A SURVEY ON MULTIDIMENSIONAL PERSISTENCE THEORY

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ABSTRACT

A SURVEY ON MULTIDIMENSIONAL PERSISTENCE THEORY

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Persistence homology is one of the commonly used theoretical methods in topological data analysis to extract information from given data using algebraic topology. Converting data to a filtered object and analyzing the topological features of each space in the filtration, we will obtain a way of representing these features called the shape of data. This will give us invariants like barcodes or persistence diagrams for the data. These invariants are stable under small perturbations. In most applications, we need multiscaled analysis of data, which is done by multidimensional persistence. This thesis is a survey that contains how to produce invariants for multiscaled filtration obtained from data and the related stability results.

Keywords: Topological Data Analysis, Persistence Homology, Multidimensional Persistence

ÖZ

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Topolojik data analizinde sıklıkla kullanılan ve teorik bir yöntem olan ısrarlı homoloji, verilen bir veriden bilgi elde etmek için cebirsel topolojiyi kullanır. Verileri filtrelenmiş bir nesneye dönüştürerek ve filtrelemede her uzayın topolojik özelliklerini analiz ederek, verilerin şekli olarak adlandırılan bu özellikleri temsil etmenin bir yolunu elde edeceğiz. Bu bize barkod veya ısrarcılık diyagramları dediğimiz verinin değişmezlerini verir. Bu değişmezler küçük oynamalar altında kararlılık gösterir. Çoğu uygulamada, verinin çok parametreli analizine ihtiyaç duyulur ve bu çok boyutlu ısrarlılık kullanılarak yapılabilir. Bu tez veriden elde edilen çok parametreli filtrasyonların değişmezlerinin belirlenmesini ve bu değişmezlerin kararlılıklarını detaylıca ortaya koyan bir incelemedir.

Anahtar Kelimeler: Topolojik Data Analizi, Israrlı Homoloji, Çok Boyulu Israrcılık

To my family

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CHAPTER 1

INTRODUCTION

The necessities of analyzing the vast amount of complex data in recent decades force scientists to develop more efficient methods to process these data. Topological data analysis, TDA for short, has been emerged as a result of these efforts. It is a field that uses pure mathematics ideas, mainly algebraic topology, to find new approaches for analyzing complicated data. In this field, the aim is to find some qualitative features of data that are computable and robust under small perturbations. To do this, the main method is known as "persistence homology" in TDA. Using persistence homology, we can study point-cloud data; which is a finite set of points in some metric space, digital images, level sets of real or vector-valued functions and networks, etc.

For example, to study point cloud data, since finite set of points are not topologically interesting, by thickening the points and using the intersection relation between them, one may obtain a shape which is explained in detail in Chapter 3. But this shape is dependent on the thickening scale. To solve this problem, one may try to look at the evolution of the shapes obtained at different scales. The qualitative features are then given by topological invariants such as Betti numbers. There is a compact way of representing such invariants in a diagram or barcode. Another way of representing the persistence homology of the above setting is to use the "persistence module". It is an algebraic way of expressing homology information in all scales. The complete classification of these modules over a field is known. It can be seen as a complete invariant; i.e. an invariant that assigns different objects to non-isomorphic structures.

In most applications, to analyze the data, we may need to use more than one parameter. In other words, we need multiscale analysis. In this case, unlike persistence diagrams or barcodes, the persistence modules can be generalized in the multiparameter case. But for multidimensional persistence modules, it is known that there is no complete discrete invariant, which is independent of the ground field. Instead Carlsson and Zomorodian [14] proposes "rank invariant", which is equivalent to barcodes in one-dimensional case. It captures the persistent features but is not a complete invariant.

This thesis aims to give a survey of "persistent homology" in a multidimensional setting. To understand the theory of multidimensional persistence, the necessary background is given in chapter 2. Chapter 3 gives an overview of the theory of one-dimensional persistence that several people in the last decade have developed. In chapter 4, we give an extensively detailed theory of multidimensional persistence. In this chapter, we mainly focus on the classification of multidimensional persistence modules and different invariant suggestions like rank invariant [14] and multigraded Hilbert series [26]. Chapter 5 gives a survey on the robustness of the qualitative features, namely; stability of the persistence diagrams and barcodes in one dimensional case. We also give stability results for general persistence modules obtained by filtration over a preordered set.

CHAPTER 2

BACKGROUND

In this section, we will introduce some definitions and facts which will be needed throughout the thesis.

A preordered set is a set with a relation which is reflexive and transitive. $\mathbb{Z}^n$ and $\mathbb{R}^n$ can be an example of a preordered set where the relation is defined as $u, v \in \mathbb{Z}^n$ or $\mathbb{R}^n$, we say $u \leq v$ if $u_i \leq v_i$ for all $1 \leq i \leq n$.

Definition 2.0.1. *Let P be a preordered abelian group. A P-graded ring R is a ring equipped with a decomposition of abelian groups $R \cong \bigoplus_{v \in P} R_v$ so that multiplication has the property $R_u R_v \subset R_{u+v}$. A P-graded module over a P-graded ring R is an abelian group M equipped with decomposition $M \cong \bigoplus_{v \in P} M_v$ together with a R-module structure so that $R_u M_v \subset M_{u+v}$. We will call it a P-graded R-module. A P-graded module over an P-graded ring R is called finitely generated if M_v is nonzero and finitely generated only for finitely many $v \in P$.*

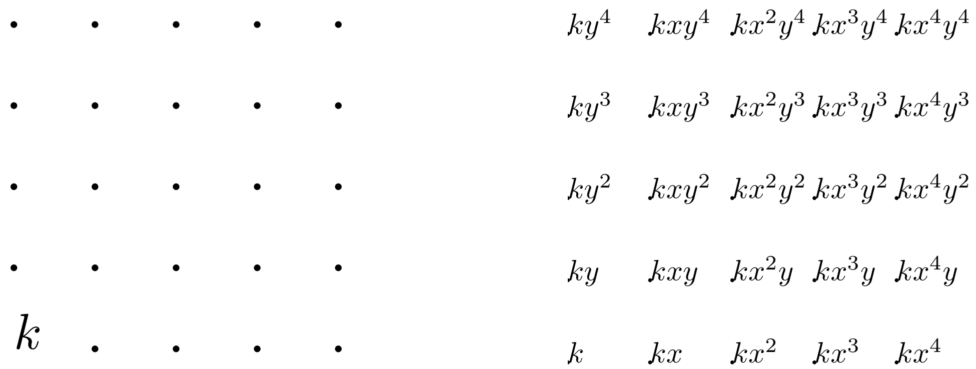
Let us illustrate the above definitions with some examples.

Example 2.0.1.

1. *The polynomial ring $S = k[x_1, \cdots, x_n]$ is $\mathbb{Z}^n$-graded ring with $S_v = kx^v$ for $v \in \mathbb{N}^n$ and $S_v = 0$ for $v \in \mathbb{Z}^n \setminus \mathbb{N}^n$ where k is a field.*

2. *The field k is a $\mathbb{Z}^n$-graded ring with $k = \bigoplus_{v \in \mathbb{Z}^n} k_v$ where $k_v = k$ for $v = (0, \cdots, 0)$ and $k_v = 0$ otherwise.*

3. *The field k is $\mathbb{Z}^n$-graded S-module where $k = \bigoplus_{v \in V} k_v$ so that $k_v = k$ if $v = (0, \cdots, 0)$ otherwise $k_v = 0$ and the action of x_i is trivial, in Fig 2.1a. In this set up, the field k is also $\mathbb{Z}^n$-graded k-vector space.*



(a) The field k as $\mathbb{Z}^2$-graded S-module

(b) S as $\mathbb{Z}^2$-graded S-module

Figure 2.1: Example 2.0.1

4. *The polynomial ring $S = k[x_1, \cdots, x_n]$ is also a $\mathbb{Z}^n$-graded S-module by $S = \bigoplus_{v\in\mathbb{Z}^n} A_v$ where $A_v = kx^v$ for $v \in \mathbb{N}^n$ and 0 otherwise,in Fig 2.1b.. The action of x^w is a map from kx^v to kx^{v+w} where $w \in \mathbb{N}^n$*

In order to describe the category of these objects, we need to define morphisms.

Definition 2.0.2. *A transition map for P-graded R-module M is a homomorphism from M_u to M_v where $u \leq v$, and it is denoted by $\varphi^M_{u,v}$. The map is defined as $\varphi^M_{u,v}(m_u) = r_{v-u}m_u$ where $m_u \in M_u$ and $r_{v-u} \in R_{v-u}$. A P-graded R-module homomorphism is a map $f : M \to N$ such that $f(M_v) \subset N_v$ for all $v \in P$ and also for all $u \leq v$, the following diagram commutes;*

$$\begin{array}{ccc} M_u & \xrightarrow{\varphi^M_{u,v}} & M_v \\ {\scriptstyle f_u}\downarrow & & \downarrow{\scriptstyle f_v} \\ N_u & \xrightarrow{\varphi^N_{u,v}} & N_v \end{array}$$

where $f_u : M_u \to N_u$ is the restriction of f and $\varphi^M_{u,v}$ and $\varphi^N_{u,v}$ are transition maps for M and N.

*Let Gr^P-$_R$**Mod** denote the category of P-graded R-modules. It contains P-graded R-module homomorphisms as morphisms. A P-graded k-vector space is a P-graded k-module and a P-graded k-vector space homomorphism is a P-graded k-module homomorphism. The category is denoted by Gr^P-$_k$**Mod**. For any P-graded R-module M, the set of homogeneous elements is denoted by $\mathbb{H}(M) = \bigcup_{v\in P} M_v$.*

Example 2.0.2. 1. *For any P-graded S-module M, we can see it as a P-graded k-vector space with linear maps $m \to x_i m$.*

2. *Consider* $(P, \leq) = (\mathbb{Z}^n, \leq)$ *and let* $(U_0, +, 0)$ *be a monoid so that* $U_0 = \{v \in \mathbb{Z}^n \mid (0, \cdots, 0) \leq v\}$ *which is also called principal up-set. Observe that* $U_0 = \mathbb{N}^n$*, which gives us the monoid ring* $R[U_0] = R[x_1, \cdots, x_n]$ *with concrete correspondence between* $v \in \mathbb{N}^n$ *and* $\boldsymbol{x}^v$*.* $R[U_0]$ *can be seen as* $\mathbb{Z}^n$*-graded by* $R[\mathbb{N}^n] = \bigoplus_{v \in \mathbb{Z}^n} R[\mathbb{N}^n]_v$ *where* $R[\mathbb{N}^n]_v$ *is the set of homogeneous elements of degree* v *if* $v \geq (0, \cdots, 0)$ *and is* 0 *otherwise.*

Now we will define free objects for *Gr*^{P}-$_R$**Mod** and *Gr*^{P}-$_k$**Mod**.

Definition 2.0.3. *Let* P *be preordered abelian group. A* P*-graded set is a pair* (X, φ) *where* X *is a set and a function* $\varphi : X \to P$*. For any* P*-graded* S*-module* M*, shifting the grading by* $v \in P$ *is defined as follows* $M(v)_w = M_{w-v}$*. A* P*-graded* S*-module* F *is called free module over* P*-graded set* (X, φ) *if* F *is isomorphic to direct sum of copies of* $S(v_i)$ *for* $v_i = \varphi(x_i)$ *where* $x_i \in X$*.* F *is called finitely generated, if it is finite direct sum of* $S(v_i)$*.*
A P*-graded* k*-vector space* V *is called free* k*-vector space with basis* P*-graded set* (X, φ) *if* V *is isomorphic to direct sum of copies of* $k(v_i)$ *for* $v_i = \varphi(x_i)$ *where* $x_i \in X$*.* V *is called finitely generated, if it is finite direct sum of* $k(v_i)$*.*

Example 2.0.3.

1. *Let* F *be free* $\mathbb{Z}^n$*-graded* S*-module over* $\mathbb{Z}^n$*-graded set* (X, φ)*, then homogeneous elements are* $\mathbb{H}(F) = \bigcup_{v \in \mathbb{Z}^n} (\bigoplus_{w \in V_v} kx^{v-w})$ *where* $V_v = \{w \in \varphi(V) \mid w \leq v\}$

2. *Let* F *be finitely generated free* $\mathbb{Z}^2$*-graded* $k[x, y]$*-module over* (X, φ) *where* $X = \{x_1, x_2, x_3, x_4\}$ *so that* $\varphi(x_1) = (0, 2)$*,* $\varphi(x_2) = (0, 2)$*,* $\varphi(x_3) = (1, 1)$ *and* $\varphi(x_4) = (3, 0)$*, illustrated in Fig2.2a. We will call this illustration block representation of free module* F*, and opacity of each block represents the number of direct sum component in that position. For example, at position* $(2, 3)$ *we have* $F_{(2,3)} = kx^2y \oplus kx^2y \oplus kxy^2$*, first two summand formed by the generators at* $(0, 2)$*, and the last one from* $(1, 1)$*.*

3. *Let* V *be finitely generated free* $\mathbb{Z}^2$*-graded* k*-vector space over the same set with the previous example, which is illustrated in Fig2.2b.*

Recall that a multiset is a set which contains an element multiple times. Any multiset can be represented by a pair (L, α) where L is the underlying set and $\alpha : L \to \mathbb{N}$ is

(a) Free $\mathbb{Z}^n$-graded S-module over (X, φ) (b) Free $\mathbb{Z}^n$-graded k-vector space over (X, φ)

Figure 2.2: Examples of free $\mathbb{Z}^2$-graded objects

the map α determines the multiplicity of an element in L.

Remark 2.0.1. *It is useful to think P-graded sets as multisets. For any P-graded set (X, φ), we have a multiset (P, α) where $\alpha(v)$ is the number of images of elements in X which maps to $v \in P$. Thus a P-graded set (X, φ) can be identified with the multiset $(P, \alpha) = \{(v, n) \mid n \leq \alpha(v)\}$.*

Definition 2.0.4. *(Rotman [36]) Let **A** and **B** be two categories where **A** is a small category. The functor category $\mathbf{B}^{\mathbf{A}}$ is a category whose objects are all covariant functors from **A** to **B** and its morphisms are all natural transformations.*

We need **A** to be a small category because the collection of natural transformation between any two functors is a set if and only if the category **A** is small. Also it is known that if **A** is a small category and **B** is an abelian category, then $\mathbf{B}^{\mathbf{A}}$ is an abelian category. The category **P** corresponding preordered set $(P, \leq)$ contains elements of P as objects and its morphisms are the inequalities $x \leq y$ where $x, y \in P$.

Definition 2.0.5. *(Persistence Module [9]) A persistence module is a functor $M : \mathbf{P} \to \mathbf{A}$. The category of persistence modules is the functor category $\mathbf{A}^{\mathbf{P}}$ where **P** is the category corresponding preordered set $(P, \leq)$. We will call $\varphi_M^{u,v} : M_u \to M_v$ as transition maps where M_u value of $u \in P$ under M and $u \leq v$. Transition map is induced by the relation in **P**. The morphism between two persistence modules is a natural transformations.*

Definition 2.0.6. *(P-filtered Space) A space X is called P-filtered, where P is a preordered abelian group, if we are given a family of subspaces $\{X_v \subset X\}_{v \in P}$ with* *inclusions $X_u \subset X_v$ whenever $u \leq v$, so that the diagrams*

$$\begin{array}{ccc} X_{v_1} & \longrightarrow & X_w \\ \uparrow & & \uparrow \\ X_u & \longrightarrow & X_{v_2} \end{array}$$

commute for $u \leq v_1$, $v_2 \leq w$.

Remark 2.0.2. *Homology of a P-filtered space is the persistence module in each dimension, and the corresponding algebraic structure is P-graded modules.*

Example 2.0.4. *$\mathbb{Z}^2$-filtered space given in Fig 2.3 which is given as an example in [13]. A cycle is named and represented by a circle, or dotted line, or colored region where they entered the filtration. For example, at $(0,1)$ there exists a new 1-cycle named ce which has a boundary $e - c$. If we apply 0-th homology group with coefficients in the field k to this bifiltration, we will have the following diagram.*

$$\begin{array}{cccc} k^2 \to & k^2 \to & k \to & k \\ \uparrow & \uparrow & \uparrow & \uparrow \\ k^3 \to & k^4 \to & k^3 \to & k^3 \\ \uparrow & \uparrow & \uparrow & \uparrow \\ k^4 \to & k^3 \to & k^2 \to & k^2 \end{array}$$

Observe that this diagram can be represented by a persistence module in ${}_k\mathbf{Mod}^{\mathbb{Z}^2}$ and transition maps are induced by the inclusions in the filtration.

The same thing can be done for 1-st homology groups with coefficients in the field k. We will have the following diagram for the first homology groups.

$$\begin{array}{cccc} 0 \to & k^2 \to & k^2 \to & k \\ \uparrow & \uparrow & \uparrow & \uparrow \\ 0 \to & k \to & k \to & 0 \\ \uparrow & \uparrow & \uparrow & \uparrow \\ 0 \to & 0 \to & 0 \to & 0 \end{array}$$

To see the relations by looking at the diagram we can represent this information by transforming to $\mathbb{Z}^2$-graded $k[x,y]$-module.

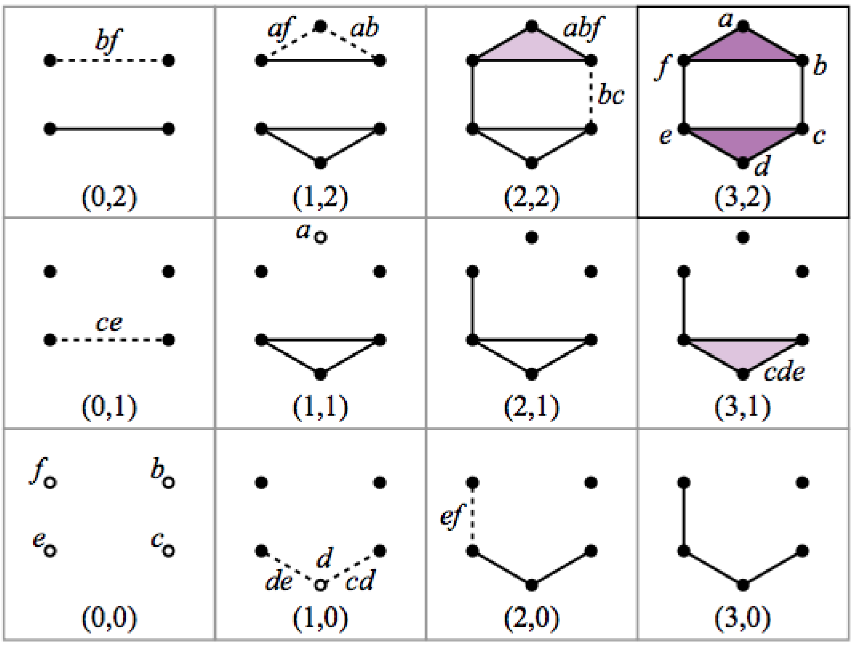


Figure 2.3: Bifiltration example in [13]

$$
\begin{array}{ccccccc}
0 & \to & ky \oplus k & \to & kxy \oplus k & \to & kx \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
0 & \to & k & \to & kx & \to & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
0 & \to & 0 & \to & 0 & \to & 0
\end{array}
$$

Observe that the 1-dimensional class born at $(1,1)$ and this class vanishes at $(3,1)$. Thus we have a component $S(1,1)/(x^2)$ which represents the life of this class. First homology groups induces persistence modules over ${}_k\mathbf{Mod}^{\mathbb{Z}^2}$ which is represented $S(1,1)/(x^2) \oplus S(1,2)/(x) \oplus S(2,2)$ as $\mathbb{Z}^2$-graded $k[x,y]$-module. We will explain this process in more detail later.

2.1 Minimal Free Resolution

In this section, we will use the definitions in [22]

Definition 2.1.1. *A free resolution of R-module M is an exact sequence*

$$\mathcal{F} : \cdots F_1 \to F_0 \to M \to 0$$

where F_i is free R-module for each i. It is called graded free resolution if R is graded ring and F_i is graded free modules. If F_i has minimal generating set for each i ,then this resolution called minimal free resolution.

Theorem 2.1.1. *(Hilbert Syzygy Theorem) If k is a field, then every finitely generated $\mathbb{N}^n$-graded S-module has a graded free resolution of length $\leq n$.*

The construction of a free resolution depends on the generating set for M. We will generate a free module F_0 with this set, which gives us a surjective map $F_0 \to M$. Kernel of this map M_1 is also finitely generated, called first syzygy of M. Now we will apply the same procedure to M_1, then we get a free module F_1. For the $\mathbb{N}^n$-graded S-module, Hilbert Syzygy Theorem implies that this process can happen only finitely many times. For any generating set, we will have different free resolutions. That is why a free resolution itself does not act like an invariant.

Theorem 2.1.2. *(Nakayama's Lemma [23]) Suppose M is finitely generated $\mathbb{N}^n$-graded S-module and $m_1, \cdots, m_n \in M$ generate $M/(x_1, \cdots, x_n)$. Then $m_1, \cdots, m_n$ generate M.*

This lemma helps us to find a minimal generating set for M. As a result, for finitely generated $\mathbb{N}^n$-graded S-module M, we can generate a free resolution formed by choosing minimal generating sets. This free resolution is called minimal free resolution.

Theorem 2.1.3. *[23] Let M be finitely generated $\mathbb{N}^n$-graded S-module. If $\mathcal{F}$ and $\mathcal{G}$ are minimal graded free resolutions of M, then there exists an isomorphism between these exact sequences. Any free resolution contains the minimal free resolution as a direct summand.*

As a consequence of this theorem, we can use the minimal free resolution as an invariant.

2.2 A Glimpse of Algebraic Geometry

In this part, we will use the book written by Robin Hartshorne, Algebraic Geometry [27].

Definition 2.2.1. *An affine space over a field k is the vector space k^n and denoted by $\mathbb{A}_k^n$ or just $\mathbb{A}^n$.*

Now we will define Zariski topology on $\mathbb{A}^n$, before that we will give some definitions to illustrate the open sets in this topology. For this section, we will denote $S_n = k[x_1, \cdots x_n]$ for $n \in \mathbb{N}$.

Definition 2.2.2. *Let $T \subset S_n$, then the zero set of T is defined as*

$$Z(T) = \{p \in \mathbb{A}^n \mid f(p) = 0 \ \ \forall f \in T\}$$

A subset Y of $\mathbb{A}^n$ is called algebraic set if there exists a subset T of S_n so that $Z(T) = Y$.

If I is an ideal of S_n generated by T, then $Z(T) = Z(I)$. By Hilbert Basis Theorem, we know that S_n is a Noetherian ring, so the ideal I is finitely generated. Thus we can represent $Z(T)$ with a finite set of polynomials. By taking open sets as the complement of algebraic sets, we will have topology on $\mathbb{A}^n$.

Observe that for any family of open sets U_α in Zariski topology, we will have an open set since $\bigcup_\alpha I_\alpha$ is an subset of S_n;

$$\bigcup_\alpha U_\alpha = \bigcup_\alpha \mathbb{A}^n \setminus Z(I_\alpha) = \mathbb{A}^n \setminus \bigcap_\alpha Z(I_\alpha) = \mathbb{A}^n \setminus Z(\bigcup_\alpha I_\alpha)$$

For finite intersection, again we have open set, $\Pi_{i=1}^m I_i$ that denotes product of m elements which come from the corresponding I_i;

$$\bigcap_{i=1}^m U_i = \bigcap_{i=1}^m \mathbb{A}^n \setminus Z(I_i) = \mathbb{A}^n \setminus \bigcup_{i=1}^m Z(I_i) = \mathbb{A}^n \setminus Z(\Pi_{i=1}^m I_i)$$

Definition 2.2.3. *An affine algebraic variety is an irreducible closed subset $\mathbb{A}^n$, with induced topology, where irreducible means it cannot be the union of the proper closed subset. An open subset of an affine variety is called quasi-affine variety.*

We can think of an affine algebraic variety as zeros of a collection of polynomials.

Definition 2.2.4. *A projective space over k is the set of equivalence classes of $(n+1)$-tuples of elements of k with equivalence relation $(a_0, \cdots, a_n) \sim (\lambda a_0, \cdots, \lambda a_n)$ for nonzero $\lambda \in k$, denoted by $\mathbb{P}_k^n$ or just $\mathbb{P}^n$. A polynomial $f \in S_{n+1}$ is called homogeneous of degree d if $f(\lambda x_0, \cdots, \lambda x_n) = \lambda^d f(x_0, \cdots, x_n)$ for any $\lambda \in k$.*

A polynomial in S_{n+1} does not define a function on $\mathbb{P}^n$ since polynomial must be the same on the equivalence class. That is why we need to focus on homogeneous polynomials to define a projective variety. Zariski topology on projective variety is defined in the same way.

Definition 2.2.5. *A projective algebraic variety is an irreducible algebraic set in $\mathbb{P}^n$, which means it cannot be union of its proper closed subsets. An open subset of a projective variety is called quasi-projective variety. A homogeneous ideal of $Y \subseteq \mathbb{P}^n$ in S_{n+1} is an ideal generated by homogeneous elements $f \in S_{n+1}$ such that $f(P) = 0$ for all $P \in Y$. The homogeneous coordinate ring of an algebraic set Y is defined as $S(Y) = S_{n+1}/I(Y)$.*

Observe that $S(Y)$ is an ideal in $\mathbb{Z}$-graded ring S_{n+1} where the grading is determined by the degree of monomials. Thus it can be seen as $\mathbb{Z}$-graded S_{n+1}-module. Using this relationship between modules and projective varieties, one can define invariants such as Hilbert polynomials and graded Betti numbers. We will explain these notions later.

The following definition will be needed to define algebraic groups.

Definition 2.2.6. *Let Y be quasi-affine variety. A function $f : Y \to k$ is regular at a point $a \in Y$ if there exists an open neighborhood U of a contained in Y so that $f = \dfrac{g}{h}$ on U where g, h are polynomials in $k[x_1, \cdots, x_n]$ and h is nonzero on U.*

If Y is a quasi-projective variety, then the polynomials $g, h \in k[x_0, \cdots, x_n]$ must be homogeneous of the same degree. For the following definition, a variety could be any affine, quasi-affine, projective or quasi-projective variety.

Definition 2.2.7. *The morphism between two varieties X and Y, $\varphi : X \to Y$, is a continuous map such that for every open set V and regular map f on this set, we have $f \circ \varphi$ is regular on $\varphi^{-1}(V)$. An algebraic group or group variety G is a variety together with morphisms $\mu : G \times G \to G$ and $\rho : G \to G$ such that the set of points of G with the operation given by μ is a group.*

Example 2.2.1. *$GL_n(k)$ is an algebraic group. It is the set of all invertible $n \times n$-matrices, so these matrices can be seen as an element of $\mathbb{A}^{n^2}$. We can identify the matrices with the relation $A = \lambda A'$ where $\lambda \in k$. Hence it forms a subvariety of $\mathbb{P}^{n^2-1}$.*

Determinant operation D is an polynomial over n^2-variable, and it is homogeneous polynomial since $D(\lambda A) = \lambda^n D(A)$. Observe that $GL_n(k) = \mathbb{P}^{n^2-1} \setminus Z(D)$, so it forms an quasi-projective variety since it is open. The function μ is defined as a composition of two functions, and ρ is an inverse map, both of which induce morphisms. Therefore $GL_n(k)$ is algebraic group.

Carlsson and Zomorodian noticed that $GL(V)$ is an algebraic group where V is a graded k-vector space, and this information helped to describe an invariant, which is an orbit of the action of $Aut(F)$ where F is graded free S-module. This can be done thanks to the relation between graded k-vector spaces and graded free S-modules. We will explain these relations in detail in Chapter 4.

It is shown that Grassmannian forms a quasi-projective variety in [25]. This result will be helpful to parametrize the invariant, which will be a subvariety of Grassmannian. This will be advantageous from the computational point of view because it will correspond in a quasi-projective variety where we can do computation easily compared to orbits.

CHAPTER 3

ONE DIMENSIONAL PERSISTENCE

The theory of one-dimensional persistence, which contains persistence modules over $\mathbb{N}$, is developed simultaneously by several people. We will focus on the ideas of Carlsson, Zomorodian in [37], and of Edelsbrunner, Letscher, Zomorodian in [21]. Like we mentioned before, for a given $\mathbb{N}$-filtered space, we can evaluate the homology of this filtered space which will give us persistence modules over $\mathbb{N}$.

3.1 Filtration Methods

There are several ways to convert a given data to a filtration. We will list some of them in this part. Mainly, there are two types of data: sublevel set of a real-valued function on a topological space and point cloud data.

Definition 3.1.1. *Let $f : \mathbb{X} \to \mathbb{R}$ be a real-valued function on a topological space $\mathbb{X}$, and $\mathbb{X}_r = f^{-1}(-\infty, r]$ be a preimage of f for $r \in \mathbb{R}$, then $\mathbb{X}_r$ is called sublevel sets.*

Observe that if $r \leq s$ then we have an inclusion $\mathbb{X}_r \subseteq \mathbb{X}_s$. The collection of sublevel sets $\{\mathbb{X}_r\}_{r\in\mathbb{R}}$ will form $\mathbb{R}$-filtered space. The p-th homology group with coefficients in the ring R of each sublevel set will form persistence module $M : \mathbb{R} \to_R$ **Mod** where $M_r = H_p(\mathbb{X}_r; R)$ and transition maps $\varphi_M^{r;s} : H_p(\mathbb{X}_r; R) \to H_p(\mathbb{X}_s; R)$ induced by inclusion for $r \leq s$. There are other ways to form filtration of real valued function, for example superlevel set $\mathbb{X}^r = f^{-1}[r, \infty)$ can be used.

From a computational point of view, we need to put some restrictions on f and homology groups of sublevel sets.

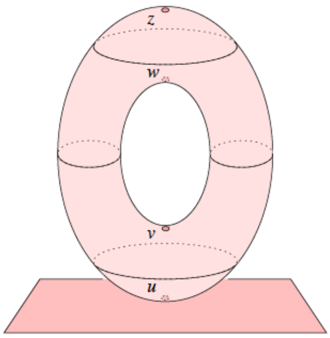


Figure 3.1: The vertical torus with points u, v, w, z in [20]

Definition 3.1.2. *A function $f : \mathbb{X} \to \mathbb{R}$ is called tame if it is continuous, $H_p(\mathbb{X}_r; R)$ has finite rank for all $p \in \mathbb{N}$ and $r \in \mathbb{R}$ and there exists only finitely many $r \in \mathbb{R}$ so that $H_p(\mathbb{X}_{r-\epsilon}; R) \to H_p(\mathbb{X}_{r+\epsilon}; R)$ is not an isomorphism for all $\epsilon > 0$.*

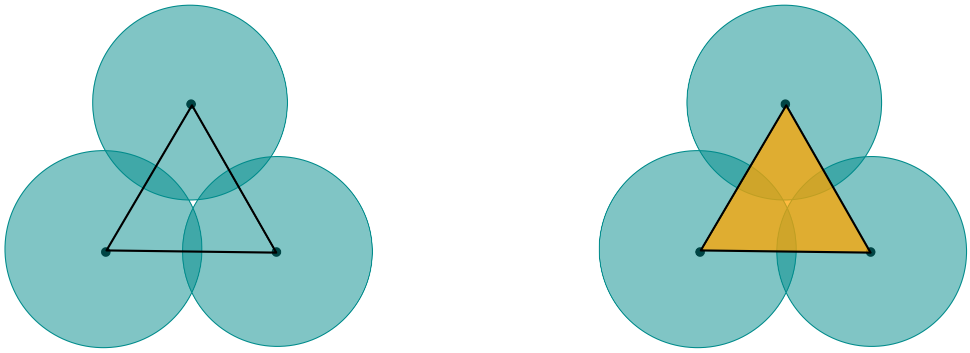
This guarantees that homology group of sublevel sets changes only finitely many places, which allow us to discretize the index of sublevel sets. Thus $\mathbb{R}$-filtered space become $\mathbb{N}$-filtered space, and homology of this filtration is persistence module in ${}_R\mathbf{Mod}^{\mathbb{N}}$.

Example 3.1.1. *Let $f : \mathbb{T}^2 \to \mathbb{R}$ be a height function on the vertical torus. Observe that we have different sublevel sets. For $a \in (f(u), f(v))$ the sublevel set $\mathbb{T}^2_a$ is a disk, for $(f(v), f(w))$ the sublevel set $\mathbb{T}^2_a$ is cylinder, and for $(f(w), f(z))$ the sublevel set $\mathbb{T}^2_a$ is capped torus. Therefore the height function f is tame since $H_p(\mathbb{T}^2_{a-\epsilon}; R) \to H_p(\mathbb{T}^2_{a+\epsilon}; R)$ is not isomorphism only for $a \in \{f(u), f(v), f(w), f(z)\}$*

Now, we will introduce filtration methods for point cloud data which can be defined as a finite set of points from a subspace of $\mathbb{R}^n$. The following concept is explained in detail in [12].

Definition 3.1.3. *Let X be topological space with a covering $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$. The nerve of $\mathcal{U}$, $\mathcal{N}(U)$ is an abstract simplicial complex with vertex set I where k different element in I forms a $k-1$-simplex if and only if the intersection of their corresponding cover sets is nonempty.*

The following theorem will allow us to evaluate the homology groups of a finite set of points.



(a) Čech Complex (b) Vietoris-Rips Complex

Figure 3.2: Difference between Čech and Vietoris-Rips Complexes

Theorem 3.1.1. *(Nerve Theorem) Let X be topological space with a covering $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$. If $\bigcap_{\alpha \in S} U_\alpha$ is contractible or empty for any $\emptyset \neq S \subseteq I$, then the nerve $\mathcal{N}(U)$ is homotopy equivalent to X.*

Using the fact that homotopy equivalent spaces have isomorphic homology groups, it is enough to find an open covering of X to compute homology groups. For the following definitions, assume that X is metric space.

Definition 3.1.4. *Let $\mathcal{B}_\epsilon(x) = \{\overline{B}_\epsilon(x)\}_{x \in X}$ be collection of closed balls with radius ϵ. Then the nerve of $\mathcal{B}_\epsilon(x)$ is called Čech complex, denoted by $\check{C}(X, \epsilon)$.*

Unfortunately, this complex is computationally expensive since we need to check all possible subsets of I, which has k elements, to see whether they form $(k-1)$-simplices or not. That is why one can use another complex construction in which the condition in the nerve definition of the k-many intersection can be dropped. To be precise, any k-many closed balls $B_\epsilon(x_i)$ will form $(k-1)$-simplex where $x_i \in X$ if and only if $d(x_i, x_j) \leq \epsilon$ for $1 \leq i, j \leq k$. This is called *Vietoris-Rips Complex* and denoted by $V(X, \epsilon)$. This would only check pairwise intersection. The following example will illustrate the difference between Čech and Vietoris-Rips complexes.

Example 3.1.2. *Let X be set of three points in $\mathbb{R}^2$. For some radius r, the pairwise intersection of balls around these three points is not empty, but the intersection of three balls is empty. In Čech complex, we have only three 1-simplexes which connect these three points like in Fig 3.2a. In the Vietoris-Rips complex, we have an additional 2-simplex in Fig 3.2b since the pairwise intersection of these balls is not empty.*

The next proposition will state that we do not lose any information by using Vietoris-Rips complex.

Proposition 3.1.1. *For any $\epsilon > 0$, we have inclusions*

$$\check{C}(X,\epsilon) \subseteq V(X,\epsilon) \subseteq \check{C}(X,2\epsilon)$$

Observe that Čech complex is a subcomplex of Vietoris-Rips complex, and any 2ϵ radius contains vertex whose distance is ϵ therefore, we have this inclusion.

The aim is to create a simplicial complex for a finite set of points in $\mathbb{R}^n$ i.e. point cloud data by taking standard metric in $\mathbb{R}^n$. There are other ways of constructing simplicial complexes, but we will not concentrate on others, see [12].

The collection of Čech or Vietoris-Rips complexes will form $\mathbb{R}$-filtered space. We can reduce this to $\mathbb{N}$-filtered space by setting an order preserving function $f : \mathbb{N} \to \mathbb{R}$. Thus homology groups with coefficients in the ring R of these complexes will form a persistence modules in ${}_R\mathbf{Mod}^{\mathbb{N}}$, where transition maps induced by the inclusion Č$(X,\epsilon_1) \subseteq$Č(X,ϵ_2) where $\epsilon_1 \leq \epsilon_2$ and $\epsilon_i \in f(\mathbb{N})$.

3.2 Visualization of Persistence

In the previous part, we examine the ways of obtaining filtration from data which gives us a persistence module in ${}_R\mathbf{Mod}^{\mathbb{N}}$. Persistence diagrams and barcodes are two ways of representing the information gained from the homology of filtration. The primary motivation is to keep track of the homology classes in the filtration and extract information about which shapes persist more. With this method, we will discover the inherent shape of the data. We will mainly use the definitions [20] in this part.

Definition 3.2.1. *For a given M persistence module in ${}_k\mathbf{Mod}^{\mathbb{N}}$ which formed by p-th homology group of a filtration, define $H^{i,j}(M)$ to be the image of the map from M_i to M_j for $i \leq j \in \mathbb{N}$ and persistence Betti number to be the rank of $H^{i,j}(M)$, denoted by $\beta_M^{i,j}$.*

We will express the birth and death time of p-dimensional homology classes in the filtration by using Betti numbers. If the persistence module M^p is the p-th homology

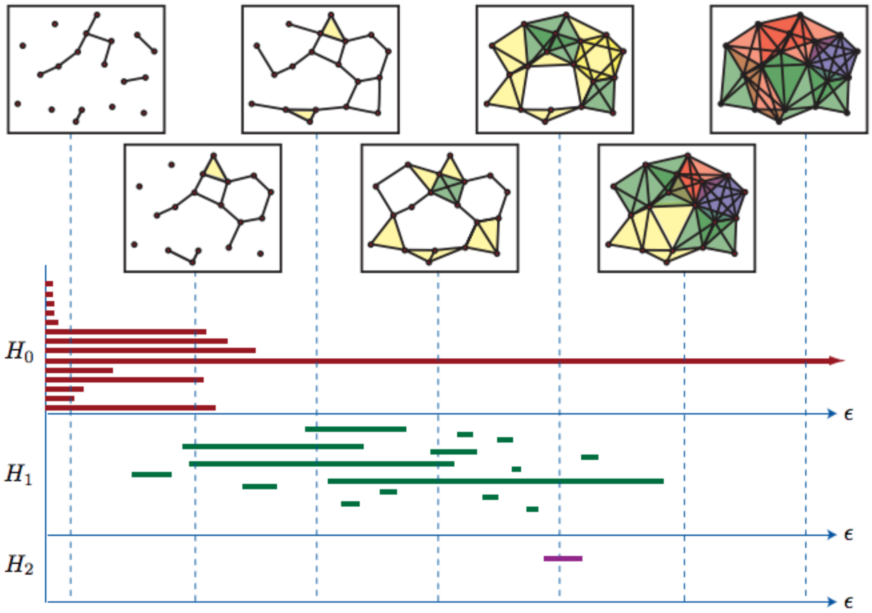


Figure 3.3: Barcodes of the filtration in [24]

of filtration, then $\beta^{i,j}_{M^p}$ is the number of p-dimensional classes which are in M^p_i and they still present in M^p_j.

Definition 3.2.2. *Let M be a persistence module in ${}_k\mathbf{Mod}^{\mathbb{N}}$ with Betti numbers $\beta^{i,j}_M$. Let $\mu^{i,j}_M$ be the number in $\mathbb{N}$ so that $\mu^{i,j}_M = (\beta^{i,j-1}_M - \beta^{i,j}_M) - (\beta^{i-1,j-1}_M - \beta^{i-1,j}_M)$. The persistence diagram of M is a collection of points (i,j) in the $\overline{\mathbb{R}}^2$ with multiplicity $\mu^{i,j}_M$, denoted by $Dgm(M)$. The barcode of M is the collection of intervals of the forms (i,j) where $\mu^{i,j}_M \neq 0$ where $i,j \in \overline{\mathbb{R}}$.*

The number $\mu^{i,j}_M$ counts the number of classes that are born at i and died entering j. We can interpret this information in the persistence diagram so that the larger the value of $j-i$, the longer the lifespan of a homology class. The same thing can be said about barcodes, too. There are some classes that are born and never die; to represent them, we will use extended $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$

Example 3.2.1. *1. Let the filtration of simplicial complex be given in the Fig 3.3. Filtration obtained by Vietoris-Rips complex from point cloud data. Yellow regions represents the 2-simplices and green ones is 3-simplices. Depending on the value of ϵ, we will get homology groups and the lifespan of the classes represented by barcodes.*

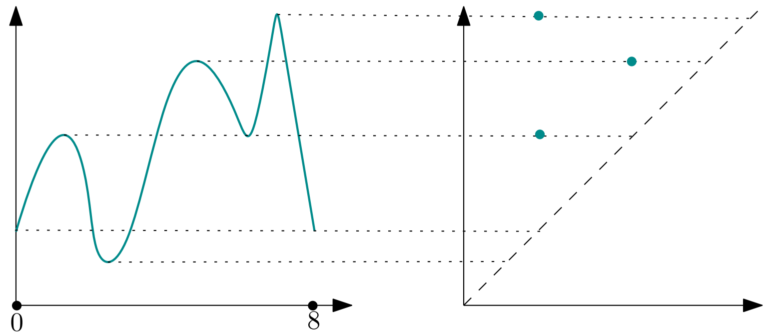


Figure 3.4: The function $f : [0, 8] \to \mathbb{R}$ with persistence diagram of 0-th homology groups

2. *Let f be a function from $[0, 8]$ to $\mathbb{R}$ which is illustrated in Fig3.4. On the left, we have persistence diagram of 0-th homology groups. We have one class which never dies.*

Remark 3.2.1. *In example 3.2.1, we have real parameter filtrations. In the first one, we can do the discretization since there are finitely many places where a new class is born, and these places are separable. In the second one, we have critical points which determine the classes are positioned as to be separable. Therefore, even though we have a persistence module in ${}_k\mathbf{Mod}^{\mathbb{R}}$, we can think of it as in ${}_k\mathbf{Mod}^{\mathbb{N}}$.*

Observe that these two methods are equivalent. In other words, they represent the same information. Therefore both can be used as an output. Also, these are complete invariants for the persistence modules, and we will explain this after examining the algebraic structure of persistence modules in ${}_k\mathbf{Mod}^{\mathbb{N}}$.

Representation of real parameter persistence modules in $\mathbf{A}^{\mathbb{R}}$ argued in [35] for small category $\mathbf{A}$ by Patel. Their main objects are constructible persistence modules which means there exist only finitely many different objects in the image of the persistence module. Let $\mathbf{Dgm}$ be the set of intervals in $\mathbb{R}$ which contains only (a, b) and (a, ∞) and containment relation between intervals induces a partial ordering.

Definition 3.2.3. *Let S be any finite set in $\mathbb{R}$ and G be an abelian group. A map $X : \mathbf{Dgm} \to G$ is called constructible if for $\overline{J} \cap S = \overline{I} \cap S$ where $J \subset I$ and $\overline{J}$ is the closure of J, we have $X(I) = X(J)$. A map $Y : \mathbf{Dgm} \to G$ is called finite if*

$X(I) \neq e_G$ for only intervals which has end points in S. A persistence diagram is a finite map $Y : \mathbf{Dgm} \to G$.

Observe that constructible map means that there is no change in the number of classes between $\overline{I} \cap S$. This allows us to reduce the endpoints in $\mathbb{R}$ to S. The following theorem guarantees that there exists a finite map that only takes nonidentity values on the intervals formed by points in S.

Theorem 3.2.1. *(Möbius Inversion Formula [35]) For any constructible map $X : \mathbf{Dgm} \to G$, there exists a finite map $Y : \mathbf{Dgm} \to G$ which satisfies the Möbius Inversion Formula $X(I) = \sum_{J \subseteq I} Y(J)$ for any $I, J \in \mathbf{Dgm}$.*

Example 3.2.2. *Let f be function in Fig3.4. Then the life of 0 dimensional classes are represented by following intervals: $(1, \infty)$, $(2, 4)$, $(2, 8)$ and $(4, 7)$. Let $S = \{1, 2, 4, 7, 8\}$ and $X : \mathbf{Dgm} \to \mathbb{N}$ so that $X(I)$ gives the number of intervals in barcodes intersects with I. This will be defined as rank invariant later. The map X is constructible with respect to S since any interval containing elements greater than any elements in S will have the same image under X. Thus, by Theorem 3.2.1, there exists a finite map Y satisfies the Möbius inversion formula by Theorem 3.2.1. This map Y is the persistence diagram for this filtration.*

Let $Y : \mathbf{Dgm} \to \mathbb{N}$ so that $Y((1, \infty)) = 4$, $Y((4, 7)) = Y((2, 4)) = 1$, $Y((2, 8)) = 3$ and it takes 0 on any other $I \in \mathbf{Dgm}$. The difference between rank function and persistence diagram is that rank function only counts the number of classes born at a and dying at b where persistence diagram gives the number of classes born and died within the interval (a, b).

The notion of a diagram is argued for a persistence diagram for $\mathbf{A}^{\mathbf{P}}$ in [29] by Kim and Memoli.

3.3 Decompositions of Persistence Modules

Even though we reached up a complete invariant for persistence modules in ${}_k\mathbf{Mod}^{\mathbb{N}}$, barcode and persistence diagrams need compatible basis for M_i and M_j, which are indeed homology groups. In this section, we will examine the decomposition of persistence modules. This would give another calculation method for barcodes and persistence diagrams. We will follow [37] for this part. Throughout this chapter, R is a commutative ring with identity and Noetherian.

Definition 3.3.1. *Let M be persistence module in ${}_R\textbf{\textit{Mod}}^P$ where P is preordered set. If M_i is finitely generated for all $i \in P$ and transition maps are not isomorphism for only finitely many $i, j \in P$, then M is called finite type.*

Theorem 3.3.1. *(Correspondence)[37] Let M be a finite type persistence module in ${}_R\textbf{\textit{Mod}}^{\mathbb{N}}$, and α be a functor from ${}_R\textbf{\textit{Mod}}^{\mathbb{N}}$ to $Gr^{\mathbb{N}}\text{-}_{R[t]}\textbf{\textit{Mod}}$ so that $\alpha(M) = \bigoplus_{i\in\mathbb{N}} M_i$ where $t \cdot \{m_i\}_{i\in\mathbb{N}} = \{\varphi^{i,i+1}(m_i)\}_{i+1\in\mathbb{N}}$. This α defines an equivalence between the categories of finite type of persistence module and finitely generated $\mathbb{N}$-graded $R[x]$-modules.*

An explicit proof of this theorem was not given in [37], and they refer to Artin–Rees theory in [22]. Lesnick and Corbet in [18] gave explicit proof of this theorem, and they observed that this statement is true for the general case, and we will examine it in the multidimensional case.

This correspondence is very useful when R is a field because of the decomposition of the $\mathbb{N}$-grade $k[x]$-modules where k is a field. Every $\mathbb{N}$-graded $k[x]$-module has a unique decomposition of the form

$$\bigoplus_{i=1}^{n} k[x](\alpha_i) \oplus \bigoplus_{j=1}^{m} k[x](\gamma_j)/(x_j)$$

where x_j are homogeneous elements so that $x_j|x_{j+1}, \alpha_i, \gamma_j \in \mathbb{N}$ and $k[x](\alpha)$ is a shifting in the grading by α. Free component of this decomposition represents the classes which never dies and the torsion part for classes with mortal life. The information about intervals in barcodes can be easily extracted from this decompositon, for example $k[x](\gamma_j)/(x_j)$ will represent the interval $(\gamma_j, \gamma_j + deg(x_j))$. So we do not need a compatible basis to optain barcodes or persistence diagram. Also the uniqueness of the corresponding $\mathbb{N}$-graded $k[x]$-module will give us to completeness of the barcode invariant.

Similar results for the partially ordered set are also true. The definition of interval in any partially ordered set can be stated as I is an interval in P if and only if $a, b \in I$ and $a \leq c \leq b$ implies $c \in I$.

Definition 3.3.2. *The functor $k_I : P \rightarrow_k Mod$, which is defined as $k_I(s) = k$ for all $s \in I$ otherwise it is zero, is called interval persistence module where I is an interval in preordered set $\mathbf{P}$.*

Decomposition into interval modules was stated for partially ordered set which contains countable dense subset in order topology by Crawley-Boevey in [19]. The containment condition on preordered set is dropped and replaced with a totally ordered set to have interval decomposition in [5] by Botnan and Crawley-Boevey.

Definition 3.3.3. *A persistence module in ${}_k\mathbf{Mod}^{\mathbf{P}}$ is called pointwise finite-dimensional if each M_u is finite dimensional for $u \in P$.*

Theorem 3.3.2. *[5] Let P be a totally ordered set. Then any pointwise finite dimensional persistence module in ${}_kMod^P$ decomposes into interval modules.*

These interval modules in the decomposition of the pointwise finite-dimensional persistence module correspond to intervals in the barcodes when $P = \mathbb{R}$.

A similar result for non totally ordered set and bifiltration is given in the same paper. For bifiltration, there exists block decomposition instead of intervals. To explain this result, we will use the definitions in [5]. Remember that we used the betti numbers in the filtration to track classes through the filtration. After that, we represent a life of a class with an interval. In bifiltration, we want to represent the life of classes with blocks. The following definition will help us in this way.

Definition 3.3.4. *Let M be a persistence module in ${}_k\mathbf{Mod}^{\mathbf{P}}$ where $P = S \times T$ is a product of two totally ordered set. M is called middle exact if $Im(\varphi^{a,b}, \varphi^{a,c}) = Ker(\varphi^{b,d}, -\varphi^{c,d})$ where $a = (x, y), b = (x, y'), c = (x', y), d = (x', y')$ with adjacent $x \leq x' \in S$ and $y \leq y' \in T$. In other words, following sequence is exact for all a, b, c, d of the form described above*

$$M_a \xrightarrow{(\varphi^{a,b}, \varphi^{a,c})} M_b \oplus M_c \xrightarrow{(\varphi^{b,d} - \varphi^{c,d})} M_d$$

Observe that exactness of this sequence guarantees that we have a preimage of a class in M_d on both sides, which are M_b and M_c, and the preimages of these will have common in M_a. Therefore a class lives across this rectangle. In the following definition, we will explain what is called a block.

Definition 3.3.5. *A nonempty subset I of $P = S \times T$ where S and T are totally ordered sets is called block if it is one of the following form;*

1. *$I = J_S \times J_T$ where J_S and J_T interval ideals in S and T*
2. *$I = J_S \times J_T$ where J_S and J_T interval filters in S and T*
3. *$I = J_S \times T$ where J_S interval in S*
4. *$I = S \times J_T$ where J_T interval in T*

where ideal inverval means $q \in I$ with $p \geq q$ implies $p \in I$ and filter interval means $q \in I$ with $p \leq q$ implies $p \in I$.

Similar to interval persistence modules, we have block persistence modules which is a functor from P to ${}_k\mathbf{Mod}$ so that it has a nonzero support on some block I.

Remark 3.3.1. *In any block persistence module, middle exactness condition will be satisfied. For an example, for $I = J_S \times J_T$ where J_S and J_T interval ideals, if all four positions are in the block then from injectivity of transition maps, this diagram will be middle exact. Another possibility is that only two of the position will be in the block. This will look like*

$$\begin{tikzcd} 0 \arrow[r] & k \\ 0 \arrow[u] \arrow[r] & k \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} k \arrow[r] & k \\ 0 \arrow[u] \arrow[r] & 0 \arrow[u] \end{tikzcd}$$

In either case, these diagrams will be middle exact since the transition map is one-to-one. There left only one case, which is k in the right-upper corner and other positions are 0. This would be the middle exact, too.

Theorem 3.3.3. *[5] Any pointwise finite-dimensional middle exact persistence module in ${}_k\mathbf{Mod}^{\mathbf{P}}$ has block decomposition where P is a product of two totally ordered sets.*

This exactness condition is loosened by Botnan, Lebovici, and Oudot in [6], and their decomposability result contains a different class of two-dimensional persistence module.

Definition 3.3.6. *A persistence module M in ${}_k\mathbf{Mod}^{\mathbb{R}^2}$ is called weak exact if the following inequalities hold for all $a \leq d \in \mathbb{R}^2$*

$$Im\varphi^{a,d} = Im\varphi^{b,d} \cap Im\varphi^{c,d}$$

$$Ker\varphi^{a,d} = Ker\varphi^{a,b} + Ker\varphi^{a,c}$$

where $a = (x, y), b = (x, y'), c = (x', y), d = (x', y')$ with $x \leq x' \in \mathbb{R}$ and $y \leq y' \in \mathbb{R}$

Rectangle, product of two interval, modules are defined as expected to be a functor from $\mathbb{R}^2$ to ${}_k\mathbf{Mod}$ so that $k_R(s) = k$ for all $s \in R$ and zero otherwise.

Remark 3.3.2. *Every middle exact persistence module in ${}_k\mathbf{Mod}^{\mathbb{R}^2}$ with finite support is weak exact. If persistence module M is middle exact then $Im(\varphi^{a,b}, \varphi^{a,c}) = Ker(\varphi^{b,d} - \varphi^{c,d})$. Let $d \in Im\varphi^{b,d} \cap Im\varphi^{c,d}$ then there exists $b \in M_b$ and $c \in M_c$ so that $\varphi^{b,d}(b) = d = \varphi^{c,d}(c)$. Thus $(b, c) \in Ker(\varphi^{b,d} - \varphi^{c,d}) = Im(\varphi^{a,b}, \varphi^{a,c})$ which means there exists $a \in M_a$ so that $b = \varphi^{a,b}(a)$ and $c = \varphi^{a,c}(a)$. So*

$$\varphi^{b,d}(b) = d = \varphi^{c,d}(c)$$

$$\varphi^{b,d}(\varphi^{a,b}(a)) = d = \varphi^{c,d}(\varphi^{a,c}(a))$$

$$\varphi^{a,d}(a) = d$$

Therefore, $Im\varphi^{b,d} \cap Im\varphi^{c,d} \subseteq Im\varphi^{a,d}$, otherside is easy to observe since we have $\varphi^{a,b}\varphi^{b,d} = \varphi^{a,d}$ so we have $Im\varphi^{a,d} = Im\varphi^{b,d} \cap Im\varphi^{c,d}$.

Similarly, we have $Ker\varphi^{a,d} = Ker\varphi^{a,b} + Ker\varphi^{a,c}$.

But the converse is not true. For example, rectangle persistence modules need not be middle exact, but they are weak exact. The following diagram can be seen in rectangle persistence modules but it can not be a part of block persistence module.

$$\begin{tikzcd} k \arrow[r] & 0 \\ 0 \arrow[u] \arrow[r] & 0 \arrow[u] \end{tikzcd}$$

This is not middle exact but it is weak exact.

Theorem 3.3.4. *[6] Any pointwise finite dimensional weak exact persistence module M over $X \times Y$,where X and Y are finite subset of $\mathbb{R}$, has rectangle decomposition. In other words, there exists unique multiset $\mathcal{R}$ of rectangles in $X \times Y$ so that*

$$M \simeq \bigoplus_{R \in \mathcal{R}} k_R$$

We can restate this theorem for bounded ${}_k\mathbf{Mod}^{\mathbb{Z}^2}$, and directly say that these modules have rectangle decomposition. The method to count the multiplicity of rectangles in the decomposition is very similar to the one in the barcodes. Recall the multiplicity in barcodes

$$\mu_M^{i,j} = (\beta_M^{i,j-1} - \beta_M^{i,j}) - (\beta_M^{i-1,j-1} - \beta_M^{i-1,j})$$

First, we will find the number of intervals which contains s as left corner and contains t where $s \leq t \in \mathbb{Z}^2$. Let $\beta^{s,t}$ be the number of classes both in M_s and M_t. The number of classes $\mu^{s,t+}$ which born at s and still alive at t is counted as

$$\mu^{s,t+} = \beta^{s,t} - \beta^{s-(1,0),t} - \beta^{s-(0,1),t} + \beta^{s-(1,1),t}$$

This is a simple inclusion-exclusion counting for these classes. To make s as a left corner, we need to make sure that $s-(0,1)$ and $s-(1,0)$ are not in the rectangle that we are counting. By doing these subtraction, we also remove the intersection twice. Thus by adding $\beta^{s-(1,1),t}$, we reinserted the intersection because a rectangle contains both $s-(0,1)$ and $s-(1,0)$ if and only if it contains $s-(1,1)$.

The next step is fixing the right-upper corner at t. Again by similar method, we will have the number of classes $\mu^{s,t}$ which born at s and died at t as

$$\mu^{s,t} = \mu^{s,t+} - \mu^{s,t+(1,0)+} - \mu^{s,t+(0,1)+} + \mu^{s,t+(1,1)+}$$

Thus, we have a barcode-like representation of ${}_k\mathbf{Mod}^{\mathbb{Z}^2}$ formed by rectangles in the decomposition with multiplicity.

For an arbitrary preordered set, there exists a decomposition into indecomposable components. However, these components will not be specified like intervals, blocks, or rectangles. An endomorphism ring $End(M)$ is local if for every $\theta \in End(M)$, θ or $id_M - \theta$ is invertible.

Theorem 3.3.5. *([5]) Any pointwise finite dimensional persistence module M in ${}_k\mathbf{Mod}^{\mathbf{P}}$ where P is arbitrary preordered set is a direct sum of indecomposable modules M_α which have local endomorphism rings.*

Botnan and Crawley-Boevey stated that this theorem is known in the theory of locally finitely presented additive categories. The shape of indecomposable components is complicated to work with. In the next chapter, we will discuss the results about finite type persistence modules in ${}_k\mathbf{Mod}^{\mathbb{R}^n}$. Finite type persistence modules in ${}_k\mathbf{Mod}^{\mathbb{R}^n}$ have finite support, which allows us to work with $\mathbb{Z}^n$-graded modules.

CHAPTER 4

MULTIDIMENSIONAL PERSISTENCE MODULE

The idea of multi-scale analysis of dataset is introduced in the paper [14] by Carlsson and Zomorodian. Their primary purpose is to give a concrete model to a multiparameter structure by using multi filtration. Their previous work gave foundations for the persistence homology of single parameter filtration, and the completeness of barcodes as an invariant was shown in [37]. Unfortunately, they showed that this is not the case for the multiparameter structures.

4.1 Corresponding Algebraic Structure for Persistence Modules

Relationship between persistence modules and graded structures first stated by Carlsson and Zomorodian in [14]. They were focused on the specific case when persistence module $M : \mathbb{N}^n \to_k$ **Mod** where $\mathbb{N}^n$ is category which corresponds to $(\mathbb{N}^n, \leq)$ and $_k$**Mod** is the category of k-modules. Later, a more general statement that includes an arbitrary preordered set as grading is given by Bubenik and Milicevic in [10]. First, that specific case will be explained, then continued with the general case.

Definition 4.1.1. *(Structure) Given a persistence module $M : \mathbb{N}^n \to_k$ **Mod**, we define an $\mathbb{N}^n$-graded module over S by*

$$\alpha(M) = \bigoplus_{v \in \mathbb{N}^n} M_v$$

where M_v is the image of v under the functor M. The k-module structure form by direct sum , and S-module structure is defined by the action $\boldsymbol{x}^u(m_v)_{v\in\mathbb{N}^n} := (\varphi_M^{v,u+v}(m_v))_{u+v\in\mathbb{N}^n}$ where $m_v \in M_v$ for $u, v \in \mathbb{N}^n$

We will call a persistence module finite if M_v is finitely generated for all $v \in \mathbb{N}^n$ and the category of this persistence modules will be denoted by ${}_k\mathbf{Mod}_F^{\mathbf{N}^n}$

Theorem 4.1.1. *(Correspondence)[14] The correspondence α defines an equivalence of categories between the category of finite persistence modules ${}_k\boldsymbol{Mod}_F^{\mathbf{N}^n}$ and the category of finitely generated $\mathbb{N}^n$-graded modules over S $Gr^{\mathbb{N}^n}$-${}_S\boldsymbol{Mod}$.*

Like we mentioned before, the proof of this statement is given in [18]. This correspondence theorem helps us capture the information in the finite multifiltered complex by thinking of finite persistence modules as finitely generated $\mathbb{N}^n$-graded modules. To do the converse, we need the realization theorem, which is stated in [14] without proof. Harrington, Otter, Schenck, and Tillman in [26] provided proof for the realization theorem. The idea here is similar to Lesnick's proof of Proposition 5.8 in [31].

Theorem 4.1.2. *(Realization) [14] If $k = \mathbb{F}_p$ for some prime p, $\mathbb{Q}$ or $\mathbb{N}$ then any finitely generated $\mathbb{N}^n$-graded module over S can be realized as the homology in degree i, for any $i > 0$ of a finite $\mathbb{N}^n$-filtered simplicial complex.*

According to Harrington, Otter, Schenck, and Tillman [26], this theorem is true for any k if we put some restriction on $\mathbb{N}^n$-graded S-module. The notion of tameness was first defined in [16] by Chazal et al.

Definition 4.1.2. *(Tameness) An $\mathbb{N}^n$-graded module over S, $M = \bigoplus_{v\in\mathbb{N}^n} M_v$ is tame if $dim M_v < \infty$ for all $v \in \mathbb{N}^n$*

Theorem 4.1.3. *[26] If M is $\mathbb{N}^n$-graded tame S-module then it can be realized as the homology in degree i, for any $i > 0$ of a finite $\mathbb{N}^n$-filtered simplicial complex.*

We will state a more general statement about the persistence module over the arbitrary preordered abelian group, given by Bubenik and Milicevic.

Theorem 4.1.4. *[10, Theorem 2.21] Let $(P, \leq, +, 0)$ be preordered set with compatible abelian group structure. Let R be commutative unitary ring. Then we have isomorphism of categories*

$$ {}_R\boldsymbol{Mod}^{\boldsymbol{P}} \simeq Gr^P\text{-}_{R[U_0]}\boldsymbol{Mod} $$

where $R[U_0]$ is the ring of polynomials whose coefficients in the monoid U_0 formed by elements greater than 0.

Proof. Similarly to the case $P = \mathbb{N}^n$, we will build the functor $\alpha :_R \mathbf{Mod}^{\mathbf{P}} \to$ Gr^P-$_{R[U_0]}$**Mod** for the correspondence. Let $M : \mathbf{P} \to \mathbf{A}$ then $\alpha(M) = \bigoplus_{a \in P} M_a$ where $M_a = M(a)$. Again R-structure given by direct sum structure and $R[U_0]$-module structure given by the action $\mathbf{x}^s(m_a)_{a\in P} = (\varphi_M^{a,a+s}(m_a))_{a+s\in P}$. □

Observe that the category $Gr^{\mathbf{Z}^n} -_S \mathbf{Mod}$ contains $Gr^{\mathbf{N}^n} -_S \mathbf{Mod}$ since we can see any $\mathbb{N}^n$-graded S-module as $\mathbb{Z}^n$-graded S-module by taking $M(v) = 0$ for $v \in \mathbb{Z}^n \setminus \mathbb{N}^n$. Therefore, Carlsson and Zomorodian's correspondence theorem is the restriction of this theorem to the $\mathbb{Z}^n$-graded S-modules.

4.2 Classification of Finitely Generated $\mathbb{N}^n$-graded S-modules

In this part, we will focus on the specific case examined by Carlsson and Zomorodian. So we will fix persistence module as $M : \mathbb{N}^n \to_k \mathbf{Mod}$.

Thanks to the correspondence theorem, it is enough to focus on the classification of finitely generated $\mathbb{N}^n$-graded S-modules up to isomorphism. Generally, a free resolution of a module is not invariant, but a minimal free resolution of a module is unique up to isomorphism. By using $Tor_0^S(M,k)$ and $Tor_1^S(M,k)$, Carlsson and Zomorodian showed that there is no neat classification of $\mathbb{N}^n$-graded S-modules.

Observe that $\mathbb{N}^n$-graded objects are $\mathbb{Z}^n$-graded objects with grading bounded below. So we will continue with bounded below $\mathbb{Z}^n$-graded objects, but we will just write $\mathbb{Z}^n$-graded in this section for the bounded below $\mathbb{Z}^n$-graded.

For any P-graded S-module M, we can relate it with a P-graded set. The pair $(\mathbb{H}(M), \varphi)$ where $\mathbb{H}(M) = \bigcup_{v\in P} M_v$ the homogeneous elements in $M = \bigoplus_{v \in P} M_v$ and $\varphi : m \mapsto v$ where $m \in M_v$, is a P-graded set. Now we will use this pairing to state the following theorem which is stated in [14] as a definition.

Proposition 4.2.1. *If F is free $\mathbb{Z}^n$-graded S-module over (X, φ) with inclusion of $\mathbb{Z}^n$-graded sets $\eta : (X, \varphi) \hookrightarrow \mathbb{H}(F) \subset F$, then for any $\mathbb{Z}^n$-graded S-module M and a map of $\mathbb{Z}^n$-graded sets $\theta : (X, \varphi) \to \mathbb{H}(M)$, there exists a unique homomorphism $\lambda : F \to M$ of $\mathbb{Z}^n$-graded S-modules so that following diagram commutes ;*

$$\begin{tikzcd}
(X,\varphi) \arrow[r, "\eta"] \arrow[dr, "\theta"'] & \mathbb{H}(F) \arrow[d, "\mathbb{H}(\lambda)"] \\
& \mathbb{H}(M)
\end{tikzcd}$$

where $\mathbb{H}(\lambda)$ is the restriction of λ to the homogeneous elements.

Proof. Let F be free $\mathbb{Z}^n$-graded S-module over (X,φ), then $F = \bigoplus_{x_i \in X} S(\varphi(x_i)) = \bigoplus_{v_i \in \varphi(X)} S(v_i) = \bigoplus_{v \in \mathbb{Z}^n} F_v$. Let M be a $\mathbb{Z}^n$-graded S-module in the hypothesis. Since η and θ are given, we have $\eta(x_i) = f^i_{v_i} \in F_{v_i}$ and $\theta(x_i) = m^i_{v_i} \in M_{v_i}$ where $\varphi(x_i) = v_i$. Since these are maps between multisets then they respect the grading i.e. η is family of functions $\{\eta_{x_i} : x_i \to F_{\varphi(x_i)}\}_{x_i \in X}$.

It is enough to define $\lambda : F \to M$ on the positions of (X,φ), call $\lambda(f^i_{v_i}) = m^i_{v_i}$. The image of the homogeneous elements which are not in the image of η determined uniquely by the definition of free $\mathbb{Z}^n$-graded S-module. If $f \in F_v$ where $v \notin \varphi(X)$ then $f = \sum_{v_i < v} a_{v_i} x^{v-v_i} f^i_{v_i}$ where $a_{v_i} \in k$. So

$$\lambda(f) = \sum_{v_i<v} a_{v_i} x^{v-v_i} \lambda(f^i_{v_i}) = \sum_{v_i<v} a_{v_i} x^{v-v_i} m^i_{v_i}$$

This map λ is a homomorphism of $\mathbb{Z}^n$-graded S-module since it commutes with transition maps. Let $v < w \in \mathbb{Z}^n$

$$\begin{aligned}
\varphi_M^{v,w}(\lambda(f)) &= \varphi^{v,w}(\sum_{v_i<v} a_{v_i} x^{v-v_i} m^i_{v_i}) \\
&= \sum_{v_i<v} a_{v_i} x^{w-v} x^{v-v_i} m^i_{v_i} \\
&= \lambda(\sum_{v_i<v} a_{v_i} x^{w-v} x^{v-v_i} f^i_{v_i}) \\
&= \lambda(\varphi_F^{v,w}(f))
\end{aligned}$$

Clearly, restriction map on the graded sets commutes with the diagram.

Let λ' be another map so that $\mathbb{H}(\lambda') \circ \eta = \theta$, then λ' and λ must agree on the elements which have the grading in $\varphi(X)$. Since other elements can be represented by using the image of η, λ and λ' must be same on the all homogeneous elements. $\square$

Let F be free $\mathbb{Z}^n$-graded S-module over (X, φ) and F' be free $\mathbb{Z}^n$-graded S-module over (X', φ'). If F is isomorphic to F' then (X, φ) is isomorphic to (X', φ') as $\mathbb{Z}^n$-graded sets. By this fact, we can define *type* of an $\mathbb{Z}^n$-graded free S-module F, denote it by $\xi(F) = (X, \varphi)$. Also we can do the same thing for $\mathbb{Z}^n$-graded vector space V by using the graded basis. It is again a $\mathbb{Z}^n$-graded set determined uniquely.

After this point, we will think graded sets as a multiset, which is explained how we made this transformation in remark 2.0.1. So by a multiset $\xi = \{(v_i, n_i)\}$ we will mean $v_i \in \mathbb{Z}^n$ and n_i is the number of elements in the generator set which maps to v_i. For the same multiset ξ, $\mathbb{Z}^n$-graded free S-module over ξ and free $\mathbb{Z}^n$-graded k-vector space is canonically isomorphic. However, for a $\mathbb{Z}^n$-graded S-module, this correspondence in vector space needs some effort to represent.

Definition 4.2.1. *Let n be a positive integer and P be a preordered set. A P-filtered k-vector space V is a k-vector space with a family $\mathcal{F}$ of subspaces $\mathcal{F}_w V$ for every $w \in P$, so that :*

1. *If $w \leq w'$, then $\mathcal{F}_w V \subseteq \mathcal{F}_{w'} V$.*
2. *There is $w_0 \in P$ such that $\mathcal{F}_{w_0} V = V$.*
3. *There is $w_1 \in P$ such that $\mathcal{F}_{w_1} V = \{0\}$.*

A morphism φ between P-filtered k-vector spaces $(V, \mathcal{F})$ and $(V', \mathcal{F}')$ is a homomorphism between underlying vector spaces which respects grading i.e. $\varphi(\mathcal{F}_w V) \subseteq \mathcal{F}'_w V'$ for all $w \in P$.

The category of P-filtered k-vector space is denoted by $\mathbf{Fil}^P({}_k\mathbf{Mod})$. If k-vector space V is finite dimensional, then the P-filtered k-vector space over V is called finite dimensional.

We can identify P-graded k-vector space V with an object in $\mathbf{Fil}^P({}_k\mathbf{Mod})$, and we will call this identification the associated P-filtered k-vector space. In particular, for finite dimensional P-graded k-vector space V, $(V, \mathcal{F}V)$ will denote the associated P-filtered k-vector space and for every $w \in P$, $\mathcal{F}_w V = \bigoplus_{w' \leq w} V_{w'}$. So morphism of P-graded k-vector space can be seen as a morphism of associated P-filtered k-vector space.

$$\begin{array}{ccccc} k^2 & k^3 & k^3 & k^4 & k^4 \\ k^2 & k^3 & k^3 & k^4 & k^4 \\ k^2 & k^3 & k^3 & k^4 & k^4 \\ \bullet & k & k & k^2 & k^2 \\ \bullet & \bullet & \bullet & k & k \end{array}$$

(a) $\mathbb{Z}^2$-filtered k-vector space k^4

$$\begin{array}{ccccc} \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ k^2 & \bullet & \bullet & \bullet & \bullet \\ \bullet & k & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & k & \bullet \end{array}$$

(b) $\mathbb{Z}^2$-graded k-vector space V'

Figure 4.1: Examples about correspondence between filtered and graded objects

We can present this correspondence as

$$\begin{tikzcd} k\text{-vector space } V \arrow[dr, "(\mathrm{V},\mathcal{F})"] & \\ & P\text{-filtered } k\text{-vector space} \\ \text{finite dimensional } P\text{-graded } k\text{-vector space} \arrow[ur, "\mathcal{F}V"] & \end{tikzcd}$$

Example 4.2.1. *Let $V = k^4$ and $P = \mathbb{Z}^2$ with the order relation. The family of subspaces defined as $\mathbb{F}_{(2,0)} = k^2$, $\mathbb{F}_{(1,1)} = k$ and $\mathbb{F}_{(3,0)} = k$. The rest of the family is defined to conform the first condition, which is illustrated Fig4.1a. Observe that $\mathbb{F}_{(0,0)} = \{0\}$ and $\mathbb{F}_{(3,2)} = k^4$.*

Let V' be $\mathbb{Z}^2$-graded k-vector space over $\{((0,2),2),((1,1),1),((3,0),1)\}$, Fig 4.1b. The associated $\mathbb{Z}^2$-filtered k-vector space will corresponds to previous $\mathbb{Z}^2$-filtered k-vector space V.

Remark 4.2.1. *For finitely generated $\mathbb{Z}^n$-graded k-vector space V, we can create a $\mathbb{Z}^n$-filtered k-vector space which is same with the associated filtered $\mathbb{Z}^n$-filtered k-vector space for V. Since V is finitely generated, we have a maximal dimensional component which will be k-vector space. Thus, with this k-vector space, the $\mathbb{Z}^n$-filtered k-vector space can be formed.*

Next, we will examine automorphism groups of these objects because the algebraic variety structure induced by this correspondence will help us to state the third invariant for complete classification which is an orbit of action of automorphism on set of subspaces.

Theorem 4.2.1. *[14] Let ξ be multiset and V be an $\mathbb{Z}^n$-graded vector space over k of type ξ. Then $Aut(V) \simeq \Pi_{i\in I} GL_{n_i}(k)$ with $\xi = \{(v_i, n_i) \mid v_i \in \mathbb{Z}^n,\ n_i \in \mathbb{N},\ \ i \in I\}$*

where all v_i's are distinct.

Proof. We know that $V = \bigoplus_{i\in I} V_{v_i}$. Since automorphism preserves direct sum decomposition, we have

$$Aut(V) = \Pi_{i\in I} Aut(V_{v_i}) = \Pi_{i\in I} Aut(k^{n_i}) = \Pi_{i\in I} GL_{n_i}(k)$$

□

Example 4.2.2. *Let V' be the $\mathbb{Z}^2$-graded k-vector space in Fig4.1b. Then the type will be the multiset $\xi = \{((0,2),2),((1,1),1),((3,0),1)\}$. Thus $Aut(V') = GL_2(k) \times GL_1(k)^2$.*

In example 2.2.1, we showed that $GL_n(k)$ is an algebraic group. So for any finite dimensional k-vector space V, $GL(V) = GL_n(k)$ is an algebraic group where n is the dimension of V. By using this information, we will state the following theorem.

Theorem 4.2.2. *[14] For any finite-dimensional $\mathbb{Z}^n$-filtered k-vector space $(V,\mathcal{F})$, the automorphism group $Aut(V,\mathcal{F})$ is an algebraic subgroup of full automorphism group $GL(V)$. For any finite dimensional $\mathbb{Z}^n$-graded k-vector space V, $Aut(V)$ is algebraic subgroup of algebraic group $Aut(\mathcal{F}V)$.*

Proof. Automorphisms of $(V,\mathcal{F})$ is automorphism of V which respects the subspace grading, therefore it forms a subgroup of algebraic group $GL(V)$. Automorphism of finite dimensional $\mathbb{Z}^n$-graded k-vector space $V = \bigoplus_{v\in\mathbb{Z}^n} V_v$ preserves the grading relation therefore it is an element in $Aut(\mathcal{F}V)$. Since $Aut(\mathcal{F}V)$ can be associated to $Aut(V,\mathcal{F})$, we have again algebraic subgroup.

□

Definition 4.2.2. *For finitely generated $\mathbb{Z}^n$-graded S-module M, we define the finite dimensional $\mathbb{Z}^n$-graded k-vector space $\rho(M) = k \otimes_S M$ where all the variables x_i act trivially.*

Theorem 4.2.3. *[14] For any finitely generated free $\mathbb{Z}^n$-graded S-module F, there is an isomorphism $Aut(F) \simeq Aut(\mathcal{F}(\rho(F)))$.*

(a) free $\mathbb{Z}^2$-graded S-module F (b) $\mathbb{Z}^2$-graded k-vector space $\rho(F)$

Figure 4.2: Free $\mathbb{Z}^n$-graded S-module F and $\mathbb{Z}^2$-graded k-vector space $\rho(F)$

Example 4.2.3. *Let F be free $\mathbb{Z}^2$-graded S-module which has two generator at $(0,2)$, one at $(1,1)$ and one at $(3,0)$, which is illustrated in Fig4.2a. The corresponding $\mathbb{Z}^2$-graded k-vector space $\rho(F)$ can be evaluated as $k\otimes_S F = S/IS\otimes_S F = F/IF$ where $I = (x_1, \cdots x_n)$ is an ideal. The operator ρ detect the positions of the generators of a module and it places k-vector space of dimension n where n is the number of generator at that position. Therefore, we will have $\rho(F)$ so that $\rho(F)_{(0,2)} = k^2$, $\rho(F)_{(1,1)} = k$, $\rho(F)_{(3,0)} = k$ and any other position is 0. This is illustrated in Fig4.2b. We already evaluated the automorphism group of $\mathcal{F}(\rho(F))$ in the example 4.2.2, so the automorphism group of F is $GL_2(k) \times GL_1(k) \times GL_1(k)$ by Theorem 4.2.3.*

We discovered the structure of automorphism group of finitely generated free $\mathbb{Z}^n$-graded S-module and it is algebraic group over k.

Now we will turn our attention to the classification of these modules. This ρ actually helps us to calculate the Tor modules. Carlsson and Zomorodian only use the first two Tor modules in [14] to come up with the classification. Knudson tried to use higher tor modules to get finer classification in [30], we will explain this approach later.

Definition 4.2.3. *For any finitely generated $\mathbb{Z}^n$-graded S-module M, a free hull is a surjective homomorphism p from finitely generated free $\mathbb{Z}^n$-graded S-module F to M so that $\rho(F)$ and $\rho(M)$ are isomorphic.*

We will use the minimal free resolution invariant which is introduced in Section 2.1. The following theorem will confirm that any finitely generated $\mathbb{Z}^n$-graded S-module admits a free hull.

Theorem 4.2.4. *[23, Proposition 1.7] If $\mathcal{F} : \cdots F_1 \to F_0 \to M \to 0$ is the minimal free resolution of finitely generated $\mathbb{Z}^n$-graded S-module then $Tor_i^S(M,k) = F_i \otimes_S k$.*

Since $Tor_0^S(M,k) = F_0 \otimes_S k$ and $Tor_0^S(M,k) = M \otimes_S k$, we can directly say that map from F_0 to M will form free hull for finitely generated $\mathbb{Z}^n$-graded S-module M. This will be induced by the minimal free resolution and we already discussed the existence in the section 2.1.

Theorem 4.2.5. *[14] Any two free hulls for M are isomorphic i.e. for any two free hull $p : F \to M$ and $p' : F' \to M$, then there exits an isomorphism $f : F \to F'$ so that $p' \circ f = p$.*

This is the direct result of the theorem 2.1.3 which states that any minimal free resolution exact sequences must be isomorphic. This uniqueness result allows us to define type sets for the classification.

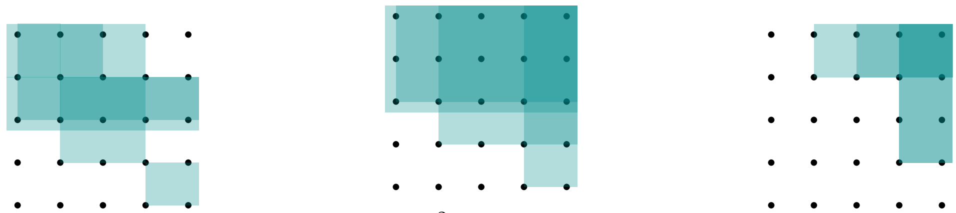
Definition 4.2.4. *For finitely generated $\mathbb{Z}^n$-graded S-module M, define $\xi_i(M) = \xi(Tor_i^S(M,k))$.*

We will focus on first two type as in [14], then we will explain higher ξ_i as in [30]. Observe that $\xi_i = 0$ for $i > n$ since minimal free resolution terminates after n. The following theorem is an extension of the theorem 8 in [14] to the types ξ_i where $0 \leq i \leq n$

Theorem 4.2.6. *ξ_i is a discrete invariant for each $0 \leq i \leq n$ which means that we will get a multiset for any field k. In this case, it is always bounded below multiset in $\mathbb{Z}^n$.*

Proof. If M is isomorphic to M' then their minimal free resolution must be isomorphic, too. By Theorem 4.2.4, we have $Tor_i^S(M,k) \simeq F_i \otimes k \simeq F_i' \otimes k \simeq Tor_i^S(M',k)$ for all $i \in \mathbb{N}$. Since $\xi_i(M) = \xi(Tor_i^S(M,k))$ and by definition of ξ, ξ_i is an invariant which is always a multiset. □

Observe that for any $\mathbb{Z}^n$-graded S-module, we have $\rho(M) = M \otimes_S k = M \otimes_S S/I = M/IM$ where $I = (x_1, \cdots, x_n)$ is a $\mathbb{Z}^n$-graded ideal in S. The computation of $\rho(M)_v$ for $v \in \mathbb{Z}^n$ is done by quotient out the translated form of the preceding components.



(a) $\mathbb{Z}^2$-graded S-module M (b) Free $\mathbb{Z}^2$-graded S-module of type ξ_0 (c) Kernel of the map p

Figure 4.3: Example of $\mathbb{Z}^2$-graded $k[x, y]$-modules for type sets

Example 4.2.4. *Let $M = S(0,2)/(xy) \oplus S(0,2)/(x^2y) \oplus S(1,1)/(x^2) \oplus S(3,0)/(y)$ which is illustrated in Fig 4.3a. We know that $\rho(M) = M/IM$ where $I = (x, y)$, then for the degrees less than $(0,2)$ the component of M will be zero therefore $\rho(M)_{(0,2)} = M_{(0,2)} = k \oplus k$. For the degree $(1,3)$,*

$$
\begin{aligned}
\rho(M)_{(1,3)} &= M_{(1,3)}/xM_{(1,2)} + yM_{(0,3)} \\
&= kxy \oplus kxy \oplus ky^2/x(ky \oplus ky) + y(kx \oplus kx \oplus ky) \\
&= kxy \oplus kxy \oplus ky^2/(kxy \oplus kxy) + (kxy \oplus kxy \oplus ky^2) \\
&= kxy \oplus kxy \oplus ky^2/kxy \oplus kxy \oplus ky^2 \\
&= 0
\end{aligned}
$$

At the end we will see $\xi_0 = \{((0,2),2),((1,1),1),((3,0),1)\}$. Let F be free $\mathbb{Z}^2$-graded S-module with type ξ_0 which is illustrated in Fig 4.3b. To calculate ξ_1, we need write the kernel of the map p from F to M illustrated in Fig 4.3c. Thus we have $\xi_1 = \{((3,1),2),((1,3),1),((2,3),1)\}$. Observe that ξ_0 will detect the places of generators and ξ_1 contains the places where we have a relation on these generators.

Carlsson and Zomorodian in [14], used only first two type invariant in their paper to give complete classification. Their intuition comes from the module theory. A module can be represented by its generators and relations on these generators. Observe that ξ_0 and ξ_1 can be used in this manner. For that reason, first we will focus on these , then we will explain the same result by using the all ξ_i where $i \geq 2$.

Definition 4.2.5. *Given multisets ξ_0 and ξ_1. Let F be a free $\mathbb{Z}^n$-graded S-module so that $\xi(\rho(F)) = \xi_0$. Let $S(F, \xi_1)$ be the set such that*

$$S(F, \xi_1) = \{L \subseteq F \mid \xi(\rho(L)) = \xi_1\}$$

Let $[M]$ denote the isomorphism class of finitely generated $\mathbb{Z}^n$-graded S-module M and K is syzygy of M which means it is the kernel of the surjection $F_0 \to M$ where F_0 in the minimal free resolution. The set $\mathcal{I}(\xi_0, \xi_1)$ is defined as

$$\mathcal{I}(\xi_0, \xi_1) = \{[M] \mid \xi(\rho(M)) = \xi_0 \text{ and } \xi(\rho(K)) = \xi_1\}$$

Theorem 4.2.7. *[14, Classification] Let F be a free $\mathbb{Z}^n$-graded S-module of type ξ_0 and $g \in Aut(F)$. The action of g on the submodule L of F is defined as $g \cdot L = g(L)$. If the map $q : S(F, \xi_1) \to \mathcal{I}(\xi_0, \xi_1)$ is defined as $q(L) = [F/L]$, then we have $q(g \cdot L) = q(L)$ and it induces a bijection $\overline{q} : S(F, \xi_1)/\sim_{Aut(F)} \to \mathcal{I}(\xi_0, \xi_1)$ where $\sim_{Aut(F)}$ is an equivalence relation induced by the action of $Aut(F)$.*

Proof. The action of g on the submodule L is just the image of L under g which is an automorphism. Thus $g(L) \simeq L$ which gives us $F/g(L) \simeq F/L$. The map q is the same on the orbits of this action. Therefore $\overline{q}$ is well-defined.

Let $[M] \in \mathcal{I}(\xi_0, \xi_1)$ then $\xi(\rho(M)) = \xi_0$ and $\xi(\rho(K)) = \xi_1$ where K is the kernel of the free hull $p : F \to M$. We know that $\rho(M) \simeq \rho(F)$ which gives us that their type is the same. Thus $K \in S(F, \xi_1)$ and F is unique up to isomorphism. So $q(K) = [F/K] = [M]$ which implies that $\overline{q}$ is surjective, too.

Let $\overline{q}(\tilde{K}) = \overline{q}(\tilde{K'})$ for $\tilde{K}, \tilde{K'} \in S(F, \xi_1)/\sim_{Aut(F)}$. Then there exists an isomorphism between F/K and F/K' for any representative of the classes $\tilde{K}$ and $\tilde{K'}$. The equivalence class $\tilde{K}$ consists of the submodules of F which are isomorphic to K. Thus $K' \in \tilde{K}$ and similarly $K \in \tilde{K'}$. Hence $\overline{q}$ is injective which gives us bijection. $\square$

As a summary, for a given finitely generated $\mathbb{Z}^n$-graded S-module M we have defined the first type $\xi(\rho(M))$ which is connected to its generator, and the second type $\xi(\rho(K))$ related to relations on its generators. By the previous theorem, we have an orbit of an action which corresponds to an isomorphism class of a $\mathbb{Z}^n$-graded S-module M. By these three invariants, we reach the complete classification.

4.3 Structure of the Classification of Finitely Generated $\mathbb{N}^n$-graded S-modules

Although there is a classification proposed by Carlsson and Zomorodian which matches isomorphism classes with an orbit of action and two multisets, the orbit part of this classification will depend on the computational field k. For the parametrization suggested by Carlsson and Zomorodian, the existence of a continuous invariant results in the nonexistence of a complete discrete invariant. Since the orbit part is always a continuous invariant, there is no complete discrete invariant for $\mathbb{N}^n$-graded S-modules.

In this section, we will investigate the structure of these orbits. After that, we will explain Knudson's suggestion for classification, but unfortunately, this classification still has continuous parts.

Like in the previous part, when we write $\mathbb{Z}^n$-graded we will mean bounded below grading.

Definition 4.3.1. *Let M be $\mathbb{Z}^n$-graded S-module. Define k-vector space as*

$$(IM)_v = \sum_{i=1}^{n} x_i M_{v-e_i} \subseteq M_v$$

where e_i denotes the ith standard basis vector $\mathbb{Z}^n$ and $I = (x_1, \cdots, x_n)$ is a $\mathbb{Z}^n$-graded ideal in S. $v \in \mathbb{Z}^n$ is called gap for $M \simeq \bigoplus_{v \in \mathbb{Z}^n} M_v$ if $(IM)_v \neq M_v$. Let $\Gamma(M)$ be the set of all gaps of M.

Observe that gamma set represents the points in $\mathbb{Z}^n$ so that in any direction toward this place, we will have a new generator. Thus $\Gamma(M)$ will determine the positions of generators of M. So for finitely generated $\mathbb{Z}^n$-graded S-module M, $\Gamma(M)$ is finite. The following theorem states the multiset type explicitly.

Theorem 4.3.1. *[14] Let M be a finitely generated $\mathbb{Z}^n$-graded S-module and $I = (x_1, \cdots, x_n)$ be a $\mathbb{Z}^n$-graded ideal in S. The type of $\rho(M)$ is the multiset $(\Gamma(M), \alpha_M)$ where $\alpha_M(v) = dim_k(M_v/(IM)_v) = dim_k(M_v) - dim_k((IM)_v)$.*

Theorem 4.3.2. *[14] Let F be a free $\mathbb{Z}^n$-graded S-module. Two submodules are the same if and only if they have the same gap set and they are the same on this gap set.*

Proof. Let L and L' have the same gap set and be the same on this gap set. Let $w \in \mathbb{Z}^n$ be an element so that $L_w \neq L'_w$ and for any $v < w$ we have $L_v = L'_v$. This element exists since $\mathbb{Z}^n$-grading is bounded below. If w is not a gap for these submodules, then we have $L_w = (IL)_w = \sum_{i=1}^n x_i L_{w-e_i} = \sum_{i=1}^n x_i L'_{w-e_i} = (IL')_w = L'_w$. Therefore there is no $w \in \mathbb{Z}^n$ so that $L_w \neq L'_w$ which implies $L = L'$. □

Our goal is to create a relation between the third invariant, orbits of the action $Aut(F)$ on $S(F, \xi_1)$, and a variety. The set $S(F, \xi_1)$ can be represented as variety and action of $Aut(F)$ become algebraic group action. Elements of $S(F, \xi_1)$ are the submodules L with $\xi(\rho(L)) = \xi_1$, therefore their gap set is the same. In order to list all possible submodules L with the same gap set, we will use the following definition.

Definition 4.3.2. *Let F be finitely generated free $\mathbb{Z}^n$-graded S-module over (X, φ). Let $\xi = (V, \alpha)$ be a multiset in $\mathbb{Z}^n$ and $\delta : V \to \mathbb{N}$ be any function. For any F, define $\mathbf{ARR}_{\xi,\delta}(F)$ to be the set of all assignments $v \to L_v$ for $v \in V$ and L_v is a k-linear subspace of F_v which satisfies the following properties;*

1. *If $v' \leq v$ then $x^{v-v'} L_{v'} \subseteq L_v$.*
2. *$dim_k(L_v) = \delta(v)$.*
3. *$dim_k(L_v / \sum_{v'<v} x^{v-v'} L_{v'}) = \alpha(v)$ for all $v \in V$.*

Observe that these assignments corresponds to submodules of F with the gap set V and α_M in the Theorem4.3.1 corresponds to α. The role of δ is to determine the number of the generators. Therefore we identified $S(F, \xi_1)$ with $\mathbf{ARR}_{\xi,\delta}(F)$. Our next goal is to show that $\mathbf{ARR}_{\xi,\delta}(F)$ is quasi-projective variety. Before that we will give an example of an assignment.

Example 4.3.1. *Let F be finitely generated free $\mathbb{Z}^2$-graded $k[x, y]$-module over (X, φ) where $|X| = 4$ so that each one of the generators placed at $(0, 2)$, $(0, 3)$, $(1, 1)$ and $(3, 0)$ illustrated in Fig 4.4.*

Let $\xi = (V, \alpha)$ be multiset in $\mathbb{Z}^2$ where $V = \{(1, 3), (2, 1), (2, 4), (4, 1)\}$. The functions are defined as $\delta(1, 3) = 1$, $\delta(2, 1) = 1$, $\delta(2, 4) = 3$, $\delta(4, 1) = 2$ and $\alpha(v) = 1$ for all $v \in V$. Observe that $dim_k(F_{(2,1)}) = 1$, $dim_k(F_{(2,4)}) = 3$ and $dim_k(F_{(4,1)}) = 2$ which implies $F_{(2,1)} = L_{(2,1)}$, $F_{(2,4)} = L_{(2,4)}$, and $F_{(4,1)} = L_{(4,1)}$. Therefore $L_{(1,3)}$ will determine the different assignments. We have three choice for $L_{(1,3)}$ since it is

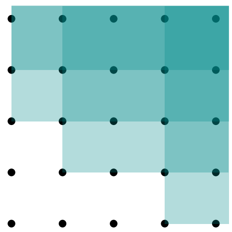


Figure 4.4: Free $\mathbb{Z}^2$-graded S-module F with generators placed at $(0,2)$, $(0,3)$, $(1,1)$ and $(3,0)$

in the range of three generators at the positions $(0,2),(0,3)$ and $(1,1)$. Each of these choices will give us $L_{(1,3)} = kxy$, $L_{(1,3)} = kx$ and $L_{(1,3)} = ky^2$ respectively. Therefore we have three different assignment in $\mathbf{ARR}_{\xi,\delta}(F)$.

Remark 4.3.1. *For finitely generated F, we know that dimension of F_v is bounded for all $v \in \mathbb{Z}^n$. This induces a bound for a dimension of subspaces. The third condition in the definition of $\mathbf{ARR}_{\xi,\delta}(F)$ assures that dimension of L_v which is $\delta(v)$ increases when v increased in the grading of V. Let W be a subset of V so that $\alpha(v) \neq 0$ for $v \in W$. This set contains the positions where dimension of subspace strictly increased, thus the set W is finite because dimension is bounded. The elements in $V \setminus W$ does not effect the type of corresponding submodule, so assignment defined on V and W would be the same. For this reason, we will consider V as a finite set.*

We will fix some notations for ease of writing. Let F be finitely generated free $\mathbb{Z}^n$-graded S-module over (X, φ), $\xi = (V, \alpha)$ be a multiset and $\delta : V \to \mathbb{Z}$ be any function where $X = \{x_1, \cdots, x_m\}$ and $V = \{v_1, \cdots, v_s\}$.

Let $Gr_k(V)$ be the Grassmannian variety of k-dimensional subspaces of V and let F, ξ and δ be given. We can relate $\mathbf{ARR}_{\xi,\delta}(F)$ and a subset of $\epsilon = \Pi_{v\in V} Gr_{\delta(v)}(F_v)$ by taking the elements in ϵ which satisfies the conditions in the definition $\mathbf{ARR}_{\xi,\delta}(F)$.

The following definition will help us to state another representation of ϵ.

Definition 4.3.3. *A (ξ, δ)-frame for finitely generated free $\mathbb{Z}^n$-graded S-module F is a family of linear embeddings $\{j_v : W_v \to F_v\}_{v\in V}$ where W_v is a vector space isomorphic to $k^{\delta(v)}$. $\mathcal{F}(F)$ will denote the the set of (ξ, δ)-frames in F for a given multiset ξ and a function δ.*

Notice that automorphisms of W_v induces an action on $\mathcal{F}(F)$. More precisely, let

$$\{\sigma_v\}_{v\in V} \in \Gamma = \Pi_{v\in V} GL(W_v) \simeq \Pi_{v\in V} GL(k^{\delta(v)}) \simeq \Pi_{v\in V} GL_{\delta(v)}(k)$$

then action of $\{\sigma_v\}_{v\in V}$ on (ξ, δ)-frame is defined as

$$\{\sigma_v\}_{v\in V} \cdot \{j_v : W_v \to F_v\}_{v\in V} = \{j_v \circ \sigma_v : W_v \to F_v\}_{v\in V}$$

Since each (ξ, δ)-frame determines a family of subspaces by $im(j_v) = L_v$, the orbits of this action contains the subspaces of the same type since σ_v does not change the type of the images. So it is exactly the set of all families of subspaces $\{L_v\}_{v\in V}$ with dimension $\delta(v)$ which matches up with $\epsilon = \Pi_{v\in V} Gr_{\delta(v)}(F_v)$. Later, we will use this description of ϵ to explain variety correspondence for $\mathbf{ARR}_{\xi,\delta}(F)$.

Since $\mathbf{ARR}_{\xi,\delta}(F)$ is a subset of ϵ, we will relate the orbits of the action of Γ on the frames $\mathcal{F}(F)$ with this specific subset. However, this representation of frames is not useful to state the correspondence explicitly. Therefore, we will use the fact (ξ, δ)-frame consists of j_v embedding which can be seen as matrix relative to the bases for W_v and F_v. The next definition will help us to express bases concretely.

Definition 4.3.4. *Let F be free finitely generated $\mathbb{Z}^n$-graded S-module. Let $v^* \in \mathbb{Z}^n$ so that $v^* \geq w_i$ for $w_i \in \varphi(X)$ and $v^* \geq v_j$ for $v_j \in V$. Embed each F_{v_i} into F_{v^*} by multiplying with $x^{v^*-v_i}$ and call it G_{v_i} where $v_i \in W$.*

It is more convenient to see subspaces $\{L_v\}_{v\in V}$ in one single k-vector space. For this reason, by using the same method in the definition of F_{v^*}, we can identify subspace L_{v_i} of F_{v_i} with $L^*_{v_i}$ of $G_{v_i} \subseteq F_{v^*}$ and this embedding respects the properties of being an element of $\mathbf{ARR}_{\xi,\delta}(F)$. Thus, there is an identification of elements of $\mathbf{ARR}_{\xi,\delta}(F)$ with the subspaces of F_{v^*} in a natural way.

Definition 4.3.5. *Let e_i be generator for $S(\varphi(x_i)) = S(w_i)$ and the basis for F_{v^*} be defined as follows:*

$$B = \{x^{v^*-w_i} e_i \mid 1 \leq i \leq m\}$$

The basis for the subspace G_{v_i} is defined as

$$B_{v_i} = \{x^{v^*-w_j} e_j \mid w_j \leq v_i\}$$

where $v_i \in V$ and $w_j \in \varphi(X)$.

$$
\begin{array}{c}
 \\
B(v_1) \\
\vdots \\
B(v_i) \\
B(v_{i+1}) \\
\vdots \\
B(v_s)
\end{array}
\begin{array}{c}
\mathcal{B}(v_i) \\
\left[\begin{array}{ccc}
 & \star & \\
 & \vdots & \\
 & \star & \\
 & 0 & \\
 & \vdots & \\
 & 0 &
\end{array}\right]
\end{array}
$$

Figure 4.5: Block decomposition of M_{v_i} where $v_i \lesssim v_j$ whenever $i \leq j$

We can construct a total order on V which is compatible with the order in $\mathbb{Z}^n$ and we will denote it by $\preceq$. There exists a partition on B induced by total ordering on V. This partition can be written as $B = \bigsqcup_{i=1}^{s}(B_{v_i} \setminus \bigcup_{v_j \prec v_i} B_{v_j}) = \bigsqcup_{i=1}^{s} B(v_i)$. Also we can choose a total order on $B(v_i)$ for each i, thus we reached a total ordered basis for F_{v^*}.

Next we will form a basis for W_v. We will use the same total ordering on V and isomorphism between $W_v \simeq k^{\delta(v)}$ induces a basis on W_v. Let $\mathcal{B}$ denote an ordered basis for $\bigoplus_{v \in V} W_v$. Like in the previous case, we have a partition on this set induced by the direct sum so that $\mathcal{B} = \bigsqcup_{v \in V} \mathcal{B}(v)$ where $\mathcal{B}(v)$ is an ordered basis for W_v.

Definition 4.3.6. *Let $m \times \delta(v)$-matrix M_v , where $v \in V$, denote the matrix representation for linear embedding $j_v : W_v \to G_v$ relative to the bases $\mathcal{B}(v)$ and B. Let M be a $m \times D$-matrix defined as*

$$
M = \left[\begin{array}{c|c|c|c} M_{v_1} & M_{v_2} & \cdots & M_{v_s} \end{array} \right]
$$

where $D = \sum_{i=1}^{s} \delta(v_i)$ and $V = \{v_1, \cdots, v_s\}$ so that $v_i \preceq v_{i+1}$.

Observe that the matrix M_v has block decomposition induced by the partition on B, and the corresponding block for $B(v)$ and $\mathcal{B}(v')$ is zero for $v \not\lesssim v'$ illustrated in Fig 4.5. Objects of $\mathcal{F}(F)$ which are (ξ, δ)-frames can be seen as a $m \times D$-matrices of the form $M = [M_{v_1} | \cdots | M_{v_s}]$ which satisfies

1. M_{v_i} has rank $\delta(v_i)$ for all $v_i \in V$, i.e. full
2. Block of M_{v_i}, which is of the form $B(v_j) \times \mathcal{B}(v_i)$, is zero if $v_j \not\lesssim v_i$.

Up to this point, we reached a matrix representation for a family of subspaces formed by frames. The action of $\Gamma = \Pi_{v_i \in V} GL_{\delta(v_i)}(k)$ on $\mathcal{F}(F)$ can be described as multiplication on the right of a corresponding matrix for $\{\sigma_{v_i}\}_{v_i \in V}$ with M which is matrix form of the (ξ, δ)-frame. Since the corresponding matrix for $\{\sigma_{v_i}\}_{v_i \in V}$ will be nonsingular matrix which means its determinant is nonzero, we can use the equality $rank(AB) = rank(A)$ where B is nonsingular matrix. Therefore this action is well-defined and it does not affect the rank of the submatrices. By the following theorem, the orbit of $\mathcal{F}(F)$ is a quasi-projective variety, which is $\epsilon = \Pi_{v \in V} Gr_{\delta(v)}(F_v)$.

Theorem 4.3.3. *[34, Theorem 1.10] The action of Γ on $\mathcal{F}(F)$ has closed orbits and satisfies stability hypothesis then action admits geometric quotient which means the set of orbits is naturally quasi-projective variety.*

$\mathbf{ARR}_{\xi,\delta}(F)$ is a proper subset of quasi-projective variety ϵ. We explained how we can visualize elements of ϵ by using matrices, and now with this information we will give matrix representation for $\mathbf{ARR}_{\xi,\delta}(F)$.

Definition 4.3.7. *Let $\mathcal{F}_0(F)$ be the set of $m \times D$-matrices in $\mathcal{F}(F)$ which satisfies the following properties where $V = \{v_1, \cdots, v_s\} \subset \mathbb{Z}^n$ and the order is the standard order in $\mathbb{Z}^2$;*

1. *The $m \times (\delta(v_i) + \delta(v_j))$-matrix defined as $\mu(v_i, v_j) = [M_{v_i} | M_{v_j}]$ has rank $\delta(v_j)$ where $v_i \leq v_j$*
2. *The $m \times \sum_{v'_i < v_j} \delta(v'_i)$-matrix defined as $\lambda(v_j) = [M_{v'_1} | M_{v'_2} | \cdots | M_{v'_k}]$ has rank exactly $\delta(v_j) - \alpha(v_j)$ where $\{v'_1, v'_2, \cdots, v'_k\} \subset V$ for some k.*

The following result is stated in [14] as an observation which gives precise matching with $\mathbf{ARR}_{\xi,\delta}(F)$.

Proposition 4.3.1. *The set of orbits of action of Γ on $\mathcal{F}_0(F)$ is in bijective correspondence with $\mathbf{ARR}_{\xi,\delta}(F) \subset \epsilon$.*

Proof. Action of Γ does not change the rank of the submatrices M_{v_i} for $v_i \in V$. So the orbits of $\mathcal{F}_0(F)$ are separated from other orbits and again it is quasi-projective variety. Let $M \in \mathcal{F}_0(F)$, now we will examine the effect of the conditions on

$\mathcal{F}_0(F)$ on frames. We have $M \in \mathcal{F}(F)$, then M_{v_i} has rank $\delta(v_i)$ for all $v_i \in V$, which means j_{v_i} is injective. If $v_i \leq v_j$ then the matrix $\mu(v_i, v_j)$ has rank $\delta(v_j)$, which means $x^{v_j - v_i} im j_{v_i} \subset im j_{v_j}$. Let $v'_i < v_j$ for $1 \leq i \leq k$. The rank of $\lambda(v_j)$ is $\delta(v_j) - \alpha(v_j)$ means that $\dim_k(\sum x^{v_j - v'_i} im j_{v'_i}) = \delta(v_j) - \alpha(v_j)$. Thus $\dim_k(im j_{v_j} / \sum x^{v_j - v'_i} im j_{v'_i}) = \delta(v_j) - (\delta(v_j) - \alpha(v_j)) = \alpha(v_j)$.

The orbits of the action of Γ on the frames satisfies these conditions are formed by the family of subspaces which satisfies the conditions of $\mathbf{ARR}_{\xi,\delta}(F)$. So the set of orbits of $\mathcal{F}_0(F)$ is a subset of $\mathbf{ARR}_{\xi,\delta}(F)$.

We already mentioned that an assignment in $\mathbf{ARR}_{\xi,\delta}(F)$ can be embedded in F_{v^*} as described in Def 4.3.4. Let $\{L_{v_i}\}_{v_i \in V}$ be a family of k-linear subspaces of F_v in $\mathbf{ARR}_{\xi,\delta}(F)$ and $L^*_{v_i} \subseteq G_{v_i} = x^{v^* - v_i} F_{v_i}$ be the corresponding subspace of F_{v^*} for each $v_i \in V$. The conditions on the assignment induces the conditions on rank of the submatrices which corresponds $L^*_{v_i}$. Therefore, it is clear that the set of orbits of $\mathcal{F}_0(F)$ contains $\mathbf{ARR}_{\xi,\delta}(F)$. □

Observe that the conditions in the definition of $\mathcal{F}_0(F)$ are invariant under the action of Γ. Therefore the orbits of $\mathcal{F}_0(F)$ is again a quasi-projective variety. Hence $\mathbf{ARR}_{\xi,\delta}(F)$ is quasi-projective variety. The following diagram will explain how we associate $S(F, \xi_1)$ with a point in the quasi-projective variety.

$$S(F, \xi_1) \Leftrightarrow \mathbf{ARR}_{\xi,\delta}(F) \Leftrightarrow \mathcal{F}_0(F) / \sim_{Aut(F)}$$

We have already shown that $Aut(F)$ is an algebraic group, thus its action on $S(F, \xi_1)$ is algebraic action. The set $\mathcal{I}(\xi_0, \xi_1)$ is identified with orbits of this action which not need to be a variety. The reason for that, algebraic group actions on a variety does not result in variety structure on orbit space.

If we summarize the information we stated up to this point, according to Carlsson and Zomorodian in [14], an isomorphism class of an $\mathbb{Z}^n$-graded S-module M can be parametrized with two multisets and a point in the set of orbits of an algebraic action. The third invariant will help us with the computation of isomorphism classes.

Now, we will explain the contribution of Knudson in [30] to the classification of $\mathbb{Z}^n$-

graded S-modules.

Definition 4.3.8. *Let $\xi_0, \cdots, \xi_n$ be multisets in $\mathbb{Z}^n$. If type of free $\mathbb{Z}^n$-graded S-module F is ξ_0 and the type of a submodule L of F is ξ_1 , then define*

$$S(\xi_2, \cdots, \xi_n) = \{M \in [F/L] \mid \xi(Tor_i^S(M,k)) = \xi_i \textit{ and } L \in S(F, \xi_1)\}$$

For a fixed ξ_0 and ξ_1, we will have finitely many nonempty $S(\xi_2, \cdots, \xi_n)$ since these two multisets create a bound for a number of multisets ξ_2 that can occur. Similarly, these three sets bound the number of possible ξ_3. By continuing like this, we will have only finitely many possibilities. For a fixed F, we can find several $n-1$ tuple of type sets $(\xi_2, \cdots, \xi_n)$, and the corresponding sets $S(\xi_2, \cdots, \xi_n)$ will be disjoint for each $n-1$ tuple. Also their union will be $S(F, \xi_1)$.

The next observation will help us to define action of $Aut(F)$ on the set $S(\xi_2, \cdots, \xi_n)$.

Let $\mathcal{F}_M$ be minimal free resolution of $\mathbb{Z}^n$-graded S-module M as ;

$$\mathcal{F}_M : 0 \to F_n \xrightarrow{d_n} F_{n-1} \xrightarrow{d_{n-1}} \cdots F_1 \xrightarrow{d_1} F_0 \to M \to 0$$

Let L be a submodule of F_0 so that $M \simeq F_0/L$. The automorphism g of F_0 induces an isomorphism between L and $g(L)$ which gives us $M \simeq F_0/L \simeq F_0/g(L)$. The action of g on M is defined as $g \cdot M = F/g(L) := g(M)$. For the minimal free resolution of $g(M)$ we will have

$$\mathcal{F}_{g(M)} : 0 \to F_n \xrightarrow{d_n} F_{n-1} \xrightarrow{d_{n-1}} \cdots F_1 \xrightarrow{d_1^{g(M)}} F_0 \to g(M) \to 0$$

In fact, $\mathcal{F}_M$ and $\mathcal{F}_{g(M)}$ differs only at d_1 so that $d_1^{g(M)} = g \circ d_1^M$. This means that for isomorphic modules we have not only the same type sets but also the same syzygies.

Type of F_i is the multiset ξ_i for each $1 \leq i \leq n$ where $\xi_i = (V_i, \alpha_i)$. We will call the syzygies $d_j(M)$ which is the image of d_j and at the same time kernel of d_{j-1}. Let $\epsilon_{\xi_2, \cdots, \xi_n}$ denote the set of subspaces which is also a variety so that

$$\epsilon_{\xi_2, \cdots, \xi_n} = \Pi_{j=1}^n \Pi_{v_i \in V_j} Gr_{\alpha_j(v_i)}((F_{j-1})_{v_i})$$

For a fixed j, this set contains $\alpha_j(v_i)$-dimensional subspaces of the homogeneous component $(F_{j-1})_{v_i}$. The next theorem will state the corresponding between isomorphism classes and this variety explicitly.

Theorem 4.3.4. *[30] Let $\varphi : S(\xi_2, \cdots, \xi_n) \to \epsilon_{\xi_2,\cdots,\xi_n}$ be a map defined as $\varphi(M) = (d_2(F_2), \cdots, d_n(F_n))$ where d_i and F_i are in the minimal free resolution of M. Then there exists a map $\overline{\varphi} : S(\xi_2, \cdots, \xi_n)/\sim_{Aut(F_0)} \to \epsilon_{\xi_2,\cdots,\xi_n}$. Also there exists an injective map $\Phi : S(F_0, \xi_1)/\sim_{Aut(F_0)} \to \bigsqcup_{\xi_2,\cdots,\xi_n} S(F_0, \xi_1) \times \epsilon_{\xi_2,\cdots,\xi_n}$ so that $\Phi([M]) = (M, (d_2(F_2), \cdots, d_n(F_n)))$ where M is the chosen representative for the orbit.*

Proof. Let M and M' be two $\mathbb{Z}^n$-graded S-modules so that $g(M) = M'$ for some $g \in Aut(F_0)$. As they are isomorphic, their syzygies will be the same, which are $d_i(F_i)$, and their representative for the isomorphism class is the same. Thus the map $\overline{\varphi}$ is well-defined.

Restricting the image by using a representative of the isomorphism class makes Φ an injective map. □

The map $\overline{\varphi}$ give a variety structure on orbits of $S(\xi_2, \cdots, \xi_n)$, and the map Φ does the same thing for orbits of $S(F, \xi_1)$. This structure is different from the one in Carlsson and Zomorodian's method. While the set of orbits of $S(F, \xi_1)$ identify with a subset of a variety in Knudson's approach, on the other side, the set $S(F, \xi_1)$ identified with a projective variety in Carlsson and Zomorodian's approach. Therefore, the former has topology induced by a quotient of Zariski topology, and the latter has Zariski topology.

The $(n-1)$ tuple will form an invariant, which is an addition to multisets ξ_0 and ξ_1. However, they will not form a complete invariant because these F_i's are not enough to define a module, information about the maps is also needed. Therefore there is a need to use a variety structure since they are directly related to the syzygies. However, it may lose some information by putting a variety structure like that.

In the next chapter, we will examine the difference between these two approaches given by Carlsson, Zomorodian, and Knudson. Also, the continuous portion of this parametrization will be explained.

4.4 Comparison of Two Different Structure on $S(F, \xi_1)$ With an Example

In this section, we will explain the continuous part of the classification by using both Carlsson & Zomorodian's and Knudson's methods on an example.

Let ξ_0 and ξ_1 be given two multisets in $\mathbb{Z}^2$ such as

$$\xi_0 = \{((0,0),2)\}$$

$$\xi_1 = \{((3,0),1),((2,1),1),((1,2),1),((0,3),1)\}$$

Finitely generated free $\mathbb{Z}^2$-graded $k[x,y]$-module is $F = k[x,y](0,0) \oplus k[x,y](0,0)$.

By theorem 4.2.3, we know that $Aut(F) \simeq Aut(\mathcal{F}(\rho(F)))$ and $\rho(F) = F \otimes_{k[x,y]} k = k(0,0) \oplus k(0,0)$. Recall that the associated $\mathbb{Z}^2$-filtered vector space $\mathcal{F}(\rho(F))$ is defined as a family of subspaces $\mathcal{F}_w(\rho(F))$ where $\mathcal{F}_w(\rho(F)) = \bigoplus_{w' \leq w} \rho(F)'_w$. Only nonzero components is $\rho(F)_{(0,0)} = k^2$, which gives us the family as $\mathcal{F}_w(\rho(F)) = k^2$ for $w \in \mathbb{N}^n$ and $\{0\}$ otherwise. This means that $Aut(\mathcal{F}(\rho(F))) = GL(k^2) = GL_2(k)$.

Next, we will investigate the structure of $S(F, \xi_1)$. This set contains submodules whose type is ξ_1. Let L be submodule in the set $S(F, \xi_1)$ then $\dim_k(L_v) = 1$ where $v \in \{(3,0),(2,1),(1,2),(0,3)\} = V_1$. The containment condition which is $L_v \subseteq F_v$ would give us the set of L_v is actually 1-dimensional subspace of 2-dimensional space F_v where $v \in V_1$. Thus this set can be identified with $\mathbb{P}^1(k)$ which is projective line. Projective line $\mathbb{P}^1(k)$ is defined as $k \times k \backslash \{(0,0)\}/\sim$ where $(a,b) \sim (\lambda a, \lambda b)$ for some nonzero $\lambda \in k$. We will denote the equivalence class of (a,b) as $[a : b]$. Observe that $(\frac{a}{b}, 1)$ is in this class if $b \neq 0$ and we will choose this element as a representative. For the case $b = 0$, we will show this class as $[\infty : 1]$. If we go back to our observation, $S(F, \xi_1)$ is exactly $\bigoplus_{v \in V_1} \mathbb{P}^1(k)$ so it is $\mathbb{P}^1(k)^4$.

Under this identification, $S(F, \xi_1)$ can be rewritten as

$$S(F, \xi_1) = \{(l_1, l_2, l_3, l_4) \in \mathbb{P}^1(k)^4 \mid l_i = [l_i : 1] \text{ for } 1 \leq i \leq 4\}$$

This set can be partition into two part such as 4-tuple of distinct lines and 4-tuples of lines at least two of them in common. Let Ω be the set of distinct lines.

Action on Ω respects the condition on this subset i.e. orbits of Ω contains elements only 4-tuple of distinct lines. Let $g \in GL_2(k)$ so that $g \cdot l_1 = 0 = [0 : 1]$,$g \cdot l_2 = \infty = [1 : 0]$ and $g \cdot l_3 = 1 = [1 : 1]$ for $(l_1, l_2, l_3, l_4) \in \Omega$. This action exists since all these lines are distinct and any two of them span k^2. Let $g' \in GL_2(k)$ be another element which does the same transformation with g. Then we have $g \cdot (l_1, l_2, l_3, l_4) = (0, \infty, 1, g \cdot l_4)$ and $g' \cdot (l_1, l_2, l_3, l_4) = (0, \infty, 1, g' \cdot l_4)$. There exists an element in $GL_2(k)$ which transform first one to the second one, this element fixes diagonal and axes, which means it is diagonal matrix. Therefore $g \cdot l_4$ determined uniquely. Each orbit contain unique $(0, \infty, 1, l'_4)$ since the fourth component determined uniquely and we will identify orbits of $(l_1, l_2, l_3, l_4) \in \Omega$ with $g \cdot l_4$. Observe that g cannot map l_4 to 0, 1 or ∞. Thus set of orbits is $\mathbb{P}^1(k) - \{0, 1, \infty\} = k - \{0, 1\}$. This will be the continuous part of the parametrization since it is dependent on the field k.

There are elements in $S(F, \xi_1)$ other than Ω. Consider the set of 4-tuple formed by three different lines, which means exactly two of them are the same. Like we did in the set Ω, we can find a representative by sending elements to 0, 1, or ∞. Because of this reason, we will have representatives determined by the similarity positions in the 4-tuple. For example, if first two positions are the same that is (l_1, l_1, l_2, l_3), then there exists an action which sends (l_1, l_2, l_3) to $(0, \infty, 1)$. So the orbit representative for this element is $(0, 0, \infty, 1)$. We have 14 orbits of this form listed in Fig4.6.

Now, we will examine the structure of these orbits in the quotient topology. If we think any 4-tuple in the matrix representation like we did in the construction, then each M_{v_i} be 2×1-matrix where the chosen total order is $(0, 3) < (1, 2) < (2, 1) < (3, 0)$. Observe that $\mathcal{F}(F)$ and $\mathcal{F}_0(F)$ are the same in this example since there is no order relation on the elements of V. Therefore ϵ will be the same with $\mathbf{ARR}_{\xi_1,\delta}(F)$ where $\delta(v) = 1$ and $\alpha(v) = 0$ for all $v \in V$. The lines in the 4-tuple correspond to columns of the matrix M formed by submatrices M_{v_i} where $v_i \in V$. Therefore, 4-tuple of different lines will correspond to matrixes M where any two columns are linearly independent, which means rank of the matrix M is 2. They form an open set since six polynomials evaluate the determinants of 2×2-minors of M, and their zero sets are the complement Ω. Therefore the orbits of Ω is open set in quotient $S(F, \xi_1)/\sim_{GL_2(k)}$.

orbit rep	ξ_2	Y_{ξ_2}	$\mathrm{im}\overline{\varphi}$
$(0, \infty, 1, \alpha)$	$(2,3),(3,2)$	$\mathbb{P}^2 \times \mathbb{P}^2$	$([1:-1:1],[1-\alpha:-1:1])$
$(0, 0, \infty, 1)$	$(1,3),(3,2)$	$\mathbb{P}^1 \times \mathbb{P}^2$	$([1:-1],[1:-1:1])$
$(0, \infty, 0, 1)$	$(2,3),(3,2)$	$\mathbb{P}^2 \times \mathbb{P}^2$	$([1:0:-1],[1:-1:-1])$
$(0, \infty, 1, 0)$	$(2,3),(3,3)$	$\mathbb{P}^2 \times \mathbb{P}^3$	$([1:-1:1],[1:0:0:-1])$
$(0, \infty, \infty, 1)$	$(2,2),(3,3)$	$\mathbb{P}^1 \times \mathbb{P}^3$	$([1:-1],[1:0:-1:1])$
$(0, \infty, 1, \infty)$	$(2,3),(3,2)$	$\mathbb{P}^2 \times \mathbb{P}^2$	$([1:-1:1],[1:0:-1])$
$(0, \infty, 1, 1)$	$(2,3),(3,1)$	$\mathbb{P}^2 \times \mathbb{P}^1$	$([1:1:-1],[1:-1])$
$(0, 0, \infty, \infty)$	$(1,3),(3,1)$	$\mathbb{P}^1 \times \mathbb{P}^1$	$([1:-1],[1:-1])$
$(0, \infty, 0, \infty)$	$(2,3),(3,2)$	$\mathbb{P}^2 \times \mathbb{P}^2$	$([1:0:-1],[1:0-1])$
$(0, \infty, \infty, 0)$	$(2,2),(3,3)$	$\mathbb{P}^1 \times \mathbb{P}^3$	$([1:-1],[1:0:0:-1])$
$(0, 0, 0, \infty)$	$(1,3),(2,2)$	$\mathbb{P}^1 \times \mathbb{P}^1$	$([1:-1],[1:-1])$
$(0, 0, \infty, 0)$	$(1,3),(3,2)$	$\mathbb{P}^1 \times \mathbb{P}^2$	$([1:-1],[1:0:-1])$
$(0, \infty, 0, 0)$	$(2,3),(3,1)$	$\mathbb{P}^2 \times \mathbb{P}^1$	$([1:0:-1],[1:-1])$
$(\infty, 0, 0, 0)$	$(2,2),(3,1)$	$\mathbb{P}^1 \times \mathbb{P}^1$	$([1:-1],[1:-1])$
$(0, 0, 0, 0)$	$(1,3),(2,2),(3,1)$	$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$	$([1:-1],[1:-1],[1:-1])$

Figure 4.6: The $GL_2(k)$ orbits [30]

Other 14 orbits will be in the closure of orbits of Ω. We will explain this only for one of the orbits. Let us take the 4-tuple, which has the same line at the first and third positions. It will correspond to the matrix M, which has the same vectors on the first and third columns. Let d_1 be the the polynomial which takes the determinant of first and second column, then M will be in the zero set of d_1. Observe that M will not be zero of other 5 polynomial which are compute the determinant of other 2×2-minors, call these polynomials $d_2, \cdots, d_6$. Since any other pair of columns of M are linearly independent, M will be in the complement of these zero set $Z(d_2, \cdots, d_6)$. Let U be any open set containing M, thus intersection of U and the $S(F, \xi_1) \setminus Z(d_2, \cdots, d_6)$ is nonempty. By using the fact $Z(d_1, d_2, \cdots, d_6) \subset Z(d_2, \cdots, d_6)$, we will have

$$S(F, \xi_1) - Z(d_2, \cdots, d_6) \subset S(F, \xi_1) - Z(d_1, d_2, \cdots, d_6) = \Omega$$

Therefore intersection of U and Ω will be nonempty for any open set containing M, which implies M is in the closure of Ω. Same thing is true for the orbits. Thus any orbit will be in the closure of orbit of Ω.

Up to this point, we have only used Carlsson and Zomorodian's method. Now to compare their approach in [14] with Knudson's in [30], we should find higher ξ_i multisets.

$\mathbb{Z}^2$-graded $k[x, y]$-module can have at most ξ_2. Therefore, we only need to find suitable ξ_2. This multiset represents the minimal relation generators on the set ξ_1, so possible elements are less than or equal to $(3, 3)$ which is least upper bound of ξ_1. Let us focus on the set Ω again. For each $g \cdot l_4$ which is identified with $\alpha \in k - \{0, 1\}$, we have different isomorphism classes of modules, thus we can fix the relation on three element of ξ_1 and parametrize the fourth one. Remember that an orbit representative in Ω is of the form $(0, \infty, 1, l_4)$, which means these modules differ by one relation only. We know that 0 represents the line $[0 : 1]$, and then we can see that this line is implies that the first generator should die in the lowest degree determined in the total order which is $(0, 3)$. Similarly, ∞ represent the line $[1 : 0]$ which means second generator should die at the second degree which is $(1, 2)$. This method would give us the corresponding relations for these modules. We can summarize these relations as;

1. First generator a dies at $(0, 3)$
2. Second generator b dies at $(1, 2)$
3. a and b are equal at $(2, 1)$
4. α multiple of a is equal to b at $(3, 0)$

We have the minimal free resolution

$$0 \to F(\xi_2) \xrightarrow{d_2} S(0,3) \oplus S(1,2) \oplus S(2,1) \oplus S(3,0) \xrightarrow{d_1} S(0,0) \oplus S(0,0) \xrightarrow{p} M \to 0$$

The kernel of p is generated by $[y^3, 0], [0, xy^2], [-x^2y, x^2y], [-x^3, \alpha x^3]$. Exactness of this sequence implies that imd_1 is also generated by these elements. Thus any relation on this set will give us the kernel of d_1. To be more precise, kernel of d_1 is of the form $\{\beta_1[y^3, 0] + \beta_2[0, xy^2] + \beta_3[-x^2y, x^2y] + \beta_4[-x^3, \alpha x^3] = 0 \mid \beta_i \in S\}$. Thus we have two equation as

$$\beta_1 y^3 - \beta_3 x^2 y - \beta_4 x^3 = 0$$

$$\beta_2 xy^2 + \beta_3 x^2 y + \beta_4 \alpha x^3$$

We have four solutions for $(\beta_1, \beta_2, \beta_3, \beta_4)$ as

$$
\begin{aligned}
R_1 &= (x^2, -xy, y^2, 0) \\
R_2 &= (0, (1-\alpha)x^2, -xy, y^2) \\
R_3 &= (x^3, -\alpha x^2 y, 0, -y^3) \\
R_4 &= (2x^3, (-1-\alpha)x^2 y, xy^2, y^3)
\end{aligned}
$$

Observe that $xR_1 + yR_2 = R_3$ and $2xR_1 + yR_2 = R_4$, so the kernel of d_1 generated by R_1 and R_2. The degree of these relations will generate $\xi_2 = (V_2, \alpha_2)$ as $V_2 = \{(2,3),(3,2)\}$ and $\alpha_2(v) = 1$ for all $v \in V_2$. For this ξ_2, we have $\epsilon_{\xi_2} = Gr_{\alpha_2((2,3))}F(\xi_1)_{(2,3)} \times Gr_{\alpha_2((3,2))}F(\xi_1)_{(3,2)} = Gr_1(k^3) \times Gr_1(k^3) = \mathbb{P}^2(k) \times \mathbb{P}^2(k)$ where $\mathbb{P}^2(k)$ is projective plane. Projective plane is defined as $k \times k \times k/\sim$ where $(a,b,c) \sim (\lambda a, \lambda b, \lambda c)$ for some nonzero $\lambda \in k$. Again, the representatives of the equivalence classes are of the form $[a:b:1]$, $[a:1:0]$ and $[1:0:0]$.

Recall that, the map $\varphi : S(\xi_2) \to \epsilon_{\xi_2}$ induces a map $\overline{\varphi} : S(\xi_2)/\sim_{GL_2(k)} \to \epsilon_{\xi_2}$. Observe that $\Omega \subset S(\xi_2)$ and we already found $\epsilon_{\xi_2} = \mathbb{P}^2(k) \times \mathbb{P}^2(k)$. The image of elements in $\Omega/\sim_{GL_2(k)}$ under $\overline{\varphi}$ can be evaluated by using parametrization on the elements of orbits of Ω, which are $(0,\infty,1,\alpha)$, $\alpha \in k \setminus \{0,1\}$. Recall that $kerd_1$ generated by R_1 and R_2, so imd_2 will be generated by these elements too. Thus we have $d_2(F_2) = \langle (x^2,-xy,y^2,0),(0,(1-\alpha)x^2,-xy,y^2)\rangle$, correspondence of these generators in $\mathbb{P}^2(k) \times \mathbb{P}^2(k)$ are $[1:-1:1],[1-\alpha:-1:1]$. Thus $\overline{\varphi}((0,\infty,1,\alpha)) = ([1:-1:1],[1-\alpha:-1:1])$. In this example $\overline{\varphi}$ is injective as we can see in Fig4.6, therefore the induced map on the orbits of $S(F,\xi_1)$ can be written as $\Phi : S(F,\xi_1)/\sim_{GL_2(k)} \to \bigsqcup_{\xi_2} \epsilon_{\xi_2}$ for nine distinct ξ_2.

The main difference in Knudson's approach, the variety structure on the orbits of $S(F,\xi_1)$ given by the map Φ. Observe that preimage of $\epsilon_{\xi_2} = \{((2,3),1),((3,2),1)\}$ is contains the orbits of Ω, the 4 tuple (l_1,l_2,l_1,l_3) , the 4 tuple (l_1,l_2,l_3,l_2) and the 4 tuple (l_1,l_2,l_1,l_2). The last three points are separated from the orbits of Ω, while they were in the closure of these orbits. Therefore this method cause the lost of the information about the relation between orbits.

4.5 Invariants for $\mathbb{N}^n$-graded S-modules

We observed that there is a field dependence problem which we argued in the section 4.4. Thus, there is no complete discrete invariant for this parametrizations of $\mathbb{N}^n$-graded S-modules which is identified with bounded below finite persistence modules over $\mathbb{Z}^n$. There are some invariant proposals which can discriminate large homology classes from others.

4.5.1 The Rank Invariant

The rank invariant suggested by Carlsson and Zomorodian in [14]. This invariant is equivalent to barcodes which is complete invariant for finite persistence modules, that is bounded and each component is finitely generated over $\mathbb{N}$.

Definition 4.5.1. *Let (u, v) and (u', v') be two elements in $\mathbb{Z}^n \times \overline{\mathbb{N}}^n$ where $\overline{\mathbb{N}}$ is extended natural numbers contains ∞. Let $\mathbb{D}^n = \{(u, v) \mid u \in \mathbb{N}^n, v \in \overline{\mathbb{N}}^n, u \leq v\}$ where the order $\lesssim$ is defined as $(u, v) \lesssim (u', v')$ if and only if $u \leq u'$ and $v' \leq v$. Let M be finitely generated $\mathbb{N}^n$-graded S-module. Then the rank invariant is defined as $\rho_M(u, v) = rank(x^{v-u} : M_u \to M_v)$ where $(u, v) \in \mathbb{D}^n$.*

The reason why we use extended version of natural numbers is that some classes born at some stage but they never dies, so in order to represent their lifespan we should use the notation ∞.

This rank invariant is also called graded or persistent Betti numbers, and the notation $\beta_{u,v}$ is used.

A persistence module over $\mathbb{N}^n$ is derived from an $\mathbb{N}^n$-filtered space. We can modify the definition of the rank invariant to its input which is homology of an $\mathbb{N}^n$-filtered space. Let $X = \{X_v\}_{v \in \mathbb{N}^n}$ be an $\mathbb{N}^n$-filtered space, then the rank invariant $\rho_{X,i}(u, v) = \text{rank}(H_i(X_u, k) \to H_i(X_v, k))$

Remark 4.5.1. *When $n = 1$, this invariant is complete for pointwise finite dimensional persistence modules which is shown in [14].*

Using the structure theorem for pointwise finite dimensional persistence modules in ${}_k\mathbf{Mod}^{\mathbb{R}}$, we observed that the rank invariant and barcodes determine each other uniquely.

However, for $n > 1$ we have an incomplete discrete invariant and like in the following example for $n = 2$.

Example 4.5.1. *Let M and N be two persistence modules in ${}_k\mathbf{Mod}^{\mathbb{Z}^2}$ so that $M = k[x,y]/(x,y) \oplus k[x,y]/(xy)$ and $N = k[x,y]/(x) \oplus k[x,y]/(y)$. These $\mathbb{Z}^2$-graded $k[x,y]$-modules are not isomorphic, but their rank function will be the same. This happens because rank function only detects the number of classes in the specific direction and it cannot distinguish the classes. In our example, rank functions for both will give us 1 at $((0,0),(1,0))$ and $((0,0),(0,1))$. For M we will have the same class, but for N these numbers represents different classes. In other words for $((0,0),(1,0))$ it will give the class $k[x,y]/(y)$ and for $((0,0),(0,1))$ we have $k[x,y]/(x)$.*

4.5.2 Multigraded Hilbert Series and Associated Prime

In this section, we will examine the approach of Harrington, Otter, Schenck and Tillman in [26]. They offered two invariants for free resolution of an $\mathbb{N}^n$-graded S-modules , one of them is multigraded Hilbert series which is inspired from Hilbert function and the other one is called support shape which uses associated primes.

Definition 4.5.2. *Let M be an $\mathbb{N}^n$-graded S-module. The Hilbert function of M is defined as $HF(M,u) = dim_k M_u$ where $u \in \mathbb{N}^n$.*

The rank definition in Commutative Algebra by Eisenbud [22] defined for arbitrary module over an integral domain. The rank of R-module M is

$$rk_Q(M) = \text{dim}_Q M \otimes_R Q$$

where Q is the field of fractions of R. In [26], they showed that for any finitely generated $\mathbb{N}^n$-graded S-module M, $rk_k(M) = HF(M,u)$ for some high enough degree of u.

Theorem 4.5.1. *[22, Theorem 1.11] If M is finitely generated $\mathbb{N}$-graded S-module, then Hilbert function $HF(M,u)$ is a polynomial of degree less than $n-1$ for large $u \in \mathbb{N}$.*

This polynomial is called Hilbert polynomial and denoted by $HP(M, u)$ where $u \in \mathbb{N}$. We can see any $\mathbb{N}^n$-graded S-module to $\mathbb{N}$-graded by assigning degree 1 for each variable and taking the sum of the exponent.

Corollary 1.3 in Geometry of Syzygies by Eisenbud [23] states that there exists a lower bound for u which creates the equality of Hilbert function and Hilbert polynomial of $\mathbb{N}$-graded S-module M. This bound is induced by the free resolution of M, and $u \geq \max_j a_{ij} - (r-1)$ where free modules $F_i = \oplus_j S(a_{ij})$. Therefore Hilbert function can be used to discriminate two $\mathbb{N}^n$-graded S-modules, however it is not as good as we expected. Since the grading is reduced to one parameter, we will lose a lot of information about the multifiltration. That is why multigraded Hilbert series is proposed to use in [26].

Definition 4.5.3. *Let M be finitely generated $\mathbb{N}^n$-graded S-module. The multigraded Hilbert series is the formal series in $\mathbb{Z}[[t_1, \cdots, t_n]]$ so that*

$$HS(M, \boldsymbol{t}) = \sum_{u \in \mathbb{N}^n} HF(M, u) \boldsymbol{t}^u$$

Observe that for a fixed degree $u \in \mathbb{N}^n$ and finitely generated $\mathbb{N}^n$-graded S-module M, we have an exact sequence of k-vector spaces in each degree with components $(F_i)_u$ and M_u induced by free resolution. So by using $(F_i)_u = Im(d_i^u) + ker(d_i^u)$ where $d_i^u : (F_i)_u \to (F_{i-1})_u$ with $0 \leq i \leq n$ and $u \in \mathbb{N}^n$

$$\begin{aligned} HF(M, u) &= dim_k M_u \\ &= dim_k(Imd_0^u) \\ &= \sum_{i=0}^{n} (-1)^i (dim_k(kerd_i^u) + dim_k(Imd_i^u)) \\ &= \sum_{i \geq 0} (-1)^i dim_k (F_i)_u \\ &= \sum_{i \geq 0} (-1)^i HF(F_i, u) \end{aligned}$$

For any free resolution, we will have the same result. Therefore Hilbert function is independent of free resolution chosen. By using this equality, we can calculate the

Hilbert series.

$$\begin{aligned} HS(M,\mathbf{t}) &= \sum_{u\in\mathbb{N}^n}\sum_{i=0}^{n}(-1)^i HF(F_i,u)\mathbf{t}^u \\ &= \sum_{i=0}^{n}(-1)^i\sum_{u\in\mathbb{N}^n}HF(F_i,u)\mathbf{t}^u \\ &= \sum_{i=0}^{n}(-1)^i HS(F_i,\mathbf{t}) \end{aligned}$$

The following proposition stated in [26].

Proposition 4.5.1. *The multigraded Hilbert Series of finitely generated $\mathbb{N}^n$-graded S-module M is of the form*

$$HS(M,\boldsymbol{t}) = \frac{P(t_1,\cdots,t_n)}{\prod_{i=1}^{n} 1-t_i}$$

and the polynomial $P(t_1,\cdots,t_n)$ is an invariant.

Proof. Let F be finitely generated free $\mathbb{N}^n$-graded S-module, then $F = \bigoplus_{j=1}^{m} S(v_j)$. Thus we have $HS(F,\mathbf{t}) = \sum_{j=1}^{m}\frac{\mathbf{t}^{v^j}}{\prod_{i=1}^{n}1-t_i}$ so $P(t_1,\cdots,t_n) = \sum_{j=1}^{m}\mathbf{t}^{v^j}$ for this free module since $HS(S(v),\mathbf{t}) = \frac{\mathbf{t}^v}{\prod_{i=1}^{n}1-t_i}$ where $v\in\mathbb{N}^n$.

For finitely generated $\mathbb{N}^n$-graded S-module M, we already stated that $HS(M,\mathbf{t}) = \sum_{i=0}^{n}(-1)^i HS(F_i,\mathbf{t})$. This means that there is a polynomial for M.

Let $\mathcal{F}$ be any free resolution of M with components F_i and maps d_i with $0\le i\le n$, then it contains minimal free resolution as a direct summand by Theorem 2.1.3. Thus $F_i = F_i^m\oplus F_i^c$ and $d_i = d_i^m\oplus d_i^c$ where d_i^m and F_i^m are in the minimal free resolution. It is known that $M = Im(d_0) = Im(d_0^m)\oplus Im(d_0^c)$ then $Im(d_0^c) = 0$ and also $Im(d_i^c) = Ker(d_{i-1}^c)$. The sequence formed by F_i^c and d_i^c is an exact sequence which means we have a free resolution of F_0^c. The multigraded Hilbert series for F_0^c is $HS(F_0^c,\mathbf{t}) = \sum_{i=0}^{n-1}(-1)^i HS(F_{i+1}^c,\mathbf{t})$.

Observe that

$$
\begin{aligned}
HS(M, \mathbf{t}) &= \sum_{i=0}^{n} (-1)^i HS(F_i, \mathbf{t}) \\
&= \sum_{i=0}^{n} (-1)^i HS(F_i^m \oplus F_i^c, \mathbf{t}) \\
&= \sum_{i=0}^{n} (-1)^i HS(F_i^m, \mathbf{t}) + HS(F_0^c, \mathbf{t}) + \sum_{i=1}^{n} (-1)^i HS(F_i^c, \mathbf{t}) \\
&= \sum_{i=0}^{n} (-1)^i HS(F_i^m, \mathbf{t}) + \sum_{i=1}^{n} (-1)^{i-1} HS(F_i^c, \mathbf{t}) + \sum_{i=1}^{n} (-1)^i HS(F_i^c, \mathbf{t}) \\
&= \sum_{i=0}^{n} (-1)^i HS(F_i^m, \mathbf{t}) + \sum_{i=1}^{n} ((-1)^{i-1} + (-1)^i) HS(F_i^c, \mathbf{t}) \\
&= \sum_{i=0}^{n} (-1)^i HS(F_i^m, \mathbf{t})
\end{aligned}
$$

Thus multigraded Hilbert series is independent of free resolution chosen, indeed it woulf only depend minimal free resolution. Any two isomorphic modules will have isomorphic minimal free resolution therefore their multigraded Hilbert series must be same which means the polynomial $P(t_1, \cdots, t_n)$ is an invariant.

□

They gave an algorithm to calculate multigraded Hilbert series in [26]. However, this invariant is not complete in one dimension. The following example will be an observation of incompleteness of this invariant.

Example 4.5.2. *1. Let M and N be two persistence modules in ${}_k\mathbf{Mod}^{\mathbb{Z}}$ so that $M = k[x]$ and $N = k[x]/(x) \oplus k[x](1)$. These modules are not isomorphic, but their Hilbert series will be the same. $HS(M, t) = \dfrac{1}{1-t}$ and $HS(N, t) = \dfrac{1+t}{1-t} - \dfrac{t}{1-t}$ since it has resolution as follows*

$$0 \to k[x](1) \to k[x] \oplus k[x](1) \to N \to 0$$

2. Let $M = S(0,2) \oplus S(0,2) \oplus S(1,1) \oplus S(0,3)$ and $N = S(0,2) \oplus S(0,2) \oplus S(1,2) \oplus S(3,0) \oplus S(1,1)/(y)$. These modules are not isomorphic since N contains torsion part but M does not. Multigraded Hilbert series of M can be evaluated easily since it is free so $HS(M, t) = \dfrac{2t_2^2 + t_1 t_2 + t_1^3}{(1-t_1)(1-t_2)}$.

Now for N, we should find a free resolution first .

$$0 \to S(1,2) \to S(0,2) \oplus S(0,2) \oplus S(1,2) \oplus S(1,1) \oplus S(3,0) \to N \to 0$$

Thus we have

$$\begin{aligned}
HS(M,t) &= HS(F_0,t) - HS(F_1,t) \\
&= \frac{2t_2^2 + t_1t_2^2 + t_1t_2 + t_1^3}{(1-t_1)(1-t_2)} - \frac{t_1t_2^2}{(1-t_1)(1-t_2)} \\
&= \frac{2t_2^2 + t_1t_2 + t_1^3}{(1-t_1)(1-t_2)}
\end{aligned}$$

Hence these modules have the same polynomial $P(t_1,t_2) = 2t_2^2 + t_1t_2 + t_1^3$ eventhough they are not isomorphic.

The polynomial P only detects the number of component which is very rough idea about the structure of the module. That is why they proposed to use associated primes to get more discriminating information in [26]. Now we will recall some definitions.

Definition 4.5.4. *Let M be a R-module where R is commutative ring. For $U \subseteq M$, the annihilator of U is defined as*

$$Ann(U) = \{r \in R \mid \forall u \in U \; ru = 0\}$$

A prime ideal $P \subset R$ is associated to M if it is the annihilator of some element of M. If R is $\mathbb{N}^n$-graded ring, an ideal P is called homogeneous if it is generated by homogeneous elements.

Observe that $U \subset V$ implies that $Ann(V) \subset Ann(U)$. So if we restrict our attention to the prime ideals associated to M, then we have partial order on the set of these ideals, denoted by $Ass(M)$. Minimal element of this set is called minimal, and other elements are called embedded. In [26], they showed that any associated prime to a finitely generated $\mathbb{N}^n$-graded S-module is homogeneous and it is of the form $P = \langle x_{i_1}, \cdots x_{i_k} \rangle$ where $1 \leq i_j \leq n$. By using these, the information about the direction of generators in which they live can be extracted.

Definition 4.5.5. *Let M be a finitely generated $\mathbb{N}^n$-graded S-module with associated prime $P = \langle x_{i_1}, \cdots x_{i_k} \rangle$. The support of P is defined as*

$$c_P = \{(u_1, \ldots, u_n) \in \mathbb{N}^n \mid u_i = 0 \forall i \in \{i_1, \cdots i_k\}\}$$

The support shape of M is a subset of $\mathbb{N}^n$ so that $ss(M) = \bigcup_{P \in Ass(M)} c_P$.

The idea behind of these definitions comes from the vanishing set of an ideal $I \subset S$, in other words $V(I) = c_I$. The next example is an illustration of this definition.

Example 4.5.3. *1. Let $M = S/(x)$ and $M = S/(y)$ where $S = k[x, y]$, these modules are not isomorphic. We have $Ass(M) = \{0, (x)\}$ and $Ass(N) = \{0, (y)\}$ which will give us $ss(M) = c_{(x)} = \{(0, a) \mid a \in \mathbb{N}\}$ and $ss(N) = c_{(y)} = \{(a, 0) \mid a \in \mathbb{N}\}$. These modules can be distinguishable by support shape.*

2. Let $M = S(0, 2) \oplus S(0, 2) \oplus S(1, 1) \oplus S(0, 3)$ and $N = S(0, 2) \oplus S(0, 2) \oplus S(1, 2) \oplus S(3, 0) \oplus S(1, 1)/(y)$. Since M is free, $Ass(M) = \{(0)\}$ and for N we have $Ass(N) = \{0, (y)\}$. We can evaluate support shape as $ss(M) = c_{(0)}$ and $ss(N) = c_{(0)} \cup c_{(y)}$. Even though we have different number of component in support shapes, $ss(M) = ss(N)$ since $c_{(0)} = \mathbb{N}^2$. This means that support shape cannot discriminate if modules contains free parts.

Overall, we can see that the support shape cannot distinguish the number or the position of generators, also the life of these generators. The next definition will help us to layer these support of prime ideals.

Definition 4.5.6. *For a given nested sequence of associated primes in $Ass(M)$ so that $P_0 \subset P_1 \subset \cdots \subset P_m$, then the stratification of $ss(M)$ is defined as to be nested sequence $c_{P_m} \subset \cdots c_{P_1} \subset c_{P_0} \subset ss(M)$.*

Example 4.5.4. *1. Let M and N be the same in the second part of Ex 4.5.3. The stratification of $ss(M)$ has only one component which is $ss(M)$ itself, but stratification of $ss(N)$ is $c_{(y)} \subset ss(N)$. Therefore, these two module can be distinguishable by this method.*

2. Let $M = (S(v_1, v_2)/(y^n))^m$ for some $n, m \in \mathbb{N}$ and $v \in \mathbb{N}^2$. Observe that (y) is an associated to M since it is annihilator of the set of elements in the degree (i, n) for $i \geq v_1$. So support of (y) is $c_{(y)} = \{(a, 0) \in \mathbb{N}^2\}$. There is no other associated prime ideal to M since the only relation is in the direction y, thus $ss(M) = c_{(y)}$. Hence the stratification is $c_{(y)}$. So for any n, m and (v_1, v_2)

values, we have a different modules but these modules will give the same result in stratification method.

Unfortunately, this method still does not give a complete invariant as we observed from the above examples.

CHAPTER 5

STABILITY

Whatever the input type will be, our primary goal is to present the information in this data with filtration. After applying the homology functor to this filtration, we will get a persistence module. This application pipeline seems perfect, but there is a problem at the beginning where we convert this data to filtration. Most of the time, data is too extensive and detailed. That is why we need to use an approximation. We need to make sure that we will not lose any information on the shape of the data by doing this. Therefore, we need to state that a small change in the data will imply a slight change in the output. This statement is called stability in persistence theory.

5.1 One Parametered Case

Up to this point, we present the given data in barcodes or persistence diagrams. There are two common types of one dimensional parametrized data: data induced by a real-valued function and a point cloud data. As you may notice, the distance on the input depends on the type of data.

The stability of persistence diagrams was studied by Cohen-Steiner, Edelsbrunner, and Harer in [17] for the first time. We will present their result first, then explain the contributions of the others.

We will start with the definitions of distance on the output. Recall that, persistence diagram is a set of points in $\overline{\mathbb{R}}^2$ with multiplicity, therefore they are multisets. We will use the L_∞ on $\mathbb{R} \times \overline{\mathbb{R}}$ which is $||(p_1, q_1) - (p_2, q_2)||_\infty = max\{|p_1 - p_2|, |q_1 - q_2|)\}$.

Definition 5.1.1. *Let X and Y be two multisets in $\mathbb{R} \times \overline{\mathbb{R}}$. Let Γ be the set of all* *bijections from X to Y. The Hausdorff distance between X and Y is*

$$d_H(X,Y) = max\{sup_{x\in X} inf_{y\in Y}||x-y||_\infty, sup_{y\in Y} inf_{x\in X}||y-x||_\infty\}$$

The bottleneck distance between X and Y defined as

$$d_B(X,Y) = inf_{\gamma\in\Gamma} sup_{x\in X}||x-\gamma(x)||_\infty$$

We know that X and Y are finite sets. Let us consider points in $X \cap \mathbb{R}^2$ and $Y \cap \mathbb{R}^2$. Then

$$\begin{aligned} ||x-y||_\infty &\leq sup_{x\in X}||x-y||_\infty \\ inf_{y\in Y}||x-y||_\infty &\leq inf_{y\in Y} sup_{x\in X}||x-y||_\infty \\ sup_{x\in X} inf_{y\in Y}||x-y||_\infty &\leq inf_{y\in Y} sup_{x\in X}||x-y||_\infty \end{aligned}$$

For any bijection from X to Y which may have domain and range as subsets of X and Y, we can say that $inf_{y\in Y} sup_{x\in X}||x-y||_\infty \leq inf_{\gamma\in\Gamma} sup_{x\in X}||x-\gamma(x)||_\infty$. Similarly we can say $sup_{y\in Y} inf_{x\in X}||y-x||_\infty \leq inf_{\gamma\in\Gamma} sup_{x\in X}||x-\gamma(x)||_\infty$. Thus for the points in $\mathbb{R}^2$, $d_H \leq d_B$. For the case (r,∞), the same thing still true. Therefore bottleneck distance is bounded below by Hausdorff distance. The computation of Hausdorff distance is easier than bottleneck distance. We were using tame functions which means they are bounded, thus supremum metric on the functions, defined on the same topological space $\mathbb{X}$, which is $||f-g||_\infty = sup_{x\in\mathbb{X}}|f(x)-g(x)|$.

Theorem 5.1.1. *[17, Stability Theorem] Let $f, g : \mathbb{X} \to \mathbb{R}$ be tame functions where $\mathbb{X}$ is triangulable space. Then we have bound for the distance between persistence diagrams $D(f)$ and $D(g)$*

$$d_B(D(f), D(g)) \leq ||f-g||_\infty$$

We already stated that $d_H(X,Y) \leq d_B(D(f), D(g))$, so the same stability statement is true for Hausdorff distance.

Example 5.1.1. *Let f be represented as blue and g as orange in the Fig5.1. In the filtration, we are interested in the points so that $H_p(\mathbb{X}_{r-\epsilon}; k) \to H_p(\mathbb{X}_{r+\epsilon}; k)$ is not isomorphism for all $\epsilon > 0$. These points are called critical points. We reduce $\mathbb{R}$-filtered space to $\mathbb{N}$-filtered with these critical points. We will use field coefficients*

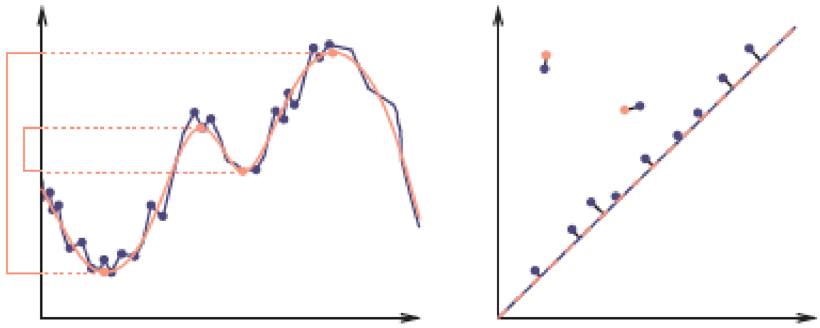


Figure 5.1: Persistence diagrams of the sublevel sets of f and g in [17]

since it is reasonable to use decomposition of $\mathbb{N}$*-graded* $k[x]$*-module, giving the information in the persistence diagram.*

Observe that f *has multiple critical points where* g *has only 4. On the right, we see that nine classes have a short life. These are called noise because they would not reflect the actual shape of the data formed by* f*. The function* g *will have two classes, and the life of these classes is shown on the left of the vertical axis in the first picture in Fig5.1. The function* g *can be seen as an approximation of* f*, and the distance* $||f - g||_\infty$ *will bound the distance between persistence diagrams. Bottleneck distance looks for the possible bijection between subsets of each diagram and chooses the least one. In our case, the matching between points is given in the right.*

This stability theorem needs two conditions: $\mathbb{X}$ needs to be triangulable, and f needs to be a tame real-valued function on $\mathbb{X}$. In [16], Chazal et al. dropped some of the restrictions on the stability theorem. They remove the continuity condition on the function, and space need not be triangulable anymore. Also, the finiteness condition on the number of critical values of the filtration is loosened, and functions can be defined on different topological spaces. The main result gained by taking input as a persistence module in ${}_kMod^{\mathbb{R}}$ where k is a field.

The following definition will give a distance between two pointwise finite-dimensional persistence module over $\mathbb{R}$ stated in [16].

Definition 5.1.2. *Let* M *and* N *be two persistence modules over* $\mathbb{R}$*. The* ϵ*-shift of* M *defined as* $M(\epsilon)_u = M_{u+\epsilon}$ *for all* $u \in \mathbb{R}$ *and* ϵ*-shift of a map* $\phi : M \to N$ *is* $\phi(\epsilon) : M(\epsilon) \to N(\epsilon)$*. The two persistence module called* ϵ*-interleaved if there exists*

$\phi : M \to N(\epsilon)$ and $\psi : N \to M(\epsilon)$ such that

$$\psi(\epsilon) \circ \phi = \varphi_M^{2\epsilon}$$

$$\phi(\epsilon) \circ \psi = \varphi_N^{2\epsilon}$$

where $\varphi_M^{2\epsilon}$ and $\varphi_N^{2\epsilon}$ are transition maps.

The motivation of this definition comes from the observation of nested relation on filtration obtained by sublevel sets of $f, g : \mathbb{X} \to \mathbb{R}$. If we have $||f - g||_\infty \leq \epsilon$, then $\mathbb{X}_r^f \subset \mathbb{X}_{r+\epsilon}^g \subset \mathbb{X}_{r+2\epsilon}^f$. This would induces these conditions in the definition of ϵ-interleaved.

Theorem 5.1.2. *[16, Algebraic Stability Theorem] If M and N are ϵ-interleaved pointwise finite-dimensional persistence modules in ${}_k\mathbf{Mod}^{\mathbb{R}}$, then the distance between persistence diagrams $D(M), D(N)$ of M and N is bounded above by ϵ i.e. $d_B(D(M), D(N)) \leq \epsilon$.*

The most important result of this theorem is that we can directly compare the persistence modules. Thus we can compare any two functions or even filtrations of different types of data.

Observe that if two pointwise finite-dimensional persistence modules over $\mathbb{R}$ are ϵ-interleaved, then they will be ϵ'-interleaved for all $\epsilon' \geq \epsilon$. The interleaving maps combined with transition maps from ϵ to ϵ' will give us ϵ'-interlaving. Since this ϵ forms a lower bound, it is reasonable to look for the least one.

Definition 5.1.3. *Let M and N be in ${}_kMod^{\mathbb{R}}$ and be pointwise finite-dimensional. Then the interleaving distance is defined as*

$$d_I(M, N) = inf\{\epsilon \in [0, \infty) \mid M \textit{ and } N \textit{ is } \epsilon\textit{-interleaved}\}$$

and $d_I(M, N) = \infty$ if M and N are not ϵ-interleaved for any $\epsilon \geq 0$

This would be extended pseudometric which means it takes values in $[0, \infty]$ and it differs from a metric in the contidion $d(x, y) = 0$ if and only if $x = y$. Instead of this condition, we have $d(x, x) = 0$ for all x.

Clearly, $d_I(M, M) = 0$ and $d_I(M, N) = d_I(N, M)$ so we need to show only triangle inequality. Let $d_I(M, N) = \epsilon$ and $d_I(N, L) = \epsilon'$, then for any $\delta \geq 0$ we know that M and N are $\epsilon + \delta$-interleaved with morphisims f_{MN} and g_{NM}. Similarly we have f_{NL} and g_{LN} for $\epsilon' + \delta$-interleaved modules N and L. Now define $f = f_{NL}(\epsilon + \delta) \circ f_{MN}$ and $g = g_{NM}(\epsilon' + \delta) \circ g_{LN}$. These morphisms will induce a $\epsilon + \epsilon' + 2\delta$-interleaving between M and L. Since it is true for all $\delta \geq 0$, we have $d_I(M, L) \geq \epsilon + \epsilon'$. Therefore, $d_I(M, N) + d_I(N, L) = \epsilon + \epsilon' \leq d_I(M, L)$.

Another observation about interleaving distance is that if two modules are isomorphic, then their distance with a third module will be the same. By combining the isomorphism map with interleaving maps, we will get this result.

Bauer and Lesnick expanded the algebraic stability theorem to an equality which is called isometry theorem in [2], and we can see the algebraic stability theorem for barcodes as a direct result of isometry theorem. The advantage of dealing with barcodes is to get rid of the multiplicity matching problem in the persistence diagram. It is easy to match two intervals rather than two points in the multiset. The following definition, the distance between two barcodes, will be needed to state the isometry theorem.

Definition 5.1.4. *Let C and D be two barcodes of persistence modules in ${}_kMod^{\mathbb{R}}$ and $\epsilon > 0$. The subset of D is defined as*

$$D_\epsilon = \{I \in D \mid [t, t + \epsilon] \subset I \; for \; some \; t \in \mathbb{R}\}$$

Let I be an interval in $\mathbb{R}$, then ϵ-extension is defined as

$$Ex^\epsilon(I) = \{t \in \mathbb{R} \mid \exists s \in I \; with \; |s - t| \leq \epsilon\}$$

An ϵ-matching between C and D is defined as a bijection between subsets $C' \subset C$ and $D' \subset D$ i.e. $\sigma : C' \to D'$ with the properties

1. $C_{2\epsilon} \subset C'$
2. $D_{2\epsilon} \subset D'$
3. *if* $\sigma(I) = J$ *then* $I \subset Ex^\epsilon(J)$ *and* $J \subset Ex^\epsilon(I)$

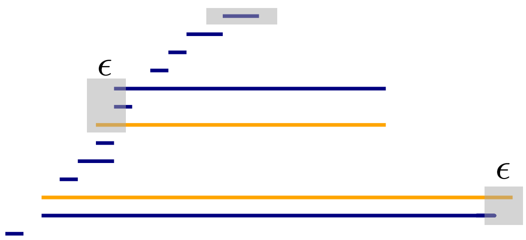


Figure 5.2: Barcodes of Example 5.1.1 with $\epsilon-$matching

The corresponding bottleneck distance between C and D is

$$d_B(C, D) = inf\{\epsilon \in [0, \infty) \mid \exists \text{ an } \epsilon\text{-matching between } C \text{ and } D\}$$

and $d_B(C, D) = \infty$ if there is no ϵ-matching for any $\epsilon \geq 0$.

The $D_{2\epsilon}$ containment condition will remove the unwanted intervals which are very small in barcodes. In the next example, we will explain how this matching works.

Example 5.1.2. *Let f and g be the same as Example 5.1.1. Barcodes of these functions given in the Fig 5.2 with the length ϵ. Blue intervals represent the 0-dimensional classes in the filtration formed by f, and orange ones represent g. Call them C and D, respectively. Observe that for the given ϵ, C' can contain two intervals that are longer than 2ϵ. Therefore C' can contain only two intervals. The same thing is true for D', and it can be equal to D. Thus we have the same cardinality for both C' and D', and the third condition will give us only one matching for these barcodes, which are illustrated in Fig5.2. The bottleneck distance between these barcodes will be the infimum of such ϵ's.*

The bottleneck distance can be defined in terms of interleaving distance. Recall that we have interval decomposition in ${}_kMod^{\mathbb{R}}$ by Theorem 3.3.2, and we can see every interval persistence module will represent the interval in the barcodes. This correspondence is used to connect the idea of ϵ-shift in persistence modules and ϵ-matching in [10]. The ϵ-matching between intervals in two barcodes will induce function f between the index intervals I of interval persistence modules $\mathcal{I}$. Thus the bottleneck distance between two barcodes can be rewritten as

$$d_B(C, D) = \inf_f \sup_I d_I(\mathcal{I}, f(\mathcal{I}))$$

Now, we can state the isometry theorem, which implies the algebraic stability theorem.

Theorem 5.1.3. *[2, Isometry Theorem] Let M and N be two pointwise finite dimensional persistence modules in ${}_k Mod^{\mathbb{R}}$ with barcodes C and D. Two persistence modules are ϵ-interleaved if and only if there exists a ϵ-matching between C and D. In other words,*

$$d_B(C, D) = d_I(M, N)$$

The isometry theorem is also true more general case. If rank invariant, which is the image of M_u in M_v, is finite, then the persistence module is called q-tame. Every pointwise finite-dimensional persistence modules and finite type modules are q-tame. The isometry theorem for q-tame modules is stated in [2] by Bauer and Lesnick.

The stability of the real parameter persistence module over arbitrary small category $\mathbb{A}$ was studied in [33] by McCleary and Patel. Recall that Patel defined a persistence diagram for the constructible persistence modules, which have only finitely many different components, in other words, $M_u \neq M_v$ for finitely many $u, v \in \mathbb{R}$. We have a similar definition for ϵ-interleaving for objects in $\mathbf{A}^{\mathbb{R}}$ stated in [35] by Patel. ϵ-shift will be the same, and it would not affect constructibility of the persistence modules since there will still be finitely many different components. Like we have in definition 5.1.2, two constructible persistence modules are called ϵ-interleaved if there exists a map between them which respects the shift operation. Thus interleaving distance is defined to be the minimum of these ϵ's.

Next, we will explain a distance between the representations of these constructible modules, which is given in Definition 3.2.3. Recall that persistence diagram for a constructible module is a map from $\mathbf{Dgm}$; the collection of the bounded below open intervals in $\mathbb{R}$ to an abelian group G with partial ordering and have the property that if $a \leq b$ then $a + c \leq b + c$ for any $c \in G$.

Definition 5.1.5. *Let $X_1, X_2 : \mathbf{Dgm} \to G$ be two persistence diagrams. A matching between X_1 and X_2 is a nonnegative map $\gamma : \mathbf{Dgm} \times \mathbf{Dgm} \to G$ which means $\gamma(I, J) \geq e_G$ for all I and $J \in \mathbf{Dgm}$ satisfying the following property*

$$X_1(I) = \sum_{J \in \mathbf{Dgm}} \gamma(I, J)$$

$$X_2(J) = \sum_{I \in \mathbf{Dgm}} \gamma(I, J)$$

for all $I, J \in \mathbf{Dgm}$. The bottleneck distance between X_1 and X_2 is defined as

$$d_B(X_1, X_2) = \inf_{\gamma} \max_{\{I,J \mid \gamma(I,J) \neq e_G\}} \{|a_1 - a_2|, |b_1 - b_2|\}$$

where $I = (a_1, b_1)$ and $J = (a_2, b_2)$.

Recall that the persistence diagram in ${}_k\mathbf{Mod}^{\mathbb{R}}$ is a finite map for a finite set S, which is a Möbius inversion of rank function. A finite map means $Y(I) \neq e_G$ if and only if I has endpoints in S. In other words, $Y : \mathbf{Dgm} \to G$ so that $X(I) = \sum_{J \subseteq I} Y(J)$ be the persistence diagram for M where X is the rank function for $M \in_k \mathbf{Mod}^{\mathbb{R}}$. The persistence diagram counts the number of classes that lived in (a, b) where the rank function only counts the number of classes born at a and die at b.

The condition on the matching guarantees that if there is no class lived in the interval, then it will not be matched. In other words, if there is no class in I, then $X(I) = e_G$ which implies $\gamma(I, J) = e_G$ for all $J \in \mathbf{Dgm}$ since $\gamma \geq e_G$.

The following theorem is given by McCleary and Patel for the bottleneck stability result in [33].

Theorem 5.1.4. *(Bottleneck stability of real parametered persistence module) Let X and Y be two constructible persistence modules in $\mathbf{A}^{\mathbb{R}}$ where $\mathbf{A}$ is a small category. Then $d_B(X', Y') \leq d_I(X, Y)$ where X', Y' are the persistence diagrams for X, Y, respectively.*

This theorem can be seen as a generalization of algebraic stability theorem for ${}_k\mathbf{Mod}^{\mathbb{R}}$ to an arbitrary abelian category. In the next section, we will examine the stability results for persistence modules over an arbitrary preordered set.

5.2 Generalized Case

Bubenik, de Silva, and Scott studied the persistence modules $\mathbf{A}^{\mathbf{P}}$ in [8] where $\mathbf{P}$ is a category of an arbitrary preordered sets P and $\mathbf{A}$ is an arbitrary category. In this general setting, there exist two types of stability theorems described by Bubenik, de Silva, and Scott: soft and hard. When we looked at the workflow in persistence theory for point cloud data, we should first convert it to a persistence module over simplicial complexes or filtered topological space. Then, we will apply the homology functor to this filtration, which results in a persistence module over some category dependent on the coefficient in homology. If we again follow these steps, we will see the pipeline described at [8]: data, generalized persistence module over $\mathbf{Simp}$ or $\mathbf{Top}$, generalized persistence module over $\mathbf{A}$ and an invariant. The soft stability theorem includes only the first two steps, and the hard stability theorem is concerned with the last step. In order words, the soft stability theorem involved the stability of transformation of the data to the persistence module. The hard stability theorem is about the representation of the persistence module.

We will start with the definition of interleaving between two generalized persistence modules. For that, we will need some transition notion in the preordered set P, as we defined in the definition 2.0.5. We will follow the definitions in [8].

Definition 5.2.1. *Let P be a preordered set with the order relation $\leq$. A translation $\Gamma : P \to P$ is defined to be a function so that $x \leq \Gamma(x)$ for every $x \in P$. $\mathbf{Trans_P}$ will denote the set of all transitions of P.*

$\mathbf{Trans_P}$ forms a monoid since composition operation is an associative binary operation and identity transition exists. Each transition defines a relation on generalized persistence modules with the action of the monoid on $\mathbf{A^P}$. The action of Γ on $F \in \mathbf{A^P}$ is $\Gamma \cdot F = F\Gamma$ where $F\Gamma_x = F_{\Gamma(x)}$ for every $x \in P$. Two consecutive operation would be $\Gamma K \cdot F = K \cdot (F\Gamma) = F(\Gamma K)$. This operation is called shift map. The relation between two translation defined as

$$\Gamma \leq K \leftrightarrow \Gamma(x) \leq K(x)\, \forall x \in P$$

Definition 5.2.2. *Let P be preordered set and $\Gamma, K \in \mathbf{Trans_P}$. Let F and G be generalized persistence modules in $\mathbf{A^P}$. F and G is called (Γ, K)-interleaving if there exists morphisms $\phi : F \to G\Gamma$ and $\psi : G \to FK$ such that*

$$\psi(\Gamma)\phi = \varphi_F^{K\Gamma}$$

$$\phi(K)\psi = \varphi_G^{\Gamma K}$$

where $\varphi_F^{K\Gamma}$ and $\varphi_F^{\Gamma K}$ are transition maps. If there exists an (Γ, K)-interleaving between F and G then they are called (Γ, K)-interleaved. If F and G are (Γ, Γ)-interleaved then they are called Γ-interleaved.

Bubenik, de Silva and Scott suggested two ways to define a metric on $\mathbf{A^P}$ in [8]. The first one is embedding $\mathbf{Trans_P}$ to $[0, \infty)$ and the second one is doing the reverse. To do the first one, we will need a function $\omega : \mathbf{Trans_P} \to [0, \infty)$ so that $\omega(I) = 0$ where I is identity transition and $\omega(\Gamma_1\Gamma_2) \leq \omega(\Gamma_1) + \omega(\Gamma_2)$. This function ω is called sublinear projection.

Definition 5.2.3. *Let $\Gamma, K \in \mathbf{Trans_P}$, $F, G \in \mathbf{A^P}$ and ω be a sublinear projection. Γ is called ϵ-translation if $\omega(\Gamma) \leq \epsilon$. F and G are called ϵ-interleaved with respects to ω if F and G are (Γ, K)-interleaved where both Γ and K are ϵ-translations. The interleaving distance between F and G is defined as*

$$d_I^{\omega}(F, G) = inf\{\epsilon \in [0, \infty) \mid F, G \ \epsilon\textit{-interleaved with respect to } \omega\}$$

and $d_I^{\omega}(F, G) = \infty$ if F and G are not ϵ-interleaved with respect to ω for any $\epsilon \in [0, \infty)$.

The following proposition stated and proved in [8]. For the completeness, we will prove it here too.

Proposition 5.2.1. *d_I^{ω} is an extended pseudometric on $\mathbf{A^P}$.*

Proof. It is clear that d_I^{ω} is reflexive and symmetric. Let F, G be (Γ_1, K_1)-interleaved and G, H be (Γ_2, K_2)-interleaved. Then we have morphisms commutes with transitions maps

$$\phi_1 : F \to G\Gamma_1 \ \psi_1 : G \to FK_1$$
$$\phi_2 : G \to H\Gamma_2 \ \psi_2 : H \to GK_2$$

Let ϕ and ψ defined as follows $\phi = \phi_2\Gamma_1 \circ \phi_1$ and $\psi = \psi_1(K_2) \circ \psi_2$. We know that ψ_1, ψ_2, ϕ_1 and ϕ_2 are morphisms and they commute with transition maps, by using this fact

$$
\begin{aligned}
\psi(\Gamma_2\Gamma_1)\circ\phi &= (\psi_1(K_2)\circ\psi_2)(\Gamma_2\Gamma_1)\circ\phi_2\Gamma_1\circ\phi_1\\
&= \psi_1(K_2\Gamma_2\Gamma_1)\circ\psi_2(\Gamma_2\Gamma_1)\circ\phi_2\Gamma_1\circ\phi_1\\
&= \psi_1(K_2\Gamma_2\Gamma_1)\circ(\varphi_G^{K_2\Gamma_2})\Gamma_1\circ\phi_1\\
&= \varphi_{FK_1}^{K_2\Gamma_2}\Gamma_1\circ\psi_1\Gamma_1\circ\phi_1\\
&= \varphi_{FK_1}^{K_2\Gamma_2}\Gamma_1\circ\varphi_F^{K_1\Gamma_1}\\
&= \varphi_F^{K_1K_2\Gamma_2\Gamma_1}
\end{aligned}
$$

Similarly, we have $\phi(K_1K_2)\circ\psi = \varphi_H^{\Gamma_2\Gamma_1K_1K_2}$, which implies that F and H are $(\Gamma_2\Gamma_1, K_1K_2)$-interleaved.

To show triangle inequality, let $d_I^\omega(F,G)=\epsilon_1$ and $d_I^\omega(G,H)=\epsilon_2$ then

$\omega(\Gamma_1),\omega(K_1)\geq\epsilon_1$ and $\omega(\Gamma_2),\omega(K_2)\geq\epsilon_2$. We know that

$$d_I^\omega(F,H)\leq max\{\omega(\Gamma_2\Gamma_1),\omega(K_1K_2)\}$$

Since ω is sublinear projection, $\omega(\Gamma_2\Gamma_1)\leq\omega(\Gamma_2)+\omega(\Gamma_1)\leq\epsilon_2+\epsilon_1$ and $\omega(K_1K_2)\leq\omega(K_1)+\omega(K_2)\leq\epsilon_1+\epsilon_2$. Therefore $d_I^\omega(F,H)\leq\epsilon_1+\epsilon_2=d_I^\omega(F,G)+d_I^\omega(G,H)$. If one of the distances ∞ , this inequality still holds. Thus d_I^ω is extended pseudo-metric on $\mathbf{A}^\mathbf{P}$. □

Theorem 5.2.1. *([8])Soft Stability Theorem for d_I^ω Let F and G be two generalized persistence modules in $\mathbf{A}^\mathbf{P}$ and ω be sublinear projection on $\mathbf{Trans_P}$. If $H:\mathbf{A}\to\mathbf{B}$ is a functor , then $d^w(HF,HG)\leq d^\omega(F,G)$.*

This is a consequence of the functoriality of interleaving, which is shown in [8]. By applying the functor H to the morphisms in the interleaving, we will get the result.

The second method which embeds $[0,\infty]$ to $\mathbf{Trans_P}$ will gives another metric. The idea is very similar to the previous one. Let Ω be a function from $[0,\infty]$ to $\mathbf{A}^\mathbf{P}$ so that $\Omega(\epsilon_1+\epsilon_2)\geq\Omega(\epsilon_1)\Omega(\epsilon_2)$ for all $\epsilon_1,\epsilon_2\in[0,\infty]$. This function is called superlinear family.

Definition 5.2.4. *The interleaving distance between $F,G\in\mathbf{A}^\mathbf{P}$ is defined as*

$$d_I^\Omega(F,G)=inf\{\epsilon\in[0,\infty)\ F,G\ are\ \Omega(\epsilon)-interleaved\}$$

and $d_I^{\Omega}(F,G) = \infty$ *if they are not interleaved.*

This will again induce pseudometric because triangle inequality is provided by the superlinearity and functoriality conditions. Functoriality directly implies stability results for this metric. The previous two theorems will give us soft stability results if the map from data to generalized persistence module is stable.

Bubenic, de Silva, and Scott compared these two metrics and found a condition that makes these two metrics equivalent.

Theorem 5.2.2. *([8]) Let* w *be a sublinear projection on* $\mathbf{P}$*. Suppose there exists* $\Omega(\epsilon) \in \mathbf{Trans_P}$ *for every* $\epsilon \geq 0$ *so that* $\omega(\Omega(\epsilon)) \leq \epsilon$ *and for any* $\Gamma \in \mathbf{Trans_P}$ *with* $\omega(\Gamma) \leq \epsilon$ *implies that* $\Gamma \leq \Omega(\epsilon)$*. Then* $\Omega(\epsilon)$ *forms a superlinear family and we have* $d_I^{\omega} = d_I^{\Omega}$*.*

Up to this point, we looked for pseudometrics which can be the same or the generalization of bottleneck distance. Thanks to the isometry theorem, we know that in one dimension, interleaving distance on pointwise finite-dimensional persistence modules in ${}_k\mathbf{Mod}^{\mathbb{R}}$ is the same with bottleneck distance. In dimension two, this is not the case. We will explain this later.

5.2.1 Multidimensional Case

In the previous section, we examined the soft stability of the persistence module over any preordered set; we would like to investigate the stability results for ${}_k\mathbf{Mod}^{\mathbb{R}^n}$.

Remember that we started with bottleneck distance for ${}_k\mathbf{Mod}^{\mathbb{R}}$ and the main goal is to find a generalization of this bottleneck distance for ${}_k\mathbf{Mod}^{\mathbb{R}^n}$. In one dimensional case, the interleaving distance is equivalent to bottleneck, so the first suggestion is interleaving distance.

Lesnick proposed an interleaving distance that is stable in [31]. Also, by the universality result, it is shown that this distance has a high value of discrimination, and it is unique in this sense.

Lesnick used the definition of transition, and the interleaving, which is similar to the ones used by Bubenik, de Silva, and Scott in [8]. These definitions are inspired from [16] stated by Chazal et al.

Recall that a transition Γ map which is a function from preordered set to itself is needed to define interleaving. In multidimensional case, Lesnick used $\Gamma = \vec{\epsilon} : \mathbb{R}^n \to \mathbb{R}^n$ so that $\vec{\epsilon}(x_1, \cdots, x_n) = (x_1 + \epsilon, \cdots, x_n + \epsilon)$ in [31]. In that paper, embedding $\mathbf{Trans}_{\mathbb{R}^n}$ to $[0, \infty)$ was preferred by sublinear projection $\omega(\vec{\epsilon}) = \epsilon$. Thus interleaving distance given as $d^\omega = d_I$. To make it clear, we will rewrite the definitions. Any two persistence module in ${}_k\mathbf{Mod}^{\mathbb{R}^n}$ is ϵ interleaved if they are $(\vec{\epsilon}, \vec{\epsilon})$-interleaved. The interleaving distance for $F, G \in_k \mathbf{Mod}^{\mathbb{R}^n}$ is defined as

$$d_I(F, G) = inf\{\epsilon \geq 0 \,|F, G \text{ are } \epsilon\text{-interleaved}\}$$

and $d_I(F, G) = \infty$ if they are not ϵ-interleaved for any $\epsilon \geq 0$.

The stability result for ${}_k\mathbf{Mod}^{\mathbb{R}^n}$ was stated only for the ones formed by sublevelset filtration.

Definition 5.2.5. *Let $\gamma^X : X \to \mathbb{R}^n$ and $\gamma^Y : Y \to \mathbb{R}^n$ be two functions on topological spaces X and Y.*

- $||\gamma||_\infty$ *defined as* $\sup_X\{||\gamma(x)||_\infty\}$ *if X is nonempty otherwise 0.*
- $d_\infty(\gamma^X, \gamma^Y) = \inf_{h \in \mathcal{H}} ||\gamma^X - \gamma^Y \circ h||_\infty$ *where $\mathcal{H}$ is the set of all homeomorphisms from X to Y, and $d_\infty(\gamma^X, \gamma^Y) = \infty$ if they are not homeomorphic.*

Theorem 5.2.3. *(Stability of interleaving distance with respect to multidimensional sublevelset filtration [31]) Let $\gamma^X : X \to \mathbb{R}^n$ and $\gamma^Y : Y \to \mathbb{R}^n$ be two functions on topological spaces X and Y. Then $d_I(X_i, Y_i) \leq d_\infty(\gamma^X, \gamma^Y)$ for all $i \geq 0$ where X_i, Y_i are i-th homology groups of sublevelset of γ^X and γ^Y, respectively.*

The proof of this theorem is straightforward in [31]; if $d_\infty(\gamma^X, \gamma^Y) = \epsilon$ then there will be a homeomorphism between sublevelsets of γ^X and $\gamma^Y \circ h$ with δ-shift where $\delta \geq \epsilon$ and h is a homeomorphism between X and Y. This induces δ-interleaving between persistence modules induces by X_i and $(X \circ h)_i$ for any $i \geq 0$. By using the fact that Y_i and $(X \circ h)_i$ are isomorphic, we have X_i and Y_i are δ-interleaved which means $d_I(X_i, Y_i) \leq \epsilon$ for all i.

Other stability results depending on different filtration methods given by Lesnick in Ph.D. Thesis [32]. Another important result about interleaving distance is the universality which means that this distance is optimal in the sense of distinguishability and stability on the sublevel set filtration. To our knowledge, the universality result is not yet stated for other filtration types.

Theorem 5.2.4. *[31] Let $k = \mathbb{Q}$ or $\mathbb{Z}_p$ for prime p, and X_i, Y_i be i-th homology group of sublevelset of γ_X and γ_Y. d' is another stable pseudometric if and only if $d'(X_i, Y_i) \leq d_I(X_i, Y_i)$ for all $i \geq 0$.*

This result reduces the stability problem checking the relation with the interleaving distance. However, there is a computational problem of interleaving distance for the multidimensional case. Bjerkevik, Botnan, and Kerber showed that computation of interleaving distance for persistence modules in ${}_k\mathbf{Mod}^{\mathbb{R}^n}$ is NP-hard for $n \geq 2$ in [4].

Another suggestion is the generalization of bottleneck distance directly for pointwise finite-dimensional ${}_k\mathbf{Mod}^{\mathbb{R}^n}$ by using the decomposition theorem 3.3.5 which is stated for any preordered set. It was first generalized by Botnan and Lesnick in [7].

Definition 5.2.6. *Let M and N be two pointwise finite dimensional persistence modules with indecomposables M_i and N_j where $i \in I = \{1, \cdots, n\}$ and $j \in J = \{1, \cdots, m\}$. M is called ϵ-trivial if $d_I(M, 0) = \epsilon$ where 0 is the trivial persistence module. A matching between two multisets I and J is a bijection σ between $I' \subseteq I$ and $J' \subseteq J$. M and N are ϵ-matched for $\epsilon \geq 0$ if there exists a matching $\sigma : I' \to J'$ between I and J so that*

1. *If $i \in I \setminus I'$, then M_i is 2ϵ-trivial.*
2. *If $j \in J \setminus J'$, then N_i is 2ϵ-trivial.*
3. *If $\sigma(i) = j$ then M_i and N_j are ϵ-interleaved.*

The bottleneck distance between M and N is defined as

$$d_B(M, N) = \inf \epsilon \geq 0 \mid M \text{ and } N \text{ are } \epsilon\text{-matching}$$

Lesnick and Botnan realized that converse algebraic stability result for interval decomposable persistence modules in ${}_k\mathbf{Mod}^{\mathbb{R}^n}$. This is the one side of the isometry theorem in one dimension.

Theorem 5.2.5. *(Converse algebraic stability theorem [7]) Let M and N be interval decomposable modules in ${}_k\mathbf{Mod}^{\mathbb{R}^n}$. Then we have*

$$d_I(M,N) \leq d_B(M,N)$$

For rectangle decomposable persistence modules in ${}_k\mathbf{Mod}^{\mathbb{R}^n}$, the converse relationship between bottleneck and interleaving distance stated in [1] by Bjerkevik. The rectangle in $\mathbb{R}^n$ defined as a product of intervals in $\mathbb{R}$, so it can be written as as $I = I_1 \times \cdots I_n$ where $I_i = (a,b)$ is an interval in $\mathbb{R}$ for all $1 \leq i \leq n$. The difference between an interval and a rectangle in $\mathbb{R}^n$ when $n \geq 2$ is that a class lived in only one direction can be represented by an interval but not by a rectangle. They will be the same when $n = 1$. Recall that, rectangle persistence module is a functor k_R from R to ${}_k\mathbf{Mod}$ where R is a rectangle in $\mathbb{R}^n$.

Theorem 5.2.6. *[1] Let M and N be rectangle decomposable persistence modules in ${}_k\mathbf{Mod}^{\mathbb{R}^n}$. Then*

$$d_B(M,N) \leq (2n-1)d_I(M,N)$$

Observe that when $n = 1$, it is the algebraic stability theorem 5.1.2. When $n = 2$, Bjerkevik gave an example in which $d_B = 3d_I$ for rectangle decomposable persistence modules in ${}_k\mathbf{Mod}^{\mathbb{R}^2}$ in [1]. This means the isometry theorem can not be generalized for rectangle decomposable persistence modules in ${}_k\mathbf{Mod}^{\mathbb{R}^2}$. This is not the case for block decomposable modules.

Theorem 5.2.7. *[1] Let M and N be block decomposable persistence modules in ${}_k\mathbf{Mod}^{\mathbb{R}^2}$. Then*

$$d_B(M,N) = d_I(M,N)$$

In general, we will not get a nice decomposition for a given filtration. For practical reasons, the matching distance is proposed to be a generalization of the bottleneck distance which is defined on the rank invariant. Stability results were shown for ${}_k\mathbf{Mod}^{\mathbb{R}}$ and ${}_k\mathbf{Mod}^{\mathbb{R}^n}$ in [15] by Cerri et al. These modules were induced by a sublevel set of continuous functions on triangulable spaces. Although the study in [15] is not the first one on the stability of multidimensional rank function, the statement of stability does not contain tameness condition on vector-valued functions.

Kerber, Lesnick, and Oudot showed that matching distance is computable in polynomial time in two-dimensional case in [28].

Observe that in one-dimensional case, if we consider barcode as a map from $\mathbb{D}^1 \to \mathbb{N}$ which maps intervals to its multiplicities, then the rank invariant is the same with barcode. So matching distance, d_{match} of sublevel sets of two real-valued functions will be the same with the bottleneck distance or the interleaving distance.

Recall that the rank function is defined as $\beta^F : \mathbf{D}^n \to \mathbb{N} \cup \infty$ and it measures the number of homology classes in the sublevel set filtration of the function $F : X \to \mathbb{R}^n$ where $\mathbb{D}^n = \{(u, v) \in \mathbb{R}^n \times \mathbb{R}^n \mid u_i < v_i \ 1 \leq i \leq n\}$. The idea in the proof of the stability of multidimensional case is to restrict it to one dimension. In order to make sure that we will not lose any information by doing this, we will need some definitions. The following one was first used in [3] by Biasotti et al. to show the stability of 0-th homology of a filtered space obtained by tame function. After that, the same definition and the idea were used in [11] by Cagliari, Di Fabio, and Ferri to get the same result for higher dimensional homology. These papers need some restrictions on the function, which is eliminated in [15].

Definition 5.2.7. *For every vector $L = (l_1, \cdots, l_n) \in \mathbb{R}^n$ so that $\sum l_i^2 = 1$ and any vector $B = (b_1, \cdots, b_n) \in \mathbb{R}^n$ with $\sum b_i = 0$, the pair $(L, B) \in \mathbb{R}^n \times \mathbb{R}^n$ is called admissible pair and the set of all admissible pair denoted by Adm_n. For a given admissible pair (L, B), $\pi_{(L,B)}$ is a half-plane of $\mathbb{R}^n \times \mathbb{R}^n$ containing (u, v) which has a parametric equation as follows*

$$u = sL + B$$

$$v = tL + B$$

where $s < l \in \mathbb{R}$.

Admissible pair is a way of parametrizing the point $(u, v) \in \mathbb{R}^n \times \mathbb{R}^n$. B is the point of intersection of the plane $\sum_{i=1}^{n} x_i = 0$ in $\mathbb{R}^n$ and the line passing through the points u and v. L is the unit vector on the point u or v decided according to the distance

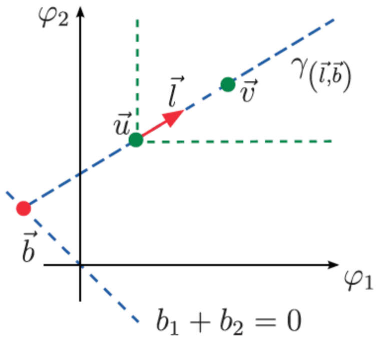


Figure 5.3: Example of (L, B) parametrization of $(u, v) \in \mathbb{D}^2$ in [15]

to B. That is why we have $s < l$ in the parametrization of the point (u, v) by using (L, B). For each $(u, v) \in \mathbb{D}^n$, we will have unique pair (L, B) in Adm_n. In the Fig 5.3, the two dimensional case is illustrated.

It was shown by Reduction Theorem in [15], by restriction the rank invariant to one dimension does not affect the value. For a given vector valued function $f : X \to \mathbb{R}^n$, they defined $f_{(u,v)} : X \to \mathbb{R}^n$ as

$$f_{(u,v)}(x) = \min l_i \max \frac{f_i(x) - b_i}{l_i}$$

Then $\beta^f(u, v) = \beta^{\min l_i^{-1} f_{(u,v)}}(s, t)$ where $u = sL + B$ and $v = tL + B$. For the following definition, instead of $\beta^{\min l_i^{-1} f_{(u,v)}}$ we will just use $\beta^f_{u,v}$.

Definition 5.2.8. *Let $F : X \to \mathbb{R}^n$, and $G : Y \to \mathbb{R}^n$ be continuous functions where X, Y are triangulable spaces. The multidimensional matching distance D_{match} between β^F and β^G is defined as*

$$D_{match}(\beta^F, \beta^G) = \sup_{(u,v)\in\mathbf{D}^n} d_{match}(\beta^F_{u,v}, \beta^G_{u,v})$$

The triangulability condition on the spaces in the following theorem is needed to make rank invariant finite. They stated that it was done to reduce the technical aspects in the proof in [15]. The distance between two vector valued function is given as

$$d_\infty(f, g) = \inf_h \sup_{x\in X} ||f(x) - g(h(x))||_\infty$$

where $f : X \to \mathbb{R}^n$, $g : Y \to \mathbb{R}^n$ and h is homeomorphism between X and Y. If they are not homeomorphic then $d_\infty(f, g) = \infty$.

Theorem 5.2.8. *(Multidimensional Stability Theorem for D_{match} [15]) If X and Y are triangulable spaces with vector valued continuous functions f and g on them, then*

$$D_{match}(\beta^f, \beta^g) \leq d_\infty(f, g)$$

Proof. Let h be a homeomorphism between X and Y. Observe that $\beta^g = \beta^{g \circ h}$. Then we have $D_{match}(\beta^f, \beta^g) = D_{match}(\beta^f, \beta^{g \circ h})$. From the definition of mathching distance and by stability result for one dimensional case,

$$D_{match}(\beta^f, \beta^{g \circ h}) = \sup_{(u,v) \in \mathbb{D}^n} d_{match}(\beta^f_{u,v}, \beta^{g \circ h}_{u,v}) \leq \sup_{x \in X} ||f_{(u,v)}(x) - (g \circ h)_{(u,v)}(x)||_\infty$$

By reduction theorem, we know that $f_{(u,v)}(x) = \min l_i \max \dfrac{f_i(x) - b_i}{l_i}$. So

$$\begin{aligned}
\sup_{x \in X} ||f_{(u,v)}(x) - (g \circ h)_{(u,v)}(x)||_\infty &\leq \min_i l_i \sup_{x \in X} \{\max_i \frac{f_i(x) - b_i}{l_i} - \max_i \frac{(g \circ h)_i(x) - b_i}{l_i}\} \\
&\leq \min_i l_i \frac{\sup_{x \in X} ||f(x) - (g \circ h)(x)||_\infty}{\min_i l_i} \\
&\leq \sup_{x \in X} ||f(x) - (g \circ h)(x)||_\infty
\end{aligned}$$

Since this is true for all homemorphisms h, we have $D_{match}(\beta^f, \beta^g) \leq d_\infty(f, g)$. $\square$

The idea in the proof is to restrict the multidimensional rank invariants to the one-dimensional case. By doing this, the stability result for the multidimensional case can be shown as a result of a one-dimensional case.

CHAPTER 6

CONCLUSIONS

In this thesis, we have served a survey on the persistence theory. In the first chapter, we gave the motivation for the persistence theory. In the second chapter, we demonstrated some background information that is helpful in the explanation of mathematical objects obtained by data. In section 3.1, we listed how to convert point cloud data or real-valued functions to a filtered object and how to get persistence modules from these filtered objects. In section 3.2, the visualization methods are mentioned, which are considered invariants. In section 3.3, we explained how underlying mathematical objects' structure helps us observe invariants for data by decomposition results on persistence modules.

In chapter 4, we focused on a specific type of persistence module that is produced by multifiltration and mainly followed Carlsson and Zomorodian's work [14]. In section 4.1, we review the algebraic structure of the multidimensional persistence modules. In section 4.2, we worked on classifying this algebraic structure to obtain invariants for multidimensional persistence modules. In section 4.3, we examined the type of these invariants, and we concluded that there are no complete discrete invariants for multifiltered data. In section 4.4, we compared two methods of classifying invariants from [14] and [30] via an example. In section 4.5, two discrete invariants, the rank function [14] and multigraded Hilbert series [26], are presented.

In chapter 5, the stability results are overviewed, which means small changes in data will produce small changes in persistence modules or invariants. In section 5.1, the results for one-dimensional data, which has complete invariant, are listed. In section 5.2, we did the same thing for filtration dependent on any preordered set. In the following subsection, we examine the results for multidimensional persistence modules which are rectangle and block decomposable.

For future work, one may try to improve the algorithms for one dimensional persistence modules. In more theoretical side, one can search for a discrete invariant which can discern non-isomorphic modules where rank invariant can not. Another study area is search for a suitable pseudometric which is defined on persistence modules. Stability and computation can be studied for this new pseudometric.

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