LECTURE 20: THE MAYER-VIETORIS SEQUENCE

1. The Mayer-Vietoris sequence

We start with some conceptions from homological algebra. A *cochain complex*

$$\cdots \xrightarrow{d_{k-2}} V^{k-1} \xrightarrow{d_{k-1}} V^k \xrightarrow{d_k} V^{k+1} \xrightarrow{d_{k+1}} V^{k+2} \xrightarrow{d_{k+2}} \cdots$$

consists of a sequence of vector spaces

$$\cdots, V^{k-1}, V^k, V^{k+1}, \cdots$$

and a sequence of linear maps $d_k : V^k \to V^{k+1}$ so that

$$d_{k+1} \circ d_k = 0$$

for all k. Obviously we have $\mathrm{Im}(d_k) \subset \mathrm{Ker}(d_{k+1})$. Such a complex is called *exact* if for all k,

$$\mathrm{Im}(d_k) = \mathrm{Ker}(d_{k+1}).$$

Note that if an exact sequence starts with 0,

$$0 \xrightarrow{d_0} V^1 \xrightarrow{d_1} V^2 \xrightarrow{d_2} V^3 \xrightarrow{d_3} \cdots,$$

then $d_1 : V^1 \to V^2$ is injective, while if an exact sequence ends with 0,

$$\cdots \xrightarrow{d_{k-2}} V^{k-1} \xrightarrow{d_{k-1}} V^k \xrightarrow{d_k} V^{k+1} \xrightarrow{d_{k+1}} 0,$$

then $d_k : V^k \to V^{k+1}$ is surjective. In particular, if we have a *short exact sequence*

$$0 \longrightarrow V^1 \xrightarrow{d_1} V^2 \xrightarrow{d_2} V^3 \longrightarrow 0,$$

then d_1 is injective, d_2 is surjective, and $V^2 \simeq V^1 \oplus V^3$.

Now suppose M is a smooth manifold, and U, V are open sets in M so that $M = U \cup V$. Then we have inclusion maps

$$\iota_1 : U \hookrightarrow M, \quad \iota_2 : V \hookrightarrow M$$

and inclusion maps

$$\jmath_1 : U \cap V \hookrightarrow U, \quad \jmath_2 : U \cap V \hookrightarrow V.$$

These inclusion maps induce linear maps between the spaces of k-forms (and also induces linear maps between corresponding de Rham cohomology groups, which we use the same notation)

$$\alpha_k : \Omega^k(M) \to \Omega^k(U) \oplus \Omega^k(V), \quad \omega \mapsto (\iota_1^*\omega, \iota_2^*\omega)$$

and

$$\beta_k : \Omega^k(U) \oplus \Omega^k(V) \to \Omega^k(U \cap V), \quad (\omega_1, \omega_2) \mapsto \jmath_1^*\omega_1 - \jmath_2^*\omega_2.$$

The following proposition is the starting point of everything below.The proof is simple and is left as an exercise.

Proposition 1.1. *For any k, the sequence*
$$0 \longrightarrow \Omega^k(M) \xrightarrow{\alpha_k} \Omega^k(U) \oplus \Omega^k(V) \xrightarrow{\beta_k} \Omega^k(U \cap V) \longrightarrow 0$$
is a short exact sequence.

It is a standard fact in homological algebra that such a short exact sequence of cochain complexes induces a long exact sequence on cohomology groups. To describe this, we need to define a linear map (called the *connecting homomorphism*)
$$\delta_k : H^k_{dR}(U \cap V) \to H^{k+1}_{dR}(M).$$
The map δ_k can be defined by the standard "diagram chasing" method. In what follows we will give an explicit construction: We fix a partition of unity $\{\rho_U, \rho_V\}$ subordinate to the cover $\{U, V\}$ of M. For any $\omega \in Z^k(U \cap V)$, we define
$$\delta_k([\omega]) := [\eta],$$
where η is the $(k+1)$-form defined to be $d(\rho_V \omega)$ on U and $-d(\rho_U \omega)$ on V.

Lemma 1.2. *The map δ_k is well-defined.*

Proof. There are many issues to be handled:

- $\rho_V \omega \in \Omega^k(U)$: This is because the function ρ_V is supported in $U \cap V$. So although ω is a k-form only defined on $U \cap V$, the form $\rho_V \omega$ is defined and smooth on U.
- $\eta \in \Omega^{k+1}(M)$: Since $\rho_U + \rho_V = 1$ and $d\omega = 0$, we see $d(\rho_V \omega) = -d(\rho_U \omega)$ on $U \cap V$, so η is a well-defined smooth $(k+1)$-form on M.
- $\eta \in Z^{k+1}(M)$: By definition we also have $d\eta = 0$ on both U and V. So $\eta \in Z^{k+1}(M)$.
- $[\eta]$ is independent of the choices of ρ_U and ρ_V: Let $\tilde\rho_U$ and $\tilde\rho_V$ be another partition of unity subordinate to the cover $\{U, V\}$, and let $\tilde\eta$ be the resulting $(k+1)$-form. Then $\tilde\rho_V - \rho_V = \rho_U - \tilde\rho_U$ is compactly supported in $U \cap V$. So if we let $\xi = (\tilde\rho_V - \rho_V)\omega$, then it is a smooth k-form defined on M, and by construction, we have $\tilde\eta - \eta = d\xi$.
- $[\eta]$ is independent of the choices of ω: Suppose $\tilde\omega = \omega + d\zeta$ and denote the resulting $(k+1)$-from by $\tilde\eta$. Then $\tilde\eta - \eta = d(\rho_V d\zeta)$ on U and $\tilde\eta - \eta = -d(\rho_U d\zeta)$ on V. We take $\xi = -d\rho_V \wedge \zeta = d\rho_U \wedge \zeta$. Then ξ is a smooth k-form on M so that
$$d\xi = \left\{ \begin{array}{rll} d\rho_V \wedge d\zeta = & d(\rho_V d\zeta) & \text{on } U \\ -d\rho_U \wedge d\zeta = & -d(\rho_U d\zeta) & \text{on } V \end{array} \right\} = \tilde\eta - \eta.$$
So $[\tilde\eta] = [\eta]$ and the conclusion follows.

$\square$

Now we can state the main theorem:

Theorem 1.3 (Mayer-Vietories)**.** *Let U, V be open sets in M so that $M = U \cup V$. Then we have a long exact sequence*
$$\cdots \xrightarrow{\delta_{k-1}} H^k_{dR}(M) \xrightarrow{\alpha_k} H^k_{dR}(U) \oplus H^k_{dR}(V) \xrightarrow{\beta_k} H^k_{dR}(U \cap V) \xrightarrow{\delta_k} H^{k+1}_{dR}(M) \xrightarrow{\alpha_{k+1}} \cdots$$

Proof. One has to show $\mathrm{Im}(\alpha_k) = \ker(\beta_k)$, $\mathrm{Im}(\beta_k) = \ker(\delta_k)$, and $\mathrm{Im}(\delta_k) = \ker(\alpha_{k+1})$. This amounts to prove 6 inclusion relations. We will prove one of them and leave the rest as exercises.

Proof of $\mathrm{Im}(\beta_k) \subset \ker(\delta_k)$: Suppose $\omega_1 \in Z^k(U)$, $\omega_2 \in Z^k(V)$. Let $\omega := \beta_k(\omega_1, \omega_2) = \jmath_1^*\omega_1 - \jmath_2^*\omega_2$. Then $\delta_k([\omega]) = [\eta]$, where

$$\eta = \begin{cases} d(\rho_V\omega) = \quad d(\rho_V\omega - \omega_1) & \text{on } U \\ -d(\rho_U\omega) = -d(\rho_U\omega + \omega_2) & \text{on } V \end{cases}$$

Note that on $U \cap V$,

$$\rho_V\omega = -\rho_U\omega + \jmath_1^*\omega_1 - \jmath_2^*\omega_2.$$

So there is a smooth k-form ξ on M so that $\xi = \rho_V\omega - \jmath_1^*\omega_1$ on U and $\xi = -\rho_U\omega - \jmath_2^*\omega_2$ on V. As a consequence, $\eta = d\xi$ and thus $[\eta] = 0$. □

2. Applications of Mayer-Vietoris sequence

Mayer-Vietories sequence is a very powerful tool which has many applications.

Application 1: The de Rham cohomology groups of spheres

Theorem 2.1. *For $n \geq 1$, $H^k_{dR}(S^n) \simeq \begin{cases} \mathbb{R}, & k = 0, n, \\ 0, & 1 \leq k \leq n-1. \end{cases}$.*

Proof. We have proved $H^0_{dR}(S^n) \simeq \mathbb{R}$ and $H^1_{dR}(S^1) \simeq \mathbb{R}$. In what follows we will prove

(1) For $n \geq 2$, $H^1_{dR}(S^n) = 0$.
(2) For $n \geq 2, k \geq 2$, $H^k_{dR}(S^n) \simeq H^{k-1}_{dR}(S^{n-1})$.

Obvious these results together implies the theorem.

For $n \geq 2$, we let $U = S^n - \{(0, \cdots, 0, -1)\}$ and $V = S^n - \{(0, \cdots, 0, 1)\}$. Then

- $M = U \cup V$,
- U and V are diffeomorphic to $\mathbb{R}^n$,
- $U \cap V$ is homotopy equivalent to S^{n-1}.

To prove (1), we look at the beginning of the Mayer-Vietories sequence

$$0 \longrightarrow H^0_{dR}(S^n) \longrightarrow H^0_{dR}(U) \oplus H^0_{dR}(V) \longrightarrow H^0_{dR}(U \cap V) \longrightarrow H^1(S^n) \to H^1_{dR}(U) \oplus H^1_{dR}(V),$$

which now becomes

$$0 \longrightarrow \mathbb{R} \xrightarrow{\alpha_0} \mathbb{R}^2 \xrightarrow{\beta_0} \mathbb{R} \xrightarrow{\delta_0} H^1_{dR}(S^n) \to 0.$$

Since α_0 is injective,

$$\dim \mathrm{Ker}(\beta_0) = \dim \mathrm{Im}(\alpha_0) = 1.$$

It follows that $\dim \mathrm{Im}(\beta_0) = 1$, i.e. β_0 is surjective. So $\mathrm{Ker}(\delta_0) = \mathbb{R}$, i.e. $\delta_0 = 0$. But by exactness, δ_0 is surjective. This implies $H^1_{dR}(S^n) = 0$.

To prove (2), we look at the following part of Mayer-Vietories sequence

$$H^{k-1}_{dR}(U) \oplus H^{k-1}_{dR}(V) \xrightarrow{\beta_{k-1}} H^{k-1}_{dR}(U \cap V) \xrightarrow{\delta_{k-1}} H^k_{dR}(S^n) \xrightarrow{\alpha_k} H^k_{dR}(U) \oplus H^k_{dR}(V),$$

which becomes

$$0 \xrightarrow{\beta_{k-1}} H^{k-1}_{dR}(S^{n-1}) \xrightarrow{\delta_{k-1}} H^k_{dR}(S^n) \xrightarrow{\alpha_k} 0.$$

By exactness, the map δ_{k-1} has to be injective and surjective, and thus a linear isomorphism. This proves (2). $\square$

As a consequence, we get

Corollary 2.2 (Topological Invariance of Dimension)**.** *If $m \neq n$, then $\mathbb{R}^n$ is not homeomorphic to $\mathbb{R}^m$.*

Proof. If $f : \mathbb{R}^n \to \mathbb{R}^m$ is a homeomorphism, then $f : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^m \setminus \{f(0)\}$ is also a homeomorphism. It follows that $H^k_{dR}(\mathbb{R}^n \setminus \{0\}) = H^k_{dR}(\mathbb{R}^m \setminus \{f(0)\}$ for all k. But $\mathbb{R}^n \setminus \{0\}$ is homotopy equivalent to S^{n-1}, while $\mathbb{R}^m \setminus \{f(0)\}$ is homotopic to S^{m-1}. It follows $H^k_{dR}(S^{m-1}) = H^k_{dR}(S^{n-1})$ for all k. This cannot happen unless $m = n$. $\square$

Application 2: $\dim H^k_{dR}(M) < \infty$ for many smooth manifolds

Definition 2.3. Let M be a smooth manifold and $\{U_\alpha\}_{\alpha\in\Lambda}$ an open cover of M. We say $\{U_\alpha\}_{\alpha\in\Lambda}$ is a *good cover* if for any finite subset $I = \{\alpha_1, \cdots, \alpha_k\} \subset \Lambda$ of indices, the intersection

$$U_I := U_{\alpha_1} \cap U_{\alpha_2} \cap \cdots \cap U_{\alpha_k}$$

is either empty or diffeomorphic to $\mathbb{R}^n$.

Using Riemannian geometry, one can show that any open cover of any smooth manifold M admits a refinement which is a good cover. (Use the so-called geodesically convex neighborhoods.) Obviously any sub-cover of a good cover is still good.

Theorem 2.4. *If M admits a finite good cover, then $H^k_{dR}(M)$ is finite dimensional.*

Proof. We proceed by induction on the number of sets in a finite good cover of M. If M admits a good cover that contains only one open set, then that open set has to be M itself, so that M is contractible. In this case M is diffeomorphic to $\mathbb{R}^n$ and the conclusion follows.

Now suppose the theorem holds for any manifold that admits a good cover containing $l-1$ open sets. Let M be a manifold with an good cover $\{U_1, \cdots, U_l\}$. We denote $U = U_1 \cup \cdots \cup U_{l-1}$ and $V = U_l$. Then $U \cap V$ admits a finite good cover $\{U_1 \cap U_l, \cdots, U_{l-1} \cap U_l\}$. It follows that all the de Rham cohomology groups of U, V and $U \cap V$ are finite dimensional. Now consider the Mayer-Vertoris sequence

$$\cdots \longrightarrow H^{k-1}_{dR}(U \cap V) \xrightarrow{\delta_{k-1}} H^k_{dR}(M) \xrightarrow{\alpha_k} H^k_{dR}(U) \oplus H^k_{dR}(V) \longrightarrow .$$

The conclusion follows since

$$\dim \mathrm{Im}(\alpha_k) \leq \dim H^k_{dR}(U) \oplus H^k_{dR}(V) < \infty$$

and

$$\dim \mathrm{Ker}(\alpha_k) = \dim \mathrm{Im}(\delta_{k-1}) \leq \dim H^{k-1}_{dR}(U \cap V) < \infty.$$

$\square$

As a consequence, we immediately get

Corollary 2.5. *If M is compact or M is homotopy equivalent to a compact manifold, then* $\dim H^k_{dR}(M) < \infty$ *for all k.*

$\boxed{\text{Application 3: The Kunneth formula}}$

Theorem 2.6. *Let M and N be manifolds with finite good covers. Then for any $0 \le k \le \dim M + \dim N$, one has*

$$H^k_{dR}(M \times N) \simeq \bigoplus_{i=0}^{k} H^i_{dR}(M) \otimes H^{k-i}_{dR}(N).$$

Sketch of proof. Let $\pi_M : M \times N \to M$ and $\pi_N : M \times N$ be the standard projections. Then we get a map

$$\Psi : \Omega^*(M) \otimes \Omega^*(N) \to \Omega^*(M \times N), \quad \omega_1 \otimes \omega_2 \mapsto \pi_M^* \omega_1 \wedge \pi_N^* \omega_2.$$

One can check that this map induces a map on cohomologies

$$\Psi : H^*_{dR}(M) \otimes H^*_{dR}(N) \to H^*_{dR}(M \times N), \quad [\omega_1] \otimes [\omega_2] \mapsto [\pi_M^* \omega_1 \wedge \pi_N^* \omega_2].$$

To prove that this Ψ is in fact an linear isomorphism, we do induction on the number l of elements in a good cover of M.

If $l = 1$, i.e. M is diffeomorphic to $\mathbb{R}^n$, then the Kunneth formula follows from the fact that $\mathbb{R}^n \times N$ is homotopy equivalent to N, and $H^k_{dR}(\mathbb{R}^n) = \mathbb{R}$ for $k = 0$ and $H^k_{dR}(\mathbb{R}^n) = 0$ for other k's.

Now suppose the Kunneth formula is proved for manifolds M admitting a good cover with no more than $l - 1$ open sets, and supppose now that

$$M = U_1 \cup \cdots \cup U_l$$

is a good cover. Again we let $U = U_1 \cup \cdots U_{n-1}$ and $V = U_n$. For simplicity we will denote

$$\widetilde{H}^k(M, N) := \bigoplus_{i=0}^{k} H^i_{dR}(M) \otimes H^{k-i}_{dR}(N).$$

Consider the following diagram

$$\begin{tikzcd}
\widetilde{H}^k(M,N) \arrow[r, "\alpha"'] \arrow[d, "\Psi"'] & \widetilde{H}^k(U,N) \oplus \widetilde{H}^k(V,N) \arrow[r, "\beta"'] \arrow[d, "\Psi"'] & \widetilde{H}^k(U \cap V, N) \arrow[r, "\delta"'] \arrow[d, "\Psi"'] & \widetilde{H}^{k+1}(M,N) \arrow[d, "\Psi"'] \\
H^k_{dR}(M \times N) \arrow[r, "\alpha"'] & H^k_{dR}(U \times N) \oplus H^k_{dR}(V \times N) \arrow[r, "\beta"'] & H^k_{dR}(U \cap V \times N) \arrow[r, "\delta"'] & H^{k+1}_{dR}(M \times N)
\end{tikzcd}$$

where the horizontal maps α, β, δ's are the ones that is induced in the obvious way from the $\alpha_k, \beta_k, \delta_k$'s that we defined above. For this diagram we have

- By using the Mayer-Vietoris sequence one can prove that the two rows are exact.
- Moreover, one can prove that the diagram is commutative, i.e. we have $\Psi \circ \alpha = \alpha \circ \Psi$, $\Psi \circ \beta = \beta \circ \Psi$ and $\Psi \circ \delta = \delta \circ \Psi$. [The first two equalities are easy to prove, while the last one is much more complicated.]

- By using the induction hypothesis, the second and the third Ψ are linear isomorphisms.

Now the result follows from the following well-known Five lemma in homological algebra. □

Lemma 2.7 (Five lemma)**.** *Suppose we have the following commutative diagram*

$$\begin{tikzcd} V_1 \arrow[r, "\alpha_1"'] \arrow[d, "\gamma_1"'] & V_2 \arrow[r, "\alpha_2"'] \arrow[d, "\gamma_2"'] & V_3 \arrow[r, "\alpha_3"'] \arrow[d, "\gamma_3"'] & V_4 \arrow[r, "\alpha_4"'] \arrow[d, "\gamma_4"'] & V_5 \arrow[d, "\gamma_5"'] \\ W_1 \arrow[r, "\beta_1"'] & W_2 \arrow[r, "\beta_2"'] & W_3 \arrow[r, "\beta_3"'] & W_4 \arrow[r, "\beta_4"'] & W_5 \end{tikzcd}$$

where each V_i is a linear space, and each map is a linear map. Suppose the two rows are exact, and the vertical maps $\gamma_1, \gamma_2, \gamma_4, \gamma_5$ are isomorphisms, then the map γ_3 is also an isomorphism.

Proof. Left as an exercise. □

As corollaries of the Kunneth formula, one can prove

Corollary 2.8. *For the n-dimensional torus $\mathbb{T}^n = S^1 \times \cdots \times S^1$,*

$$H^k(\mathbb{T}^n) \simeq \mathbb{R}^{\binom{n}{k}}.$$

Suppose M is a manifold whose de Rham cohomology groups are finite dimensional ,and we consider the polynomial

$$p_M(t) = \sum_{i=0}^{n} \beta_i(M) t^i,$$

where $\beta_i(M)$ is the ith betti number of M. Note that the Euler characteristic $\chi(M) = p_M(-1)$. A direct consequence of Kunneth formula is

Corollary 2.9. *If M and N are manifolds admitting finite good cover. Then*

$$p_{M\times N}(t) = p_M(t) p_N(t).$$

In particular, $\chi(M \times N) = \chi(M)\chi(N)$.