. Regard α as an element $[\alpha] \in \pi_k(X_k, X_{k-1} \cup A)$. Looking at $f_*[\alpha] = [f \circ \alpha] \in \pi_k(Y, B)$ and using the fact that $\pi_k(Y, B) = 0$ by our hypothesis (and since $e^k \in X \backslash A$), we get a homotopy $H : (D^k, S^{k-1}) \times I \to (Y, B)$ such that $H_0 = f \circ \alpha$ and $\mathrm{Im} H_1 \subseteq B$.

Performing this process for all the k cells in $X \backslash A$ simultaneously, we get a homotopy H_k from f to f' such that $f'(X_k \cup A) \subseteq B$. Using the homotopy extension property, regard this as a homotopy on all of X. We hence get $f \underset{H_1}{\simeq} f_1$, such that $f_1(X_1 \cup A) \subseteq B$, $f_1 \underset{H_2}{\simeq} f_2$ such that $f_2(X_2 \cup B)$, and so on.

Define $H : X \times I \to Y$ as $H = H_i$ on$[1 - \frac{1}{2^{i-1}}, 1 - \frac{1}{2^i}]$. H is continuous by CW topology, thus giving us the required homotopy. $\square$

Proof of Whitehead's Theorem: We can assume f is an inclusion (by using cellular approximation and the mapping cylinder M_f). Let (Y, X) be a CW pair. By assumption and the long exact sequence of the pair, we have that $\pi_n(Y, X) = 0$ for all n. By applying the compression lemma to the identity map of (Y, X), we get the desired deformation retract $r : Y \to X$. $\square$

Example: Let $X = \mathbb{R}P^2$, $Y = S^2 \times \mathbb{R}P^\infty$. We know that $\pi_1(X) = \mathbb{Z}/2\mathbb{Z} = \pi_1(Y)$, $\pi_n(\mathbb{R}P^2) = \pi_n(S^2)$ for all $n \geq 2$.

Also note that, $\pi_n(S^2 \times \mathbb{R}P^\infty) = \pi_n(S^2) \times \pi_n(\mathbb{R}P^\infty)$, and that $\pi_n(\mathbb{R}P^\infty) = \pi_n(S^\infty) = 0$, since S^∞ is contractible. We hence have that $\pi_n(S^2 \times \mathbb{R}P^\infty) = \pi_n(S^2)$

So $\pi_n(X) = \pi_n(Y)$ for all $n \geq 1$. However, $X \not\simeq Y$ since their second homology groups are unequal, as $H_2(\mathbb{R}P^2) = 0$ and $H_2(S^2 \times \mathbb{R}P^\infty) \neq 0$. $\square$

Theorem: Weak homotopy equivalence induce isomorphisms on $H_*(\_, G)$ and $H^*(\_, G)$ for any coefficient ring G.

Proof in the simply connected case:

Let $f : X \to Y$ be a weak homotopy equivalence. By the Universal Coefficient theorem, it is sufficient to show that f induces isomorphisms on integral homology $H_*(\_, \mathbb{Z})$. We can also assume f is the inclusion map. Since f is a weak homotopy, via the long exact sequence, we have $\pi_n(Y, X) = 0$ for all n. Combining this fact along with the the fact that the fundamental group of X is 0, by using the Hurewicz theorem we get that $H_n(Y, X) = 0$ for all n. Hence, using the long exact sequence for homology, we get $H_n(X) \underset{f_*}{\cong} H_n(Y)$.

$\square$

Exercise: Show that any finitely generated (abelian when $n \geq 2$) group can be realized as the n^{th} homotopy group for some space X.

$\square$

2 Cellular approximation.

Cellular Approximation: If $f : X \to Y$ is a continuous map of CW complexes then f has a cellular approximation $f' : X \to Y$, i.e. $f \underset{he}{\simeq} f'$ such that $f(X_n) \subseteq Y_n$ for all $n \geq 0$. Moreover, if f is already cellular on some subcomplex $A \subseteq X$, then we can perform cellular approximation relative to A, i.e. $f'|_A = f|_A$.

The proof of this relies on the technical lemma which states that if $f : X \cup e^n \to Y \cup e^k$ (where e^n, e^k are n cells and k cells respectively) such that $f(X) \subseteq Y$ and $f|_X$ is cellular and if $n \leq k$, then $f \underset{h.e.}{\simeq} f'$, with $Im(f') \subseteq Y$. The technical lemma is used along with induction on skeleta to prove the above result on cellular approximation.

Remark: If X and Y are points in the statement of the above lemma then we get that $S^n \hookrightarrow S^k$ can be homotoped into the constant map $S^n \to \{s_0\}$ for some point $s_0 \in S^k$.

$\square$

Relative cellular approximation:

Any map $f : (X, A) \to (Y, B)$ of CW pairs has a cellular approximation by a homotopy through such maps of pairs.

Proof: First use cellular approximation for $f|_A : A \to B, f|_A \underset{H}{\simeq} f'$ where $f' : A \to B$ is a cellular map. Using the Homotopy Extension Property, we can regard H as a homotopy on all of X, so we get a map $f' : X \to Y$ such that $f'|_A$ is a cellular map. By the second statement of cellular approximation, $f' \underset{h.e.}{\simeq} f''$ where $f'' : X \to Y$ is a cellular map $f'|_A = f''|_A$. $\square$

3 CW approximation

We are going to show that given any space X there exists a (unique up to homotopy) CW complex Z, with a weak homotopic equivalence $f : Z \to X$. Such a Z is called a CW approximation of X.

Definition: Given a pair (X, A) with $\emptyset \neq A \subseteq X$, where A is a CW complex, an n- connected model of (X, A) is an n-connected CW pair (Z, A), together with a map $f : Z \to X$, $f|_A = id|_A$ so that $f_* : \pi_i(Z) \to \pi_i(X)$ is an isomorphism for $i > n$ and is injective when $i = n$.

Remark: If such models exist, we can take A to be some point on X, let $n = 0$ and we get a cellular approximation Z of X.

Theorem: Such n-connected models (Z, A) of a pair (X, A) (with A is a CW complex) exist. Moreover, Z can be obtained from A by attaching cells of dimension greater than n. (Note that from cellular approximation we then have that $\pi_i(Z, A) = 0$ for $i \leq n$).

We will prove this theorem after proving two following corollaries:

Corollary: Any pair of spaces (X, X_0) has a CW approximation (Z, Z_0).

Proof: Let $f_0 : Z_0 \to X_0$ be a CW approximation of X_0. Consider the map $g : Z_0 \to X$ defined by the composition of f_0 and the inclusion map $X_0 \hookrightarrow X$. Consider the mapping cylinder M_g of g, i.e. $M_g = (Z_0 \times I) \sqcup X/(z_0, 1) \sim g(z_0)$. We hence get the sequence of maps $Z_0 \hookrightarrow M_g \to X$ where the map $M_g \to X$ is a deformation retract.

Now, let (Z, Z_0) be a 0-connected CW model of (M_g, Z_0). Consider the triangle:

$$\begin{array}{ccc} (Z, Z_0) & \longrightarrow & (M_g, Z_0) \\ & \searrow & \downarrow \\ & & (X, X_0). \end{array}$$

This gives us a map $f : Z \to X$ that is obtained by composing the weak homotopy equivalence $Z \to M_g$ and the deformation retract (hence homotopy equivalence) $M_g \to X$. In other words, f is a weak homotopy equivalence and $f|_{Z_0} = f_0$, thus proving the result.

□

Corollary: For each n-connected CW pair (X, A) there is a CW pair (Z, A) that is homotopy equivalent to (X, A) relative to A such that Z is built from A by attaching cells of dim $> n$.

Proof: take (Z, A) to be an n-connected model of (X, A). We claim: $Z \underset{h.e.}{\simeq} X$ relative A. In fact, there exists $f : Z \to X$ such that f_* is an isomorphism on π_i when $i > n$ and is injective on π_n. For $i < n$, by the n-connectedness of the given model, $\pi_i(X) \cong \pi_i(A) \cong \pi_i(Z)$ where the isomorphisms are induced by f since the followig diagram commutes,

$$\begin{array}{ccc} Z & \xrightarrow{f} & X \\ \uparrow & & \uparrow \\ A & \xrightarrow{id} & A \end{array}$$

(where the maps $A \hookrightarrow Z$ and $A \hookrightarrow X$ are inclusion maps.) For $i = n$, by n-connectedness we have the surjective maps:

$$\pi_n(A) \xrightarrow{onto} \pi_n(Z) \xrightarrow{inj} \pi_n(X)$$

So the induced map $f_* : \pi_n(A) \to \pi_n(X)$ is surjective. This, coupled with the Whitehead's Theorem allows us to conclude that $f : Z \to X$ is a homotopy equivalence.

We make f stationary on A in the following way: define $W_f := M_f/\{\{a\} \times I \sim \text{pt}, \forall a \in A\}$ and look at the map $h : Z \to X$ given by the composition of $Z \hookrightarrow W_f \to X$ where the map from $W_f \to X$ is a deformation retract.

Claim: Z is a deformation retract of W_f, thus giving us that h is a homotopy equivalence relative to A. Proof of claim: We have $\pi_i(W_f) \cong \pi_i(X)$ (since W_f is a deformation retract of X) and $\pi_i(X) \cong \pi_i(Z)$ since X is homotopy equivalent to Z. Using Whitehead's theorem, we conclude that Z is a deformation retract of W_f. □

Proof of the theorem on cellular approximation: Construct Z as a union of subcomplexes $A = Z_n \subseteq Z_{n+1} \subseteq ...$ such that for each $k \geq n+1$, Z_k is obtained from Z_{k-1} by attaching k-cells.

We will show by induction that we can construct Z_k with a map $f_k : Z_k \to X$ such that $f_k|_A = id|_A$ and f_{k*} is injective on π_i for $n \leq i < k$ and onto on π_i for $n < i \leq k$. We start the induction at $k = n$, $Z_n = A$, in which case the conditions on π_i are void.

Induction step: $k \to k+1$. Consider $\{\phi_\alpha\}_\alpha$ where $\phi_\alpha : S^k \to Z_k$ is the generator of $ker[f_{k*} : \pi_k(Z_k) \to \pi_k(X)]$. Define

$$Y_{k+1} := Z_k \underset{\alpha}{\cup} \underset{\phi_\alpha}{\cup} e_\alpha^{k+1},$$

where e_α^{k+1} is a $(k+1)$ cell attached to Z_k along ϕ_α.

Then $f_k : Z_k \to X$ extends to Y_{k+1}. Indeed, $f \circ \phi_\alpha : S^k \to Z_k \to X$ is nullhomotopic, since $[f_k \circ \phi_\alpha] = f_{k*}[\phi_\alpha] = 0$. We hence get a map $g : Y_{k+1} \to X$. It's easy to check that the map is injective on π_i for $n \leq i \leq k$, and onto on π_k. In fact, since we extend f_k on $(k+1)$-cells, we only need to check the effect on π_k. The elements of $\ker(g)$ on π_k are represented by nullhomotopic maps (by construction) $S^k \to Z_k \subset Y_{k+1} \to X$. So g_* is one-to-one on π_k. Moreover, it is onto on π_k since, by hypothesisthe composition $\pi_k(Z_k) \to \pi_k(Y_{k+1}) \to \pi_k(X)$ is onto.

Let $\{\phi_\beta : S^{k+1} \to X\}$ be a generator of $\pi_{k+1}(X, x_0)$ and let $Z_k = Y_{k+1} \underset{\beta}{\vee} S_\beta^{k+1}$. We now get a map $f_{k+1} : Z_{k+1} \to X$, by defining $f_{k+1}|_{S_\beta^{k+1}} = \phi_\beta$. This implies that f_{k+1} induces an epimorphism on π_{k+1}. The remaining conditions on homotopy groups are easy to check. $\square$