Chapter 3. Homological Algebra

The aim of this chapter is to introduce the fundamental results of homological algebra. Homological algebra appeared in the 1800's and is nowadays a very useful tool in several branches of mathematics, such as algebraic topology, commutative algebra, algebraic geometry, and, of particular interest to us, group theory.

Throughout this chapter R denotes a ring, and unless otherwise specified, all rings are assumed to be *unital* and *associative*.

Reference:

[Rot09] J. J. Rotman, *An introduction to homological algebra. Second ed.*, Universitext, Springer, New York, 2009.

[Wei94] C. A. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994.

8 Chain and Cochain Complexes

Definition 8.1 (*Chain complex*)

(a) A **chain complex** (or simply a **complex)** of R-modules is a sequence

$$(C_\bullet, d_\bullet) = \left(\cdots \longrightarrow C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \longrightarrow \cdots \right),$$

where for each $n \in \mathbb{Z}$, C_n is an R-modules and $d_n \in \mathrm{Hom}_R(C_n, C_{n-1})$ satisfies $d_n \circ d_{n+1} = 0$. We often write simply $C_\bullet$ instead of $(C_\bullet, d_\bullet)$.

(b) The integer n is called the **degree** of the R-module C_n.

(c) The R-linear maps d_n ($n \in \mathbb{Z}$) are called the **differential maps**.

(d) A complex $C_\bullet$ is called **non-negative** (resp. **positive**) if $C_n = 0$, for all $n \in \mathbb{Z}_{<0}$ (resp. for all $n \in \mathbb{Z}_{\leqslant 0}$).

Notice that sometimes we will omit the indices and write d for all differential maps, and thus the condition $d_n \circ d_{n+1} = 0$ can be written as $d^2 = 0$. If there is an integer N such that $C_n = 0$ for all $n \leqslant N$, then we omit to write the zero modules and zero maps on the right-hand side of the complex:

$$\cdots \longrightarrow C_{N+2} \xrightarrow{d_{N+2}} C_{N+1} \xrightarrow{d_{N+1}} C_N$$

Similarly, if there is an integer N such that $C_n = 0$ for all $n \geqslant N$, then we omit to write the zero modules and zero maps on the left-hand side of the complex:

$$C_N \xrightarrow{d_N} C_{N-1} \xrightarrow{d_{N-1}} C_{N-2} \longrightarrow \cdots$$

Definition 8.2 (*Morphism of complexes*)

A **morphism of complexes** (or a **chain map**) between two chain complexes $(C_\bullet, d_\bullet)$ and $(D_\bullet, d'_\bullet)$, written $\varphi_\bullet : C_\bullet \longrightarrow D_\bullet$, is a familiy of R-linear maps $\varphi_n : C_n \longrightarrow D_n$ $(n \in \mathbb{Z})$ such that $\varphi_n \circ d_{n+1} = d'_{n+1} \circ \varphi_{n+1}$ for each $n \in \mathbb{Z}$, that is such that the following diagram commutes:

$$\begin{tikzcd}
\cdots \arrow[r, "d_{n+2}"] & C_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "\varphi_{n+1}"'] & C_n \arrow[r, "d_n"] \arrow[d, "\varphi_n"'] & C_{n-1} \arrow[r, "d_{n-1}"] \arrow[d, "\varphi_{n-1}"'] & \cdots \\
\cdots \arrow[r, "d'_{n+2}"] & D_{n+1} \arrow[r, "d'_{n+1}"] & D_n \arrow[r, "d'_n"] & D_{n-1} \arrow[r, "d'_{n-1}"] & \cdots
\end{tikzcd}$$

Notation. Chain complexes together with morphisms of chain complexes (and composition given by degreewise composition of R-morphisms) form a category, which we will denote by $\mathbf{Ch}({}_R\mathbf{Mod})$.

Definition 8.3 (*Subcomplex / quotient complex*)

(a) A **subcomplex** $C'_\bullet$ of a chain complex $(C_\bullet, d_\bullet)$ is a family of R-modules $C'_n \leqslant C_n$ $(n \in \mathbb{Z})$, such that $d_n(C'_n) \subset C'_{n-1}$ for every $n \in \mathbb{Z}$.
In this case, $(C'_\bullet, d_\bullet)$ becomes a chain complex and the inclusion $C'_\bullet \hookrightarrow C_\bullet$ given by the canonical inclusion of C'_n into C_n for each $n \in \mathbb{Z}$ is a chain map.

(b) If $C'_\bullet$ is a subcomplex of $C_\bullet$, then the **quotient complex** $C_\bullet/C'_\bullet$ is the familiy of R-modules C_n/C'_n $(n \in \mathbb{Z})$ together with the differential maps $\overline{d}_n : C_n/C'_n \longrightarrow C_{n-1}/C'_{n-1}$ uniquely determined by the universal property of the quotient.
In this case, the quotient map $\pi_\bullet : C_\bullet \longrightarrow C_\bullet/C'_\bullet$ defined for each $n \in \mathbb{Z}$ by the canonical projection $\pi_n : C_n \longrightarrow C_n/C'_n$ is a chain map.

Definition 8.4 (*Kernel / image / cokernel*)

Let $\varphi_\bullet : C_\bullet \longrightarrow D_\bullet$ be a morphism of chain complexes between $(C_\bullet, d_\bullet)$ and $(D_\bullet, d'_\bullet)$. Then,

(a) the **kernel** of $\varphi_\bullet$ is the subcomplex of $C_\bullet$ defined by $\ker \varphi_\bullet := (\{\ker \varphi_n\}_{n\in\mathbb{Z}}, d_\bullet)$;

(b) the **image** of $\varphi_\bullet$ is the subcomplex of $D_\bullet$ defined by $\operatorname{Im} \varphi_\bullet := (\{\operatorname{Im} \varphi_n\}_{n\in\mathbb{Z}}, d'_\bullet)$; and

(c) the **cokernel** of $\varphi_\bullet$ is the quotient complex $\operatorname{coker} \varphi_\bullet := D_\bullet / \operatorname{Im} \varphi_\bullet$.

With these notions of kernel and cokernel, one can show that **Ch($_R$Mod)** is in fact an abelian category.

Definition 8.5 (*Cycles, boundaries, homology*)

Let $(C_\bullet, d_\bullet)$ be a chain complex of R-modules.

(a) An n-**cycle** is an element of $\ker d_n =: Z_n(C_\bullet) := Z_n$.

(b) An n-**boundary** is an element of $\operatorname{Im} d_{n+1} =: B_n(C_\bullet) := B_n$.
[Clearly, since $d_n \circ d_{n+1} = 0$, we have $0 \subseteq B_n \subseteq Z_n \subseteq C_n$ $\forall n \in \mathbb{Z}$.]

(c) The n-**th homology module** (or simply **group**) of $C_\bullet$ is $H_n(C_\bullet) := Z_n/B_n$.

In fact, for each $n \in \mathbb{Z}$, $H_n(-) : \mathbf{Ch}({}_R\mathbf{Mod}) \longrightarrow {}_R\mathbf{Mod}$ is a covariant additive functor (see Exercise 1, Exercise Sheet 6), which we define on morphisms as follows:

Lemma 8.6

Let $\varphi_\bullet : C_\bullet \longrightarrow D_\bullet$ be a morphism of chain complexes between $(C_\bullet, d_\bullet)$ and $(D_\bullet, d'_\bullet)$. Then $\varphi_\bullet$ induces an R-linear map

$$\begin{array}{rccc} H_n(\varphi_\bullet): & H_n(C_\bullet) & \longrightarrow & H_n(D_\bullet) \\ & z_n + B_n(C_\bullet) & \mapsto & \varphi_n(z_n) + B_n(D_\bullet) \end{array}$$

for each $n \in \mathbb{Z}$. To simplify, this map is often denoted by φ_* instead of $H_n(\varphi_\bullet)$.

Proof: Fix $n \in \mathbb{Z}$, and let $\pi_n : Z_n(C_\bullet) \longrightarrow Z_n(C_\bullet)/B_n(C_\bullet)$, resp. $\pi'_n : Z_n(D_\bullet) \longrightarrow Z_n(D_\bullet)/B_n(D_\bullet)$, be the canonical projections.
First, notice that $\varphi_n\big(Z_n(C_\bullet)\big) \subset Z_n(D_\bullet)$ because if $z \in Z_n$, then $d'_n \circ \varphi_n(z) = \varphi_{n-1} \circ d_n(z) = 0$. Hence, we have $\varphi_n(z) \in Z_n(D)$.
Similarly, we have $\varphi_n\big(B_n(C_\bullet)\big) \subset B_n(D_\bullet)$. Indeed, if $b \in B_n(C_\bullet)$, then $b = d_{n+1}(a)$ for some $a \in C_{n+1}$, and because $\varphi_\bullet$ is a chain map we have $\varphi_n(b) = \varphi_n \circ d'_{n+1}(a) = d_{n+1} \circ \varphi_{n+1}(a) \in B_n(D_\bullet)$.
Therefore, by the universal property of the quotient, there exists a unique R-linear map $\overline{\pi'_n \circ \varphi_n}$ such that the following diagram commutes:

$$\begin{tikzcd} Z_n(C_\bullet) \arrow[r, "\varphi_n"] \arrow[d, "\pi_n"'] & Z_n(D_\bullet) \arrow[r, "\pi'_n"] & Z_n(D_\bullet)/B_n(D_\bullet) \\ Z_n(C_\bullet)/B_n(C_\bullet) \arrow[urr, dashed, "\overline{\pi'_n\circ\varphi_n}"'] & & \end{tikzcd}$$

Set $H_n(\varphi_\bullet) := \overline{\pi'_n \circ \varphi_n}$. The claim follows. ■

It should be thought that the homology module $H_n(C_\bullet)$ measures the "non-exactness" of the sequence

$$C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} .$$

Moroever, the functors $H_n(-)$ ($n \in \mathbb{Z}$) are neither left exact, nor right exact in general. As a matter of fact, using the Snake Lemma, we can use s.e.s. of complexes to produce so-called "long exact sequences" of R-modules.

Theorem 8.7 (*Long exact sequence in homology*)

Let $0_\bullet \longrightarrow C_\bullet \xrightarrow{\varphi_\bullet} D_\bullet \xrightarrow{\psi_\bullet} E_\bullet \longrightarrow 0_\bullet$ be a s.e.s. of chain complexes. Then there is a long exact sequence

$$\cdots \xrightarrow{\delta_{n+1}} H_n(C_\bullet) \xrightarrow{\varphi_*} H_n(D_\bullet) \xrightarrow{\psi_*} H_n(E_\bullet) \xrightarrow{\delta_n} H_{n-1}(C_\bullet) \xrightarrow{\varphi_*} H_{n-1}(D_\bullet) \xrightarrow{\psi_*} \cdots,$$

where for each $n \in \mathbb{Z}$, $\delta_n : H_n(E_\bullet) \longrightarrow H_{n-1}(C_\bullet)$ is an R-linear map, called **connecting homomorphism**.

Note: Here $0_\bullet$ simply denotes the **zero complex**, that is the complex

$$\cdots \longrightarrow 0 \xrightarrow{0} 0 \xrightarrow{0} 0 \longrightarrow \cdots$$

consisting of zero modules and zero morphisms. We often write simply 0 instead of $0_\bullet$.

Proof: To simplify, we denote all differential maps of the three complexes $C_\bullet$, $D_\bullet$, $E_\bullet$ with the same letter d, and we fix $n \in \mathbb{Z}$. First, we apply the "non-snake" part of the Snake Lemma to the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & C_n \arrow[r, "\varphi_n"] \arrow[d, "d_n"] & D_n \arrow[r, "\psi_n"] \arrow[d, "d_n"] & E_n \arrow[r] \arrow[d, "d_n"] & 0 \\
0 \arrow[r] & C_{n-1} \arrow[r, "\varphi_{n-1}"] & D_{n-1} \arrow[r, "\psi_{n-1}"] & E_{n-1} \arrow[r] & 0,
\end{tikzcd}$$

and we obtain two exact sequences

$$0 \longrightarrow Z_n(C_\bullet) \xrightarrow{\varphi_n} Z_n(D_\bullet) \xrightarrow{\psi_n} Z_n(E_\bullet)\ ,$$

and

$$C_{n-1}/\operatorname{Im} d_n \xrightarrow{\overline{\varphi_{n-1}}} D_{n-1}/\operatorname{Im} d_n \xrightarrow{\overline{\psi}_{n-1}} E_{n-1}/\operatorname{Im} d_n \longrightarrow 0.$$

Shifting indices in both sequences we obtain similar sequences in degrees $n-1$, and n respectively. Therefore, we have a commutative diagram with exact rows of the form:

$$\begin{tikzcd}
 & C_n/\operatorname{Im} d_{n+1} \arrow[r, "\overline{\varphi_n}"] \arrow[d, "\overline{d_n}"] & D_n/\operatorname{Im} d_{n+1} \arrow[r, "\overline{\psi_n}"] \arrow[d, "\overline{d_n}"] & E_n/\operatorname{Im} d_{n+1} \arrow[r] \arrow[d, "\overline{d_n}"] & 0 \\
0 \arrow[r] & Z_{n-1}(C_\bullet) \arrow[r, "\varphi_{n-1}"] & Z_{n-1}(D_\bullet) \arrow[r, "\psi_{n-1}"] & Z_{n-1}(E_\bullet), &
\end{tikzcd}$$

where $\overline{d_n} : C_n/\operatorname{Im} d_{n+1} \longrightarrow Z_{n-1}(C_\bullet)$ is the unique R-linear map induced by the universal property of the quotient by $d_n : C_n \longrightarrow C_{n-1}$ (as $\operatorname{Im} d_{n+1} \subseteq \ker d_n$ by definition of a chain complex), and similarly for $D_\bullet$ and $E_\bullet$. Therefore, the Snake Lemma yields the existence of the connecting homomorphisms

$$\delta_n : \underbrace{\ker \overline{d_n}(E_\bullet)}_{=H_n(E_\bullet)} \longrightarrow \underbrace{\operatorname{coker} \overline{d_n}(C_\bullet)}_{=H_{n-1}(C_\bullet)}$$

for each $n \in \mathbb{Z}$ as well as the required long exact sequence:

$$\cdots \xrightarrow{\delta_{n+1}} \underbrace{H_n(C_\bullet)}_{=\ker \overline{d_n}} \xrightarrow{\varphi_*} \underbrace{H_n(D_\bullet)}_{=\ker \overline{d_n}} \xrightarrow{\psi_*} \underbrace{H_n(E_\bullet)}_{=\ker \overline{d_n}} \xrightarrow{\delta_n} \underbrace{H_{n-1}(C_\bullet)}_{=\operatorname{coker} \overline{d_n}} \xrightarrow{\varphi_*} \underbrace{H_{n-1}(D_\bullet)}_{=\operatorname{coker} \overline{d_n}} \xrightarrow{\psi_*} \cdots$$

■

We now describe some important properties of chain maps and how they relate with the induced morphisms in homology.

Definition 8.8 (*Quasi-isomorphism*)

A chain map $\varphi_\bullet : C_\bullet \longrightarrow D_\bullet$ is called a **quasi-isomorphism** if $H_n(\varphi_\bullet)$ is an isomorphism for all $n \in \mathbb{Z}$.

Warning: A quasi-isomorphism does not imply that the complexes $C_\bullet$ and $D_\bullet$ are isomorphic as chain complexes. See Exercise 2, Sheet 5 for a counter-example.

In general complexes are not exact sequences, but if they are, then their homology vanishes, so that there is a quasi-isomorphism from the zero complex.

Exercise [Exercise 3, Exercise Sheet 5]

Let $C_\bullet$ be a chain complex of R-modules. Prove that TFAE:

(a) $C_\bullet$ is **exact** (i.e. exact at C_n for each $n \in \mathbb{Z}$);

(b) $C_\bullet$ is **acyclic**, that is, $H_n(C_\bullet) = 0$ for all $n \in \mathbb{Z}$;

(c) The map $0_\bullet \longrightarrow C_\bullet$ is a quasi-isomorphism.

Definition 8.9 (*Homotopic chain maps / homotopy equivalence*)

Two chain maps $\varphi_\bullet, \psi_\bullet : C_\bullet \longrightarrow D_\bullet$ between chain complexes $(C_\bullet, d_\bullet)$ and $(D_\bullet, d'_\bullet)$ are called **(chain) homotopic** if there exists a familiy of R-linear maps $\{s_n : C_n \longrightarrow D_{n+1}\}_{n\in\mathbb{Z}}$ such that

$$\varphi_n - \psi_n = d'_{n+1} \circ s_n + s_{n-1} \circ d_n$$

for each $n \in \mathbb{Z}$.

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "\varphi_{n+1}-\psi_{n+1}"'] & C_n \arrow[r, "d_n"] \arrow[dl, "s_n"'] \arrow[d, "\varphi_n-\psi_n"] & C_{n-1} \arrow[r] \arrow[dl, "s_{n-1}"] \arrow[d, "\varphi_{n-1}-\psi_{n-1}"] & \cdots \\
\cdots \arrow[r] & D_{n+1} \arrow[r, "d'_{n+1}"'] & D_n \arrow[r, "d'_n"'] & D_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

In this case, we write $\varphi_\bullet \sim \psi_\bullet$.
Moreover, a chain map $\varphi_\bullet : C_\bullet \longrightarrow D_\bullet$ is called a **homotopy equivalence** if there exists a chain map $\sigma : D_\bullet \longrightarrow C_\bullet$ such that $\sigma_\bullet \circ \varphi_\bullet \sim \mathrm{id}_{C_\bullet}$ and $\varphi_\bullet \circ \sigma_\bullet \sim \mathrm{id}_{D_\bullet}$.

Note: One easily checks that $\sim$ is an equivalence relation on the class of chain maps.

Proposition 8.10

If $\varphi_\bullet, \psi_\bullet : C_\bullet \longrightarrow D_\bullet$ are homotopic morphisms of chain complexes, then they induce the same morphisms in homology, that is

$$H_n(\varphi_\bullet) = H_n(\psi_\bullet) : H_n(C_\bullet) \longrightarrow H_n(D_\bullet) \quad \forall n \in \mathbb{Z}.$$

Proof: Fix $n \in \mathbb{Z}$ and let $z \in Z_n(C_\bullet)$. Then, with the notation of Definition 8.9, we have

$$\big(\varphi_n - \psi_n\big)(z) = \big(d'_{n+1}s_n + s_{n-1}d_n\big)(z) = \underbrace{d'_{n+1}s_n(z)}_{\in B_n(D_\bullet)} + \underbrace{s_{n-1}d_n(z)}_{=0} \in B_n(D_\bullet).$$

Hence, for every $z + B_n(C_\bullet) \in H_n(C_\bullet)$, we have

$$\left(H_n(\varphi_\bullet) - H_n(\psi_\bullet)\right)(z + B_n(C_\bullet)) = H_n(\varphi_\bullet - \psi_\bullet)(z + B_n(C_\bullet)) = 0 + B_n(D_\bullet)\,.$$

In other words $H_n(\varphi_\bullet) - H_n(\psi_\bullet) \equiv 0$, so that $H_n(\varphi_\bullet) = H_n(\psi_\bullet)$. ∎

Remark 8.11

(Out of the scope of the lecture!)
Homotopy of complexes leads to considering the so-called **homomotopy category** of R-modules, denoted $\mathbf{Ho}({}_R\mathbf{Mod})$, which is very useful in algebraic topology or representation theory of finite groups for example. It is defined as follows:

- The objects are the chain complexes, i.e. $\operatorname{Ob}\mathbf{Ho}({}_R\mathbf{Mod}) = \operatorname{Ob}\mathbf{Ch}({}_R\mathbf{Mod})$.
- The morphisms are given by $\operatorname{Hom}_{\mathbf{Ho}({}_R\mathbf{Mod})} := \operatorname{Hom}_{\mathbf{Ch}({}_R\mathbf{Mod})}/\sim$.

It is an additive category, but it is not abelian in general though. The isomorphisms in the homotopy category are exactly the classes of the homotopy equivalences.

Dualizing the objects and concepts we have defined above yields the so-called "cochain complexes" and the notion of "cohomology".

Definition 8.12 (*Cochain complex / cohomology*)

(a) A **cochain complex** of R-modules is a sequence

$$(C^\bullet, d^\bullet) = \left(\cdots \longrightarrow C^{n-1} \xrightarrow{d_{n-1}} C^n \xrightarrow{d^n} C^{n+1} \longrightarrow \cdots\right),$$

where for each $n \in \mathbb{Z}$, C^n is an R-module and $d^n \in \operatorname{Hom}_R(C^n, C^{n+1})$ satisfies $d^{n+1} \circ d^n = 0$. We often write simply $C^\bullet$ instead of $(C^\bullet, d^\bullet)$.

(b) The elements of $Z^n := Z^n(C^\bullet) := \ker d^n$ are the n-**cocycles**.

(c) The elements of $B^n := B^n(C^\bullet) := \operatorname{Im} d^{n-1}$ are the n-**coboundaries**.

(d) The n**-th cohomology module** (or simply **group**) of $C^\bullet$ is $H^n(C^\bullet) := Z_n/B_n$.

Similarly to the case of chain complexes, we can define:

- **Morphisms of cochain complexes** (or simply **cochain maps**) between two cochain complexes $(C^\bullet, d^\bullet)$ and $(D^\bullet, \tilde{d}^\bullet)$, written $\varphi^\bullet : C^\bullet \longrightarrow D^\bullet$, as a familiy of R-linear maps $\varphi^n : C^n \longrightarrow D^n$ ($n \in \mathbb{Z}$) such that $\varphi^n \circ d^{n-1} = \tilde{d}^{n-1} \circ \varphi^{n-1}$ for each $n \in \mathbb{Z}$, that is such that the following diagram commutes:

$$\begin{tikzcd}
\cdots \arrow[r, "d^{n-2}"] & C^{n-1} \arrow[r, "d^{n-1}"] \arrow[d, "\varphi^{n-1}"'] & C^n \arrow[r, "d^n"] \arrow[d, "\varphi^n"'] & C^{n+1} \arrow[r, "d^{n+1}"] \arrow[d, "\varphi^{n+1}"'] & \cdots \\
\cdots \arrow[r, "\tilde{d}^{n-2}"] & D^{n-1} \arrow[r, "\tilde{d}^{n-1}"] & D^n \arrow[r, "\tilde{d}^n"] & D^{n+1} \arrow[r, "\tilde{d}^{n+1}"] & \cdots
\end{tikzcd}$$

- subcomplexes, quotient complexes;
- kernels, images, cokernels of morphisms of cochain complexes.
- Cochain complexes together with morphisms of cochain complexes (and composition given by degreewise composition of R-morphisms) form an abelian category, which we will denote by **CoCh($_R$Mod)**.

Exercise: formulate these definitions in a formal way.

Theorem 8.13 (*Long exact sequence in cohomology*)

Let $0^\bullet \longrightarrow C^\bullet \xrightarrow{\varphi^\bullet} D^\bullet \xrightarrow{\psi^\bullet} E^\bullet \longrightarrow 0^\bullet$ be a s.e.s. of cochain complexes. Then, for each $n \in \mathbb{Z}$, there exists a connecting homomorphism $\delta^n : H^n(E^\bullet) \longrightarrow H^{n+1}(C^\bullet)$ such that the following sequence is exact:

$$\cdots \xrightarrow{\delta^{n+1}} H^n(C^\bullet) \xrightarrow{\varphi^*} H^n(D^\bullet) \xrightarrow{\psi^*} H^n(E^\bullet) \xrightarrow{\delta^n} H^{n+1}(C^\bullet) \xrightarrow{\varphi^*} H^{n+1}(D^\bullet) \xrightarrow{\psi^*} \cdots$$

Proof: Similar to the proof of the long exact sequence in homology (Theorem 8.7). Apply the Snake Lemma. ■

9 Projective Resolutions

Definition 9.1 (*Projective resolution*)

Let M be an R-module. A **projective resolution** of M is a non-negative complex of projective R-modules

$$(P_\bullet, d_\bullet) = \left(\cdots \xrightarrow{d_3} P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \right)$$

which is exact at P_n for every $n \geqslant 1$ and such that $H_0(P_\bullet) = P_0/\operatorname{Im} d_1 \cong M$.
Moreover, if P_n is a free R-module for every $n \geqslant 0$, then $P_\bullet$ is called a **free resolution** of M.

Notation: Letting $\varepsilon : P_0 \twoheadrightarrow M$ denote the quotient homomorphism, we have a so-called **augmented complex**

$$\cdots \xrightarrow{d_3} P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \overset{\varepsilon}{\twoheadrightarrow} M \longrightarrow 0\,,$$

associated to the projective resolution, and this augmented complex is exact. Hence we will also denote projective resolutions of M by $P_\bullet \overset{\epsilon}{\twoheadrightarrow} M$.

Example 8

The $\mathbb{Z}$-module $M = \mathbb{Z}/n\mathbb{Z}$ admits the following projective resolution: $0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z}$.

We now prove that projective resolutions do exist, and consider the question of how "unique" they are.

Proposition 9.2

Any R-module has a projective resolution. (It can even chosen to be free.)

Proof: We use the fact that every R-module is a quotient of a free R-module (Proposition 6.4). Thus there exists a free module P_0 together with a surjective R-linear map $\varepsilon : P_0 \twoheadrightarrow M$ such that $M \cong P_0/\ker\varepsilon$. Next, let P_1 be a free R-module together with a surjective R-linear map $d_1 : P_1 \twoheadrightarrow \ker\varepsilon \subseteq P_0$ such that $P_1/\ker d_1 \cong \ker\varepsilon$:

$$\begin{tikzcd} P_1 \arrow[rr, dashed, "d_1"] \arrow[dr, "d_1"'] & & P_0 \arrow[r, two heads, "\varepsilon"] & M\,. \\ & \ker\varepsilon \arrow[ur, hook] & & \end{tikzcd}$$

Inductively, assuming that the R-homomorphism $d_{n-1} : P_{n-1} \longrightarrow P_{n-2}$ has already been defined, then there exists a free R-module P_n and a surjective R-linear map $d_n : P_n \twoheadrightarrow \ker d_{n-1} \subseteq P_{n-1}$ with $P_n/\ker d_n \cong \ker d_{n-1}$. The claim follows. ∎

Theorem 9.3 (*Lifting Theorem*)

Let $(P_\bullet, d_\bullet)$ and $(Q_\bullet, d'_\bullet)$ be two non-negative chain complexes such that

1. P_n is a projective R-module for every $n \geqslant 0$;
2. $Q_\bullet$ is exact at Q_n for every $n \geqslant 1$ (that is $H_n(Q_\bullet) = 0$, for all $n \geqslant 1$).

Let $\varepsilon : P_0 \twoheadrightarrow H_0(P_\bullet)$ and $\varepsilon' : Q_0 \twoheadrightarrow H_0(Q_\bullet)$ be the quotient homomorphims.
If $f : H_0(P_\bullet) \longrightarrow H_0(Q_\bullet)$ is an R-linear map, then there exists a chain map $\varphi_\bullet : P_\bullet \longrightarrow Q_\bullet$ inducing the given map f in degree-zero homology, that is such that $H_0(\varphi_\bullet) = f$ and $f \circ \varepsilon = \varepsilon' \circ \varphi_0$. Moreover, such a chain map $\varphi_\bullet$ is unique up to homotopy.

In the situation of the Theorem, it is said that $\varphi_\bullet$ **lifts** f.

Proof: Existence. Beacuse P_0 is projective and ε' is surjective, by definition (Def. 6.7), there exists an R-linear map $\varphi_0 : P_0 \longrightarrow Q_0$ such that the following diagram commutes

$$\begin{tikzcd} \cdots \arrow[r, "d_1"] & P_0 \arrow[r, two heads, "\varepsilon"] \arrow[d, dashed, "\exists\varphi_0"] \arrow[dr, phantom, "\circlearrowleft"] & H_0(P_\bullet) = P_0/\operatorname{Im} d_1 \arrow[d, "f"] \\ \cdots \arrow[r, "d'_1"] & Q_0 \arrow[r, two heads, "\varepsilon'"] & H_0(Q_\bullet) = Q_0/\operatorname{Im} d'_1\,, \end{tikzcd}$$

that is $f \circ \varepsilon = \varepsilon' \circ \varphi_0$. But then, $\varepsilon' \circ \varphi_0 \circ d_1 = f \circ \underbrace{\varepsilon \circ d_1}_{=0} = 0$, so that $\operatorname{Im}(\varphi_0 \circ d_1) \subseteq \ker\varepsilon' = \operatorname{Im} d'_1$. Again by Definition 6.7, since P_1 is projective and d'_1 is surjective onto its image, there exists an R-linear map $\varphi_1 : P_1 \longrightarrow Q_1$ such that $\varphi_0 \circ d_1 = d'_1 \circ \varphi_1$:

$$\begin{tikzcd} P_1 \arrow[rr, "d_1"] \arrow[dr, "\varphi_0\circ d_1"] \arrow[d, dashed, "\exists\varphi_1"'] & & P_0 \arrow[d, "\varphi_0"] \\ Q_1 \arrow[r, two heads, "d'_1"'] & \operatorname{Im} d'_1 = \ker\varepsilon' \arrow[r, hook, "\mathrm{inc}"'] & Q_0 \end{tikzcd}$$

The morphisms $\varphi_n : P_n \longrightarrow Q_n$ are constructed similarly by induction on n. Hence the existence of a chain map $\varphi_\bullet : P_\bullet \longrightarrow Q_\bullet$ as required.

Uniqueness. For the uniqueness statement, suppose $\psi_\bullet : P_\bullet \longrightarrow Q_\bullet$ also lifts the given morphism f. We have to prove that $\varphi_\bullet \sim \psi_\bullet$ (or equivalently that $\varphi_\bullet - \psi_\bullet$ is homotopic to the zero chain map).

For each $n \geqslant 0$ set $\sigma_n := \varphi_n - \psi_n$, so that $\sigma_\bullet : P_\bullet \longrightarrow Q_\bullet$ is becomes a chain map. In particular $\sigma_0 = \varphi_0 - \psi_0 = H_0(\varphi_\bullet) - H_0(\psi_\bullet) = f - f = 0$. Then we let $s_{-2} : 0 \longrightarrow H_0(Q_\bullet)$ and $s_{-1} : H_0(P_\bullet) \longrightarrow Q_0$ be the zero maps. Therefore, in degree zero, we have the following maps:

$$\begin{tikzcd}
P_0 \arrow[r, two heads, "\varepsilon"] & H_0(P_\bullet) \arrow[r, "0"] \arrow[d, "0"] \arrow[dl, "s_{-1}"'] & 0 \arrow[dl, "s_{-2}"'] \\
Q_0 \arrow[r, two heads, "\varepsilon'"'] & H_0(Q_\bullet) \arrow[r, "0"'] & 0\,,
\end{tikzcd}$$

where clearly $0 = s_{-2} \circ 0 + \varepsilon' \circ s_{-1}$. This provides us with the starting point for constructing a homotopy $s_n : P_n \longrightarrow Q_{n+1}$ by induction on n. So let $n \geqslant 0$ and suppose $s_i : P_i \longrightarrow Q_{i+1}$ is already constructed for each $-2 \leqslant i \leqslant n-1$ and satisfies $d'_{i+1} \circ s_i + s_{i-1} \circ d_i = \sigma_i$ for each $i \geqslant -1$, and where we identify

$$P_{-1} = H_0(P_\bullet), \quad Q_{-1} = H_0(Q_\bullet), \quad P_{-2} = 0 = Q_{-2}, \quad d_0 = \varepsilon, \quad d'_0 = \varepsilon', \quad d_{-1} = 0 = d'_{-1}\,.$$

Now, we check that the image of $\sigma_n - s_{n-1} \circ d_n$ is contained in $\ker d'_n = \operatorname{Im} d'_{n+1}$:

$$\begin{aligned}
d'_n \circ \big(\sigma_n - s_{n-1} \circ d_n\big) &= d'_n \circ \sigma_n - d'_n \circ s_{n-1} \circ d_n \\
&= d'_n \circ \sigma_n - (\sigma_{n-1} - s_{n-2} \circ d_{n-1}) \circ d_n \\
&= d'_n \circ \sigma_n - \sigma_{n-1} \circ d_n \\
&= \sigma_{n-1} \circ d_n - \sigma_{n-1} \circ d_n = 0\,,
\end{aligned}$$

where the last-nut-one equality holds because both $\sigma_\bullet$ is a chain map. Therefore, again by Definition 6.7, since P_n is projective and d'_{n+1} is surjective onto its image, there exists an R-linear map $s_n : P_n \longrightarrow Q_{n+1}$ such that $d'_{n+1} \circ s_n = \sigma_n - s_{n-1} \circ d_n$:

$$\begin{tikzcd}
 & P_n \arrow[r, "d_n"] \arrow[d, "\sigma_n - s_{n-1}\circ d_n"'] \arrow[dl, dashed, "\exists s_n"'] & P_{n-1} \arrow[r, "d_{n-1}"] \arrow[d, "\sigma_{n-1}"'] \arrow[dl, "s_{n-1}"'] & P_{n-2} \arrow[r] \arrow[d, "\sigma_{n-2}"'] \arrow[dl, "s_{n-2}"'] & \cdots \arrow[dl] \\
Q_{n+1} \arrow[r, "d'_{n+1}"'] & Q_n \arrow[r, "d'_n"'] & Q_{n-1} \arrow[r, "d'_{n-1}"'] & Q_{n-2} \arrow[r] & \cdots
\end{tikzcd}$$

Hence we have $\varphi_n - \psi_n = \sigma_n = d'_{n+1} \circ s_n + s_{n-1} \circ d_n$, as required. ∎

As a corollary, we obtain the required statement on the uniqueness of projective resolutions:

Theorem 9.4 (*Comparison Theorem*)

Let $P_\bullet \overset{\varepsilon}{\twoheadrightarrow} M$ and $Q_\bullet \overset{\varepsilon'}{\twoheadrightarrow} M$ be two projective resolutions of an R-module M. Then $P_\bullet$ and $Q_\bullet$ are homotopy equivalent. More precisely, there exist chain maps $\varphi_\bullet : P_\bullet \longrightarrow Q_\bullet$ and $\psi_\bullet : Q_\bullet \longrightarrow P_\bullet$ lifting the identity on M and such that $\psi_\bullet \circ \varphi_\bullet \sim \mathrm{Id}_{P_\bullet}$ and $\varphi_\bullet \circ \psi_\bullet \sim \mathrm{Id}_{Q_\bullet}$.

Proof: Consider the identity morphism $\mathrm{Id}_M : M \longrightarrow M$.

By the Lifting Theorem, there exists a chain map $\varphi_\bullet : P_\bullet \longrightarrow Q_\bullet$, unique up to homotopy, such that $H_0(\varphi_\bullet) = \mathrm{Id}_M$ and $\mathrm{Id}_M \circ \varepsilon = \varepsilon' \circ \varphi_0$. Likewise, there exists a chain map $\psi_\bullet : Q_\bullet \longrightarrow P_\bullet$, unique up to homotopy, such that $H_0(\varphi_\bullet) = \mathrm{Id}_M$ and $\mathrm{Id}_M \circ \varepsilon' = \varepsilon \circ \psi_0$.

$$\begin{tikzcd}
\cdots \arrow[r] & P_n \arrow[r, "d_n"] \arrow[d, dashed, shift left, "\exists\varphi_n"] & \cdots \arrow[r] & P_1 \arrow[r, "d_1"] \arrow[d, dashed, shift left, "\exists\varphi_1"] \arrow[dr, phantom, "\circlearrowleft"] & P_0 \arrow[r, "\varepsilon"] \arrow[d, dashed, shift left, "\exists\varphi_0"] \arrow[dr, phantom, "\circlearrowleft"] & M \arrow[r] \arrow[d, shift left, "\mathrm{Id}_M"] & 0 \\
\cdots \arrow[r] & Q_n \arrow[r, "d'_n"'] \arrow[u, dashed, shift left, "\exists\psi_n"] & \cdots \arrow[r] & Q_1 \arrow[r, "d'_1"'] \arrow[u, dashed, shift left, "\exists\psi_1"] & Q_0 \arrow[r, "\varepsilon'"] \arrow[u, dashed, shift left, "\exists\psi_0"] & M \arrow[r] \arrow[u, shift left, "\mathrm{Id}_M"] & 0
\end{tikzcd}$$

Now, $\psi_\bullet \circ \varphi_\bullet$ and $\mathrm{Id}_{P_\bullet}$ are both chain maps that lift the identity map $\mathrm{Id}_M : H_0(P_\bullet) \longrightarrow H_0(P_\bullet)$. Therefore, by the uniqueness statement in the Lifting Theorem, we have $\psi_\bullet \circ \varphi_\bullet \sim \mathrm{Id}_{P_\bullet}$. Likewise, $\varphi_\bullet \circ \psi_\bullet$ and $\mathrm{Id}_{Q_\bullet}$ are both chain maps that lift the identity map $\mathrm{Id}_M : H_0(Q_\bullet) \longrightarrow H_0(Q_\bullet)$, therefore they are homotopic, that is $\varphi_\bullet \circ \psi_\bullet \sim \mathrm{Id}_{Q_\bullet}$. ■

Another way to construct projective resolutions is given by the following Lemma, often called the *Horseshoe Lemma*, because it requires to fill in a horseshoe-shaped diagram:

Lemma 9.5 (*Horseshoe Lemma*)

Let $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ be a short exact sequence of R-modules. Let $P'_\bullet \overset{\varepsilon'}{\twoheadrightarrow} M'$ be a resolution of M' and $P''_\bullet \overset{\varepsilon''}{\twoheadrightarrow} M''$ be a **projective** resolution of M''.

$$\begin{tikzcd}
 & \vdots \arrow[d] & & \vdots \arrow[d] & \\
 & P'_1 \arrow[d] & & P''_1 \arrow[d] & \\
 & P'_0 \arrow[d, "\varepsilon'"'] & & P''_0 \arrow[d, "\varepsilon''"'] & \\
0 \arrow[r] & M' \arrow[r] \arrow[d] & M \arrow[r] & M'' \arrow[r] \arrow[d] & 0 \\
 & 0 & & 0 &
\end{tikzcd}$$

Then, there exists a resolution $P_\bullet \overset{\varepsilon}{\twoheadrightarrow} M$ of M such that $P_n \cong P'_n \oplus P''_n$ for each $n \in \mathbb{Z}_{\geqslant 0}$ and the s.e.s. $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ lifts to a s.e.s. of chain complexes $0_\bullet \longrightarrow P'_\bullet \xrightarrow{i_\bullet} P_\bullet \xrightarrow{\pi_\bullet} P''_\bullet \longrightarrow 0_\bullet$ where $i_\bullet$ and $\pi_\bullet$ are the canonical injection and projection. Moreover, if $P'_\bullet \overset{\varepsilon'}{\twoheadrightarrow} M'$ is a projective resolution, then so is $P_\bullet \overset{\varepsilon}{\twoheadrightarrow} M$.

Proof: Exercise 3, Exercise Sheet 6.
[Hint: Proceed by induction on n, and use the Snake Lemma.] ■

Finally, we note that dual to the notion of a projective resolution is the notion of an injective resolution:

Definition 9.6 (*Injective resolution*)

Let M be an R-module. An **injective resolution** of M is a non-negative cochain complex of injective

R-modules

$$(I^\bullet, d^\bullet) = \big(I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} I^2 \xrightarrow{d^2} \cdots\big)$$

which is exact at I^n for every $n \geqslant 1$ and such that $H^0(I^\bullet) = \ker d^0/0 \cong M$.

Notation: Letting $\iota : M \hookrightarrow I^0$ denote the natural injection, we have a so-called **augmented complex**

$$M \xrightarrow{\iota} I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} I^2 \longrightarrow \cdots$$

associated to the injective resolution $(I^\bullet, d^\bullet)$, and this augmented complex is exact. Hence we will also denote injective resolutions of M by $M \overset{\iota}{\hookrightarrow} I^\bullet$.

Similarly to projective resolutions, one can prove that an injective resolution always exists. There is also a Lifting Theorem and a Comparison Theorem for injective resolutions, so that they are unique up to homotopy (of cochain complexes).

10 Ext and Tor

We now introduce the Ext and Tor groups, which are cohomology and homology groups obtained from applying Hom and tensor product functors to projective/injective resolutions. We will see later that Ext groups can be used in group cohomology to classify abelian group extensions.

Definition 10.1 (Ext-*groups*)

Let M and N be two left R-modules and let $P_\bullet \overset{\varepsilon}{\twoheadrightarrow} M$ be a projective resolution of M. For $n \in \mathbb{Z}_{\geqslant 0}$, the **n-th Ext-group** of M and N is

$$\operatorname{Ext}^n_R(M, N) := H^n\big(\operatorname{Hom}_R(P_\bullet, N)\big),$$

that is, the n-th cohomology group of the cochain complex $\operatorname{Hom}_R(P_\bullet, N)$.

Recipe:

1. Choose a projective resolution $P_\bullet$ of M.
2. Apply the left exact contravariant functor $\operatorname{Hom}_R(-, N)$ to the *projective resolution*

$$P_\bullet = \big(\cdots \xrightarrow{d_3} P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0\big)$$

   to obtain a *cochain* complex

$$\operatorname{Hom}_R(P_0, N) \xrightarrow{d_1^*} \operatorname{Hom}_R(P_1, N) \xrightarrow{d_2^*} \operatorname{Hom}_R(P_3, N) \xrightarrow{d_3^*} \cdots.$$

   of *abelian groups* (which is not exact in general).

3. Compute the cohomology of this new complex.

First of all, we have to check that the definition of the abelian groups $\operatorname{Ext}^n_R(M, N)$ is independent from the choice of the projective resolution of M.

Proposition 10.2

If $P_\bullet \overset{\varepsilon}{\twoheadrightarrow} M$ and $Q_\bullet \overset{\varepsilon'}{\twoheadrightarrow} M$ are two projective resolutions of M, then the groups $H^n\big(\mathrm{Hom}_R(P_\bullet, N)\big)$ and $H^n\big(\mathrm{Hom}_R(Q_\bullet, N)\big)$ are (canonically) isomorphic, via the homomorphisms induced by the chain maps between $P_\bullet$ and $Q_\bullet$ given by the Comparison Theorem applied to the identity morphism Id_M.

Proof: By the Comparison Theorem, there exist chain maps $\varphi_\bullet : P_\bullet \longrightarrow Q_\bullet$ and $\psi_\bullet : Q_\bullet \longrightarrow P_\bullet$ lifting the identity on M and such that $\psi_\bullet \circ \varphi_\bullet \sim \mathrm{Id}_{P_\bullet}$ and $\varphi_\bullet \circ \psi_\bullet \sim \mathrm{Id}_{Q_\bullet}$.
Now, applying the functor $\mathrm{Hom}_R(-, N)$ yields morphisms of cochain complexes

$$\varphi^* : \mathrm{Hom}_R(Q_\bullet, N) \longrightarrow \mathrm{Hom}_R(P_\bullet, N) \quad \text{and} \quad \psi^* : \mathrm{Hom}_R(P_\bullet, N) \longrightarrow \mathrm{Hom}_R(Q_\bullet, N)\,.$$

Since $\varphi_\bullet \circ \psi_\bullet \sim \mathrm{Id}_{Q_\bullet}$ and $\psi_\bullet \circ \varphi_\bullet \sim \mathrm{Id}_{P_\bullet}$, it follows that $\varphi^* \circ \psi^* \sim \mathrm{Id}_{\mathrm{Hom}_R(P_\bullet,N)}$ and $\psi^* \circ \varphi^* \sim \mathrm{Id}_{\mathrm{Hom}_R(Q_\bullet,N)}$.
But then, passing to cohomology, φ^* induces a group homomorphism

$$\overline{\varphi^*} : H^n\big(\mathrm{Hom}_R(Q_\bullet, N)\big) \longrightarrow H^n\big(\mathrm{Hom}_R(P_\bullet, N)\big)$$

(see Exercise 1, Exercise Sheet 5). Since $\varphi_\bullet$ is unique up to homotopy, so is φ^*, and hence $\overline{\varphi^*}$ is unique because homotopic chain maps induce the same morphisms in cohomology. Likewise, there is a unique homomorphism $\overline{\psi^*} : H^n\big(\mathrm{Hom}_R(P_\bullet, N)\big) \longrightarrow H^n\big(\mathrm{Hom}_R(Q_\bullet, N)\big)$ of abelian groups induced by $\psi_\bullet$. Finally, $\varphi^* \circ \psi^* \sim \mathrm{Id}$ and $\psi^* \circ \varphi^* \sim \mathrm{Id}$ imply that $\overline{\varphi^*} \circ \overline{\psi^*} = \mathrm{Id}$ and $\overline{\psi^*} \circ \overline{\varphi^*} = \mathrm{Id}$. Therefore, $\overline{\varphi^*}$ and $\overline{\psi^*}$ are canonically defined isomorphisms. ■

Proposition 10.3 (*Properties of* Ext^n_R)

Let M, M_1, M_2 and N, N_1, N_2 be R-modules and let $n \in \mathbb{Z}_{>0}$ be an integer. The following holds:

(a) $\mathrm{Ext}^0_R(M, N) \cong \mathrm{Hom}_R(M, N)$.

(b) Any morphism of R-modules $\alpha : M_1 \longrightarrow M_2$ induces a group homomorphism

$$\alpha^* : \mathrm{Ext}^n_R(M_2, N) \longrightarrow \mathrm{Ext}^n_R(M_1, N)\,.$$

(c) Any morphism of R-modules $\beta : N_1 \longrightarrow N_2$ induces a group homomorphism

$$\beta_* : \mathrm{Ext}^n_R(M, N_1) \longrightarrow \mathrm{Ext}^n_R(M, N_2)\,.$$

(d) If P is a projective R-module, then $\mathrm{Ext}^n_R(P, N) = 0$ for all $n \geqslant 1$.

(e) If I is an injective R-module, then $\mathrm{Ext}^n_R(M, I) = 0$ for all $n \geqslant 1$.

Proof: (a) Let $P_\bullet \overset{\varepsilon}{\twoheadrightarrow} M$ be a projective resolution of M. Applying the left exact functor $\mathrm{Hom}_R(-, N)$ to the resolution $P_\bullet$ yields the cochain complex

$$\mathrm{Hom}_R(P_0, N) \xrightarrow{d_1^*} \mathrm{Hom}_R(P_1, N) \xrightarrow{d_2^*} \mathrm{Hom}_R(P_3, N) \xrightarrow{d_3^*} \cdots\,.$$

Therefore,

$$\mathrm{Ext}^0_R(M, N) = H^0\big(\mathrm{Hom}_R(P_\bullet, N)\big) = \ker d_1^*/0 \cong \ker d_1^*\,.$$

Now, the tail $\cdots \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} M \longrightarrow 0$ of the augmented complex $P_\bullet \overset{\varepsilon}{\twoheadrightarrow} M$ is an exact sequence of R-modules, so that the induced sequence

$$0 \longrightarrow \mathrm{Hom}_R(M, N) \xrightarrow{\varepsilon^*} \mathrm{Hom}_R(P_0, N) \xrightarrow{d_1^*} \cdots$$

is exact at $\mathrm{Hom}_R(M,N)$ and at $\mathrm{Hom}_R(P_0,N)$ and it follows that

$$\mathrm{Ext}^0_R(M,N) \cong \ker d_1^* = \mathrm{Im}\,\varepsilon^* \cong \mathrm{Hom}_R(M,N)$$

because ε^* is injective.

(b) Let $P_\bullet \overset{\varepsilon}{\twoheadrightarrow} M_1$ be a projective resolution of M_1 and $P'_\bullet \overset{\varepsilon'}{\twoheadrightarrow} M_2$ be a projective resolution of M_2. The Lifting Theorem implies that α lifts to a chain map $\varphi_\bullet : P_\bullet \longrightarrow P'_\bullet$. Then, $\varphi_\bullet$ induces a morphism of chain complexes $\varphi^* : \mathrm{Hom}_R(P'_\bullet,N) \longrightarrow \mathrm{Hom}_R(P_\bullet,N)$ and then φ^* induces a morphism in cohomology

$$\overline{\varphi^*} : \mathrm{Ext}^n_R(M_2,N) \longrightarrow \mathrm{Ext}^n_R(M_1,N)$$

for each $n \geqslant 0$ and we set $\alpha^* := \overline{\varphi^*}$.

(c) Let $P_\bullet \overset{\varepsilon}{\twoheadrightarrow} M$ be a projective resolution of M. Then, there is a morphism of cochain complexes $\beta^\bullet : \mathrm{Hom}_R(P_\bullet,N_1) \longrightarrow \mathrm{Hom}_R(P_\bullet,N_2)$ induced by β, which, in turn, induces a homomorphism of abelian groups β_* in cohomology.

(d) Let $P_\bullet \overset{\varepsilon}{\twoheadrightarrow} M$ be a projective resolution of M.Choose $\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow P$ as a projective resolution of P (i.e. $P_0 := P$, $P_1 = 0, \ldots$), augmented by the identity morphism $\mathrm{Id}_P : P \longrightarrow P$. Then the induced cochain complex is

$$\mathrm{Hom}_R(P,N) \xrightarrow{0} 0 \xrightarrow{0} 0 \xrightarrow{0} \cdots,$$

so that clearly $\mathrm{Ext}^n_R(P,N) = 0$ for each $n \geqslant 1$.

(e) Let $P_\bullet \overset{\varepsilon}{\twoheadrightarrow} M$ be a projective resolution of M. Since I is injective, the functor $\mathrm{Hom}_R(-,I)$ is exact. Therefore the induced cochain complex

$$\mathrm{Hom}_R(P_0,I) \xrightarrow{d_1^*} \mathrm{Hom}_R(P_1,I) \xrightarrow{d_2^*} \mathrm{Hom}_R(P_2,I) \xrightarrow{d_3^*} \cdots$$

is exact and its cohomology is zero. The claim follows. ■

Remark 10.4

Using the proposition one can prove that for every $n \in \mathbb{Z}_{\geqslant 0}$, $\mathrm{Ext}^n_R(-,N) : {}_R\mathbf{Mod} \longrightarrow \mathbf{Ab}$ is a contravariant additive functor, and $\mathrm{Ext}^n_R(M,-) : {}_R\mathbf{Mod} \longrightarrow \mathbf{Ab}$ is a covariant additive functor.

Theorem 10.5 (*Long exact sequences of* Ext*-groups*)

(a) Any s.e.s. $0 \longrightarrow N_1 \xrightarrow{\varphi} N_2 \xrightarrow{\psi} N_3 \longrightarrow 0$ of R-modules induces a long exact sequence of abelian groups

$$0 \longrightarrow \mathrm{Ext}^0_R(M,N_1) \xrightarrow{\varphi_*} \mathrm{Ext}^0_R(M,N_2) \xrightarrow{\psi_*} \mathrm{Ext}^0_R(M,N_3) \xrightarrow{\delta^0} \mathrm{Ext}^1_R(M,N_1) \longrightarrow \ldots$$
$$\ldots \longrightarrow \mathrm{Ext}^n_R(M,N_1) \xrightarrow{\varphi_*} \mathrm{Ext}^n_R(M,N_2) \xrightarrow{\psi_*} \mathrm{Ext}^n_R(M,N_3) \xrightarrow{\delta^n} \mathrm{Ext}^{n+1}_R(M,N_1) \longrightarrow \ldots.$$

(b) Any s.e.s. $0 \longrightarrow M_1 \xrightarrow{\alpha} M_2 \xrightarrow{\beta} M_3 \longrightarrow 0$ of R-modules induces a long exact sequence of abelian groups

$$0 \longrightarrow \mathrm{Ext}^0_R(M_3,N) \xrightarrow{\beta^*} \mathrm{Ext}^0_R(M_2,N) \xrightarrow{\alpha^*} \mathrm{Ext}^0_R(M_1,N) \xrightarrow{\delta^1} \mathrm{Ext}^1_R(M_3,N) \longrightarrow \ldots$$
$$\ldots \longrightarrow \mathrm{Ext}^n_R(M_3,N) \xrightarrow{\beta^*} \mathrm{Ext}^n_R(M_2,N) \xrightarrow{\alpha^*} \mathrm{Ext}^n_R(M_1,N) \xrightarrow{\delta^n} \mathrm{Ext}^{n+1}_R(M_3,N) \longrightarrow \ldots.$$

Proof: (a) Let $P_\bullet$ be a projective resolution of M. Then there is an induced short exact sequence of cochain complexes

$$0 \longrightarrow \mathrm{Hom}_R(P_\bullet, N_1) \xrightarrow{\varphi^\bullet} \mathrm{Hom}_R(P_\bullet, N_2) \xrightarrow{\psi^\bullet} \mathrm{Hom}_R(P_\bullet, N_3) \longrightarrow 0$$

because each module P_n is projective. Indeed, at each degree $n \in \mathbb{Z}_{\geqslant 0}$ this sequence is

$$0 \longrightarrow \mathrm{Hom}_R(P_n, N_1) \xrightarrow{\varphi_*} \mathrm{Hom}_R(P_n, N_2) \xrightarrow{\psi_*} \mathrm{Hom}_R(P_n, N_3) \longrightarrow 0$$

obtained by applying the functor $\mathrm{Hom}_R(P_n, -)$, which is exact as P_n is projective. It is then easily checked that this gives a s.e.s. of cochain complexes, that is that the induced differential maps commute with the induced homomorphisms φ_*. Thus, applying Theorem 8.13 yields the required long exact sequence in cohomology.

(b) Let $P_\bullet$ be a projective resolution of M_1 and let $Q_\bullet$ be a projective resolution of M_3. By the Horseshoe Lemma (Lemma 9.5), there exists a projective resolution $R_\bullet$ of M_2 and a short exact sequence of chain complexes

$$0 \longrightarrow P_\bullet \longrightarrow R_\bullet \longrightarrow Q_\bullet \longrightarrow 0\,,$$

lifting the initial s.e.s. of R-modules. Since Q_n is projective for each $n \geqslant 0$, the sequences

$$0 \longrightarrow P_n \longrightarrow R_n \longrightarrow Q_n \longrightarrow 0$$

are split exact for each $n \geqslant 0$. Therefore applying $\mathrm{Hom}_R(-, N)$ yields a split exact s.e.s.

$$0 \longrightarrow \mathrm{Hom}_R(Q_n, N) \longrightarrow \mathrm{Hom}_R(R_n, N) \longrightarrow \mathrm{Hom}_R(P_n, N) \longrightarrow 0$$

for each for each $n \geqslant 0$. It follows that there is a s.e.s. of cochain complexes

$$0 \longrightarrow \mathrm{Hom}_R(Q_\bullet, N) \longrightarrow \mathrm{Hom}_R(R_\bullet, N) \longrightarrow \mathrm{Hom}_R(P_\bullet, N) \longrightarrow 0.$$

The associated long exact sequence in cohomology (Theorem 8.13) is the required long exact sequence. ■

The above results show that the Ext groups "measure" and "repair" the non-exactness of the functors $\mathrm{Hom}_R(M, -)$ and $\mathrm{Hom}_R(-, N)$.

The next result is called "dimension-shifting" in the literature (however, it would be more appropriate to call it "degree-shifting"); it provides us with a method to compute Ext-groups by induction.

Remark 10.6 (*Dimension shifting*)

Let N be an R-module and consider a s.e.s.

$$0 \longrightarrow L \xrightarrow{\alpha} P \xrightarrow{\beta} M \longrightarrow 0$$

of R-modules, where is P projective (if M is given, take e.g. P free mapping onto M, with kernel L). Then $\mathrm{Ext}^n_R(P, N) = 0$ for all $n \geqslant 1$ and applying the long exact sequence of Ext-groups yields at each degree $n \geqslant 1$ an exact sequence of the form

$$0 \xrightarrow{\alpha^*} \mathrm{Ext}^n_R(L, N) \xrightarrow{\delta^n} \mathrm{Ext}^{n+1}_R(M, N) \xrightarrow{\beta^*} 0,$$

where the connecting homomorphism δ^n is therefore forced to be an isomorphism:

$$\mathrm{Ext}_R^{n+1}(M,N) \cong \mathrm{Ext}_R^n(L,N)\,.$$

Note that the same method applies to the second variable with a short exact sequence whose middle term is injective.

A consequence of the dimension shifting argument is that it allows us to deal with direct sums and products of modules in each variable of the Ext-groups. For this we need the following lemma:

Lemma 10.7

Consider the following commutative diagram of R-modules with exact rows:

$$\begin{tikzcd} A' \arrow[r, "\alpha"] \arrow[d, "f"'] & A \arrow[r, "\beta"] \arrow[d, "g"] & A'' \arrow[r] & 0 \\ B' \arrow[r, "\varphi"'] & B \arrow[r, "\psi"'] & B'' \arrow[r] & 0 \end{tikzcd}$$

Then there exists a morphism $h \in \mathrm{Hom}_R(A'', B'')$ such that $h \circ \beta = \psi \circ g$. Moreover, if f and g are isomorphisms, then so is h.

Proof: Exercise 5, Exercise Sheet 7. ■

Proposition 10.8 (Ext *and direct sums*)

(a) Let $\{M_i\}_{i\in I}$ be a family of R-modules and let N be an R-module. Then

$$\mathrm{Ext}_R^n\left(\bigoplus_{i\in I} M_i, N\right) \cong \prod_{i\in I} \mathrm{Ext}_R^n(M_i,N) \quad \forall\, n \geqslant 0\,.$$

(b) Let M be an R-module and let $\{N_i\}_{i\in I}$ be a family of R-modules. Then

$$\mathrm{Ext}_R^n\left(M, \prod_{i\in I} N_i\right) \cong \prod_{i\in I} \mathrm{Ext}_R^n(M,N_i) \quad \forall\, n \geqslant 0.$$

Proof: (a) **Case** $n = 0$. By Proposition 10.3(a) and the universal property of the direct sum (Proposition 4.2), we have

$$\mathrm{Ext}_R^0\left(\bigoplus_{i\in I} M_i, N\right) \cong \mathrm{Hom}_R\left(\bigoplus_{i\in I} M_i, N\right) \cong \prod_{i\in I} \mathrm{Hom}_R(M_i,N) \cong \prod_{i\in I} \mathrm{Ext}_R^n(M_i,N)\,.$$

Now, suppose that $n \geqslant 1$ and choose for each $i \in I$ a s.e.s. of R-modules

$$0 \longrightarrow L_i \longrightarrow P_i \longrightarrow M_i \longrightarrow 0,$$

where P_i is projective (e.g. choose P_i free with quotient isomorphic to M_i and kernel L_i). These sequences induce a s.e.s.

$$0 \longrightarrow \bigoplus_{i\in I} L_i \longrightarrow \bigoplus_{i\in I} P_i \longrightarrow \bigoplus_{i\in I} M_i \longrightarrow 0\,.$$

Case $n \geqslant 1$: We proceed by induction on n.
First, for n=1, using a long exact sequence of Ext-groups, we obtain a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_R(\bigoplus_{i\in I} P_i, N) \arrow[r] \arrow[d, "\cong"] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{Hom}_R(\bigoplus_{i\in I} L_i, N) \arrow[r, "\delta^0"] \arrow[d, "\cong"] & \mathrm{Ext}^1_R(\bigoplus_{i\in I} M_i, N) \arrow[r] & \mathrm{Ext}^1_R(\bigoplus_{i\in I} P_i, N) \\
\prod_{i\in I} \mathrm{Hom}_R(P_i, N) \arrow[r] & \prod_{i\in I} \mathrm{Hom}_R(L_i, N) \arrow[r] & \prod_{i\in I} \mathrm{Ext}^1_R(M_i, N) \arrow[r] & \prod_{i\in I} \mathrm{Ext}^1_R(P_i, N)
\end{tikzcd}$$

with the following properties:

- the morphisms of the bottom row are induced componentwise;
- both rows are exact; and
- the two vertical isomorphisms are given by the case $n = 0$.

Since P_i is projective for every $i \in I$, so is $\bigoplus_{i\in I} P_i$, thus Proposition 10.3 yields

$$\mathrm{Ext}^1_R\left(\bigoplus_{i\in I} P_i, N\right) \cong 0 \cong \prod_{i\in I} \mathrm{Ext}^1_R(P_i, N) \quad \forall\, i \in I\,.$$

Therefore Lemma 10.7 yields

$$\mathrm{Ext}^1_R\left(\bigoplus_{i\in I} M_i, N\right) \cong \prod_{i\in I} \mathrm{Ext}^1_R(M_i, N)\ .$$

Now assume that $n \geqslant 2$ and assume that the claim holds for the $(n-1)$-th Ext-groups, that is

$$\mathrm{Ext}^{n-1}_R\left(\bigoplus_{i\in I} L_i, N\right) \cong \prod_{i\in I} \mathrm{Ext}^{n-1}_R(L_i, N)\ .$$

Then, applying the Dimension Shifting argument yields

$$\mathrm{Ext}^{n-1}_R\left(\bigoplus_{i\in I} L_i, N\right) \cong \mathrm{Ext}^{n}_R\left(\bigoplus_{i\in I} M_i, N\right)\ .$$

and

$$\mathrm{Ext}^{n-1}_R(L_i, N) \cong \mathrm{Ext}^{n}_R(M_i, N) \quad \forall i \in I\,,$$

so that

$$\prod_{i\in I} \mathrm{Ext}^{n-1}_R(L_i, N) \cong \prod_{i\in I} \mathrm{Ext}^{n}_R(M_i, N)\ .$$

Hence the required isomorphism

$$\mathrm{Ext}^{n}_R\left(\bigoplus_{i\in I} M_i, N\right) \cong \prod_{i\in I} \mathrm{Ext}^{n}_R(M_i, N)\ .$$

(b) Similar to (a): proceed by induction and apply a dimension shift. (In this case, we use s.e.s.'s with injective middle terms.) ■

To end this chapter, we introduce the Tor-groups, which "measure" the non-exactness of the functors $M \otimes_R -$ and $-\otimes_R N$.

Definition 10.9 (Tor-*groups*)

Let M be a right R-module and N be a left R-module. Let $P_\bullet$ be a projective resolution of N. For $n \in \mathbb{Z}_{\geqslant 0}$, the **n-th Tor-group** of M and N is

$$\mathrm{Tor}_n^R(M,N) := H_n(M \otimes_R P_\bullet),$$

that is, the n-th homology group of the chain complex $M \otimes_R P_\bullet$.

Proposition 10.10

Let M, M_1, M_2, M_3 be right R-modules and let N, N_1, N_2, N_3 be left R-modules.

(a) The group $\mathrm{Tor}_n^R(M,N)$ is independant of the choice of the projective resolution of N.

(b) $\mathrm{Tor}_0^R(M,N) \cong M \otimes_R N$.

(c) $\mathrm{Tor}_n^R(-,N)$ is an additive covariant functor.

(d) $\mathrm{Tor}_n^R(M,-)$ is an additive covariant functor.

(e) $\mathrm{Tor}_n^R(M_1 \oplus M_2, N) \cong \mathrm{Tor}_n^R(M_1,N) \oplus \mathrm{Tor}_n^R(M_2,N)$.

(f) $\mathrm{Tor}_n^R(M, N_1 \oplus N_2) \cong \mathrm{Tor}_n^R(M,N_1) \oplus \mathrm{Tor}_n^R(M,N_2)$.

(g) If either M or N is flat (so in particular if either M or N is projective), then $\mathrm{Tor}_n^R(M,N) = 0$ for all $n \geqslant 1$.

(h) Any s.e.s. $0 \longrightarrow M_1 \xrightarrow{\alpha} M_2 \xrightarrow{\beta} M_3 \longrightarrow 0$ of right R-modules induces a long exact sequence

$$\cdots \longrightarrow \mathrm{Tor}_{n+1}^R(M_3,N) \xrightarrow{\delta_{n+1}} \mathrm{Tor}_n^R(M_1,N) \xrightarrow{\alpha_*} \mathrm{Tor}_n^R(M_2,N) \xrightarrow{\beta_*} \mathrm{Tor}_n^R(M_3,N) \xrightarrow{\delta_n} \cdots$$
$$\cdots \longrightarrow \mathrm{Tor}_1^R(M_3,N) \xrightarrow{\delta_1} M_1 \otimes_R N \xrightarrow{\alpha \otimes \mathrm{Id}_N} M_2 \otimes_R N \xrightarrow{\beta \otimes \mathrm{Id}_N} M_3 \otimes_R N \longrightarrow 0$$

of abelian groups.

(i) Any s.e.s. $0 \longrightarrow N_1 \xrightarrow{\alpha} N_2 \xrightarrow{\beta} N_3 \longrightarrow 0$ of left R-modules induces a long exact sequence

$$\cdots \longrightarrow \mathrm{Tor}_{n+1}^R(M,N_3) \xrightarrow{\delta_{n+1}} \mathrm{Tor}_n^R(M,N_1) \xrightarrow{\alpha_*} \mathrm{Tor}_n^R(M,N_2) \xrightarrow{\beta_*} \mathrm{Tor}_n^R(M,N_3) \xrightarrow{\delta_n} \cdots$$
$$\cdots \longrightarrow \mathrm{Tor}_1^R(M,N_3) \xrightarrow{\delta_1} M \otimes_R N_1 \xrightarrow{\mathrm{Id}_M \otimes \alpha} M \otimes_R N_2 \xrightarrow{\mathrm{Id}_M \otimes \beta} M \otimes_R N_3 \longrightarrow 0$$

of abelian groups.

The proof of the above results are in essence similar to the proofs given for the Ext-groups.