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The determination of distances between images of objects based on persistent spectra of eigenvalues of Laplace matrices

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Abstract. The work uses the method of filtering simplicial complexes, similar to the method used in the formation of persistent homology. The filtering process creates a number of nested simplicial complexes encoded with topological information. In papers [1-6] algorithms for the formation of persistent barcodes are used to compare images of objects. The use of persistent homology in relation to the methods of traditional algebraic topology provides additional information about the image of an object. To increase the diversity of information and the number of machine learning features, this work proposes algorithms for the formation of persistent spectra of eigenvalues of Laplace matrices for comparing images of objects. When comparing the shapes of objects, it is proposed to construct a modified Wasserstein distance based on the determination of the spectra of the eigenvalues of the Laplace matrix of the compared shapes.

Keywords: persistent spectra of eigenvalues, Laplace matrices, distance between images, Vietoris–Rips complex, Wasserstein distance

1. Introduction

Geometric data analysis methods can characterize information about the local structure, but lead to data complication. Features created from topological models retain information about the global internal structure, but they reduce information about the local structure. The basic concept of persistent homology is to use a filtering procedure; the filtering process creates a number of nested simplicial complexes encoded with topological information. The lifespan of topological invariants in the process of changing the filtering parameter is associated with the topological and geometric properties of the object image.

The Wasserstein distance is used to measure the fit from one persistent homology to another. Special topological properties, such as Betti numbers at a certain filtering value, can be considered features. In paper [1, 2] algorithms for the formation of persistent barcodes [3-6] are used to compare images of objects. Using persistent homology in relation to traditional algebraic topology methods [4] provides additional information about the shape of an object.

To increase the diversity of information and the number of machine learning features when using the persistent homology method, this paper proposes algorithms for the formation of persistent spectra of eigenvalues of Laplace matrices for comparing images of objects.

Determining the persistent spectrum of eigenvalues of Laplace matrices for simplicial complexes allows you to expand the number of features for machine learning. When comparing the shapes of



objects, it is proposed to construct a modified Wasserstein distance based on the determination of the spectra of the eigenvalues of the Laplace matrix of the compared shapes. An example of finding a modified Wasserstein distance that takes into account the spectra of the eigenvalues of the Laplace matrices under affine transformations is considered.

2. Simplicial complexes

Graphs can be generalized to combinatorial objects known as simplicial complexes. Simplicial complexes are a class of topological spaces consisting of simplexes of various dimensions. For a set of points V , a k -simplex σ_k is a subset $\{v_0, v_1, \dots, v_k\}; v_i \in V; v_i \neq v_j, \forall i \neq j$. A vertex is a 0-simplex, an edge 1-simplex, a triangle 2-simplex, etc. The faces of this k -simplex consist of all $(k-1)$ -simplexes of the form $\{v_0, \dots, v_{i-1}, v_{i+1}, \dots, v_k\}; 0 \leq i \leq k$. A simplicial complex is a set of simplexes that is closed under the inclusion of faces. The order of the vertices corresponds to the orientation. k -simplex $\{v_0, v_1, \dots, v_k\}$ together with the ordinal type is an oriented k -simplex, denoted $[v_0, \dots, v_k]$, where the change in orientation corresponds to a change in the sign of the coefficient: $[v_0, \dots, v_i, \dots, v_j, \dots, v_k] = -[v_0, \dots, v_j, \dots, v_i, \dots, v_k]$.

Two k -simplexes σ_i and σ_j a simplicial complex X are upper adjacent if they are both faces of some $(k+1)$ -simplex in X . Let's denote this adjacency by $\sigma_i \cap \sigma_j$. The upper degree of a k -simplex, denoted by $\deg_u(\sigma)$, is the number of $k+1$ simplexes in X which σ is a face. Now we define the orientation of X and assume that $\sigma_i \cap \sigma_j$ with a common $(k+1)$ -simplex ξ . If the orientations σ_i and σ_j coincide with the orientations induced ξ , then we say that σ_i and σ_j are oriented similarly with respect to ξ . Otherwise, we say that the simplexes are oriented differently. Similarly, we also define the lower adjacency and lower degree of the simplexes. Two k -simplexes σ_i and σ_j of simplicial complex X are lower adjacent if both have a common face. Let's denote it through $\sigma_i \cup \sigma_j$. The lower degree of the k -simplex, denoted $\deg_l(\sigma)$, is equal to the number of faces in σ .

For any $k \geq 0$, we denote by $C_k(X)$ a vector space, the basis of which is the set of oriented k -simplexes X . For k larger than dimension X : $C_k(X) = 0$. The elements of these vector spaces are called chains, which are linear combinations of base elements. Boundary mappings are defined as linear transformations $\partial_k: C_k(X) \rightarrow C_{k-1}(X)$ that act on the basis elements $[v_0, \dots, v_k]$ as

$\partial_k[v_0, \dots, v_k] = \sum_{j=0}^k (-1)^j [v_0, \dots, v_{j-1}, v_{j+1}, \dots, v_k]$. These boundary maps define a chain complex as a sequence of vector spaces and linear transformations:

$$0 \rightarrow C_n(X) \xrightarrow{\partial_{n+1}} \dots \xrightarrow{\partial_k} C_k(X) \xrightarrow{\partial_k} C_{k-1}(X) \xrightarrow{\partial_{k-1}} \dots \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \rightarrow 0. \quad (1)$$

When dealing with a finite simplicial complex X , vector spaces $C_i(X)$ also have finite dimension. Therefore, we can represent the boundary mappings in matrix form $\partial_i: C_i(X) \rightarrow C_{i-1}(X)$. After performing the calculation $\partial_k \partial_{k-1}$, we get that $C_k(X) \xrightarrow{\partial_k} C_{k-1}(X) \xrightarrow{\partial_{k-1}} C_{k-2}(X)$ is zero. Hence it follows that $\text{im} \partial_k \subset \ker \partial_{k-1}$. k -th homology group of the space X is defined as $H_k(X) = \frac{\ker \partial_k}{\text{im} \partial_{k-1}}$.

Homology groups are used to distinguish topological spaces from each other by determining the number of "holes" of different dimensions contained in these spaces. Each non-trivial homology class in a particular dimension helps to identify the corresponding hole in that dimension. Roughly

speaking, dimension of $H_0(X)$ is the number of connected components X (0-dimensional holes). The dimension of $H_1(X)$ is the number of non-contracting cycles in X . The dimension of $H_2(X)$ determines the number of three-dimensional voids in space and so on.

3. Combinatorial k -Laplacian matrix

Using the above definitions of boundary and coboundary operators (see Appendix 1), it is possible to define operators $\Delta_k : C^k(X; \mathbf{R}) \rightarrow C^k(X; \mathbf{R})$, $L_k : C_k(X; \mathbf{R}) \rightarrow C_k(X; \mathbf{R})$ using the relations:

$$\begin{aligned}\Delta_k &= \delta_{k-1} \delta_{k-1}^* + \delta_k^* \delta_k, \\ L_k &= \partial_{k+1} \partial_{k+1}^* + \partial_k^* \partial_k,\end{aligned}$$

where the operators Δ_k, L_k have matrix representations and $\delta_k^* = \partial_{k+1}$. Both operators are representations of the same transformation, one in the original space $C_k(X; \mathbf{R})$ and the other in its dual space $C^k(X; \mathbf{R})$. We will use L_k on chains $C_k(X; \mathbf{R})$ for explicit calculations and interpret the results in cochains $C^k(X; \mathbf{R})$.

These operators were introduced by B. Eckmann [8] and have been studied since then under the name of the combinatorial Laplacian matrix [9]. B. Eckmann showed that $C_k(X; \mathbf{R})$ decomposes into orthogonal subspaces (Hodge decomposition): $C_k(X; \mathbf{R}) = H_k(X) \oplus \text{im}(\partial_{k+1}) \oplus \text{im}(\partial_k^*)$, where $H_k(X) = \{c \in C_k(X) : L_k c = 0\} = \ker L_k$. This means that to calculate the homology (dual cohomology) groups of a simplicial complex, it suffices to study the zero space of the matrix L_k . Eigenvectors L_k corresponding to zero eigenvalues are representative cycles (or cocycles) of a certain class of homology (or cohomology). Any chain $c \in C_k(X; \mathbf{R})$ is called harmonic if $L_k c = 0$; a dual cochain $\omega \in C^k(X; \mathbf{R})$ is called harmonic if $\Delta_k \omega = 0$.

Since the mappings ∂_0 and ∂_0^* are zero mappings, then $L_0 = \partial_1 \partial_1^*$. For the boundary mapping $\partial_1 : C_1(X) \rightarrow C_0(X)$, we have a matrix representation $L_0 = \partial_1 \partial_1^*$ in component form:

$$(L_0)_{ij} = \begin{cases} \deg(v_i), & \text{if } i = j, \\ -1, & \text{if } v_i \sim v_j, \\ 0, & \text{otherwise,} \end{cases}$$

where $\deg(v_i)$ (the degree of a vertex v_i in the graph) is the same as $\deg_u(v_i)$ and the adjacency relation $v_i \sim v_j$ (the adjacency of two vertices on an edge).

Let us generalize these representations for $k > 0$. Let $\sigma_1, \dots, \sigma_n$ are oriented k -simplexes. Then [10]

$$(\partial_{k+1} \partial_{k+1}^*)_{ij} = \begin{cases} \deg_u(\sigma_i) & \text{if } i = j, \\ 1 & \text{if } i \neq j, \sigma_i \cap \sigma_j \text{ with dissimilar orientation,} \\ -1 & \text{if } i \neq j, \sigma_i \cap \sigma_j \text{ with similar orientation,} \\ 0 & \text{otherwise,} \end{cases} \quad (2)$$

$$(\partial_k^* \partial_k)_{ij} = \begin{cases} k+1 & \text{if } i = j, \\ 1 & \text{if } i \neq j, \sigma_i \cup \sigma_j \text{ with similar orientation,} \\ -1 & \text{if } i \neq j, \sigma_i \cup \sigma_j \text{ with dissimilar orientation,} \\ 0 & \text{otherwise.} \end{cases} \quad (3)$$

For the components of the Laplacian matrix L_k we get: $(L_k)_{ij} = (\partial_k^* \partial_k)_{ij} + (\partial_{k+1} \partial_{k+1}^*)_{ij}$.

If n_k is the number of k -simplexes in X , $D = \text{diag}(\deg(\sigma_1), \dots, \deg(\sigma_{n_k}))$ and A_u, A_l denote the upper and lower adjacency matrices between the k -simplexes, respectively, then $L_k = D - A_u + (k+1)I_{n_k} + A_l, k > 0$.

Since the Laplacian k -th order L_k is symmetric and positively semidefinite, its spectrum consists only of real and non-negative eigenvalues [11, 12]. We denote the spectrum L_k as $\text{Spec}(L_k) = \{\lambda_{1,k}, \lambda_{2,k}, \dots, \lambda_{N_k,k}\}$.

The multiplicity of zero in the spectrum (harmonic spectrum) reveals topological information – the Betti numbers: $\beta_k = \dim(L_k) - \text{rank}(L_k)$, that is, it is equal to the number of zero eigenvalues of the Laplace matrix L_k : $\beta_k = \#0 \text{ Spec}(L_k)$. The nonharmonic spectrum L_k encodes additional geometric information.

Example 1. Let's give a simple example to illustrate these formulas. Consider the simplicial complex shown in Figure 1. Boundary operators are given below.

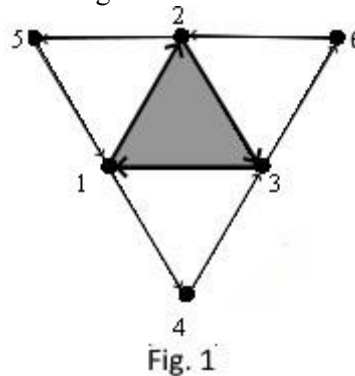


Figure 1. The simplicial complex

$$\partial_1 = \begin{bmatrix} [12] & [23] & [31] & [14] & [43] & [36] & [62] & [25] & [51] \\ 1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} [1] \\ [2] \\ [3] \\ [4] \\ [5] \\ [6] \end{bmatrix}; \partial_2 = \begin{bmatrix} [123] \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} [12] \\ [23] \\ [31] \\ [14] \\ [43] \\ [36] \\ [62] \\ [25] \\ [51] \end{bmatrix}$$

Let us define Laplacian matrix L_k in matrix form:

$$L_0 = \partial_1 \partial_1^* = \begin{pmatrix} 4 & -1 & -1 & -1 & -1 & 0 \\ -1 & 4 & -1 & 0 & -1 & -1 \\ -1 & -1 & 4 & -1 & 0 & -1 \\ -1 & 0 & -1 & 2 & 0 & 0 \\ -1 & -1 & 0 & 0 & 2 & 0 \\ 0 & -1 & -1 & 0 & 0 & 2 \end{pmatrix}, L_1 = \partial_2 \partial_2^* + \partial_1^* \partial_1 = \begin{pmatrix} 3 & 0 & 0 & 1 & 0 & 0 & 1 & -1 & -1 \\ 0 & 3 & 0 & 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 3 & -1 & -1 & 1 & 0 & 0 & 1 \\ 1 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & -1 \\ 0 & 1 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & -1 & 2 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & 2 \end{pmatrix},$$

$$L_2 = \partial_2^* \partial_2 = 3$$

The spectrum of the eigenvalues of the Laplacian matrix are:

$$\text{Spec}(L_0) = (0 \ 1.6972 \ 1.6972 \ 4 \ 5.3028 \ 5.3028)^T;$$

$$\text{Spec}(L_1) = (0 \ 0 \ 0 \ 1.6972 \ 1.6972 \ 3 \ 4 \ 5.3028 \ 5.3028)^T;$$

$$\text{Spec}(L_2) = 3.$$

Let's calculate the expansion in terms of eigenvectors L_1 ; it has three eigenvalues equal to zero. Therefore, zero space has dimension three and encompasses vectors:

$$v_1 = (-0.1287 \ -0.3173 \ 0.4461 \ 0.5385 \ 0.5385 \ -0.2249 \ -0.2249 \ -0.0363 \ -0.0363)^T;$$

$$v_2 = (0.0279 \ 0.0753 \ -0.1032 \ 0.2944 \ 0.2944 \ 0.4730 \ 0.4730 \ 0.4255 \ 0.4255)^T;$$

$$v_3 = (-0.4526 \ 0.3404 \ 0.1123 \ 0.1104 \ 0.1104 \ 0.3386 \ 0.3386 \ -0.4544 \ -0.4544)^T.$$

Hodge decomposition indicates that the null space of L_1 is isomorphic to the first homology group $H_1(K; \mathbf{R})$. Since the dimension of the zero space is three, there are three nontrivial 1-cycles in it K .

There are three "holes" in K , one of which is bounded by edges [14], [43], [31], the other by edges [12], [25], [53], and the third by edges [23], [36], [62].

4. Filtering the spectrum of the eigenvalues of the Laplace matrix

To form signs of point landmarks, it is proposed to build a Vietoris–Rips complex of point clouds relative to these landmarks. For a set of point landmarks $S = \{q_1, \dots, q_k\}; q_i \in \mathbf{R}^2$ the Vietoris–Rips complex $R(r)$ is a simplicial complex built on the set S ; the complex is formed provided that all pairs of points are at a distance not exceeding $2r$:

$$R(r) = \{\sigma = S \mid \|q_i - q_j\| \leq 2r, \forall i, j\}. \quad (4)$$

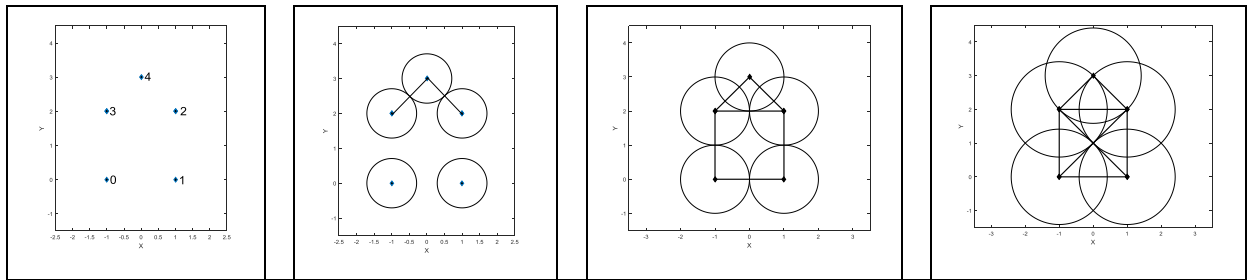
When the distance $r \in \mathbf{R}^+$ changes, the topological characteristics of the Vietoris–Rips complex change in accordance with the transformations. There are points of appearance r_b and points of disappearance r_d of topological characteristics. The values of these points are used to construct barcodes $\{[b_i \dots d_i] \mid d_i \geq b_i \geq 0\}$ (i – barcode number). The computational topology methods (see Appendix 3) determine the values b_i and d_i classes of topology (homology/cohomology) when filtering a simplicial complex by values r_i , due to the ordering of barcodes by sorting barcodes for each class. In each filtration value r_i , Betti numbers and Euler characteristics are determined for each homology / cohomology class. To increase the amount of information when filtering a simplicial complex, the spectrum of the eigenvalues of the Laplace matrices for each topology class is also determined.

Example 2. Let's form the image "House" [13], consisting of five points with coordinates $[-1, 1; -1, 0; 1, 2; -1, 2; 0, 3]$. Using the Java-Plex package, define barcodes in dimension 0: $2[0 \ 1, 41), 2[0 \ 2)$; in dimension 1: $[2 \ 2, 82)$.

Consider simplicial complexes K_0, K_1, K_2, K_3 built from point clouds when the radius changes from the value of the radius of point clouds from $r = 0$ to $r = 1, 4142$ (see figures in Table 1).

Table 1. Simplicial complexes K_0, K_1, K_2, K_3 .

$K_0; r = 0$	$K_1; r = 0, 707$	$K_2; r = 1$	$K_3; r = 1, 41$
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Let us find the boundary and coboundary operators in matrix form and the Laplace matrices.

Case $K_1; r = 0, 707$.

Table 2. Boundary operators for case K_1 .

k	$k = 0$
	24 34
∂_{k+1}^1	$\begin{matrix} 0 & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ 1 & \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} \\ 2 & \begin{bmatrix} 0 & -1 \\ 0 & -1 \end{bmatrix} \\ 3 & \begin{bmatrix} 1 & 1 \end{bmatrix} \end{matrix}$
∂_k^1	$\begin{matrix} 0 & 1 & 2 & 3 & 4 \\ / & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{matrix}$
L_k^1	$L_0^1 = \partial_1^1 (\partial_1^1)^T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & -1 & 2 \end{bmatrix}$

For the case $k = 1$: $L_1^1 = (\partial_1^1)^T \partial_1^1 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$; $\partial_1^1 = \partial_{0+1}^1$. Dimensions of Laplacian matrix:

$\dim(L_0^1) = 5; \dim(L_1^1) = 2$. Ranks of Laplacian matrix: $\text{rank}(L_0^1) = 2; \text{rank}(L_1^1) = 2$. Betti numbers:

$\beta_0^1 = 3; \beta_1^1 = 0$. Spectrum of eigenvalues of Laplacian matrix:

$\text{Spec}(L_0^1) = \{0 \ 0 \ 1 \ 0 \ 3\}; \text{Spec}(L_1^1) = \{1 \ 3\}$.

Case $K_2; r = 1$.

Table 3. Boundary operators for case K_2 .

k	$k = 0$	$k = 1$
	01 12 23 03 24 34	234
∂_{k+1}^2	$\begin{matrix} 0 & \begin{bmatrix} -1 & 0 & 0 & -1 & 0 & 0 \end{bmatrix} \\ 1 & \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \end{bmatrix} \\ 2 & \begin{bmatrix} 0 & 1 & -1 & 0 & -1 & 0 \end{bmatrix} \\ 3 & \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & -1 \end{bmatrix} \\ 4 & \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \end{matrix}$	$\begin{matrix} 01 & \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \\ 12 & \\ 23 & \\ 03 & \\ 24 & \\ 34 & \end{matrix}$
∂_k^2	$\begin{matrix} 0 & 1 & 2 & 3 & 4 \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{matrix}$	$\partial_2^2 = \partial_{1+1}^2$

L_k^2	$L_0^2 = \partial_1^2 (\partial_1^2)^T = \begin{bmatrix} 2 & -1 & 0 & -1 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 3 & -1 & -1 \\ -1 & 0 & -1 & 3 & -1 \\ 0 & 0 & -1 & -1 & 2 \end{bmatrix}$	$L_1^2 = \partial_2^2 (\partial_2^2)^T + (\partial_1^2)^T \partial_1^2 = \begin{bmatrix} 2 & -1 & 0 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & -1 & 0 \\ 0 & -1 & 3 & 1 & 0 & 0 \\ 1 & 0 & 1 & 2 & 0 & -1 \\ 0 & -1 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -1 & 0 & 3 \end{bmatrix}$
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For the case: $k=2$: $L_3^2 = (\partial_2^2)^T \partial_2^2 = 3$. Dimensions of Laplacian matrix: $\dim(L_0^2) = 5; \dim(L_1^2) = 6; \dim(L_2^2) = 1$. Ranks of Laplacian matrix: $\text{rank}(L_0^2) = 4; \text{rank}(L_1^2) = 5; \text{rank}(L_2^2) = 1$. Betti numbers: $\beta_0^2 = 1; \beta_1^2 = 1; \beta_2^2 = 0$. Spectrum of eigenvalues of Laplacian matrix:

$$\text{Spec}(L_0^2) = \{2, 38 \ 4, 62 \ 1, 38 \ 3, 62 \ 0\};$$

$$\text{Spec}(L_1^2) = \{2, 38 \ 3 \ 3, 62 \ 1, 38 \ 4, 62 \ 0\};$$

$$\text{Spec}(L_2^2) = \{3\}.$$

Since $\beta_1^2 = 1$, then the eigenvector v corresponding to the zero eigenvalue of the Laplacian L_1^2 : $v = (0, 522 \ 0, 522 \ 0, 348 \ -0, 522 \ 0, 174 \ -0, 174)^T$ and the cycle $c_1 \in C_1(K_2)$ can be represented as $c_1 = 0, 522[01] + 0, 522[12] + 0, 348[23] - 0, 522[03] + 0, 174[24] - 0, 174[34]$. Then $\partial_2^2 c_1 = 0, 348 - 0, 174 - 0, 174 = 0$. The cycle c_1 or eigenvector v_1 represents the homology class of a simplicial complex $H_1(K_2)$. The cycle c_1 has a "hole" that is bounded by the edges $[23], [24], [34]$.
Case $K_3; r=1, 41$.

Table 4. Boundary operators for case K_3 .

k	$k=0$	$k=1$	$k=2$
∂_{k+1}^3	$\begin{matrix} & 01 & 12 & 23 & 03 & 24 & 02 & 13 & 34 \\ 0 & \begin{bmatrix} -1 & 0 & 0 & -1 & 0 & -1 & 0 & 0 \end{bmatrix} \\ 1 & \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix} \\ 2 & \begin{bmatrix} 0 & 1 & -1 & 0 & -1 & 1 & 0 & 0 \end{bmatrix} \\ 3 & \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 1 & -1 \end{bmatrix} \\ 4 & \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \end{matrix}$	$\begin{matrix} & 012 & 023 & 013 & 123 & 234 \\ 01 & \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \end{bmatrix} \\ 12 & \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \end{bmatrix} \\ 23 & \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \end{bmatrix} \\ 03 & \begin{bmatrix} 0 & -1 & -1 & 0 & 0 \end{bmatrix} \\ 24 & \begin{bmatrix} 0 & 0 & 0 & 0 & -1 \end{bmatrix} \\ 02 & \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \end{bmatrix} \\ 13 & \begin{bmatrix} 0 & 0 & 1 & -1 & 0 \end{bmatrix} \\ 34 & \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{matrix}$	$\begin{matrix} & 0123 \\ 012 & \begin{bmatrix} -1 \end{bmatrix} \\ 023 & \begin{bmatrix} -1 \end{bmatrix} \\ 013 & \begin{bmatrix} 1 \end{bmatrix} \\ 123 & \begin{bmatrix} 1 \end{bmatrix} \\ 234 & \begin{bmatrix} 0 \end{bmatrix} \end{matrix}$
∂_k^3	$\begin{matrix} & 0 & 1 & 2 & 3 & 4 \\ / & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{matrix}$	$\partial_1^3 = \partial_{0+1}^3$	$\partial_2^3 = \partial_{1+1}^3$

L_k^3	$L_0^3 = \partial_1^3 (\partial_1^3)^T = \begin{bmatrix} 3 & -1 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 & 0 \\ -1 & -1 & 4 & -1 & -1 \\ -1 & -1 & -1 & 4 & -1 \\ 0 & 0 & -1 & -1 & 2 \end{bmatrix}$	$L_1^3 = \partial_2^3 (\partial_2^3)^T + (\partial_1^3)^T \partial_1^3 = \begin{bmatrix} 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 3 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & -1 & 3 \end{bmatrix}$	$L_2^3 = \partial_3^3 (\partial_3^3)^T + (\partial_2^3)^T \partial_2^3 = \begin{bmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 1 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 4 & 1 \\ 0 & 1 & 0 & 1 & 3 \end{bmatrix}$
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Dimensions of Laplacian matrix: $\dim(L_0^3)=5; \dim(L_1^3)=8; \dim(L_2^3)=5$. Ranks of Laplacian matrix: $\text{rank}(L_0^3)=4; \text{rank}(L_1^3)=8; \text{rank}(L_2^3)=5$. Betti numbers: $\beta_0^3=1; \beta_1^3=0; \beta_2^3=0$. Spectrum of eigenvalues of Laplacian matrix:

$$\text{Spec}(L_0^3) = \{4 \ 5 \ 5 \ 2 \ 0\};$$

$$\text{Spec}(L_1^3) = \{4 \ 5 \ 5 \ 5 \ 4 \ 4 \ 2 \ 2\};$$

$$\text{Spec}(L_2^3) = \{4 \ 4 \ 4 \ 5 \ 2\}.$$

The received features are the spectrum of Laplacian matrix.

Table 5. The spectra of Laplacian matrix.

	$k=0$	$k=1$	$k=2$
$\text{Spec}(L_k^0) (r=0)$	$\{0 \ 0 \ 0 \ 0 \ 0\}$	—	—
$\text{Spec}(L_k^1) (r=0,707)$	$\{0 \ 0 \ 1 \ 0 \ 3\}$	$\{1 \ 3\}$	—
$\text{Spec}(L_k^2) (r=1)$	$\{2,38 \ 4,62 \ 1,38 \ 3,62 \ 0\}$	$\{2,38 \ 3 \ 3,62 \ 1,38 \ 4,62 \ 0\}$	$\{3\}$
$\text{Spec}(L_k^3) (r=1,41)$	$\{4 \ 5 \ 5 \ 2 \ 0\}$	$\{4 \ 5 \ 5 \ 5 \ 4 \ 4 \ 2 \ 2\}$	$\{4 \ 4 \ 4 \ 5 \ 2\}$

Corresponding Betti numbers: $\beta_k = \dim(L_k) - \text{rank}(L_k)$.

Table 6. Betti numbers.

	$k=0$	$k=1$	$k=2$
$\beta_k (r=0)$	5	0	0
$\beta_k (r=0,707)$	3	0	0
$\beta_k (r=1)$	1	1	0
$\beta_k (r=1,41)$	1	0	0

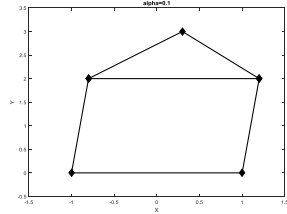
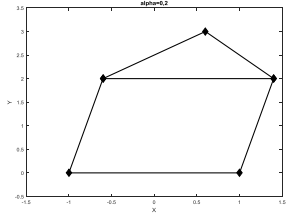
The "birth" and "death" radii and the eigenvalues of the Laplacian matrix do not change during the translations and rotations of the simplicial complex. When scaling a simplicial complex, the eigenvalues of the Laplacian matrix do not change, and the radii "birth" and "death" change in proportion to the scaling factor.

5. Using the spectrum of Laplace matrices for comparing images with affine transformations

Let us consider algorithms for the formation of spectrum of Laplace matrices for comparing images with affine transformations.

Example 3. Let's consider the change in the spectrum of the Laplace matrices of simplicial complexes under affine transformations – a shear parallel to the axis x : $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, for $\alpha = 0.1$ and $\alpha = 0.2$ (see Figure 2).

Table 7. Changes in the spectra of Laplace matrices with a shear.

α	0.1	0.2
Vertex coordinates	$x = [-1 \ 1 \ 1.2 \ 0.3 \ -0.8 \ -1]$; $y = [0 \ 0 \ 2 \ 3 \ 2 \ 0]$.	$x = [-1 \ 1 \ 1.4 \ 0.6 \ -0.6 \ -1]$; $y = [0 \ 0 \ 2 \ 3 \ 2 \ 0]$.
Barcodes with Dim = 0	$[0, 1.3)$; $[0, 1.45)$; $2[0, 2)$; $[0.0, \infty)$	$[0, 1.25)$; $[0, 1.55)$; $2[0, 2)$; $[0, \infty)$
Barcodes with Dim = 1	$[2, 2.65)$	$[2, 2.55)$
Simplex $\sigma_0 \cup \sigma_1$ images		
	Figure 2a.	Figure 2b.
$\text{Spec}(L_k^0)$	$\text{Spec}(L_{k=0}^0)_{(r=0)} = \{0 \ 0 \ 0 \ 0 \ 0\}$	$\text{Spec}(L_{k=0}^0)_{(r=0)} = \{0 \ 0 \ 0 \ 0 \ 0\}$
$\text{Spec}(L_k^1)$	$\text{Spec}(L_{k=0}^1)_{(r=0.725)} = \{0 \ 0 \ 1 \ 0 \ 3\}$ $\text{Spec}(L_{k=1}^1)_{(r=0.725)} = \{1 \ 3\}$	$\text{Spec}(L_{k=0}^1)_{(r=0.775)} = \{0 \ 0 \ 1 \ 0 \ 3\}$ $\text{Spec}(L_{k=1}^1)_{(r=0.775)} = \{1 \ 3\}$
$\text{Spec}(L_k^2)$	$\text{Spec}(L_{k=0}^2)_{(r=1)} =$ $= \{2,38 \ 4,62 \ 1,38 \ 3,62 \ 0\}$ $\text{Spec}(L_{k=1}^2)_{(r=1)} =$ $= \{2,38 \ 3 \ 3,62 \ 1,38 \ 4,62 \ 0\}$ $\text{Spec}(L_{k=2}^2)_{(r=1)} = \{3\}$	$\text{Spec}(L_{k=0}^2)_{(r=1)} =$ $= \{2,38 \ 4,62 \ 1,38 \ 3,62 \ 0\}$ $\text{Spec}(L_{k=1}^2)_{(r=1)} =$ $= \{2,38 \ 3 \ 3,62 \ 1,38 \ 4,62 \ 0\}$ $\text{Spec}(L_{k=2}^2)_{(r=1)} = \{3\}$
$\text{Spec}(L_k^3)$	$\text{Spec}(L_{k=0}^3)_{(r=1.325)} = \{4 \ 5 \ 5 \ 2 \ 0\}$ $\text{Spec}(L_{k=1}^3)_{(r=1.325)} =$ $= \{4 \ 5 \ 5 \ 5 \ 4 \ 4 \ 2 \ 2\}$ $\text{Spec}(L_{k=2}^3)_{(r=1.325)} = \{4 \ 4 \ 4 \ 5 \ 2\}$	$\text{Spec}(L_{k=0}^3)_{(r=1.275)} = \{4 \ 5 \ 5 \ 2 \ 0\}$ $\text{Spec}(L_{k=1}^3)_{(r=1.275)} =$ $= \{4 \ 5 \ 5 \ 5 \ 4 \ 4 \ 2 \ 2\}$ $\text{Spec}(L_{k=2}^3)_{(r=1.275)} = \{4 \ 4 \ 4 \ 5 \ 2\}$
Wasserstein distance Δ_W	0.075	0.125

The last row of the table gives the values of the Wasserstein distances Δ_W (see Appendix 3) from the terminal image of the object (at $\alpha = 0.1$ and $\alpha = 0.2$) to the original image of the object at $\alpha = 0$.

6. Persistent spectral theory for creating a sequence of simplicial complexes

Consider a persistent spectral theory for creating a sequence of simplicial complexes induced by a change in the filtration parameter, which is induced by persistent homology. The filtration of an oriented simplicial complex is a sequence of subcomplexes $(K_t)_{t=0}^m$ of the complex K : $\emptyset = K_0 \subseteq K_1 \subseteq K_2 \subseteq \dots \subseteq K_m = K$. The filtration induces a sequence of chain complexes:

$$\begin{array}{ccccccc}
 L & C_{k+1}^1 & \xleftrightarrow[\partial_{k+1}^{1*}]{\partial_{k+1}^1} & C_k^1 & \xleftrightarrow[\partial_k^{1*}]{\partial_k^1} & L & \xleftrightarrow[\partial_2^{1*}]{\partial_2^1} & C_1^1 & \xleftrightarrow[\partial_1^{1*}]{\partial_1^1} & C_0^1 & \xleftrightarrow[\partial_0^{1*}]{\partial_0^1} & C_{-1}^1 \\
 & \mathfrak{B} & & \mathfrak{B} & & & & \mathfrak{B} & & \mathfrak{B} & & \mathfrak{B} \\
 L & C_{k+1}^2 & \xleftrightarrow[\partial_{k+1}^{2*}]{\partial_{k+1}^2} & C_q^2 & \xleftrightarrow[\partial_q^{2*}]{\partial_q^2} & L & \xleftrightarrow[\partial_2^{2*}]{\partial_2^2} & C_1^2 & \xleftrightarrow[\partial_1^{2*}]{\partial_1^2} & C_0^2 & \xleftrightarrow[\partial_0^{2*}]{\partial_0^2} & C_{-1}^2 \\
 & M & & M & & & & M & & M & & M \\
 & \mathfrak{B} & & \mathfrak{B} & & & & \mathfrak{B} & & \mathfrak{B} & & \mathfrak{B} \\
 L & C_{k+1}^m & \xleftrightarrow[\partial_{k+1}^{m*}]{\partial_{k+1}^m} & C_q^m & \xleftrightarrow[\partial_q^{m*}]{\partial_q^m} & L & \xleftrightarrow[\partial_2^{m*}]{\partial_2^m} & C_1^m & \xleftrightarrow[\partial_1^{m*}]{\partial_1^m} & C_0^m & \xleftrightarrow[\partial_0^{m*}]{\partial_0^m} & C_{-1}^m
 \end{array} \quad (5)$$

where $C_k^t := C_k(K_t)$ and $\partial_k^t : C_k(K_t) \rightarrow C_{k-1}(K_t)$.

Each K_t is an oriented simplicial complex with dimension $\dim(K_t)$. If $k < 0$, then $C_k(K_t) = \{\emptyset\}$ and ∂_k^t is a null mapping. For the general case $0 < k \leq \dim(K_t)$, if σ_k is an oriented k -simplex K_t , then

$\partial_k^t(\sigma_k) = \sum_{i=1}^k (-1)^i \sigma_{k-1}^i, \sigma_k \in K_t$, with $\sigma_k = [v_0, L, v_k]$ – an k -oriented simplex and $\sigma_{k-1}^i = [v_0, L, \hat{v}_i, L, v_k]$ – an oriented $(k-1)$ -simplex in which the vertex v_i is removed.

Let C_k^{t+p} be a subset C_k^{t+p} , whose boundary is in C_{k-1}^t : $C_k^{t+p} := \{\alpha \in C_k^{t+p} \mid \partial_k^{t+p}(\alpha) \in C_{k-1}^t\}$. Let's define $\partial_k^{t+p} : C_k^{t+p} \rightarrow C_{k-1}^t$. Based on the k -combinatorial Laplace operator, the p -persistent k -combinatorial Laplace operator defined along the filtration, which can be expressed in the form of the Laplace matrix: $L_k^{t+p} = \partial_{k+1}^{t+p} (\partial_{k+1}^{t+p})^* + \partial_k^{t*} \partial_k^t$.

Suppose there is an oriented simplicial complex K_t and an oriented simplicial complex K_{t+p} constructed by adding simplexes of different sizes to K_t with $\dim(K_{t+p}) = k+1$. Since $K_t \subset K_{t+p}$,

then: $L_k^{t+0} = L_k^t = \partial_{k+1}^t (\partial_{k+1}^t)^* + \partial_k^{t*} \partial_k^t$; $L_k^{t+p} = \partial_{k+1}^{t+p} (\partial_{k+1}^{t+p})^* + \partial_k^{t*} \partial_k^t$.

If the spectrum $\text{Spec}(L_k^{t+0})$ and $\text{Spec}(L_k^{t+p})$ coincide, then the topological structure K_t does not change during the filtering process. If the spectrum $\text{Spec}(L_k^{t+0})$ and $\text{Spec}(L_k^{t+p})$ do not coincide, then the topological structure K_t builds a connection with "external" topological structures; by calculating the spectrum $\text{Spec}(L_k^{t+p})$, one can discover information about the disappeared and preserved structure of the filtration process. The numbers of zero eigenvalues of the p -persistent k -combinatorial Laplace matrix L_k^{t+p} determine the p -persistent k -th Betti numbers: β_k^{t+p} .

Example 4. Case $K_1 \rightarrow K_2; (r=0, 707) \rightarrow (r=1)$

Table 8. Boundary operators for case $K_1 \rightarrow K_2$

k	$k=0$	$k=1$
-----	-------	-------

∂_{k+1}^{1+1}	$\begin{array}{c} 01 \quad 12 \quad 23 \quad 03 \quad 24 \quad 34 \\ 0 \begin{bmatrix} -1 & 0 & 0 & -1 & 0 & 0 \end{bmatrix} \\ 1 \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \end{bmatrix} \\ 2 \begin{bmatrix} 0 & 1 & -1 & 0 & -1 & 0 \end{bmatrix} \\ 3 \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & -1 \end{bmatrix} \\ 4 \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$	$\begin{array}{c} 234 \\ 01 \begin{bmatrix} 0 \end{bmatrix} \\ 12 \begin{bmatrix} 0 \end{bmatrix} \\ 23 \begin{bmatrix} 1 \end{bmatrix} \\ 03 \begin{bmatrix} 0 \end{bmatrix} \\ 24 \begin{bmatrix} -1 \end{bmatrix} \\ 34 \begin{bmatrix} 1 \end{bmatrix} \end{array}$
∂_k^1	$\begin{array}{c} 0 \quad 1 \quad 2 \quad 3 \quad 4 \\ [0 \quad 0 \quad 0 \quad 0 \quad 0] \end{array}$	$\begin{array}{c} 01 \quad 12 \quad 23 \quad 03 \quad 24 \quad 34 \\ 0 \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ 1 \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ 2 \begin{bmatrix} 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix} \\ 3 \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \\ 4 \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$
L_k^{1+1}	$L_0^2 = \partial_1^2 (\partial_1^2)^T = \begin{bmatrix} 2 & -1 & 0 & -1 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 3 & -1 & -1 \\ -1 & 0 & -1 & 3 & -1 \\ 0 & 0 & -1 & -1 & 2 \end{bmatrix}$	$L_1^2 = \partial_2^{1+1} (\partial_2^{1+1})^T + (\partial_1^1)^T \partial_1^1 = \begin{bmatrix} 2 & -1 & 0 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & -1 & 0 \\ 0 & -1 & 3 & 1 & 0 & 0 \\ 1 & 0 & 1 & 2 & 0 & -1 \\ 0 & -1 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -1 & 0 & 3 \end{bmatrix}$

Dimensions of Laplacian matrix $\dim(L_0^{1+1})=5; \dim(L_1^{1+1})=6$. Laplacian ranks: $\text{rank}(L_0^{1+1})=4; \text{rank}(L_1^{1+1})=5$. Betti numbers: $\beta_0^{1+1}=1; \beta_1^{1+1}=1$. Spectrum of eigenvalues of Laplacian matrix: $\text{Spec}(L_0^{1+1})=\{2,38 \quad 4,62 \quad 1,38 \quad 3,62 \quad 0\}$; $\text{Spec}(L_1^{1+1})=\{2,38 \quad 3 \quad 3,62 \quad 1,38 \quad 4,62 \quad 0\}$.

Appendix 1. Cochains and cohomology [1]

Consider a functional $Y(X)$ that takes values on a chain $X: Y(X)=\langle Y, X \rangle$; the coboundary operator $\delta_n: C^n \rightarrow C^{n+1}$ can be determined based on the relation:

$$\langle \delta_n Y, X \rangle = \langle Y, \partial_n X \rangle, X \in C_n, Y \in C^{n-1}.$$

If vector spaces $C_i(X)$ are defined on $\mathbf{R}$, then each $C_i(X; \mathbf{R})$ can be given the structure of inner products. This allows you to define a dual space $C^i(X; \mathbf{R})$. A member of this dual space is a real mapping of chains $c_i(X)$ to $\mathbf{R}$. These maps are cochains and the dual space $\text{Hom}_{\mathbf{R}}(C_i(X; \mathbf{R}), \mathbf{R})$ for dimension i is denoted as $C^i(X; \mathbf{R})$. Since chains are linear combinations of simplexes, each mapping is described by specifying its value on each simplex. By using duality, one can identify $C^i(X; \mathbf{R})$ with $C_i(X; \mathbf{R})$ using the inner product. Therefore, a dual operator can be defined, which is called a coboundary mapping: $\delta_i: C^{i+1}(X; \mathbf{R}) \leftarrow C^i(X; \mathbf{R})$. δ_i is the adjoint operator ∂_{i+1}^* with respect to the boundary operator ∂_{i+1} .

If $\varphi \in C^i(X)$, then for a simplex $\sigma = [v_0, \dots, v_{i+1}] \in C_{i+1}(X)$:

$$\delta_i \varphi(\sigma) = \sum_{j=0}^{i+1} (-1)^j \varphi([v_0, \dots, v_{j-1}, v_{j+1}, \dots, v_{i+1}]),$$

that is, the mapping $\delta_i \varphi$ when evaluating on a simplex σ is equal to the sum of evaluations φ on all faces σ .

Since $\delta_i \delta_{i+1} = 0$ holds for coboundary mappings, the coboundary operators form a chain complex:

$$0 \leftarrow C^n(X; \mathbf{R}) \xleftarrow{\delta_n} \cdots \xleftarrow{\delta_k} C^k(X; \mathbf{R}) \xleftarrow{\delta_{k-1}} C^{k-1}(X; \mathbf{R}) \cdots \xleftarrow{\delta_1} C^1(X; \mathbf{R}) \xleftarrow{\delta_0} C^0(X; \mathbf{R}) \leftarrow 0,$$

which generates cohomology groups $H^k(X) = \frac{\ker \delta_k}{\text{im } \delta_{k+1}}$. When the homology and cohomology coefficients are chosen in $\mathbf{R}$, the cohomology groups are dual with respect to the homology groups: $H^k(X; \mathbf{R}) \cong \text{Hom}_{\mathbf{R}}(H_k(X; \mathbf{R}), \mathbf{R})$.

Let's pretend that $\omega \in C^k(X, \mathbf{R})$. $\delta_1 \omega \in C^{k+1}(X, \mathbf{R})$ is the mapping from k -simplexes belonging to X , to $\mathbf{R}$. Let $s = \sum_j \alpha_j \sigma_j \in C_{k+1}(X, \mathbf{R})$, where $\alpha_j \in \mathbf{R}$ and σ_j are k -simplexes. Let's perform the calculation $\delta_k \omega(s) = \sum_j \alpha_j \delta_1 \omega(\sigma_j)$. It follows from duality $\delta_k = \partial_{k+1}^*$ that $\delta_k \omega = \omega \partial_{k+1}$. Hence,

$$\sum_j \alpha_j \delta_k \omega(\sigma_j) = \sum_j \alpha_j \omega(\partial_{k+1} \sigma_j) = \omega \left(\partial_{k+1} \sum_j \alpha_j \sigma_j \right) = \omega \partial_{k+1} s.$$

This calculation can be viewed as a discretization of Stokes' theorem for integration over the region D , $\int_{\partial D} \omega = \int_D d\omega$, where ∂D is the region boundary and $d\omega$ is the differential ω . This suggests that $C^i(X, \mathbf{R})$ can also be interpreted as a set of discrete differential forms on a combinatorial manifold X .

Appendix 2. Construction of the Vietoris–Rips complex

A feature of computational topology methods is the presence of invariance with respect to Euclidean coordinate transformations [3]. To form signs of point landmarks, it is proposed to build a Vietoris–Rips complex of point clouds relative to these landmarks. For a set of point landmarks $S = \{q_1, \dots, q_k\}; q_i \in \mathbf{R}^2$ the Vietoris–Rips complex $R(r)$ is a simplicial complex built on the set S ; the complex is formed provided that all pairs of points are at a distance not exceeding $2r$:

$$R(r) = \{ \sigma = S \mid \|q_i - q_j\| \leq 2r, \forall i, j \}.$$

When the distance $r \geq 0$ changes, the Betti numbers of the Vietoris–Rips complex change in accordance with the transformations. You can define the spawn points (birth) r_b and the vanishing points (dead) r_d of Betti numbers. Based on the values of these points, barcodes $\{[b \dots d_i] \mid d_i \geq b_i \geq 0\}$ are built (i – barcode number). The barcodes of the Vietoris–Rips complex can be determined using the JavaPlex [13] program. JavaPlex defines b_i and d_i values homology classes when filtering a simplicial complex after sorting $\{x_1 < x_2 < \dots < x_i\}$.

Appendix 3. Wasserstein distance

Consider a barcode, which consists of: the radius of the appearance of the barcode b (birth), the radius of the disappearance of the barcode b (death), the value of the function on a given chain of the simplex: f . For barcodes of the same dimension: $B = \{b_i, d_i, f_i\}_{i \in I}$; $B' = \{b'_j, d'_j, f'_j\}_{j \in J}$; distance between B and B' :

$$\Delta_b([b, d], [b', d']) = \max\{|b - b'|, |d - d'|\},$$

and distance between functions f_j and f'_j :

$$\Delta_f(f, f') = \left| (d - b)^{-1} \int_b^d f(x) dx - (d' - b')^{-1} \int_{b'}^{d'} f'(x) dx \right|.$$

For $f(x) = \text{const}; f'(x) = \text{const} : \Delta_f(f_i, f'_j) = |f_i - f'_j|$.

Wasserstein distance between objects, due to the mismatch of the boundaries of the barcodes of the source and the terminal images: $d_{w,2}^2(B, B') = \sum_{i \in I} \left(\Delta_b([b_i, d_i], [b'_{\phi(i)}, d'_{\phi(i)}]) \right)^2$. The distance between the objects due to the discrepancy between the functions of the simplexes of the source and the terminal images: $d_f^2(B, B') = \sum_{i \in I} \left(\Delta_f(f_i, f'_{\phi(i)}) \right)^2$. The modified Wasserstein distance can be determined from the ratio:

$$d_{MW}^2(\mu_b, \mu_f, B, B') = \mu_b \cdot d_{w,2}^2(B, B') + \mu_f \cdot d_f^2(B, B'),$$

where μ_b and μ_f are the weight parameters.

To determine the Wasserstein distance, taking into account the spectrum of the eigenvalues of the Laplace matrix, the expression $d_{MW}^2(\mu_b, \mu_f, B, B')$ is added with $d_L^2(B, B') = \sum_{i \in I} \left(\Delta_L(\lambda_i, \lambda'_{\phi(i)}) \right)^2$ with

the weight parameter μ_L , where $\Delta_L(\lambda_i, \lambda'_{\phi(i)}) = 1 - \exp\left(-\frac{(\lambda_i - \lambda'_{\phi(i)})^2}{c^2}\right)$.

Conclusion

The method for constructing persistent spectrum of eigenvalues of combinatorial Laplace matrices for estimating simplicial complexes is presented in the paper. The method allows combining topological stability and geometric analysis of multidimensional datasets. A number of persistent combinatorial Laplace matrices are induced by filtering based on the determination of the radii of appearance and disappearance of the Laplacian eigenvalues. The numbers of zero eigenvalues of persistent k -combinatorial Laplace matrices are k -dimensional persistent Betti numbers for the same filtering. The proposed persistent spectral analysis can be combined with machine learning algorithms [14, 15]. The algorithms for the formation of the spectrum of Laplace matrices for comparing images with affine transformations are used in the paper.

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