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Classifying spaces for sheaves of simplicial groupoids

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To Alex Heller, in friendship and gratitude

Abstract

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We study the problem of the construction of classifying spaces for sheaves of simplicial groupoids, showing that these exist iff the groupoid is amenable. We show further that any groupoid has an amenable completion, which in the case of the abelian group $K(\pi, n)$, yields a representation theorem for sheaf cohomology in terms of isomorphism classes of certain principal bundles.

Introduction

In this paper we begin the systematic study of homotopy theory in an arbitrary Grothendieck topos $\mathscr{E}$. More precisely, we develop a theory of classifying spaces for bundles over simplicial groups in $\mathscr{E}$. Such a theory is essential for the development of tools like Postnikov systems, characteristic classes, and cohomology operations in sheaf cohomology. Let us recall first the classical theory when $\mathscr{E}$ is the category of sets. For a simplicial group G, there is a simplicial set B and a principal G-bundle $E \to B$, which is universal in the following sense. B is a Kan complex, and for any simplicial set X, the map $f \mapsto f^*(E)$ from $\hom(X, B)$ to the set of principal G-bundles over X induces a 1–1 correspondence between the set $[X, B]$ of homotopy classes of maps, and the set $H^1(X, G)$ of isomorphism classes of principal G-bundles over X.

Replacing the category of sets by an arbitrary Grothendieck topos $\mathscr{E}$, we consider the analogous problem for the category of simplicial sheaves $S(\mathscr{E})$. Concerning the homotopy theory of $S(\mathscr{E})$, recall [6] that there is a Quillen homotopy structure [10] on $S(\mathscr{E})$, in which the weak equivalences are maps

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$f : X \to Y$ inducing isomorphisms on the homotopy sheaves, the cofibrations are the monomorphisms, and the fibrations are maps having the right lifting property with respect to the cofibration weak equivalences. We call a cofibration weak equivalence an *anodyne extension*. Formally inverting the weak equivalences yields the *homotopy category* of $S(\mathscr{E})$. (We obtain the same category by inverting only the anodyne extensions.) If X and Y are simplicial sheaves, we denote by $\mathrm{ho}(X, Y)$ the set of morphisms from X to Y in the homotopy category. If Y is a fibrant simplicial sheaf, then $\mathrm{ho}(X, Y)$ is in 1–1 correspondence with the set $[X, Y]$ of *homotopy classes* of maps from X to Y. (A homotopy is a mapping $X \times I \to Y$ with I the constant simplicial sheaf on the 1-simplex $\Delta[1]$ of the category S of simplicial sets.) As to the concept of principal G-bundle, recall that if G is a group in any topos, a *G-torsor over* X is an object $E \to X$ over X provided with a free (right) G-action $\theta : E \times G \to E$ (free means the map $(\theta, \pi) : E \times G \to E \times E$ is injective), such that the map $E/G \to X$ is an isomorphism. In the topos of simplicial sets S, this concept of principal G-bundle coincides with the usual one [9].

Now, returning to the original problem, given a group G in $S(\mathscr{E})$, we want to know if there is a universal G-torsor $E \to B$ with B fibrant, as in the case of simplicial sets. We find that the answer is negative, unless G is *amenable*, meaning that G-torsors extend uniquely along anodyne extensions of $S(\mathscr{E})$. However, we prove that for any group G there is an anodyne homomorphism $G \hookrightarrow G^*$ with G^* amenable. Moreover, $H^1(X, G^*)$ can be interpreted as the set $\mathscr{H}^1(X, G)$ of *G-pseudo torsors* over X.

When the group G is equal to $K(\pi, n)$, abelian pseudo torsors turn out to be closely connected to the Yoneda interpretation of Ext^n in terms of n-fold extensions. From this we obtain a representation theorem for sheaf cohomology in terms of isomorphism classes of principal bundles.

The amenable completion of a group G is obtained via a certain groupoid completion of G. This, together with the fact that connected groupoids in a topos cannot, in general, be replaced by groups, leads us to develop the theory, from the beginning, in terms of groupoids, and not just groups.

We will often use the principle of *boolean localization* to transfer results in the homotopy theory of simplicial sets to simplicial sheaves. Namely, for any Grothendieck topos $\mathscr{E}$ there exists a surjective geometric morphism $p : \mathscr{B} \to \mathscr{E}$ such that $\mathscr{B}$ is boolean and satisfies the axiom of choice [2]. The inverse image functor p^* provides a faithful embedding of $\mathscr{E}$ into $\mathscr{B}$, whose logic is classical (i.e. $\mathscr{B}$ is a boolean valued model of ZF set theory). Not all constructions are preserved by p^*, but those using only colimits and finite limits are (the so-called *geometric constructions*). For example, p^* preserves the construction of the homotopy groups of a simplicial sheaf. Thus, a mapping $f : X \to Y$ of $S(\mathscr{E})$ is a weak equivalence iff $p^*(f)$ is. As a result, geometric constructions yielding weak equivalences in S, yield weak equivalences in $S(\mathscr{E})$.

1. Groupoids

In this section we collect some fundamental definitions and results concerning groupoids in a Grothendieck topos $\mathscr{E}$.

A *reflexive graph* in $\mathscr{E}$ is a diagram

$$G_0 \xrightarrow{u} G_1 \underset{t}{\overset{s}{\rightrightarrows}} G_0$$

in $\mathscr{E}$ with $su = tu = \mathrm{id}$. A morphism of reflexive graphs is a pair of mappings $f_0 : G_0 \to H_0$ and $f_1 : G_1 \to H_1$, which respect, in an obvious sense, the given mappings s, t and u.

A *groupoid* $\mathbb{G}$ in $\mathscr{E}$ is a reflexive graph provided with an associative composition

$$c : G_1 \times_{G_0} G_1 \to G_1 ,$$

for which the elements of G_0 are units (via u), and each element of G_1 is invertible. (These statements, of course, are interpreted in the internal language of $\mathscr{E}$.) If $\mathbb{G}$ and $\mathbb{H}$ are groupoids, a functor $f : \mathbb{G} \to \mathbb{H}$ is morphism of reflexive graphs, which respects composition. f is a *categorical equivalence* if it is full, faithful, and representative in the sense that each object of $\mathbb{H}$ is isomorphic to the image of an object of $\mathbb{G}$ under f. (See [3] for a discussion of various different notions of equivalence for internal categories.)

Let $\mathbb{G}$ be a groupoid in $\mathscr{E}$. A (right) $\mathbb{G}$-*torsor* is a non-empty object E (i.e. $E \to 1$ is surjective) over G_0 equipped with a free (contravariant) action $a : E \times_{G_0} G_1 \to E$, which is transitive. A $\mathbb{G}$-torsor $E \to X$ over X is a $\mathbb{G}$-torsor in $\mathscr{E}/X$. The set of isomorphism classes of $\mathbb{G}$-torsors over X is denoted by $H^1(X, \mathbb{G})$. $H^1(X, G)$ is contravariant in X and covariant in $\mathbb{G}$. In fact, if $g : Y \to X$ is a map, and $E \to X$ is a $\mathbb{G}$-torsor over X, then $g^*(E)$ is a $\mathbb{G}$-torsor over Y, and if $f : \mathbb{G} \to \mathbb{H}$ is a functor then $E \otimes_{\mathbb{G}} \mathbb{H}$ is an $\mathbb{H}$-torsor over X. Note that $H^1(X, \mathbb{G})$ is invariant under categorical equivalence of groupoids.

The *nerve* of a groupoid $\mathbb{G}$ in $\mathscr{E}$ is a simplicial sheaf $N\mathbb{G}$ described as follows. $(N\mathbb{G})_n$ is the object G_n of composable strings of length n, $\sigma = x_n \to x_{n-1} \to \cdots \to x_1 \to x_0$, of arrows of $\mathbb{G}$. Faces and degeneracies are given by $d^0\sigma = x_n \to x_{n-1} \to \cdots \to x_1$, $d^n\sigma = x_{n-1} \to \cdots \to x_0$, and $d^i\sigma = x_n \to \cdots \to x_{i+1} \to x_{i-1} \to \cdots \to x_0$ for $0 < i < n$, where $x_{i+1} \to x_{i-1}$ is the composite of the pair $x_{i+1} \to x_i \to x_{i-1}$. $s^i\sigma = x_n \to \cdots \to x_i \to x_i \to \cdots \to x_0$, where $x_i \to x_i$ is the identity on x_i. In particular, $(N\mathbb{G})_0 = G_0$, the objects of $\mathbb{G}$, $(N\mathbb{G})_1 = G_1$, the morphisms of $\mathbb{G}$, and $d^0(\alpha : x_1 \to x_0) = x_1 = s\alpha$, $d^1(\alpha : x_1 \to x_0) = x_0 = t\alpha$, and $s^0x = (\mathrm{id} : x \to x) = ux$.

When $\mathbb{G}$ is a groupoid in $S(\mathscr{E})$, $N\mathbb{G}$ is a double simplicial object of $\mathscr{E}$, whose nth column is the simplicial object G_n. Letting $d : S^2(\mathscr{E}) \to S(\mathscr{E})$ denote the *diagonal complex* defined by $d(X)_n = X_{nn}$, we write $B\mathbb{G}$ for $d(N\mathbb{G})$. In arguments involving $B\mathbb{G}$, we often use the following, fundamental property of the diagonal complex. Namely, let $f : X \to Y$ be a mapping in $S^2(\mathscr{E})$. We call f a *vertical weak equivalence* if $f_{n*} : X_{n*} \to Y_{n*}$ is a weak equivalence for each $n \geq 0$, and a *horizontal weak equivalence* if $f_{*m} : X_{*m} \to Y_{*m}$ is a weak equivalence for each $m \geq 0$. Then it follows by boolean localization, and the corresponding fact about double simplicial sets, that if $f : X \to Y$ is either a vertical or horizontal weak equivalence, $df : dX \to dY$ is a weak equivalence.

The following construction is used throughout the rest of the paper. Let G be a groupoid of $S(\mathscr{E})$, and denote by $\mathbb{G}'$ the groupoid of arrows of $\mathbb{G}$. If the domain and codomain of $\mathbb{G}$ are $(s, t) : G_1 \to G_0 \times G_0$, then the domain and codomain of $\mathbb{G}'$ are $(\pi_1, c) : G_1 \times_{G_0} G_1 \to G_1 \times G_1$, where π_1 is the first projection, and c is the composition of $\mathbb{G}$. That is, a morphism $f \to g$ in $\mathbb{G}'$ is an arrow h such that $hf = g$. The codomain t defines a functor $\mathbb{G}' \to \mathbb{G}$ inducing a morphism $N\mathbb{G}' \to N\mathbb{G}$. The domain s defines a mapping $N\mathbb{G}' \to hG_0$, where hG_0 is the object G_0 considered as a constant double simplicial object of $\mathscr{E}$. $N\mathbb{G}' \to hG_0$ is a horizontal weak equivalence. In fact, $N\mathbb{G}'$ is simply $(N\mathbb{G})^*$ as defined below. The groupoid $h\mathbb{G}$ acts freely on the right of $N\mathbb{G}' \to hG_0$, with quotient $N\mathbb{G}$. Thus, if we write $E\mathbb{G}$ for $B\mathbb{G}'$, applying d to the above yields a (right) $\mathbb{G}$-torsor $E\mathbb{G}$ over $B\mathbb{G}$ such that $E\mathbb{G} \to G_0$ is a weak equivalence.

If A is a reflexive graph in $\mathscr{E}$, we denote by $F(A)$ the free groupoid on A. For example, the *path space* of X in $S(\mathscr{E})$ is the free groupoid $F(\gamma X)$ on a reflexive graph γX in $S(\mathscr{E})$ described as follows. Let X be an object in $S(\mathscr{E})$, and denote by X^* the simplicial object defined by $(X^*)_n = X_{n+1}$ for $n \geq 0$ and $d(X^*)^i = d(X)^{i+1}$, $s(X^*)^j = s(X)^{j+1}$, $0 \leq i, j \leq n$. Then the 0th face operator of X defines a mapping $q : X^* \to X$, and the mapping $\delta X_0 \to X$ corresponding to the identity on X_0 is an anodyne extension, where δX_0 is the object of 0-simplices of X considered as a discrete object of $S(\mathscr{E})$. Let γX be defined by the pushout

$$\begin{tikzcd} \delta X_0 \arrow[r, hook] \arrow[d, hook] & X \arrow[d, hook] \\ X^* \arrow[r, hook] & \gamma X \end{tikzcd}$$

and let $d^0, d^1 : \gamma X \to X$ be given by, respectively, q on X^* and id on X, and the 0th vertex mapping $X^* \to \delta X_0$ followed by $\delta X_0 \to X$ on X^*, and id on X. We write $P(X)$ for the free groupoid $F(\gamma X)$ in $S(\mathscr{E})$. Since the inclusion $X \hookrightarrow P(X)$ of the units of $P(X)$ is a weak equivalence in S, the same is true in $S(\mathscr{E})$ by boolean localization. (See [8] for a full discussion of path spaces and related topics.)

2. Locally transitive groupoids

Let $p : \mathscr{B} \to \mathscr{E}$ be a surjective geometric morphism with $\mathscr{B}$ boolean as above. We call a map $f : X \to Y$ of $S(\mathscr{E})$ a *local fibration*, respectively a *local bundle*, if

$p^*(f)$ is a Kan fibration, respectively a bundle. If $\mathbb{G}$ is a groupoid in $S(\mathscr{E})$, the *simplicial object of categorical components* of $\mathbb{G}$ is the coequalizer of $s,t : G_1 \to G_0$, and is denoted by $c(\mathbb{G})$.

Definition 1. $\mathbb{G}$ is said to be *locally transitive* if $c(\mathbb{G})$ is simplicially discrete.

Equivalent conditions are: $\mathbb{G}$ is locally transitive iff $(s,t) : G_1 \to G_0 \times G_0$ is a local fibration, or iff pulling G_1 back over the 0-skeleton of G_0 yields an equivalence of categories. See [8] for proofs. As an example, $P(X)$ is locally transitive.

Proposition 2. *Let $\mathbb{G}$ be a locally transitive groupoid in $S(\mathscr{E})$. Then any $E \to G_0$ acted on by $\mathbb{G}$, as well as any $\mathbb{G}$-torsor $E \to X$, is a local bundle.*

Proof. A map $E \to X$ in the category S of simplicial sets is a bundle iff $E \to X$ admits a (left or right) action of the path space $P(X)$ [8]. If $\mathbb{G}$ is locally transitive (in S), then there is a dotted filler in the diagram

$$\begin{tikzcd} G_0 \arrow[r, hook] \arrow[d, hook] & G_1 \arrow[d, "(s,t)"] \\ \gamma G_0 \arrow[r] \arrow[ur, dashed] & G_0 \times G_0 \end{tikzcd}$$

so that $\mathbb{G}$ admits a functor $P(G_0) \to \mathbb{G}$. Thus, if $\mathbb{G}$ acts on $E \to G_0$ so does $P(G_0)$, making $E \to G_0$ a bundle. For the same reason, both $s : G_1 \to G_0$ and $t : G_1 \to G_0$ are bundles, making any $\mathbb{G}$-torsor $E \to X$ a bundle. The result now follows by boolean localization. $\square$

Corollary 3. *If $\mathbb{G}$ is a locally transitive groupoid in $S(\mathscr{E})$, weak equivalences are preserved by pullback along any $E \to G_0$ acted on by $\mathbb{G}$. They are preserved and reflected by pullback along any $\mathbb{G}$-torsor $E \to X$.*

Proof. The result follows by boolean localization, since weak equivalences are preserved by pullback along bundles in S. They are reflected by pullback along a torsor, since torsors are surjective. $\square$

Proposition 4. *Let $\mathbb{G}$ and $\mathbb{H}$ be groupoids in $S(\mathscr{E})$, and suppose $f : \mathbb{G} \to \mathbb{H}$ is a functor. Then $Bf : B\mathbb{G} \to B\mathbb{H}$ is a weak equivalence if either*

(1) *f is a categorical equivalence, or*

(2) *$\mathbb{G}$ and $\mathbb{H}$ are locally transitive, and f_0 and f_1 are weak equivalences of $S(\mathscr{E})$.*

Proof. In case (1), $Nf : N\mathbb{G} \to N\mathbb{H}$ is a horizontal weak equivalence by boolean localization.

Case (2) requires a further argument. First, let $(s,t) : G \to X \times X$ and

$(\sigma, \tau) : H \to X \times X$ be two (not necessarily reflexive) graphs in $S(\mathscr{E})$ with the same object of vertices X. Denote the composite of H and G by $H \circ G$ (its elements are pairs (b, a) with $b \in H$, $a \in G$ and $\tau b = sa$). It can be constructed as the pullback

$$\begin{tikzcd} H \circ G \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] & X \times G \arrow[d, "1 \times (s,t)"] \\ H \times H \arrow[r, "(\sigma,\tau) \times 1"'] & X \times X \times X \end{tikzcd}$$

Now suppose $(s'. t') : G' \to Y \times Y$, $(\sigma', \tau') : H' \to Y \times Y$ is another such pair of graphs, and $f : G \to G'$, $g : H \to H'$ are mappings of graphs with $f_0 = g_0 : X \to Y$. Suppose further that (s, t) and (s', t') are local fibrations, and f_1, g_1 and $f_0 = g_0$ are weak equivalences. Then we obtain a map of the pullback defining $H \circ G$ into the pullback defining $H' \circ G'$, in which the front and back faces are pullbacks with one leg a local fibration and three of the horizontal mappings weak equivalences. Hence, by boolean localization and the corresponding fact for simplicial sets, the induced map $H \circ G \to H' \circ G'$ is a weak equivalence.

Now in case (2), we have a functor $f : \mathbb{G} \to \mathbb{H}$ with $\mathbb{G}$ and $\mathbb{H}$ locally transitive and f_0 and f_1 weak equivalences. For $n \geq 2$, G_n is $G_{n-1} \circ G_1$. Thus, by the above and induction, it follows that $f_n : G_n \to H_n$ is a weak equivalence for all n, so that Nf is a vertical weak equivalence and the proposition is proved. □

Theorem 5. *Let $f : \mathbb{G} \to \mathbb{H}$ be a functor between locally transitive groupoids of $S(\mathscr{E})$. Then Bf is a weak equivalence iff f is representative, and, pulling back the arrows of $\mathbb{H}$ over $G_0 \times G_0$ we obtain a diagram*

$$\begin{tikzcd} G_1 \arrow[dr] \arrow[drr, "f_1"] \arrow[ddr] & & \\ & P \arrow[r] \arrow[d] & H_1 \arrow[d] \\ & G_0 \times G_0 \arrow[r, "f_0 \times f_0"] & H_0 \times H_0 \end{tikzcd}$$

in which the induced map $G_1 \to P$ is a weak equivalence.

Proof. If f satisfies the condition, then Bf is a weak equivalence since, in the above diagram, the inside square is a categorical equivalence, and $BG_1 \to BP$ is a weak equivalence by Proposition 2(2).

In the other direction, suppose Bf is a weak equivalence. $f : \mathbb{G} \to \mathbb{H}$ is representative iff $c(f) : c(\mathbb{G}) \to c(\mathbb{H})$ is surjective. An easy argument based on Lemma 0.17 of [5] shows that, in general, $\pi_0 B\mathbb{G}$ is the coequalizer of $\pi_0 s, \pi_0 t : \pi_0 G_1 \to \pi_0 G_0$. But π_0 preserves coequalizers, so we obtain the formula

$\pi_0 BG \simeq \pi_0(c(\mathbb{G}))$. When $\mathbb{G}$ is locally transitive, $c(\mathbb{G})$ is simplicially discrete so $\pi_0 B\mathbb{G} \simeq c(\mathbb{G})$. Thus, if $f : \mathbb{G} \to \mathbb{H}$ is a functor such that $\mathbb{G}$ and $\mathbb{H}$ are locally transitive and Bf is a weak equivalence, then, in fact, $c(f)$ is an isomorphism.

To finish the argument, it suffices to show that if $f : G \to \mathbb{H}$ is a functor between locally transitive groupoids such that f_0 is the identity on $G_0 = H_0$ and Bf is a weak equivalence, then f_1 is a weak equivalence. For X in $S(\mathscr{C})$, let dis X denote the object X considered as a discrete groupoid in $S(\mathscr{C})$. Let dis $G_0 \to \mathbb{G}$, dis $G_0 \to \mathbb{H}$, dis $G_1 \to \mathbb{G}'$, and dis $H_1 \to \mathbb{H}'$ be the functors corresponding to the identity on objects. Applying B yields pullback diagrams

$$\begin{tikzcd} G_1 \arrow[r] \arrow[d, "t"'] & E\mathbb{G} \arrow[d] \\ G_0 \arrow[r] & B\mathbb{G} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} H_1 \arrow[r] \arrow[d, "t"'] & E\mathbb{H} \arrow[d] \\ G_0 \arrow[r] & B\mathbb{H} \end{tikzcd}$$

Consider the diagram

$$\begin{tikzcd} G_1 \arrow[rr] \arrow[dr, "f_1"] \arrow[dd, "t"'] & & E\mathbb{G} \arrow[dr, "Ef"] \arrow[dd] & \\ & H_1 \arrow[rr] \arrow[dl, "t"] & & E\mathbb{H} \arrow[dd] \\ G_0 \arrow[rr] \arrow[drrr] & & B\mathbb{G} \arrow[dr, "Bf"] & \\ & & & B\mathbb{H} \end{tikzcd}$$

Ef is a weak equivalence since $E\mathbb{G} \to G_0$ and $E\mathbb{H} \to G_0$ are, and

$$\begin{tikzcd} E\mathbb{G} \arrow[dr] \arrow[dd, "Ef"'] & \\ & G_0 \\ E\mathbb{H} \arrow[ur] & \end{tikzcd}$$

commutes. $E\mathbb{G}$ and $E\mathbb{H}$ are torsors for locally transitive groupoids, so by Corollary 3 the induced map of $E\mathbb{G}$ into the pullback of $E\mathbb{H}$ along Bf is a weak equivalence. f_1 is the pullback of this map, so it too is a weak equivalence (factor $G_0 \to B\mathbb{G}$ into an anodyne extension followed by a fibration and pull back over each piece). $\square$

3. Amenable groupoids

If $\mathbb{G}$ is a groupoid in $S(\mathscr{C})$, the existence of a universal $\mathbb{G}$-torsor $E \to B$ as in the Introduction is equivalent to the existence of a fibrant simplicial sheaf B such that

$H^1(X, \mathbb{G}) \simeq [X, B]$ naturally in X. An obvious necessary condition for this is that $H^1(X, \mathbb{G})$ should invert weak equivalences, or, what is the same thing, anodyne extensions. More precisely, if $A \hookrightarrow B$ is an anodyne extension and $T \to A$ is a $\mathbb{G}$-torsor over A, there should exist a $\mathbb{G}$-torsor $S \to B$ whose restriction to A is isomorphic to T. Moreover, S should be unique up to isomorphism.

Definition 6. A groupoid $\mathbb{G}$ in $S(\mathscr{E})$, whose torsors have the above property, is said to be *amenable*.

Proposition 7. *An amenable groupoid in $S(\mathscr{E})$ is locally transitive.*

Proof. We remark first that if $\mathbb{G}$ is any groupoid in $S(\mathscr{E})$, $t : G_1 \to G_0$ is a $\mathbb{G}$-torsor, and two maps $f, g : X \to G_0$ satisfy $f^*(G_1) \simeq g^*(G_1)$ iff there is a map $k : X \to G_1$ such that

$$\begin{tikzcd} & G_1 \arrow[d, "{(s,t)}"] \\ X \arrow[ur, "k"] \arrow[r, "{(f,g)}"'] & G_0 \times G_0 \end{tikzcd}$$

commutes. This is because $\mathbb{G} = G_1 \to G_0 \times G_0$ is canonically isomorphic to $\mathrm{Iso}_{\mathbb{G}}(G_1, G_1) \to G_0 \times G_0$—the groupoid of $\mathbb{G}$-isomorphisms of the fibres of $t : G_1 \to G_0$ (the Yoneda Lemma).

Let I be the constant simplicial sheaf on the 1-simplex $\Delta[1]$ of the category S of simplicial sets. There are two maps $1 \to I$ corresponding to the endpoints of I, and two maps $f, g : X \to G_0$ are homotopic iff there is a map $h : X \to G_0^I$ such that the diagram

$$\begin{tikzcd} & G_0^I \arrow[d] \\ X \arrow[ur, "h"] \arrow[r, "{(f,g)}"'] & G_0 \times G_0 \end{tikzcd}$$

commutes. Now the diagonal $G_0 \hookrightarrow G_0^I$ is an anodyne extension, and a splitting of each of the two maps $G_0^I \to G_0$. Thus, if $\mathbb{G}$ is amenable, the two pullbacks of $t : G_1 \to G_0$ over G_0^I are isomorphic. It follows that there is a map $k : G_0^I \to G_1$ such that

$$\begin{tikzcd} & G_1 \arrow[d, "{(s,t)}"] \\ G_0^I \arrow[ur, "k"] \arrow[r] & G_0 \times G_0 \end{tikzcd}$$

commutes, which expresses universally the fact that if two maps $f, g : X \to G_0$ are homotopic, then $f^*(G_1) \simeq g^*(G_1)$. Clearly, the existence of k implies that the coequalizer of the two endpoint maps $G_0^I \to G_0$, which is $\delta\pi_0(G_0)$, maps surjectively onto the coequalizer $c(\mathbb{G})$ of $s,t : G_1 \to G_0$. But then $c(\mathbb{G})$ is simplicially discrete, and $\mathbb{G}$ is locally transitive. □

Note that amenability is invariant under categorical equivalence of groupoids.

Examples. (a) Any group G in S is amenable.

(b) A locally transitive groupoid in S is amenable. In fact, if $\mathbb{G}$ is locally transitive in S then $\mathbb{G}$ is categorically equivalent to a groupoid with discrete object of objects (the pullback of G_1 over the 0-skeleton of G_0). But such a groupoid is categorically equivalent to a coproduct of groups, which is amenable in S. Thus, amenable is equivalent to locally transitive for simplicial sets. This is not true for a general $\mathscr{E}$, however. To see this, let X be an object of $\mathscr{E}$, and define $\bar{X}$ in $S(\mathscr{E})$ by $\bar{X}_0 = X$, $\bar{X}_1 = X \times X$, $\bar{X}_2 = X \times X \times X$, etc., with faces and degeneracies given by projections and diagonals respectively. A map $W \to \bar{X}$ in $S(\mathscr{E})$ is equivalent to a map $W_0 \to X$ in $\mathscr{E}$. If G is a group in $\mathscr{E}$, then $\bar{G}$ is a group in $S(\mathscr{E})$, which is locally transitive since all groups are. But $\bar{G}$ is not, in general, amenable. For, let Y be a non-trivial G-torsor in $\mathscr{E}$. Then $\bar{Y}$ is a non-trivial $\bar{G}$-torsor in $S(\mathscr{E})$ (a point of $\bar{Y}$ is the same as a point of Y). However, $\bar{Y} \to 1$ is a weak equivalence, and $\bar{Y} \times \bar{G} \simeq \bar{Y} \times \bar{Y}$ over $\bar{Y}$, but $\bar{Y}$ is not isomorphic to $\bar{G}$ over 1. Also, note that $\bar{G}$ is not fibrant unless, say, G is an injective object of $\mathscr{E}$. Thus, groups in $S(\mathscr{E})$ are not, in general, fibrant either.

Proposition 8. *A simplicially discrete groupoid in $S(\mathscr{E})$ is amenable, where simplicially discrete means a groupoid of the form $\delta\mathbb{G}$, where $\mathbb{G}$ is a groupoid in $\mathscr{E}$.*

Proof. The proof requires a discussion of coverings in $S(\mathscr{E})$.

In [4], a *covering* of X in S is defined to be a map $p : E \to X$ such that for any vertex i of $\Delta[n]$, any diagram

$$\begin{tikzcd} 1 \arrow[r] \arrow[d, "i"'] & E \arrow[d, "p"] \\ \Delta[n] \arrow[r] \arrow[ur, dashed] & X \end{tikzcd}$$

has a unique dotted filler. This can be expressed by saying that if $i : X_n \to X_0$ denotes the ith vertex mapping, then

$$\begin{tikzcd} E_n \arrow[r, "i"] \arrow[d, "p_n"'] & E_0 \arrow[d, "p_0"] \\ X_n \arrow[r, "i"'] & X_0 \end{tikzcd}$$

is a pullback for any $n \geq 0$ and $0 \leq i \leq n$, which we take as the definition of covering in $S(\mathscr{E})$.

If $p : E \to Y$ and $f : X \to Y$ are mappings of $S(\mathscr{E})$, let

$$\begin{tikzcd} E' \arrow[r] \arrow[d, "p'"'] & E \arrow[d, "p"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

be a pullback. Then, if p is a covering so is p'. Furthermore, if f is surjective and p' is a covering, so is p, which is what we mean by saying the notion of covering is *local*. Clearly, for any mapping $g : W \to Z$ in $\mathscr{E}$, $\delta_g : \delta W \to \delta Z$ is a covering in $S(\mathscr{E})$. As a result, if $\mathbb{G}$ is a groupoid in $\mathscr{E}$, any $\delta\mathbb{G}$-torsor $p : E \to X$ is a covering in $S(\mathscr{E})$. For, if $a : E \times_{\delta G_0} \delta G_1 \to E$ denotes the action of $\delta\mathbb{G}$ on E, then both of the following diagrams

$$\begin{tikzcd} E \times_{\delta G_0} \delta G_1 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & \delta G_1 \arrow[d, "\delta t"] \\ E \arrow[r] & \delta G_0 \end{tikzcd} \qquad \begin{tikzcd} E \times_{\delta G_0} \delta G_1 \arrow[r, "a"] \arrow[d, "\pi_1"'] & E \arrow[d, "p"] \\ E \arrow[r, "p"'] & X \end{tikzcd}$$

are pullbacks, δt is a covering, and p is surjective. Hence, p is a covering.

If X is in $S(\mathscr{E})$, let πX denote the fundamental groupoid of X. As in S, the functor $\pi : S(\mathscr{E}) \to \mathrm{Gpd}(\mathscr{E})$ is left adjoint to the nerve $N : \mathrm{Gpd}(\mathscr{E}) \to S(\mathscr{E})$. Write $\mathrm{Cov}(X)$ for the full subcategory of $S(\mathscr{E})/X$ whose objects are the coverings of X, and $\mathscr{E}^{\pi X}$ for the category of internal, $\mathscr{E}$-valued functors on πX. In [4] it is shown that the adjoint pair

$$\mathrm{Cov}(X) \underset{N_X}{\overset{\pi_X}{\rightleftarrows}} \mathscr{E}^{\pi X}$$

defined by $\pi_X(E \to X) = \pi E \to \pi X$, and $N_X(F \to \pi X) =$ the left vertical mapping p in the pullback

$$\begin{tikzcd} E \arrow[r] \arrow[d, "p"'] & NF \arrow[d] \\ X \arrow[r, "\eta X"'] & N\pi X \end{tikzcd}$$

ηX being the unit of the adjunction $\pi \dashv N$, is an equivalence of categories. The same is true for $S(\mathscr{E})$ by boolean localization. Thus, if $f : X \to Y$ is such that $\pi f : \pi X \to \pi Y$ is a categorical equivalence, it follows that the change of base functor $f^* : \mathrm{Cov}(Y) \to \mathrm{Cov}(X)$ is also an equivalence of categories. Furthermore, for a simplicially discrete groupoid $\delta\mathbb{G}$, $\delta\mathbb{G}$-torsors over Y are $\delta\mathbb{G} \times Y$-torsors in $S(\mathscr{E})/Y$. But such torsors are coverings, and $\delta\mathbb{G} \times Y$ is a groupoid in $\mathrm{Cov}(Y)$, which is taken to $\delta\mathbb{G} \times X$ by f^*, so the category of $\delta\mathbb{G}$-torsors over Y is equivalent, under f^*, to the category of $\delta\mathbb{G}$-torsors over X. In particular, this holds when f is an anodyne extension, so $\delta\mathbb{G}$ is amenable. $\square$

Returning to the question of the existence of universal torsors, we give sufficient conditions for a $\mathbb{G}$-torsor to be universal in the case that $\mathbb{G}$ is amenable. Namely, we prove the following theorem:

Theorem 9. *Let $\mathbb{G}$ be an amenable groupoid in $S(\mathscr{E})$, and suppose $E \to B$ is a $\mathbb{G}$-torsor over B. Then, if B is fibrant and $E \to G_0$ is a weak equivalence, $E \to B$ is universal.*

Proof. Suppose $E \to B$ is such a $\mathbb{G}$-torsor. For an arbitrary $\mathbb{G}$-torsor $T \to X$, we want to produce a map $f : X \to B$ such that $f^*E \simeq T$. In the pullback

$$\begin{tikzcd} E' \arrow[r] \arrow[d, "w"'] & E \arrow[d, "u"] \\ T \arrow[r] & G_0 \end{tikzcd}$$

the map w is a weak equivalence by Proposition 7 and Corollary 3 since u is. Dividing by the action of $\mathbb{G}$ we obtain a diagram

$$\begin{tikzcd} T \arrow[d] & E' \arrow[l, "w"'] \arrow[r] \arrow[d] & E \arrow[d] \\ X & X' \arrow[l, "v"] \arrow[r, "f'"'] & B \end{tikzcd}$$

in which both squares are not only commutative, but pullbacks due to the freeness of the $\mathbb{G}$ actions. v is a weak equivalence by Corollary 3 since w is. B is fibrant, so there is a map $f : X \to B$ such that fv is homotopic to f'. The projection $X \times I \to X$ is a weak equivalence, so by the amenability of $\mathbb{G}$, any $\mathbb{G}$-torsor over $X \times I$ is constant in I. As a result, any two homotopic maps into B induce isomorphic pullbacks of E. Thus, $v^*(T) \simeq f'^*E \simeq v^*(f^*E)$. Since v is a weak equivalence and $\mathbb{G}$ is amenable, it follows that $T \simeq f^*E$ as desired.

To prove the theorem, it remains to show that if two maps $f, g : X \to B$ are such that $f^*E \simeq g^*E$ then f is homotopic to g. To do this, recall that the groupoid $\mathrm{Iso}_{\mathbb{G}}(E, E) \to B \times B$ of $\mathbb{G}$-isomorphisms of the fibres of $E \to B$ is constructed by dividing $E \times_{G_0} E$ by the action of $\mathbb{G}$, the two maps into B being induced by $E \to B$. The inclusion $B \hookrightarrow \mathrm{Iso}_{\mathbb{G}}(E, E)$ of the units of $\mathrm{Iso}_{\mathbb{G}}(E, E)$ is obtained by dividing the diagonal $E \hookrightarrow E \times_{G_0} E$ by the action of $\mathbb{G}$ as in

$$\begin{tikzcd} E \arrow[r, hook] \arrow[d] & E \times_{G_0} E \arrow[d] \\ B \arrow[r, hook] & \mathrm{Iso}_{\mathbb{G}}(E, E) \end{tikzcd}$$

which is a pullback as above. Since $E \to G_0$ is a weak equivalence so is $E \hookrightarrow E \times_{G_0} E$, which implies that $B \hookrightarrow \mathrm{Iso}_{\mathbb{G}}(E, E)$ is also. Now $f, g : X \to B$ are such that $f^*E \simeq g^*E$ iff the map $(f, g) : X \to B \times B$ lifts to $\mathrm{Iso}_{\mathbb{G}}(E, E)$. Since $B \hookrightarrow \mathrm{Iso}_{\mathbb{G}}(E, E)$ is a weak equivalence, if (f, g) lifts to $\mathrm{Iso}_{\mathbb{G}}(E, E)$, f and g are identified in the homotopy category of $S(\mathscr{E})$. But then, since B is fibrant, f is homotopic to g, which finishes the proof. □

Theorem 10. *Let $\mathbb{G}$ be a groupoid in $S(\mathscr{E})$. Then $\mathbb{G}$ has a universal torsor iff $\mathbb{G}$ is amenable.*

Proof. We have seen that amenability is necessary for the existence of a universal torsor. On the other hand, for any groupoid $\mathbb{G}$ in $S(\mathscr{E})$ we have a $\mathbb{G}$-torsor $E\mathbb{G} \to B\mathbb{G}$ with $E\mathbb{G} \to G_0$ a weak equivalence as in Section 1. Let $B\mathbb{G} \hookrightarrow B$ be an anodyne extension with B fibrant. If $\mathbb{G}$ is amenable, $E\mathbb{G} \to B\mathbb{G}$ extends to a $\mathbb{G}$-torsor $E \to B$ with $E \to G_0$ a weak equivalence. By Theorem 5, $E \to B$ is universal. □

4. Fibrant stacks

Definition 11. A groupoid $\mathbb{G}$ in $S(\mathscr{E})$ is said to be a *strong stack* if for every categorical equivalence of groupoids $\mathbb{A} \hookrightarrow \mathbb{B}$, which is injective on objects, each diagram

$$\begin{tikzcd} \mathbb{A} \arrow[r] \arrow[d, hook] & \mathbb{G} \\ \mathbb{B} \arrow[ur, dashed] & \end{tikzcd}$$

has a dotted filler. (See [7] for a full discussion of strong stacks.)

Definition 12. A groupoid $\mathbb{G}$ in $S(\mathscr{E})$ is called a *fibrant stack* if it is a strong stack such that $(s, t) : G_1 \to G_0 \times G_0$ is a fibration, and G_0 is fibrant in $S(\mathscr{E})$.

Proposition 13. *Let $\mathbb{G}$ be a fibrant stack in $S(\mathscr{E})$. Then $\mathbb{G}$ is amenable, and $t : G_1 \to G_0$ is a universal $\mathbb{G}$-torsor.*

Proof. Since $\mathbb{G}$ is a strong stack, any $\mathbb{G}$-torsor $T \to X$ has a section $X \to T$ [7]. But then, if $f : X \to G_0$ is the composite $X \to T \to G_0$, it follows that $f^*(t : G_1 \to G_0) \simeq T \to X$, so that every $\mathbb{G}$-torsor has a classifying map into G_0. Since G_0 is fibrant, this implies that $\mathbb{G}$-torsors extend along anodyne extensions. To prove these extensions are unique, suppose $A \hookrightarrow B$ is an anodyne extension, and T, T' are $\mathbb{G}$-torsors over B, whose restrictions to A are isomorphic. Taking classifying maps yields a diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook] & G_1 \arrow[d, "(s,t)"] \\ B \arrow[r] \arrow[ur, dashed] & G_0 \times G_0 \end{tikzcd}$$

which has a dotted filler since (s, t) is a fibration. Hence, the isomorphism over A can be extended to an isomorphism over B. Thus, we have shown that $\mathbb{G}$ is amenable.

By Theorem 9, $t : G_1 \to G_0$ is universal if $s : G_1 \to G_0$ is a weak equivalence. In fact, we show s is a trivial fibration. That is, we show any diagram of the form

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook] & G_1 \arrow[d, "s"] \\ B \arrow[r] \arrow[ur, dashed] & G_0 \end{tikzcd}$$

with $A \hookrightarrow B$ a cofibration has a dotted filler. For this, let $\mathbb{I}$ denote the groupoid in Sets having two objects 0 and 1 with one isomorphism between them, and write $\mathbb{I}$ again for the same groupoid considered as a simplicially discrete, constant groupoid in $S(\mathscr{E})$. If $A \hookrightarrow B$ is a cofibration of $S(\mathscr{E})$, write $(\text{dis } A \times \mathbb{I}) \cup \text{dis } B$ for the pushout of groupoids

$$\begin{tikzcd} \text{dis } A \arrow[r, hook, "1\times 0"] \arrow[d, hook] & \text{dis } A \times \mathbb{I} \arrow[d, hook] \\ \text{dis } B \arrow[r, hook] & (\text{dis } A \times \mathbb{I}) \cup \text{dis } B \end{tikzcd}$$

$1 \times 0 : \text{dis } B \to \text{dis } B \times \mathbb{I}$, together with $\text{dis } A \times \mathbb{I} \hookrightarrow \text{dis } B \times \mathbb{I}$, induces a categorical equivalence $(\text{dis } A \times \mathbb{I}) \cup \text{dis } B \hookrightarrow \text{dis } B \times \mathbb{I}$, which is injective on objects. Furthermore, the original diagram above is equivalent to one of the form

$$\begin{tikzcd} (\text{dis } A \times \mathbb{I}) \cup \text{dis } B \arrow[r] \arrow[d, hook] & \mathbb{G} \\ \text{dis } B \times \mathbb{I} \arrow[ur, dashed] & \end{tikzcd}$$

Since $\mathbb{G}$ is a strong stack, this diagram has a dotted filler, hence so does the first. As a result, s is a weak equivalence, and the proposition is proved. □

A reflexive graph

$$X_0 \xrightarrow{u} X_1 \overset{s}{\underset{t}{\rightrightarrows}} X_0$$

in $S(\mathscr{E})$ is a simplicial object in the topos of reflexive graphs in $\mathscr{E}$. Thus, by [6], there is a Quillen homotopy structure on the topos of reflexive graphs in $S(\mathscr{E})$, in which a morphism $k: A \to B$ of graphs is a cofibration iff k_0 and k_1 are monomorphisms, a weak equivalence iff k_0 and k_1 are weak equivalences of $S(\mathscr{E})$, and a fibration iff k has the right lifting property with respect to the cofibration weak equivalences.

Proposition 14. *A reflexive graph X is fibrant iff $(s, t): X_1 \to X_0 \times X_0$ is a fibration, and X_0 is fibrant in $S(\mathscr{E})$.*

Proof. For necessity, suppose X is a fibrant reflexive graph and $A \hookrightarrow B$ is an anodyne extension of $S(\mathscr{E})$. We write dis A for the reflexive graph, in which u, s and t are all the identity. This dis A is the underlying graph of the discrete groupoid dis A as defined above. A diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X_0 \\ B \arrow[ur, dashed] & \end{tikzcd}$$

is equivalent to a diagram

$$\begin{tikzcd} \operatorname{dis} A \arrow[r] \arrow[d, hook] & X \\ \operatorname{dis} B \arrow[ur, dashed] & \end{tikzcd}$$

of graphs. Since X is a fibrant graph, and dis $A \hookrightarrow$ dis B is an anodyne extension of graphs, the second diagram has a dotted filler. Hence so does the first, and we see that X_0 is fibrant in $S(\mathscr{E})$.

Consider a diagram of the form

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook] & X_1 \arrow[d, "(s,t)"] \\ B \arrow[r] & X_0 \times X_0 \end{tikzcd}$$

and let

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook] & X_1 \arrow[d, hook] \\ B \arrow[r, "b"'] & P \end{tikzcd}$$

be a pushout. There is a map $P \to X_0 \times X_0$ making P a reflexive graph with units X_0, and $X \hookrightarrow P$ an anodyne extension of graphs. Since X is a fibrant graph, $X \hookrightarrow P$

has a retraction $r: P \to X$. The map $rb: B \to X_1$ provides a dotted filler for the original diagram above, so $(s, t): X_1 \to X_0 \times X_0$ is a fibration in $S(\mathscr{E})$.

For sufficiency, let X be a reflexive graph with (s, t) a fibration, and X_0 fibrant. Suppose we are given a diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook] & X \\ B & \end{tikzcd}$$

in which $A \hookrightarrow B$ is an anodyne extension of graphs. There is a commutative diagram

$$\begin{tikzcd} \operatorname{dis} A_0 \arrow[r] \arrow[d, hook] & A \arrow[d, hook] \\ \operatorname{dis} B_0 \arrow[r] & B \end{tikzcd}$$

which gives rise to a factorization of $A \hookrightarrow B$ as in

$$\begin{tikzcd} \operatorname{dis} A_0 \arrow[r] \arrow[d, hook] & A \arrow[r] \arrow[d, hook] & X \\ \operatorname{dis} B_0 \arrow[r] & C \arrow[d, hook] \arrow[ur, dashed] & \\ & B & \end{tikzcd}$$

where the upper left-hand square is a pushout of graphs. The map $\operatorname{dis} A_0 \to X$ can be extended to $\operatorname{dis} B_0$, since X_0 is fibrant. Thus, $A \to X$ can be extended to C, and we are left with extending $C \to X$ to B, where $C \hookrightarrow B$ is the identity on objects. That is, we have a diagram

$$\begin{tikzcd} & C_1 \arrow[rr] \arrow[dl, hook] \arrow[dd] & & X \arrow[dd, "(s,t)"] \\ B_1 \arrow[dr] \arrow[urrr, dashed] & & & \\ & B_0 \times B_0 \arrow[rr] & & X_0 \times X_0 \end{tikzcd}$$

which has a dotted filler since (s, t) is fibrant and $C_1 \hookrightarrow B_1$ is anodyne. The filler is clearly compatible with the units, so the proposition is proved. $\square$

As a result of Proposition 14, we see that a groupoid $\mathbb{G}$ in $S(\mathscr{E})$ is a fibrant stack iff it is a strong stack, and its underlying graph is fibrant. That is, for an anodyne extension $A \hookrightarrow B$ of reflexive graphs, each diagram of groupoids

$$\begin{tikzcd} F(A) \arrow[r] \arrow[d] & \mathbb{G} \\ F(B) \arrow[ur, dashed] & \end{tikzcd}$$

has a dotted filler.

Our principal theorem is the following:

Theorem 15. *A groupoid $\mathbb{G}$ in $S(\mathscr{E})$ has a completion $a : \mathbb{G} \hookrightarrow \mathbb{G}^{\#}$ such that $\mathbb{G}^{\#}$ is a fibrant stack, $Ba : B\mathbb{G} \hookrightarrow B\mathbb{G}^{\#}$ is a weak equivalence, and a_0 and a_1 are injective.*

The proof of Theorem 15 will be divided into a series of lemmas. To begin, if A is a reflexive graph in $\mathscr{E}$, denote by $\mathrm{sk}^1 A$ the value at A on the left adjoint to the functor, which truncates a simplicial object at level 1. Let $\mathrm{sk}^1 A \to NF(A)$ be the transpose of the isomorphism $\pi(\mathrm{sk}^1 A) \to F(A)$.

Lemma 16. $\mathrm{sk}^1 A \to NF(A)$ *is a weak equivalence in $S(\mathscr{E})$.*

Proof. By boolean localization, we need only prove the lemma when A is a reflexive graph in Sets. But then $\mathrm{sk}^1 A \to NF(A)$ is a weak equivalence because it induces an isomorphism on fundamental groupoids, and both $\mathrm{sk}^1 A$ and $NF(A)$ have vanishing homotopy in dimensions greater than 1. □

Lemma 17. *With notation as above, let $A \hookrightarrow B$ be a monomorphism of reflexive graphs in $\mathscr{E}$, and*

$$\begin{tikzcd} \mathrm{sk}^1 A \arrow[r] \arrow[d, hook] & X \arrow[d, hook] \\ \mathrm{sk}^1 B \arrow[r] & Y \end{tikzcd}$$

a pushout in $S(\mathscr{E})$. Then if the homotopy of X vanishes in dimensions greater than 1, *the same is true of Y.*

Proof. Again, by boolean localization, it suffices to prove the lemma for simplicial sets. In that case, however, by transfinite induction, it is enough to prove the result when B arises from A by the addition of a single vertex, or edge, in which case Y arises from X by adding a vertex, or attaching an edge. The case of a vertex is clear. If we attach an edge whose vertices lie in the same component of X, Y, up to homotopy, is the result of wedging that component of X with a circle. Attaching an edge whose vertices lie in different components of X yields, up to homotopy, the wedge of those two components. Each component of X is a $K(\pi, 1)$ space, and the wedge of two such is another one, which proves the lemma. □

Lemma 18. *Let $A \hookrightarrow B$ be an anodyne extension of reflexive graphs in $S(\mathscr{E})$. Then the induced map $\mathrm{sk}^1 A \to \mathrm{sk}^1 B$ is a vertical weak equivalence of double simplicial objects of $\mathscr{E}$.*

Proof. It is not hard to see that $(\mathrm{sk}^1 A)_n$ is the n-fold pushout of $u : A_0 \to A_1$ with itself, so the result follows by boolean localization, and a standard fact about simplicial sets. □

Lemma 19. *Let $A \hookrightarrow B$ be an anodyne extension of reflexive graphs in $S(\mathscr{E})$. Then if*

$$\begin{tikzcd} F(A) \arrow[r] \arrow[d] & \mathbb{G} \arrow[d] \\ F(B) \arrow[r] & \mathbb{G}' \end{tikzcd}$$

is a pushout of groupoids in $S(\mathscr{E})$, $B\mathbb{G} \to B\mathbb{G}'$ is a weak equivalence.

Proof. Suppose, first, that $A \hookrightarrow B$ is simply a monomorphism of reflexive graphs in Sets. The pushout in the statement of the lemma is then as ordinary groupoids (in Sets). Consider the following diagram of simplicial sets, in which the squares labelled po are pushouts.

$$\begin{tikzcd} \mathrm{sk}^1 A \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\mathrm{po}"] & NF(A) \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\mathrm{po}"] & N\mathbb{G} \arrow[d, hook] \\ \mathrm{sk}^1 B \arrow[r, hook] \arrow[dr, hook] & B' \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\mathrm{po}"] & Y \arrow[d, hook] \\ & NF(B) \arrow[r] & P \end{tikzcd}$$

$\mathrm{sk}^1 A \hookrightarrow NF(A)$ and $\mathrm{sk}^1 B \hookrightarrow NF(B)$ are weak equivalences by Lemma 16, hence so are $\mathrm{sk}^1 B \hookrightarrow B'$ and $B' \hookrightarrow NF(B)$. The latter is a cofibration since $NF(A) \hookrightarrow NF(B)$ is, making B' the union of $NF(A)$ and $\mathrm{sk}^1 B$ in $NF(B)$. As a result, $Y \hookrightarrow P$ is a weak equivalence. By Lemma 17, the homotopy of Y vanishes in dimensions greater than 1, thus the same is true of P. The diagram

$$\begin{tikzcd} NF(A) \arrow[r] \arrow[d] & N\mathbb{G} \arrow[d] \\ NF(B) \arrow[r] & N\mathbb{G}' \end{tikzcd}$$

commutes, yielding a canonical map $P \to N\mathbb{G}'$, which is a weak equivalence, since it induces an isomorphism on fundamental groupoids, and both P and $N\mathbb{G}'$ have vanishing homotopy in dimensions greater than 1.

Returning to the original statement, by boolean localization we may assume $A \hookrightarrow B$ is an anodyne extension of graphs of simplicial sets. The pushout

$$\begin{tikzcd} F(A) \arrow[r] \arrow[d] & \mathbb{G} \arrow[d] \\ F(B) \arrow[r] & \mathbb{G}' \end{tikzcd}$$

is then as groupoids in S (simplicial sets). Taking nerves, we obtain a diagram

$$\begin{tikzcd} NF(A) \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathrm{po}"] & N\mathbb{G} \arrow[d] & \\ NF(B) \arrow[r] & P \arrow[dr] & \\ & & N\mathbb{G}' \end{tikzcd}$$

of double simplicial sets, with $P \to N\mathbb{G}'$ a horizontal weak equivalence. The maps $\mathrm{sk}^1 A \to NF(A)$ and $\mathrm{sk}^1 B \to NF(B)$ are also horizontal weak equivalences by Lemma 16. $\mathrm{sk}^1 A \to \mathrm{sk}^1 B$ is a vertical weak equivalence by Lemma 18. Thus, taking diagonal complexes in the diagram

$$\begin{tikzcd} \mathrm{sk}^1 A \arrow[r] \arrow[d, hook] & NF(A) \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\mathrm{po}"] & N\mathbb{G} \arrow[d, hook] \arrow[dr] & \\ \mathrm{sk}^1 B \arrow[r] & NF(B) \arrow[r] & P \arrow[r] & N\mathbb{G}' \end{tikzcd}$$

yields a weak equivalence $BF(A) \hookrightarrow BF(B)$, whose pushout $B\mathbb{G} \hookrightarrow dP$ is a weak equivalence. Since $dP \to B\mathbb{G}'$ is also a weak equivalence, we see that $B\mathbb{G} \to B\mathbb{G}'$ is a weak equivalence as claimed. $\square$

Proof of Theorem 15. Let $\mathbb{G}$ be a groupoid in $S(\mathscr{E})$. If $A \hookrightarrow B$ is an anodyne extension of reflexive graphs in $S(\mathscr{E})$, then, by Lemma 18, in the pushout

$$\begin{tikzcd} F(A) \arrow[r] \arrow[d] & \mathbb{G} \arrow[d] \\ F(B) \arrow[r] & \mathbb{G}' \end{tikzcd}$$

$\mathbb{G} \hookrightarrow \mathbb{G}'$ is injective on objects and morphisms, and B of it is a weak equivalence. If $\mathbb{A} \hookrightarrow \mathbb{B}$ is a categorical equivalence of groupoids in $S(\mathscr{E})$, which is injective on objects, then in the pushout

$$\begin{tikzcd} \mathbb{A} \arrow[r] \arrow[d, hook] & \mathbb{G} \arrow[d, hook] \\ \mathbb{B} \arrow[r] & \mathbb{G}'' \end{tikzcd}$$

the same is true of $\mathbb{G} \hookrightarrow \mathbb{G}''$ by boolean localization and the corresponding fact for groupoids in Sets. Thus, by Proposition 2, $B\mathbb{G} \hookrightarrow B\mathbb{G}''$ is a weak equivalence.

As in [6], there is a small set of generators for the anodyne extensions of graphs, as well as a small set of generators for the categorical equivalences injective on objects. Thus, we can use the 'small object argument' [10] to obtain the desired completion by repeated pushouts and ordinal colimits. □

Theorem 20. *A locally transitive groupoid $\mathbb{G}$ in $S(\mathscr{E})$ has a completion $a : \mathbb{G} \hookrightarrow \mathbb{G}^*$ such that $\mathbb{G}^*$ is amenable, a_0 is the identity on objects, a_1 is an anodyne extension, and $(s, t) : G_1^* \to G_0 \times G_0$ is a fibration in $S(\mathscr{E})$.*

Proof. Let $\mathbb{G} \hookrightarrow \mathbb{G}^\#$ be a fibrant stack completion of $\mathbb{G}$ as in Theorem 15, and let G_1^* be the pullback of $G_1^\#$ over $G_0 \times G_0$. By Proposition 13, $\mathbb{G}^\#$ is amenable, so the result follows from Theorem 5, and the fact that amenability is invariant under categorical equivalence. □

5. Pseudo torsors

In this section we define a relaxed notion of torsor, or principal bundle, whose equivalence classes can be represented in the homotopy category for an arbitrary locally transitive groupoid $\mathbb{G}$.

Definition 21. Let $\mathbb{G}$ be a locally transitive groupoid in $S(\mathscr{E})$. A *$\mathbb{G}$-pseudo torsor* over X is a simplicial sheaf T over X, on which $\mathbb{G}$ acts freely, and is such that the map $T \to X$ factors as

$$\begin{tikzcd} & T \arrow[dl] \arrow[dd] \\ T/\mathbb{G} \arrow[dr] & \\ & X \end{tikzcd}$$

with $T/\mathbb{G} \to X$ a weak equivalence. A *morphism* of pseudo torsors is a commutative triangle

$$\begin{tikzcd} T_1 \arrow[rr] \arrow[dr] & & T_2 \arrow[dl] \\ & X & \end{tikzcd}$$

where the map $T_1 \to T_2$ respects the action of $\mathbb{G}$.

Note that in the diagram

$$\begin{tikzcd} T_1 \arrow[r] \arrow[d] & T_2 \arrow[d] \\ T_1/\mathbb{G} \arrow[r] \arrow[dr] & T_2/\mathbb{G} \arrow[d] \\ & X \end{tikzcd}$$

the map $T_1/\mathbb{G} \to T_2/\mathbb{G}$ is a weak equivalence, and the square is a pullback, so the map $T_1 \to T_2$ is a weak equivalence by Corollary 3.

Let $\mathcal{H}(X, \mathbb{G})$ denote the category of $\mathbb{G}$-pseudo torsors over X. If $f : \mathbb{G} \to \mathbb{H}$ is a functor, and $T \to X$ is a $\mathbb{G}$-pseudo torsor over X, then $T \otimes_{\mathbb{G}} \mathbb{G}$ is an $\mathbb{H}$-torsor over $T/\mathbb{G}$, so f induces a functor (extension of scalars) $\mathcal{H}(X, \mathbb{G}) \to \mathcal{H}(X, \mathbb{H})$. Let $\mathcal{H}^1(X, \mathbb{G})$ be the set of connected components of $\mathcal{H}(X, \mathbb{G})$. It can be shown that two pseudo torsors over X are in the same component iff there is a third pseudo torsor T_3 over X and a pair of mappings $T_1 \to T_3 \leftarrow T_2$.

Proposition 22. *$\mathcal{H}^1(X, \mathbb{G}) \simeq H^1(X, \mathbb{G})$ if $\mathbb{G}$ is amenable.*

Proof. There is an obvious function $H^1(X, \mathbb{G}) \to \mathcal{H}^1(X, \mathbb{G})$. If $T \to X$ is a $\mathbb{G}$-pseudo torsor over X, by amenability of $\mathbb{G}$, let $T' \to X$ be the unique (up to isomorphism) $\mathbb{G}$-torsor over X, whose pullback along $T/\mathbb{G} \to X$ is T. It is easy to see that this is well defined on components, and provides an inverse to the function above. □

Let $a : \mathbb{G} \hookrightarrow \mathbb{H}$ be a functor between locally transitive groupoids in $S(\mathscr{E})$ such that a_0 is the identity, and a_1 is an anodyne extension.

Proposition 23. *a induces an isomorphism $\mathcal{H}^1(X, \mathbb{G}) \simeq \mathcal{H}^1(X, \mathbb{H})$.*

Proof. Let $S \to C \to X$ be an $\mathbb{H}$-pseudo torsor over X, where $S \to C$ is an $\mathbb{H}$-torsor over C, and $C \to X$ is a weak equivalence. We can restrict the action of $\mathbb{H}$ on S along a to obtain an action of $\mathbb{G}$ on S, which is free, since a_1 is injective. However, the quotient of S by $\mathbb{G}$ is, in general, larger than its quotient C by $\mathbb{H}$. So we obtain a diagram

$$\begin{tikzcd} S \arrow[r] \arrow[dr] & C \arrow[r] & X \\ & S/\mathbb{G} \arrow[u] & \end{tikzcd}$$

in which S is a $\mathbb{G}$-torsor over $S/\mathbb{G}$ and $C \to X$ is a weak equivalence. We will show that $S/\mathbb{G} \to C$ is a weak equivalence, so restriction of scalars becomes a functor $\mathscr{H}(X, \mathbb{H}) \to \mathscr{H}(X, \mathbb{G})$, which is a right adjoint to extension of scalars. On connected components, then, extension of scalars will induce an isomorphism as claimed.

Consider first the special case of the canonical $\mathbb{H}$-torsor $t : H_1 \to G_0$. Restricting the $\mathbb{H}$ action along a_1 yields a map of $\mathbb{G}$-objects $a_1 : G_1 \hookrightarrow H_1$. Dividing by the $\mathbb{G}$ actions we obtain a diagram

$$\begin{tikzcd}
G_1 \arrow[r, hook, "a_1"] \arrow[d, "t"'] & H_1 \arrow[d] \arrow[dr, "t"] & \\
G_0 \arrow[r, hook, "w"'] & H_1/\mathbb{G} \arrow[r] & G_0
\end{tikzcd}$$

in which the square is, in fact, a pullback by the freeness of the actions. Since $\mathbb{G}$ is locally transitive, pulling back along $H_1 \to H_1/\mathbb{G}$ preserves and reflects weak equivalences by Corollary 3. It follows that w is a weak equivalence, making $H_1/\mathbb{G} \to G_0$ a weak equivalence as well.

In the case of an $\mathbb{H}$-torsor $S \to C$ with a splitting $C \to S$, we obtain a map $C \to G_0$ such that in the diagram

$$\begin{tikzcd}
S \arrow[rr] \arrow[dr] \arrow[dd] & & H_1 \arrow[dd, "t"] & \\
& S/\mathbb{G} \arrow[rr] \arrow[dl] & & H_1/\mathbb{G} \\
C \arrow[rr] & & G_0 &
\end{tikzcd}$$

the back and front faces are pullbacks—the front since pulling back preserves the passage to quotients. But $\mathbb{H}$ acts on the left of $H_1/\mathbb{G} \to G_0$, so $H_1/\mathbb{G} \to G_0$ is a local bundle weak equivalence by Proposition 2. By boolean localization, these are preserved by pullback along an arbitrary map, so $S/\mathbb{G} \to C$ is a weak equivalence.

In the case of a general $\mathbb{H}$-torsor $S \to C$, we pull back over S to obtain the split case:

$$\begin{tikzcd}
P \arrow[rr] \arrow[dr] \arrow[dd] & & S \arrow[dd] \arrow[dr] & \\
& P/\mathbb{G} \arrow[rr] \arrow[dl] & & S/\mathbb{G} \arrow[dl] \\
S \arrow[rr] & & C &
\end{tikzcd}$$

$P \to S$ is split, so $P/\mathbb{G} \to S$ is a weak equivalence as above. But pulling back over the $\mathbb{H}$-torsor $S \to C$ reflects weak equivalences by Corollary 3, so $S/\mathbb{G} \to C$ is a weak equivalence as desired. □

The main result of this section now follows immediately from Propositions 22 and 23.

Theorem 24. *Let* $\mathbb{G}$ *be a locally transitive groupoid in* $S(\mathscr{E})$, *and* $a : \mathbb{G} \hookrightarrow \mathbb{G}^*$ *an amenable completion of* $\mathbb{G}$ *as in Theorem* 20. *Then* $\mathscr{H}^1(X, \mathbb{G}) \simeq H^1(X, \mathbb{G}^*)$. □

Remarks. Let $a : \mathbb{G} \hookrightarrow \mathbb{G}^*$ be an amenable completion of $\mathbb{G}$ as above, and suppose $B\mathbb{G}^* \hookrightarrow B$ is an anodyne extension with B fibrant in $S(\mathscr{E})$. As in the proof of Theorem 10, we have $H^1(X, \mathbb{G}^*) \simeq [X, B] \simeq \mathrm{ho}(X, B\mathbb{G}^*) \simeq \mathrm{ho}(X, B\mathbb{G})$, since $Ba : B\mathbb{G} \to B\mathbb{G}^*$ is a weak equivalence. By Theorem 24, then, we see that $\mathscr{H}^1(X, \mathbb{G}) \simeq \mathrm{ho}(X, B\mathbb{G})$. On the other hand, this can be proved directly, yielding an alternate proof of Theorem 24.

6. The abelian case

Here we consider the abelian version of the previous theory. Recall first that, as in [6], the category of (positive) chain complexes of abelian groups in $\mathscr{E}$ has a Quillen homotopy structure, in which the cofibrations are monomorphisms, the weak equivalences are homology isomorphisms, and the fibrations have the right lifting property with respect to the cofibration weak equivalences. Translating this to abelian groups in $S(\mathscr{E})$ via the Dold–Puppe Theorem yields a Quillen structure on the category of abelian groups in $S(\mathscr{E})$, in which the cofibrations are monomorphisms, the weak equivalences are homomorphisms which are weak equivalences in $S(\mathscr{E})$, and the fibrations have the right lifting property as above. These fibrations are also fibrations in $S(\mathscr{E})$.

The appropriate notion of 'amenable' for an abelian group is the following.

Definition 25. An abelian group A of $S(\mathscr{E})$ is said to be *amenable* if $\mathrm{Ext}^1(\ , A)$ inverts anodyne homomorphisms.

Considering an abelian group A of $S(\mathscr{E})$ as a groupoid in $S(\mathscr{E})$ with one object, we can compare this abelian definition of amenable with the previous one. Namely, if X is an object of $S(\mathscr{E})$, denote by $\mathbb{Z}X$ the free abelian group on X. Then we have $\mathrm{Ext}^1(\mathbb{Z}X, A) \simeq H^1(X, A)$. In fact, the correspondence is given as follows. Let $X \hookrightarrow \mathbb{Z}X$ denote the inclusion of the generators of $\mathbb{Z}X$. Then if $E \twoheadrightarrow \mathbb{Z}X$ is a surjective homomorphism with kernel A, its pullback over $X \hookrightarrow \mathbb{Z}X$ is an A-torsor over X (see [5, Theorem 8.33] for a proof). Since an anodyne extension $X \hookrightarrow Y$ of $S(\mathscr{E})$ gives rise to an anodyne homomorphism $\mathbb{Z}X \hookrightarrow \mathbb{Z}Y$ of abelian groups, we see that if A is amenable as an abelian group, it is amenable as a groupoid in $S(\mathscr{E})$.

We begin the discussion of amenable completions with the following proposition:

Proposition 26. *Let $p: E \to B$ be a fibration of abelian groups in $S(\mathscr{E})$ with B fibrant, and E contractible (meaning $E \to 0$ is a weak equivalence). Then the kernel of p is amenable.*

Proof. Let $A = \text{kernel } p$. p is a surjective homomorphism (any fibration is), so we obtain, for any C, a natural homomorphism $\hom(C, B) \to \mathrm{Ext}^1(C, A)$ by sending φ into $\varphi^*(E)$. Let $A \hookrightarrow F \twoheadrightarrow C$ be a short exact sequence. E is fibrant, and $E \to 0$ is a weak equivalence, so $E \to 0$ is a trivial fibration, i.e. E is injective. Thus, we can extend $A \hookrightarrow E$ to F, yielding a commutative diagram

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, hook] & A \arrow[d, hook] \\ F \arrow[r] \arrow[d] & E \arrow[d] \\ C \arrow[r] & B \end{tikzcd}$$

such that the bottom square is a pullback. Thus, $\hom(C, B) \to \mathrm{Ext}^1(C, A)$ is surjective. In the diagram above, $F \twoheadrightarrow C$ is split iff $C \to B$ lifts to E, so the kernel of $\hom(C, B) \to \mathrm{Ext}^1(C, A)$ is the image of $\hom(C, E)$. But it follows from [10], since C is cofibrant and B is fibrant, that the cokernel of $\hom(C, E) \to \hom(C, B)$ is $[C, B]_{\mathrm{ab}}$—the set of abelian (right) homotopy classes of homomorphisms from C to B. As a result,

$$\mathrm{Ext}^1(C, A) \simeq [C, B]_{\mathrm{ab}}$$

so that $\mathrm{Ext}^1(\ , A)$ inverts anodyne homomorphisms, and A is amenable. $\square$

Theorem 27. *An abelian group has an abelian amenable completion.*

Proof. Let A be an abelian group in $S(\mathscr{E})$. Taking the cone on A yields a monomorphism $A \hookrightarrow E(A)$ with $E(A)$ contractible. Dividing by A gives an exact sequence $A \hookrightarrow E(A) \to B(A)$. Let $B(A) \hookrightarrow B$ be an anodyne homomorphism with B fibrant, and factor $E(A) \to B(A) \hookrightarrow B$ as

$$\begin{tikzcd} E(A) \arrow[r, hook] \arrow[d] & E \arrow[d, "p"] \\ B(A) \arrow[r, hook] & B \end{tikzcd}$$

where $E(A) \hookrightarrow E$ is anodyne, and $p : E \to B$ is a fibration. Let A^* be the kernel of p. A^* is amenable by Proposition 23. $A \hookrightarrow A^*$ is an anodyne homomorphism, so A^* is an abelian amenable completion of A. Note that $p : E \to B$ is a universal A^*-torsor by Theorem 9. □

In the abelian case, pseudo torsors are replaced by pseudo extensions.

Definition 28. Let A and B be abelian groups in $S(\mathscr{E})$. A *pseudo extension* of B by A is a pair of homomorphisms $A \hookrightarrow E \to B$, such that $A \hookrightarrow E$ injective, the composite is zero, and the induced homomorphism $E/A \to B$ is a weak equivalence. A *morphism* of pseudo extensions is a commutative diagram

$$\begin{tikzcd} & E_1 \arrow[dd] \arrow[dr] & \\ A \arrow[ur] \arrow[dr] & & B \\ & E_2 \arrow[ur] & \end{tikzcd}$$

The homomorphism $E_1 \to E_2$ is a weak equivalence as before. We write $\mathscr{E}xt(B, A)$ for the category of pseudo extensions of B by A, and $\mathscr{E}xt^1(B, A)$ for its set (in fact abelian group) of connected components.

Proposition 29. $\mathscr{E}xt^1(B, A) \simeq \mathrm{Ext}^1(B, A)$ *if* A *is amenable.*

Proof. The proof is similar to that of Proposition 22. In fact, there is an obvious function $\mathrm{Ext}^1(B, A) \to \mathscr{E}xt^1(B, A)$. If $A \hookrightarrow E \to B$ is a pseudo extension of B by A, by amenability of A, let $A \hookrightarrow E' \twoheadrightarrow B$ be the unique, up to isomorphism, extension of B by A, whose pullback along $E/A \to B$ is E. This is well defined on components, and provides an inverse to the function above. □

Proposition 30. *An anodyne homomorphism* $A \hookrightarrow A'$ *induces an isomorphism* $\mathscr{E}xt^1(B, A) \simeq \mathscr{E}xt^1(B, A')$.

Proof. The proof is similar in spirit to that of Proposition 23, though easier. Thus, let $A \hookrightarrow E \to B$ be a pseudo extension of B by A. In the pushout

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d, hook] & E \arrow[d, hook] \\ A' \arrow[r, hook] & E' \end{tikzcd}$$

the cokernel of $A' \hookrightarrow E'$ is isomorphic to E/A, so pushing out along $A \hookrightarrow A'$ yields a functor ('extension of scalars') $\mathscr{E}xt(B, A) \to \mathscr{E}xt(B, A')$. On the other hand, if $A' \hookrightarrow E' \to B$ is a pseudo extension of B by A', consider the composite $A \hookrightarrow A' \hookrightarrow E'$. The quotient of E' by A is larger than its quotient by A', so we obtain a diagram

$$\begin{tikzcd} A \arrow[r, hook] & A' \arrow[r, hook] & E' \arrow[r] \arrow[d] & B \\ & & E'/A \arrow[r] & E'/A' \arrow[u] \end{tikzcd}$$

in which $E'/A' \to B$ is a weak equivalence. If we show that $E'/A \to E'/A'$ is a weak equivalence, then composing with $A \hookrightarrow A'$ becomes a functor ('restriction of scalers'), which is a right adjoint to pushing out along $A \hookrightarrow A'$. On connected components, then, $A \hookrightarrow A'$ will induce an isomorphism as claimed. However, we have a commutative diagram

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d, hook] & A' \arrow[d, hook] \\ E' \arrow[r, equal] \arrow[d] & E' \arrow[d] \\ E'/A \arrow[r] & E'/A' \end{tikzcd}$$

in which the first two horizontal homomorphisms are weak equivalences. Considering, say, in chain complexes the mapping induced on the long exact homology sequence, we see that the third is also a weak equivalence by a five lemma argument. □

The next result follows immediately from Propositions 29 and 30.

Proposition 31. *Let A be an abelian group in $S(\mathscr{E})$, and $A \hookrightarrow A^*$ an abelian amenable completion of A as in Theorem 27. Then $\mathscr{E}xt^1(B, A) \simeq \mathrm{Ext}^1(B, A^*)$.* □

Let A and π be abelian groups in $\mathscr{E}$, and write A again for A considered as a simplicially discrete abelian group in $S(\mathscr{E})$.

Theorem 32. $\mathrm{Ext}^n(A, \pi) \simeq \mathscr{E}xt^1(A, K(\pi, n-1))$.

Proof. We carry out the proof in chain complexes, using the Yoneda interpretation of $\mathrm{Ext}^n(A, \pi)$ as connected components of the category of n-fold extensions of A by π. Thus, let $0 \to \pi \to E_{n-1} \to E_{n-2} \to \cdots \to E_0 \to A \to 0$ be such an n-fold extension. Rewriting the extension as

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & \\
\pi \arrow[r] \arrow[d] & E_{n-1} \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] \arrow[d, dotted] & E_{n-2} \arrow[r] \arrow[d, dotted] & 0 \arrow[d, dotted] \\
0 \arrow[r] & E_0 \arrow[r] & A
\end{tikzcd}$$

yields a pseudo extension of A by $K(\pi, n-1)$, and a functor from the category of n-fold extensions of A by π to $\mathscr{E}xt(A, K(\pi, n-1))$.

In the other direction, suppose $K(\pi, n-1) \hookrightarrow E \to A$ is a pseudo extension of A by $K(\pi, n-1)$. Since $E/K(\pi, n-1) \to A$ is a weak equivalence, we see that

$$Z_{n-1}/B_{n-1} \simeq H_{n-1}(E) \simeq \pi\,, \qquad H_0(E) \simeq A$$

and $H_i(E) = 0$ otherwise. It follows that

$$0 \to \pi \to E_{n-1}/B_{n-1} \to E_{n-2} \to \cdots \to E_0 \to A \to 0$$

is exact, yielding a functor from $\mathscr{E}xt(A, K(\pi, n-1))$ to the category of n-fold extensions of A by π. This functor is easily seen to be left adjoint to the previous one, so we obtain an isomorphism on connected components as desired. □

The results of this section yield a representation theorem for sheaf cohomology in terms of (one-dimensional) torsors. Namely, let π be an abelian group in $\mathscr{E}$, and $K(\pi, n-1) \hookrightarrow K^*(\pi, n-1)$ an abelian amenable completion of the abelian group $K(\pi, n-1)$ in $S(\mathscr{E})$.

Theorem 33. $H^n(\mathscr{E}, \pi) \simeq H^1(K^*(\pi, n-1))$.

Proof.

$$\begin{aligned} H^n(\mathscr{E}, \pi) &\simeq \operatorname{Ext}^n(\mathbb{Z}, \pi) \simeq \mathscr{E}xt^1(\mathbb{Z}, K(\pi, n-1)) \\ &\simeq \operatorname{Ext}^1(\mathbb{Z}, K^*(\pi, n-1)) \simeq H^1(K^*(\pi, n-1))\,. \qquad \square \end{aligned}$$

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