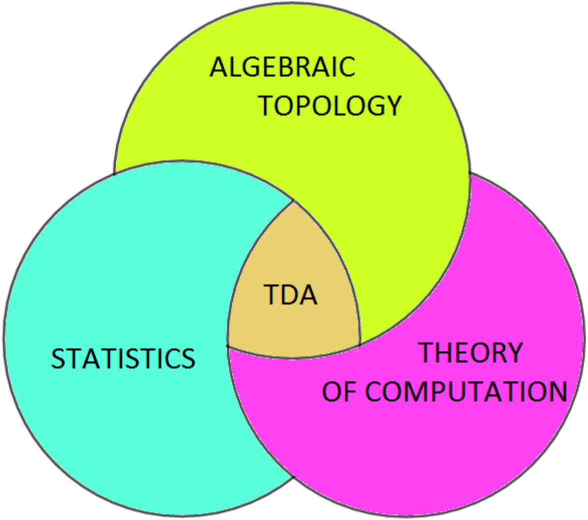
Persistent Homology and Entropy

N. Atienza, R. Gonzalez-Diaz, **M. Soriano-Trigueros.**

June 2018

Topological Data Analysis



Data Structure

directed multigraphs

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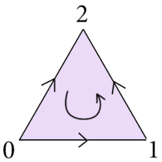
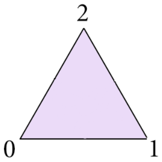
simplicial sets



undirected graphs

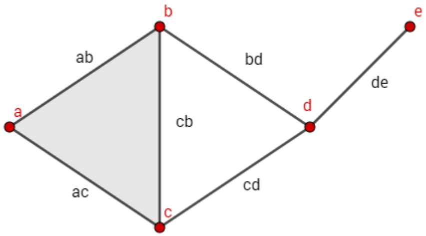
↓

abstract simplicial complexes



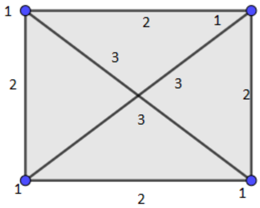
Abstract simplicial complexes are more suitable for computations.

Simplicial Complexes

	Simplicial Complex
2-simplexes	$\{a, b, c\}$
1-simplexes	$\{a, b\}, \{b, c\}, \{a, c\}$ $\{b, d\}, \{c, d\}, \{d, e\}$
0-simplexes	$\{a\}, \{b\}, \{c\}, \{d\}, \{e\}$
	

Filtration

- Fix a simplicial complex SC.
- Define the **filter function** $f : SC \to \mathbb{R}$ s.t. $\sigma \lhd \tau \Rightarrow f(\sigma) \leq f(\tau)$.
- A **filtration** is the assignment $a \mapsto f^{-1}(-\infty, a]$.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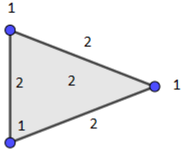
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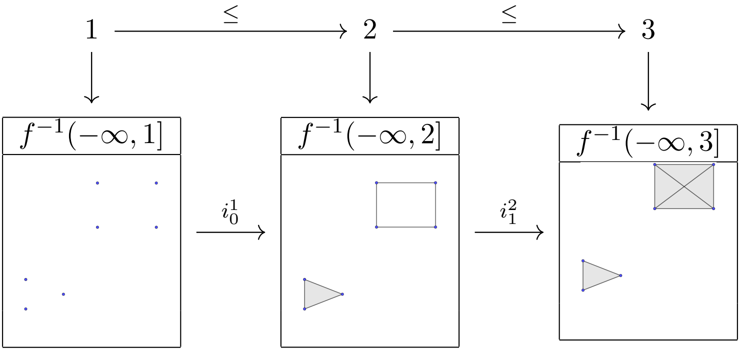
Filtration

Each simplicial complex is related with the following ones by inclusion



Filtration (Abstract Concept)

- $\mathbb{K}$ are all subcomplexes of a simplicial complex seen as a category with morphism the inclusion.
- $\mathbb{P}$ is a poset seen as a category with morphism $\leq$.

Filtration

A filtration is a functor $F : \mathbb{P} \to \mathbb{K}$

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Filtration

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In practice, $\mathbb{P}$ will be a subset of $\mathbb{R}$.

Persistent Homology

Let $\mathbf{Vec}$ be the category of vector space over a field k.

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Persistence Module

A persistence module is a functor $M : \mathbb{P} \to \mathbf{Vec}$

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We call n-persistent homology to the functor

$$M = H_n(-, k) \circ F$$

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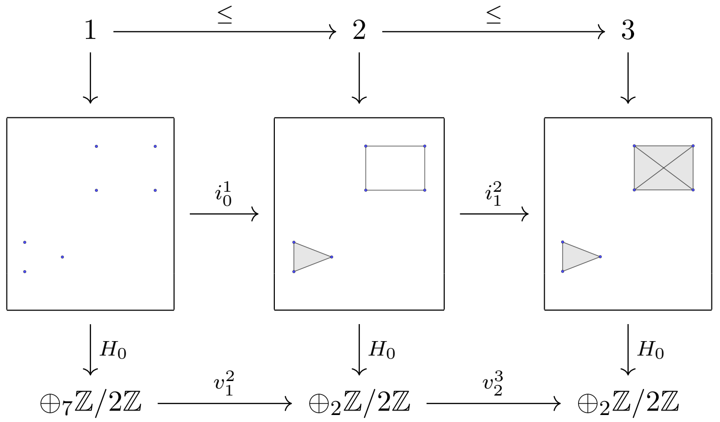
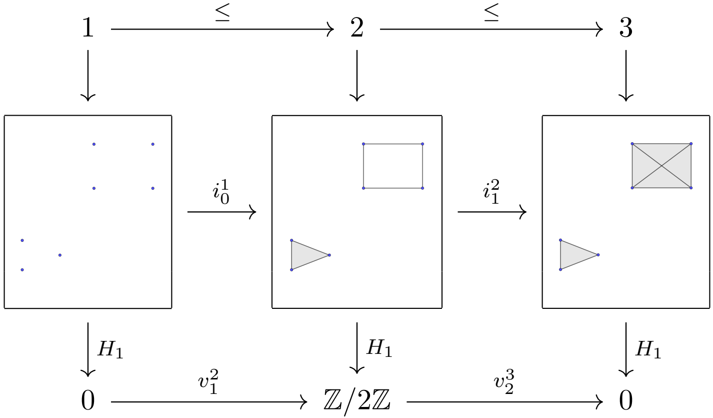
Where H_n is the n-homology and usually $k = \mathbb{Z}/2\mathbb{Z}$.

Persistent Homology



Persistent Homology



Barcodes & Persistent Diagrams

If J is an interval in $\mathbb{R}$, a **interval module** is k^J is:

$$v_t = \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{if } t \in J \\ 0 & \text{otherwise.} \end{cases} \qquad v_t^l = \begin{cases} Id & \text{if } [t,l] \subset J \\ 0 & \text{otherwise.} \end{cases}$$

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Theorem (simplified)

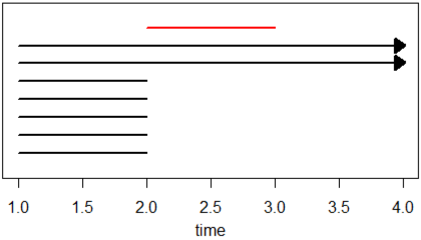
If M is a *nice* persistent homology module in $\mathbb{R}$ then

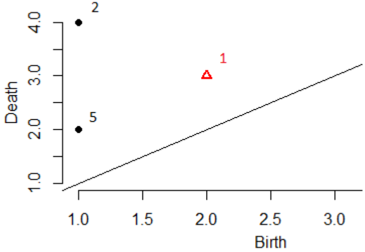
$$M \cong \oplus_{i=1}^{n} k^{J_i}$$

where $J_i = (x_i, y_i)$

Barcodes & Persistent Diagrams

We can represent the multiset $\{(x_i, y_i)\}_{i=1}^{n}$ in several ways

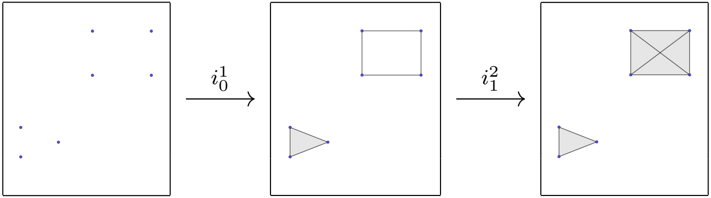




Short bars and points near the diagonal are usually considered noise

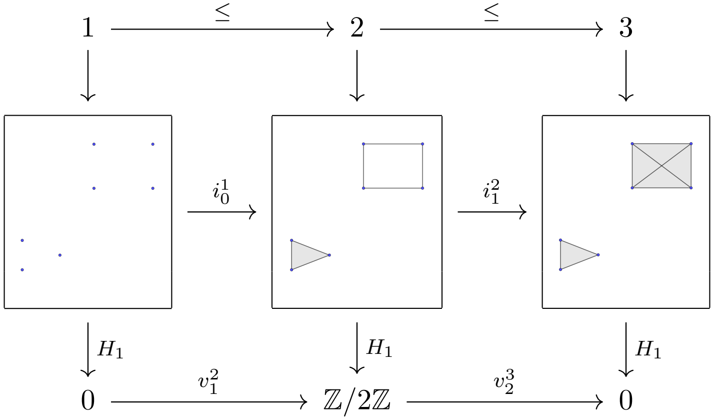
Persistent Homology

$$1 \xrightarrow{\leq} 2 \xrightarrow{\leq} 3$$



$$\oplus_7 \mathbb{Z}/2\mathbb{Z} \xrightarrow{v_1^2} \oplus_2 \mathbb{Z}/2\mathbb{Z} \xrightarrow{v_2^3} \oplus_2 \mathbb{Z}/2\mathbb{Z}$$

Persistent Homology



Barcodes & Persistent Diagrams

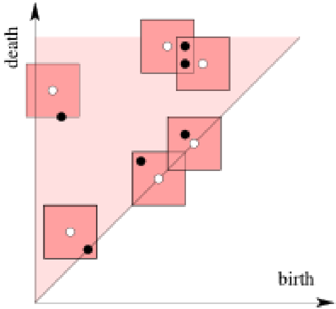


Figure: Computational Topology: An Introduction. H. Edelsbrunner, J. Harer.

Barcodes & Persistent Diagrams

- A, B persistent diagrams,
- $M \subset A \times B$ is a partial matching if satisfies
  - $\forall$ in A there is at most one β with $(\alpha, \beta) \in M$
  - $\forall \beta$ there is at most one α with $(\alpha, \beta) \in M$

There exist a δ-matching if:

- for all pairs (α, β), $||\alpha - \beta||_\infty \leq \delta$
- for all α unmatched, $|\alpha_2 - \alpha_1| \leq \delta$. Idem for β.

Stability Theorem

Bottleneck Distance

Let A, B be two persistent diagrams, then

$$d(A,B) = \inf\{\delta | \text{ exists a } \delta\text{-matching}\}$$

is their bottleneck distance

Stability theorem (simplified version)

Let K be a simplicial complex and $f, g : K \to \mathbb{R}$ filter functions. If A, B are the corresponding persistent diagrams,

$$d(A,B) \leq ||f - g||_\infty$$

Statistical disadvantage

Problems of barcodes:

- There are not obvious algebraic operation defined on them,

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- There are not obvious algebraic operation defined on them,
- They do not have unique Fréchet mean,
- They are uncomfortable for applying the null hypothesis significance test and confidence intervals.

We can associate to each barcode a **real number** and use it to obtain statistical information.

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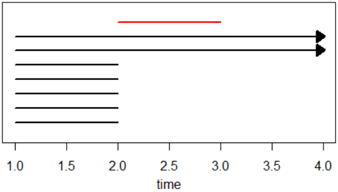
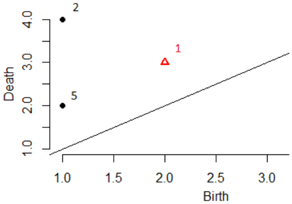
In some applications, counting the number of bars/points seems to work.





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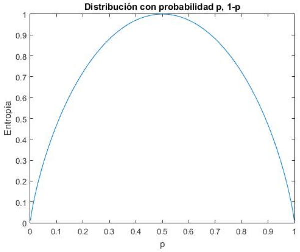
This is not **stable** respect to noise

Our research

Definition

given a finite random distribution $A = \{p_i : \sum_{i=1}^{n} p_i = 1\}$ we define its shannon entropy as

$$E(A) = -\sum_{i=1}^{n} p_i \log(p_i).$$

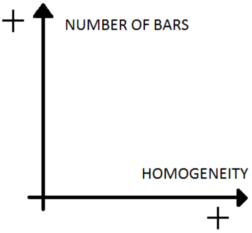


Our Research

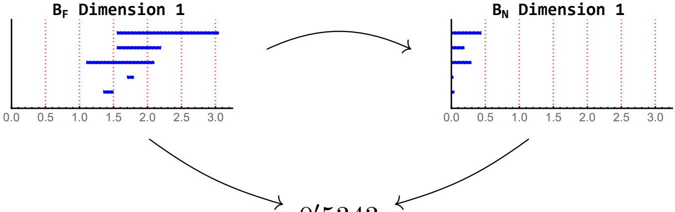
Persistent Entropy

Let $\{\ell_i\}$ be the length of the bars and $L = \ell_1 + \ldots + \ell_n$. The persistent entropy of a barcode is

$$E = -\sum_{i=1}^{n} \frac{\ell_i}{L} \log\left(\frac{\ell_i}{L}\right).$$



Our Research



$$0'5343$$

Results

Stability Theorem

Let K be a simplicial complex and let $f, g : K \to \mathbb{R}$ be two monotonic functions. Let A, B be their barcodes.

$$||f - g||_\infty \leq \delta \Rightarrow |E(A) - E(B)| \leq \frac{4\delta}{\ell_{\max}} \left[\log(n_{\max}) - \log\left(\frac{4\delta}{\ell_{\max}}\right)\right].$$

- n_{max} is the maximum number of bars of the barcodes.
- $\ell_{\max}$ is the maximum normalized length of the bars.

N. Atienza et al. Stability of persistent entropy and new summary functions for TDA.

Applications

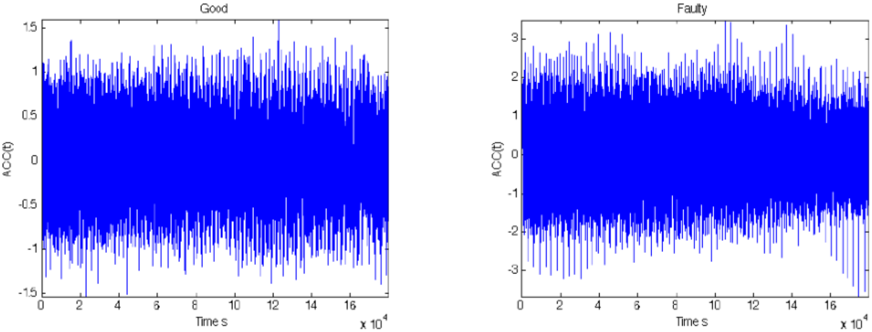
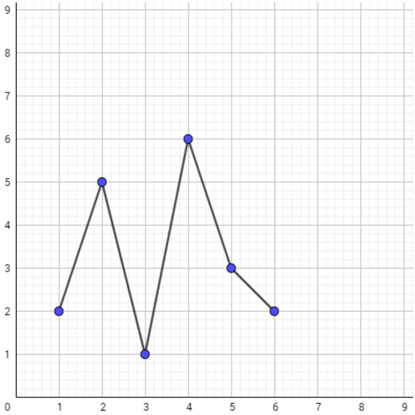


Figure: R. Gonzalez-Diaz et al. A new topological entropy-based approach for measuring similarities among piecewise linear functions.

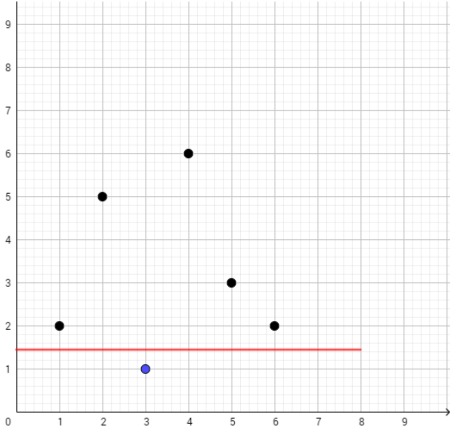
Applications

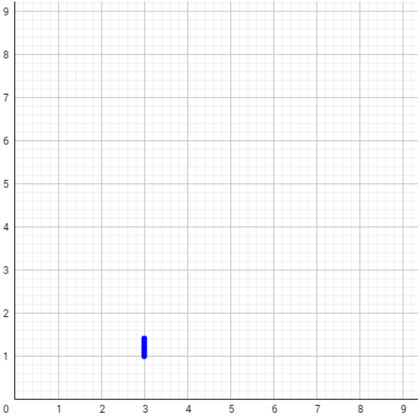


1. Vertex $\{p_i = (x_i, y_i)\}$.
   Edges $\{(p_i, p_{i+1})\}$.
2. Define the filter function
   $f(p_i) = y_i$
   $f(p_i, p_{i+1}) = \max\{y_i, y_{i+1}\}$

Applications

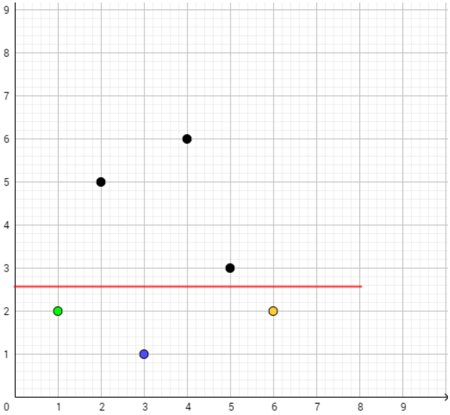
$f^{-1}(-\infty, 1'5)$

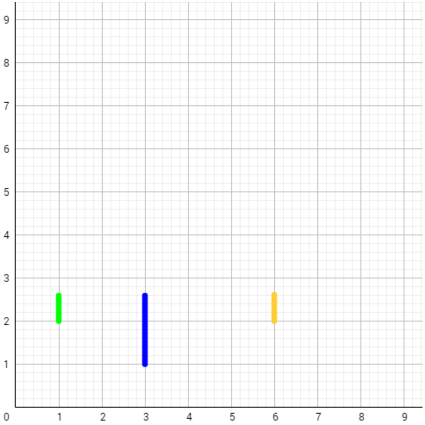




Applications

$f^{-1}(-\infty, 2'7)$

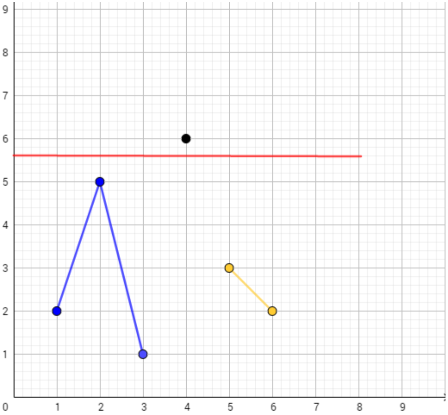


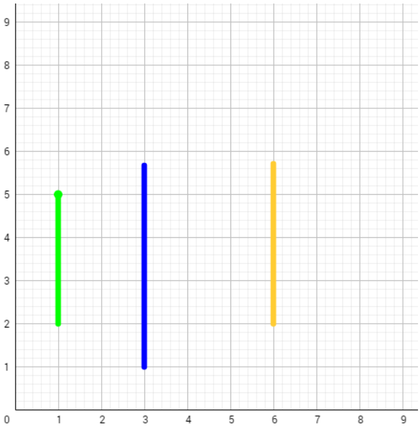


Applications

Applications

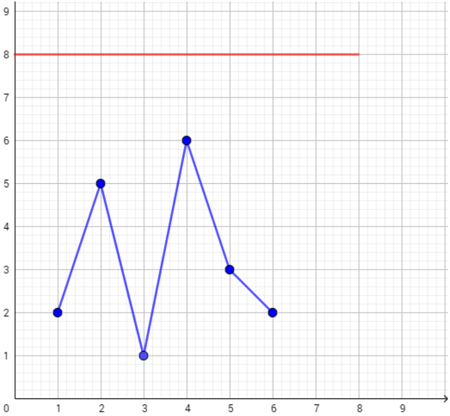
$f^{-1}(-\infty, 5'7)$

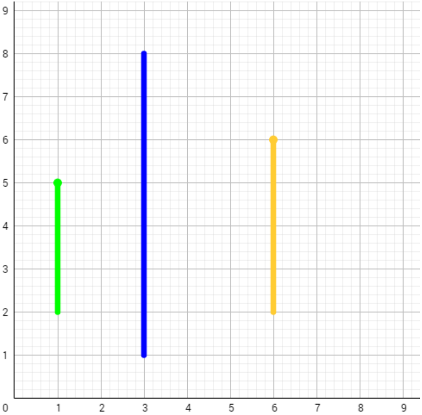




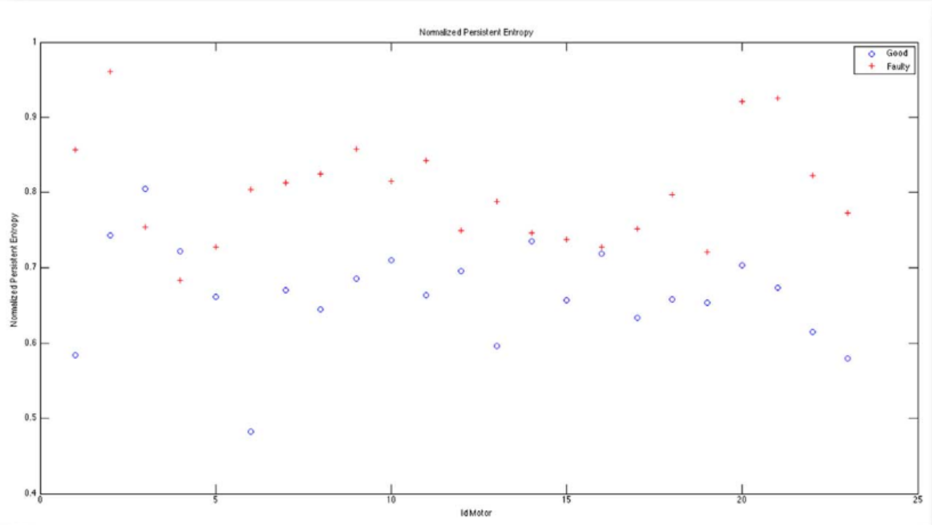
Applications

$f^{-1}(-\infty, 8)$





Our Research



Delving into TDA

- Developing representation and stability results for new posets, specially $\mathbb{R}^n$.
- Finding computational efficient functors, apart from homology, suitable for applications. (E.g. A_∞-coalgebras).
- Understanding the homotopy types of most common filtrations. (E.g. What are the homotopy types of the Vietoris-Rips complex of $\mathbb{S}^n$?)

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Email: msoriano4@us.es

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