QUIVER REPRESENTATIONS: GABRIEL'S THEOREM AND KAC'S THEOREM

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Abstract. This is an expository paper written to discuss representations of quivers. We prove Gabriel's Theorem, which was the beginning theorem to spark interest in studying quivers. We also prove a weaker form of Kac's Theorem.

Contents

1. Introduction

The aim of this paper will be to present two important theorems that have created interest in the study of quivers. We will do so in a mainly self-contained way. First, we will give a proof of Gabriel's Theorem which is the following.

Theorem 1.1. *(Gabriel's Theorem)*

1. *A quiver Q is of finite type if and only if its underlying graph, $\Gamma(Q)$ is the union of Dynkin diagrams.*
2. *If Q has finite representation type, then the map $V \mapsto \underline{dim}V$ gives a bijection between Ind(Q) and the set of positive roots.*

The roots in this theorem correspond to the roots of a semi-simple Lie algebra. However, we will define roots in a self-contained way. Interestingly, we will use the second statement to prove the first. Gabriel's Theorem sparked interest in the theory of quiver representations. In 1973, Dlab and Ringel extended the second part of the result to find Dynkin diagrams corresponding to all finite-dimensional semi-simple Lie algebras. In 1983, Kac extended the result in a much more general direction. He extended the correspondence of indecomposable representations to Kac-Moody Lie algebras, which can be thought of as the infinite-dimensional analog to semisimple Lie algebras. His proof counts the number of absolutely indecomposable isomorphism classes over a finite field using polynomials with integer

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coefficients [4]. There is no known proof that does not use finite fields. The theorem is the following

Theorem 1.2. *(Kac's Theorem) For a quiver Q,*

(1) There exists an indecomposable representation of dimension α if and only if $\alpha \in \Phi^+$, the positive roots of Q.
(2) If $\alpha \in \Phi^+_{re}$, the positive real roots, then there exists exactly one indecomposable representation of dimension α.
(3) If $\alpha \in \Phi^+_{im}$, there are infinitely many indecomposable representations of dimension α.

Here $\Phi^+, \Phi^+_{re}, \Phi^+_{im}$, denote the positives roots, positive real roots, and positive imaginary roots respectively. In this paper, we will prove a weaker version of Kac's Theorem, which we will refer to as weak Kac's theorem. It will use mainly deformed preprojective algebras and properties of roots.

Theorem 1.3. *(Weak Kac's Theorem) Suppose α is indivisible. Then there exists an indecomposable representation of dimension α if and only if $\alpha \in \Phi^+$.*

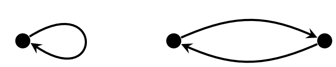
We generally follow Chapters 4 and 8 from *An Introduction to Quiver Representations* by Harm Derksen and Jerzy Weyman [1].

2. Quiver Representations Basics

We begin with the definition of a quiver.

Definition 2.1. A **quiver** $Q = (Q_0, Q_1, h, t)$ is a directed graph. In particular, it has a set of vertices Q_0 and set of arrows Q_1 with functions $h, t : Q_1 \to Q_0$ taking the arrow to its head or tail vertex respectively.

Here are some examples of quivers. We call the quiver on the left A_0 and the quiver on the right A_1.



The next needed definition is a representation of a quiver.

Definition 2.2. A **representation** V of a quiver is an assignment of a finite $\mathbb{C}-$vector space $V(x)$ to each vertex $x \in Q_0$ and a $\mathbb{C}-$linear map $V(a) : V(ta) \to V(ha)$ to each arrow $a \in Q_1$.

A representation V has dimension vector $\underline{\dim}V$ which is the column vector of the dimensions of the $V(x)$ for $x \in Q_0$. A notation we will frequently use is that $\underline{\dim}V \in \mathbb{N}^{Q_0} = \mathbb{N}^{|Q_0|}$. We want to study the space of possible representations of a given quiver Q. We call this space $\text{Rep}(Q)$ or if we are considering only representations of dimension vector α, $\text{Rep}_\alpha(Q)$. In order to make this space a category with representations as its objects, we define morphisms of representations as follows.

Definition 2.3. Let V, W be representations of $Q = (Q_0, Q_1, h, t)$. A **morphism** $\phi : V \to W$ is an attachment of a $\mathbb{C}-$linear map $\phi(x) : V(x) \to W(x)$ of vertices such that for all arrows $a \in Q_1$, the diagram commutes

$$\begin{array}{ccc} V(ta) & \xrightarrow{V(a)} & V(ha) \\ {\scriptstyle \phi(ta)}\downarrow & & \downarrow{\scriptstyle \phi(ha)} \\ W(ta) & \xrightarrow[W(a)]{} & W(ha) \end{array}$$

Note that this is the same as $\phi(ha)V(a) = W(a)\phi(ta)$. For two morphisms $\phi : V \to W$ and $\psi : W \to Z$, the composition $\phi \circ \psi : V \to Z$ is defined on x by $\phi \circ \psi(x) = \phi(\psi(x))$. One of the most important questions to ask in any kind of representation theory is what are our indecomposable representations. These in a sense can be considered building blocks of representations. To define this, we first note the definition of a subrepresentation.

Definition 2.4. Given a representation V of a quiver Q, a representation W is a **subrepresentation** of V if for each vertex x, $W(x)$ is a subspace of $V(x)$ and each arrow a, $W(a) : W(ta) \to W(ha)$ is a restriction of $V(a) : V(ta) \to V(ha)$.

For any two representations V and W, we can form another representation by $V \oplus W$ by the natural construction $(V \oplus W)(x) = V(x) \oplus W(x)$ and $(V \oplus W)(a) = V(a) \oplus W(a)$. Now we can say that a representation Z is **indecomposable** if there does not exist nontrivial subrepresentations V and W such that $V \oplus W = Z$. Let $\mathrm{Ind}(Q)$ equal the set of isomorphism classes of indecomposable representations of Q. To get an idea of what this means, we present a nonexample. Consider the following quiver B_2.

$$\overset{x_1}{\bullet} \xrightarrow{\quad \text{a} \quad} \overset{x_2}{\bullet}$$

Let V be the representation defined by $V(x_1) = \mathbb{C}^2, V(x_2) = \mathbb{C}$, and $V(a) = p_1$, projection on the first variable. We first note that $V(a)$ is indeed a linear transformation from $V(x_1) = \mathbb{C}^2$ to $V(x_2) = \mathbb{C}$. Notice that $V(a)$ has a nontrivial kernel. Let W be the representation defined by

$$W(x_1) = \text{span}\Big\{ \begin{bmatrix} x \\ 0 \end{bmatrix} \Big\}, W(x_2) = \mathbb{C}, W(a) = p_1$$

. Let Y be the representation defined by

$$Y(x_1) = \text{span}\Big\{ \begin{bmatrix} 0 \\ y \end{bmatrix} \Big\}, Y(x_2) = 0, Y(a) = p_1$$

Then $V(x_1) = W(x_1) \oplus Y(x_1)$ and $V(x_2) = W(x_2) \oplus Y(x_2)$. Additionally the restriction of $V(a)$ to either $W(a)$ or $Y(a)$ matches the representations. So we say that $V = W \oplus Y$. Thus V is decomposable.

A type of representation that will be important to us later is the simple representations S_x for a vertex x. This is the representation where $S_x(x) = \mathbb{C}$ and $S_x(y) = 0$ for $y \neq x$. We denote the dimension vector $\underline{\dim} S_x$ as ϵ_x. We will use these to define our roots.

An important way to view these representations is as modules acting on the algebra. Representations of quivers are in fact the same as the representation of an algebra, called the path algebra. This algebra we will denote $\mathbb{C}Q$ as it is an associative $\mathbb{C}-$algebra. The **path** p in a quiver formally is a finite sequence of arrows $a_k ... a_1$ that make sense, meaning that the tails and heads of the arrows match up as $ta_{i+1} = ha_i$. The tail and head of p are defined to be ta_1 and ha_k respectively. Denote the trivial path e_x at $x \in Q_0$ as the path of length 0 at that vertex.

Definition 2.5. The **path algebra** $\mathbb{C}Q$ is a $\mathbb{C}-$algebra with a basis corresponding to paths in Q. Let $\langle p \rangle$ denote the element of $\mathbb{C}Q$ corresponding to p. Multiplication is given by

$$\langle p \rangle \cdot \langle q \rangle = \begin{cases} \langle pq \rangle & \text{if } tp = hq \\ 0 & \text{if } tp \neq hq \end{cases}$$

where pq is the concatenation of paths.

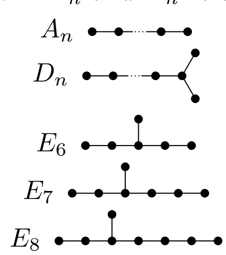
Note that the identity of this algebra is $\sum_{x \in Q_0} e_x$. Our reason for defining this is to claim that the representations of a quiver Q correspond to left $\mathbb{C}Q$ modules, denoted $\mathbb{C}Q-$mod. These are considered to be the representations of the algebra. The following theorem gives us this correspondence.

Theorem 2.6. *The categories Rep(Q) and $\mathbb{C}Q-$mod are equivalent.*

We omit the proof as it can be found in most quiver representation related books. See [1] or [3]. We will not use this characterization much.

3. Gabriel's Theorem

Gabriel's Theorem makes a statement about Dynkin diagrams of type ADE. We have them listed below where n in A_n and D_n refers to the number of vertices.

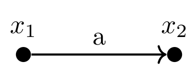


3.1. **Reflection and Coexter functor.**

To prove Gabriel's Theorem, we must first define notions to understand the theorem. Gabriel's Theorem is concerned with determining whether a quiver has finite representation type.

Definition 3.1. A quiver Q has **finite representation type** if it only has finitely many isomorphism classes of indecomposable representations.

It is not clear whether a quiver has finite representation type by inspection even for simple quivers, so we provide an example of both finite and nonfinite type. An example of a quiver with finite type is the bar graph B_2 which we used earlier.



An indecomposable representation would need to be bijective, otherwise we could use the cokernel or kernel to decompose it like we did earlier. Then we see that the only possible nontrivial dimension vectors are $[0, 1]^T, [1, 0]^T, [1, 1]^T$. Clearly there is only one possible representation with the first two dimension vectors. Let V and W be representations with dimension vector $[1, 1]^T$. Let

$$V(x_1) = \text{span}\{y_1\}, V(x_2) = \text{span}\{y_2\}$$

where $V(a)$ is defined by $V(a)(y_1) = y_2$ and

$$W(x_1) = \text{span}\{z_1\}, W(x_2) = \text{span}\{z_2\}$$

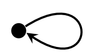
where $W(a)$ is defined by $W(a)(z_1) = z_2$.

Then we have the isomorphism $\phi : V \to W$ defined by $\phi(x_1)(y_1) = z_1$ and $\phi(x_2)(y_2) = z_2$ which satisfies the commutative diagram

$$\begin{array}{ccc} V(x_1) & \xrightarrow{V(a)} & V(x_2) \\ \downarrow{\scriptstyle \phi(x_1)} & & \downarrow{\scriptstyle \phi(x_2)} \\ W(x_1) & \xrightarrow[W(a)]{} & W(x_2) \end{array}$$

Thus all representations of dimension $[1, 1]^T$ are isomorphic. Thus we know that B_2 has finite representation type since it has 3 isomorphism classes of indecomposable representations.

We consider a quiver of nonfinite representation type. Let A_0 be the quiver with one vertex, x, and one arrow a to itself. We find a family of nonisomorphic representations.



Let V_α for nonzero $\alpha \in \mathbb{C}$ be the representation defined by $V_\alpha(x) = \mathbb{C}$ and $V_\alpha(a)(z) = \alpha z$. Clearly these representations are indecomposable since at the one vertex $\mathbb{C}$ can only decompose trivially. Suppose that $V_\alpha \cong V_\beta$. Then we have the following commutative diagram for some linear map $\phi(x)$.

$$\begin{array}{ccc} V_\alpha(x) & \xrightarrow{\times\alpha} & V_\alpha(x) \\ \downarrow{\scriptstyle \phi(x)} & & \downarrow{\scriptstyle \phi(x)} \\ V_\beta(x) & \xrightarrow[\times\beta]{} & V_\beta(x) \end{array}$$

This means that $\beta\phi(x)(y) = \phi(x)(\alpha y)$ for all y. We know that by definition ϕ must be a bijective linear map, so then $\phi(x)(\beta y) = \phi(x)(\alpha y)$ implies that $\beta = \alpha$. Thus for $\alpha \neq \beta$ we know that the two representations are nonisomorphic. Therefore A_0 does not finite representation type.

We want to be able to tell whether or not a quiver has finite representation type through easier means than the arguments given above. The following definition will give us a useful way to determine this.

Definition 3.2. The **Tits form** for $Q = (Q_0, Q_1, h, t)$ is defined on $\mathbb{N}^{Q_0}$ by

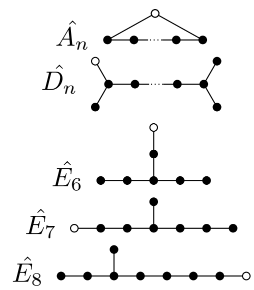
$$B_Q(\alpha) = \langle \alpha, \alpha \rangle_Q = \langle \alpha, \alpha \rangle = \sum_{x \in Q_0} \alpha(x)\alpha(x) - \sum_{a \in Q_1} \alpha(ta)\alpha(ha)$$

where $\alpha = \underline{\dim} V$ for some representation V of Q.

Note that the Tits form is the same as $\dim \mathrm{GL}_\alpha - \dim \mathrm{Rep}_\alpha(Q)$. A useful lemma that uses this form is the following

Lemma 3.3. *If Q has finite representation type, then $B_Q(\alpha) \geq 1$ for every nonzero dimension vector α.*

Refer to [1]. We will use this lemma and the next to show that a quiver of finite representation type must be a Dynkin diagram of type ADE. We do this by asserting that such a quiver must not have any of the following graphs as a subgraph. These are called the extended Dynkin diagrams of type ADE.



Each of these can be easily shown to have a dimension vector such that $B_Q(\alpha) = 0$ regardless of orientation. For example, $\hat{A_n}$ with the dimension vector consisting of all 1's.

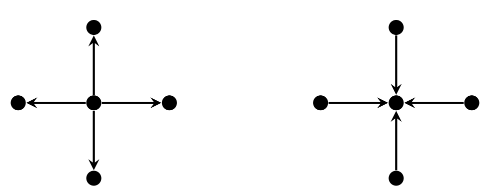
Lemma 3.4. *If Q' is a subquiver of Q and $B_Q(\alpha) \geq 1$ for every nonzero dimension vector $\alpha \in \mathbb{N}^{Q_0}$, then $B_{Q'}(\alpha) \geq 1$ for every nonzero dimension vector $\alpha \in \mathbb{N}^{Q_0}$.*

Proof. Let $Q = (Q_0, Q_1, h, t)$ and $Q' = (Q'_0, Q'_1, h, t)$ where Q' is a subquiver of Q. Assume $\alpha \in \mathbb{N}^{Q'_0}$ is nonzero. Let $\overline{\alpha} \in \mathbb{N}^{Q_0}$ be defined by $\overline{\alpha}(x) = \alpha(x)$ if $x \in Q'_0$ and $\overline{\alpha}(x) = 0$ if $x \notin Q'_0$. Then

$$B_Q(\alpha) = \sum_{x \in Q'_0} \alpha(x)\alpha(x) - \sum_{\alpha \in Q'_1} \alpha(ta)\alpha(ha) \geq \sum_{x \in Q'_0} \alpha(x)\alpha(x) - \sum_{\alpha \in Q_1} \alpha(ta)\alpha(ha) = B_Q(\overline{\alpha})$$

Thus $B_{Q'}(\alpha) \geq 1$. □

We can now show the first part of Gabriel's theorem. We will put off the proof until later. We will prove the third part of Gabriel's theorem using reflection functors. In particular, for a quiver Q, $\sigma_x(Q)$ is the quiver where every arrow attached to x is switched. Our reflection functors are then going to allow us to put a representation on $\sigma_x(Q)$ based on a representation of Q when x is either a sink or a source. A vertex is considered a **sink (resp. source)** if it is not the tail (resp. head) of any arrows. The definition makes sense when the following picture is in mind. On the left is a source and on the right is a sink.



We can also define σ_x on $\mathbb{R}^{Q_0}$ as for $\alpha \in \mathbb{R}^{Q_0}$

$$\sigma_x(\alpha)(y) = \begin{cases} \alpha(y) & \text{if } y \neq x \\ \sum_{a \in Q_1, ha = x} \alpha(ta) + \sum_{a \in Q_1, ta = x} \alpha(ha) - \alpha(x) & \text{if } y = x \end{cases}$$

For the dimension vector of a representation, this will be equivalent to the dimension of the new representation we will define next on $\sigma_x(Q)$.

We now define the reflection functors C_x^+, C_x^-, which act on x where x is a sink or source respectively. They are defined to allow a natural diagram, shown to commute. The idea is to make the new vector space on x based on the kernel of all the maps on the arrows adjacent to it, then use the natural inclusion. This will allow us to go from representations of Q to representations of $\sigma_x(Q)$. Given an arrow a, let $a^\star$ be the opposite arrow with $h(a^\star) = t(a)$ and $t(a^\star) = h(a)$.

Definition 3.5. Let $x \in Q_0$ be a sink. Let $A = \{a_1, ..., a_k\}$ be the set of arrows or edges in Q whose head is x. Let $V \in \text{Rep}_\alpha(Q)$. Define $C_x^+(V) = W$ as follows. For $y \in Q_0 - \{x\}$ and $b \in Q_1 - A$, define $W(y) = V(y)$ and $W(b) = V(b)$.

Let $in_x = \begin{bmatrix} ta_1 \\ \vdots \\ ta_k \end{bmatrix}$ and $a = (a_1 ... a_k) : in_x \to x$ and $a^\star = \begin{bmatrix} a_1^\star \\ \vdots \\ a_k^\star \end{bmatrix}$.
Consider the map $V(a) : V(in_x) \to V(x)$. Let $W(x) = \ker V(a)$ so then $W(a^\star) : W(x) \to V(in_x)$ is inclusion.

Now if we consider a morphism of quiver representations $\phi : V \to V'$ where $W = C_x^+(V)$ and $W' = C_x^+(V')$, we can define $\psi : C_x^+(\phi) : W \to W'$ as follows. If $y \neq x$, then $\phi(y) = \psi(y)$. Then $\psi(x)$ is the unique linear map $W(x) \to W'(x)$ such that the following diagram commutes.

$$\begin{array}{ccc} W(x) & \xrightarrow{W(a^\star)} & V(\text{in}_x) \\ {\scriptstyle \psi(x)} \downarrow & & \downarrow {\scriptstyle \phi(\text{in}_x)} \\ W'(x) & \xrightarrow[W'(a^\star)]{} & V'(\text{in}_x) \end{array}$$

The reflection functor C_x^- for when x is a source is defined similarly, except using the cokernel instead of kernel and we have out_x instead of in_x. We will want to know how these functors act on indecomposable representations in relation to our simple representations S_x. The following result gives us this.

Theorem 3.6. *(Bernstein-Gelfand-Ponomarev)*

(1) Let $x \in Q_0$ be a sink. Let $V \in Rep(Q)$ be an indecomposable object of dimension α.
   (a) Then $V \cong S_x$ if and only if $C_x^+(V) = 0$.
   (b) If $V \ncong S_x$, then $C_x^+(V)$ is indecomposable of dimension $\sigma_x(\alpha)$ and $C_x^- C_x^+(V) \cong V$

(2) Let $x \in Q_0$ be a source. Let $V \in Rep(Q)$ be an indecomposable object of dimension α.
   (a) We have $V \cong S_x$ if and only if $C_x^-(V) = 0$.
   (b) If $V \ncong S_x$, then $C_x^-(V)$ is indecomposable of dimension $\sigma_x(\alpha)$ and $C_x^+ C_x^-(V) \cong V$.

Proof. We only prove 1 as 2 is very similar. We first prove $1a$. Assume that $V \cong S_x$. Then by definition, we know that $C_x^+(V) = 0$. Assume that $C_x^+(V) = 0$. We assume V is nontrivial so then only $\underline{\dim}(V)(x) \neq 0$. Since V is incomposable, then $V \cong S_x$.

Now we prove 1b. Assume x is a sink and V is a representation of V. Use the same notation from our definition of reflection functors and let $Z = C_x^-(W)$ where $W = C_x^+(V)$. We know that $W(out_x) = V(in_x)$ by definition of $\sigma_x(Q)$. Thus

$$0 \to W(x) \xrightarrow{W(a^\star)} V(in_x) \xrightarrow{V(a)} V(x)$$

is exact. By definition of $Z(x)$ as the cokernel, there exists a unique linear map $\pi(x) : V(x) \to Z(x)$ such that we have the diagram

$$\begin{array}{ccc} V(in_x) & \xrightarrow{V(a)} & V(x) \\ {\scriptstyle Z(a)}\downarrow & \swarrow {\scriptstyle \pi(x)} & \\ Z(x) & & \end{array}$$

For $y \neq x$, we can take $\pi(y) = 1_{V(y)}$. It follows from the definition of π that $\pi : V \to Z$ is a morphism of quivers. Suppose V is indecomposable with $V \ncong S_x$. If $V(a)$ is not onto we could use the cokernel to decompose $V(x)$ because we assume x is a sink. Then $\pi(x)$ is an isomorphism which implies that π is an isomorphism. Thus $V \cong C_x^- C_x^+(V)$. By the exactness of the sequence above, we then get that

$$\dim W(x) = \dim V(\mathrm{in}_x) - \dim V(x) = \sum_{i=1}^{k} \alpha(ta_i) - \alpha(x) = \sigma_x(\alpha)(x)$$

so $\underline{\dim} W = \sigma_x(\alpha)$.

Assume $W = W_1 \oplus ... \oplus W_r$. Suppose that $W_i \cong S_x$ for the source vertex x. Then it follows that $C_x^-(W_i) = 0$. Since $C_x^-(W) = C_x^-(W_1) \oplus ... \oplus C_x^-(W_i) \oplus ... \oplus C_x^-(W_r)$, we know that the dimension of $C_x^-(W)$ is strictly less than W. This implies that $W(b)$ where b is the vector of adjacent arrows to x is not injective. However, by the exact sequence shown earlier this is not the case so $C_x^-(W_i) \neq 0$ by 2a. So W is indecomposable. Thus we have showed that if $V \ncong S_x$, then $C_x^-(V)$ is indecomposable of dimension $\sigma_x(\alpha)$ and $C_x^+ C_x^-(V) \cong V$. □

We unfortunately still need to do more work to understand what the Coxeter functors are. We will want to define C^+ to be a sequence of reflection functors $C_{x_i}^+$ over all vertices x_i. There are a couple of steps we will want to take. All the following will assume that Q is acyclic. This assumption makes sense since no finite representation type quiver can contain $\hat{A}_n$, which is exactly all cycles. We will want to be able to say that such a sequence exists and is not reliant on the ordering of reflection. Additionally, we can use this to show that under these reflections all quivers with the same underlying graph are related.

Let Q be a quiver and V a representation of Q. We call $x_k, ..., x_1$ an **admissible sequence** of sinks (resp. sources) if x_i is a sink in $\sigma_{x_{i-1}} ... \sigma_{x_1}(Q)$. This allows $C_{x_k}^+ ... C_{x_1}^+(V)$ (resp. $C_{x_k}^- ... C_{x_1}^-(V)$) to be well defined.

The following is an easy useful lemma following from the Bernstein-Gelfand-Ponomarev theorem.

Lemma 3.7. *Let Q be a quiver and let $x_1, .., x_m$ be an admissible sequence of sinks, and define $Q' = \sigma_{x_m} ... \sigma_{x_1}(Q)$. Define $C^+ = C_{x_m}^+ ... C_{x_1}^+ : Rep(Q) \to Rep(Q')$ and $C^- = C_{x_1}^- ... C_{x_m}^- : Rep(Q') \to Rep(Q)$.*

(1) *For every indecomposable representation V' of Q, $C^+(V) = 0$ or $C^+(V)$ is indecomposable. If $C^+(V) = 0$, then $V \cong C^-_{x_1}...C_{x_{i-1}}(S_{x_i})$ for some i. If $C^+(V)$ is indecomposable, then $C^-(C^+(V)) \cong V$.*
(2) *For every indecomposable representation V' of Q', we have $C^-(V') = 0$ or $C^-(V)$ is indecomposable. If $C^-(V) = 0$, then $V' \cong C^+_{x_m}...C^+_{x_{i+1}}(S_{x_i})$ for some i. If $C^-(V')$ is indecomposable, then $C^+(C^-(V')) \cong V'$.*
(3) *Let $\mathcal{M} \subseteq Ind(Q)$ be the set of isomorphism classes of representations $C^-_{x_1}...C^-_{x_{i-1}}(S_{x_i})$ and $\mathcal{M}'$ similarly defined for $C^+_{x_k}...C^+_{x_{i+1}}(S_{x_i})$ for $1 \leq i \leq m$. Then C^+ and C^- gives a bijection between $Ind(Q)\backslash\mathcal{M}$ and $Ind(Q')\backslash\mathcal{M}'$.*

The first two statements follow from induction on the Bernstein-Gelfand-Ponomarev theorem and the third follows from the other two. We now want to know that only the underlying graph $\Gamma(Q)$ matters for determining whether or not a quiver has finite representation type. First, we show the existence of admissible sequences.

Lemma 3.8. *Suppose Q is a quiver with n vertices and without cycles. There exists an admissible sequence of sinks $x_1, ..., x_n$ with $\{x_1, ..., x_n\} = Q_0$.*

Proof. We first show that without loss of generality, we can assume that $Q_0 = \{1, ..., n\}$ and $ha < ta$ for all $a \in Q_1$ by the following. We show this by induction. If $|Q_0| = 1$, this is obvious. Assume that $|Q_0| = n > 1$. Then remove a leaf x and order the vertices. If x is a sink, then label $x = 1$ and add one to each of the other labels. If x is a source, then keep the ordering and say $x = n$. Then $1, ..., n$ is an admissible sequence of sinks. □

Notice that $\sigma_{x_n}...\sigma_{x_1}(Q) = Q$ since each arrow is reversed twice.

Lemma 3.9. *Let Q be a quiver without cycles. Let Q' be a quiver that differs from Q only by orientation. Then there exists an admissible sequence $x_1, ..., x_k \in Q_0$ of sinks such that $Q' = \sigma_{x_k}...\sigma_{x_1}(Q)$.*

Proof. We prove this by induction on the number of vertices. Let $Q = (Q_0, Q_1, h, t)$ and $Q' = (Q_0, Q_1, h', t')$ be such quivers. If $|Q_0| = 1$, this is obvious. Assume $n = |Q_0| > 1$. Let $z \in Q_0$ be a leaf of $\Gamma(Q)$ with a its edge and y the vertex its adjacent to. Let $\overline{Q}$ be the quiver with z and a deleted. By our induction hypothesis, there exists a sequence of sinks $v_1, ..., v_l$ such that $\sigma_{v_l}...\sigma_{v_1}(\overline{Q}) = \overline{Q'}$. We will now apply this sequence to Q composed with the admissible sequence in Lemma 3.8. Then we know that y is in this admissible sequence. If a is not in the correct orientation and is not a sink, we replace y by y, z in the sequence. Otherwise, if a is not in the correct orientation and is a sink, we append z to the sequence. □

It follows from Lemma 3.9 and 3.7 that whether Q has finite representation type is reliant solely on $\Gamma(Q)$. We can now define the Coexter functor.

Definition 3.10. Let Q be a quiver with n vertices and without cycles. Let $x_1, ..., x_n$ be an admissible sequence of sinks (resp. sinks) such that $Q_0 = \{x_1, ..., x_n\}$. Define

$$C^+ = C^+_{x_n}...C^+_{x_1}, C^- = C^-_{x_1}...C^-_{x_n}$$

and call them the **Coexeter functors**.

Our last step will be to make sure that this is well defined. Note that an equivalent statement for C^-_x can be proven the same way.

Lemma 3.11. *Suppose $x_1, .., x_k$ and $y_1, ..., y_k$ are admissible sequences such that they are rearrangements of each other. Then the functors $C^+_{x_k} ... C^+_{x_1}$ and $C^+_{y_k} ... C^+_{x_1}$ are naturally equivalent.*

Proof. If $v, w \in Q_0$ are sinks not joined by an arrow, then $C^+_v C^+_w$ and $C^+_w C^+_v$ are naturally equivalent. We will use induction on k. For $k = 1$, this is clearly true. Now assume the statement is true for $k - 1 \geq 1$. Choose m minimal such that $x_1 = y_m$. If there is an arrow between x_1 and y_i for some $i \in \{1, ..., m - 1\}$, then it must go from y_i to x_1. However since $x \notin \{y_1, ..., y_{m-1}\}$ by assumption, a will not be reversed and y_i is not a sink. So $C^+_{x_1}$ commutes with $C^+_{y_1} ... C^+_{y_m}$. Then we have that $C^+_{y_k} ... C^+_{y_1}$ is naturally equivalent to $C^+_{y_k} ... C^+_{y_{m+1}} C^+_{m-1} ... C^+_{y_1} C^+_{x_1}$ since by assumption $y_m = x_1$. By our induction hypothesis we have that $C^+_{y_k} ... C^+_{y_{m+1}} C^+_{m-1} ... C^+_{y_1}$ is naturally equivalent to $C^+_{x_k} ... C^+_{x_2}$. Thus the functors $C^+_{x_k} ... C^+_{x_1}$ and $C^+_{y_k} ... C^+_{y_1}$ are naturally equivalent. □

Now we introduce our last definitions before proving Gabriel's Theorem. These are what connect quivers to lie algebras. Given that $\Gamma(Q)$ is the union of Dynkin diagrams of type ADE, we define the **Weyl group** $\mathcal{W}$ as the group generated by $\sigma_x, x \in Q_0$. By the previously mentioned definition of σ_x on $\mathbb{R}^{Q_0}$, $\mathcal{W}$ acts on $\mathbb{Z}_0$ and $\mathbb{R}^{Q_0}$. An $\alpha \in \mathbb{Z}^{Q_0}$ is a **root** if it lies in the $\mathcal{W}-$orbit of ϵ_i, the dimension vector of a simple representation, for some i. This will correspond to the roots of a lie algebra.

Here are some properties of these definitions [1].

Lemma 3.12.
(1) The Tits form B_q is positive definite.
(2) If α is a root, then $B_q(\alpha) = 1$.
(3) There are only finitely many roots.
(4) $\mathcal{W}$ is finite.
(5) For every root α, we have $\alpha \geq 0$ or $\alpha \leq 0$.

In order to prove Gabriel's Theorem, we will want to use the following lemma. The **Cartan form** is then $(\alpha, \beta) = \langle \alpha, \beta \rangle + \langle \beta, \alpha \rangle$. In the following proof, we use that it is nondegenerate [1].

Lemma 3.13.
(1) There are no nonzero vectors in $\mathbb{R}^{Q_0}$ fixed by $c = \sigma_{x_n} ... \sigma_{x_1}$.
(2) For every nonzero $\alpha \in \mathbb{R}^{Q_0}$ there exists $k \in \mathbb{N}$ such that $c^k(\alpha)$ has a negative coordinate.

Proof. Let $\alpha \in \mathbb{R}^{Q_0}$ be a vector such that $\alpha = c(\alpha) = \sigma_{x_1} ... \sigma_{x_n}(\alpha)$. We know that $\sigma_{x_i}(\alpha)(x_j) = \alpha(x_j)$ for $i \neq j$ in general. So then if $c(\alpha) = \alpha$, it follows that $\sigma_{x_i}(\alpha)(x_i) = \alpha(x_i)$ so $(\epsilon_i, \alpha) = \alpha(x_i) - \sigma_{x_i}(\alpha)(x_i) = 0$. Since the Cartan form is nondegenerate, then $\alpha = 0$. We know that $\mathcal{W}$ is finite so for c, there exists an order h, called the Coexeter number. Notice that c fixes $\alpha + c(\alpha) + ... + c^{h-1}(\alpha)$. So if α is nonzero, then α or $c^j(\alpha)$ for some j must have a negative coordinate. □

3.2. **Proof of Gabriel's Theorem.** At last, we prove Gabriel's Theorem in three steps. First the "if" of the first statement through some casework.

Theorem 3.14. *If Q has finite representation type, then $\Gamma(Q)$ is a union of Dynkin graphs of type ADE.*

Proof. Let Q be a connected quiver with finite representation type. Then we know that $\Gamma(Q)$ cannot have a subgraph isomorphic to any Dynkin diagram of type ADE.

We know that because $\Gamma(Q)$ does not contain any $\hat{A_n}$ or equivalently cycles, multi-edges, or loops, it must be a tree. Since $\Gamma(Q)$ does not contain $\hat{D_5}$ as a subgraph, all vertices must have degree ≤ 3. We know that there can be at most one vertex of degree 3 since if there were two, then $\Gamma(Q)$ would have $\hat{D_n}$ as a subgraph. If there are no vertices of degree 3, then $\Gamma(Q) \cong A_n$ for some n. Assume that there is one vertex, v, of degree 3 and call the lengths of the branches of v, p, q, and r. Note that these lengths will count v so they are each ≥ 2.

Suppose $p, q, r \geq 3$. Then $\Gamma(Q)$ contains $\hat{E_6}$. Without loss of generality, assume that $p = 2$. If $q, r \geq 4$, then $\Gamma(Q)$ contains $\hat{E_7}$ so assume $q \leq 3$. If $q = 2$, then $\Gamma(Q) \cong D_n$. If $q = 3$, then to avoid having $\hat{E_8}$ as a subgraph, $r \leq 5$. However then for $r = 3, 4, 5$, $\Gamma(Q)$ is isomorphic to E_6, E_7, E_8 respectively. Thus $\Gamma(Q)$ is a Dynkin diagram of type ADE. Since this is true for each connected component, then for any quiver of finite type, $\Gamma(Q)$ is the union of Dynkin diagrams. □

Now we show a slightly modified version of the second statement,

Theorem 3.15. *If $\Gamma(Q)$ is the union of Dynkin diagrams of type ADE, then the map $V \mapsto \underline{dim}V$ gives a bijection between $Ind(Q)$ and the set of positive roots.*

Proof. Assume $\Gamma(Q)$ is the union of Dynkin diagrams of type ADE. Without loss of generality, assume that $Q_0 = \{1, ..., n\}$ with $ha < ta$ for every $a \in Q_1$. Let $c = \sigma_n ... \sigma_1$ be the Coxeter tranformation. Let V be an indecomposable representation of Q with $\underline{dim}V = \alpha$. Then by Lemma 3.13, there exists a minimal l such that $c^l(\alpha)$ has a negative coordinate. Choose j minimal such that $\sigma_j ... \sigma_1 c^{l-1}(\alpha)$ has a negative coordinate. This is only possible if $C_j^+ ... C_1^+ (C^+)^{l-1}(V)$ is no longer an indecomposable quiver meaning that it must be 0. Since $C_{j-1}^+ ... C_1^+ (C^+)^{l-1}(V) \neq 0$, we know that $V \cong (C^-)^{l-1} C_1^- ... C_{j-1}^+(S_j)$ by Lemma 3.7. Therefore $\alpha = c^{l-1}\sigma_1 ... \sigma_{j-1}(\epsilon_j)$ is a root.

Now suppose α is a root. Let l be the minimal coordinate such that $c^l(\alpha)$ has a negative coordinate again and similarly choose j. Then $\beta = \sigma_{j-1} ... \sigma_1 c^{l-1}(\alpha)$ is a positive root and $\sigma_j(\beta)$ is a negative root. Then $\beta = \epsilon_j$ using the same argument as above. Thus $V = (C^-)^{l-1} C_1^- ... C_{j-1}^-(S_j)$ is an indecomposable representation of dimension α. □

This implies the "only if" of the first statement because we know that there are finitely many roots for such Q by Lemma 3.12. Thus we have proven Gabriel's Theorem.

4. Kac's Theorem

We will now prove weak Kac's Theorem, which as a reminder is the following with no specific restrictions on the quiver itself.

Theorem 4.1. *(Weak Kac's Theorem) Suppose α is indivisible. Then there exists an indecomposable representation of dimension α if and only if $\alpha \in \Phi^+$.*

4.1. **Reflections and Root Systems.** We first will define our roots. These are defined to correspond to the roots of a Kac-Moody Lie Algebra. They are more complicated than the roots of the semisimple Lie Algebras before because we must introduce the concept of real and imaginary roots.

Our previous definition of reflection for $\alpha \in \mathbb{R}^{Q_0}$ can also be written as $\sigma_x(\alpha) = \alpha - (\alpha, \epsilon_x)\epsilon_x$. We can then extend this for $\alpha \in \mathbb{C}^{Q_0}$ as $(\sigma_x^* \lambda)(y) = \lambda(y) - (\epsilon_x, \epsilon_y)\lambda(x)$. We can now define our set of **real roots** where

$$\Phi_{re} = \bigcup_{x \in Q_0} \mathcal{W}\epsilon_x$$

with $\Phi_{re}^+ = \Phi_{re} \cap \mathbb{N}^{Q_0}$ and $\Phi_{re}^- = \Phi_{re} \cap -\mathbb{N}^{Q_0}$.

In order to define our set of complex roots, we define the **support** of α as $\mathrm{sub}(\alpha)$ to be all nonzero dimension vectors β such that $\alpha - \beta$ is a nonzero dimension vector. Let $K \subseteq \mathbb{N}^{Q_0}$ be the set of all nonzero dimension vectors α such that the support of α is connected and $(\alpha, \epsilon_x) \leq 0$ for all Q_0.

Definition 4.2. Define the **imaginary roots** by

$$\Phi_{im}^+ = \mathcal{W}K, \Phi_{im}^- = -\Phi_{im}^+, \Phi_{im} = \Phi_{im}^+ \cup \Phi_{im}^-$$

Note that for $(\alpha, \alpha) = (\epsilon_x, \epsilon_x) = 2$ for all $\alpha \in \Phi_{re}$ and $(\alpha, \alpha) \leq 0$ for all $\alpha \in \Phi_{im}$ so $\Phi_{re} \cap \Phi_{im} = \emptyset$.

We will prove a weak version of Kac's Theorem for the particular case where α is indivisible, meaning that it cannot be written as $\alpha = k\alpha'$ for $k \in \mathbb{N} - \{1\}$ and α' a dimension vector. We show then there exists an indecomposable representation of dimension α if and only if $\alpha \in \Phi^+ = \Phi_{im}^+ \cup \Phi_{re}^+$.

4.2. **Preprojective Algebras.** The first step to proving weak Kac's Theorem will be to show that the existence of an indecomposable representation of indivisible dimension α is independent of orientation. For Gabriel's Theorem, we showed this when Q is acyclic with no restriction on the dimension vector.

For a quiver $Q = (Q_0, Q_1, h, t)$, define the **double quiver** $\overline{Q} = (Q_0, \overline{Q_1}, h, t)$ as follows. For each arrow $a \in Q_1$, define the opposite arrow $a^\star$ such that $h(a^\star) = t(a)$ and $t(a^\star) = h(a)$. Then $Q_1^\star = \{a^\star | a \in Q_1\}$ and $\overline{Q_1} = Q_1 \cup Q_1^\star$. We denote the opposite quiver, the quiver with the vertices of Q and only the opposite arrows $Q^\star$.

Definition 4.3. For $\lambda \in \mathbb{C}^{Q_0}$, define the **deformed preprojective algebra** by

$$\Pi_\lambda = \mathbb{C}\overline{Q}/(r_\lambda)$$

where

$$r_\lambda = \sum_{a \in Q_1} (a^\star a - a a^\star) - \lambda$$

We see this as an element of $\mathbb{C}\overline{Q}$ by identifying λ with $\sum_{x \in Q_0} \lambda(x) e_x$.

A way to view $\mathrm{Rep}_\alpha(\Pi_\lambda)$ is as the subset of $\mathrm{Rep}_\alpha(Q) \oplus \mathrm{Rep}_\alpha(Q^\star)$ consisting of pairs of representations (V, W) with

$$\sum_{a \in Q_1, ta = x} W(a^\star)V(a) - \sum_{a \in Q_1, ha = x} V(a)W(a^\star) = \lambda(x) 1_{\alpha(x)}$$

for all $x \in Q_0$. This allows for the follows the following definition to be useful.

Definition 4.4. The **moment map**

$$\mu_\alpha : \mathrm{Rep}_\alpha(Q) \oplus \mathrm{Rep}_\alpha(Q^\star) \to \oplus_{x \in Q_0} \mathrm{End}(\mathbb{C}^{\alpha(x)})$$

is defined by

$$\mu_\alpha(V, W) = \sum_{a \in Q_1, ta = x} W(a^\star)V(a) - \sum_{a \in Q_1, ha = x} V(a)W(a^\star)$$

This makes $\mu_\alpha^{-1}(\lambda) = \text{Rep}_\alpha(\Pi_\lambda)$ where λ has been identified with $(\lambda(x)1_{\alpha(x)}, x \in Q_0)$. Note that we continue to use this identification.

Lemma 4.5. *If $\lambda(\alpha) \neq 0$, then $Rep_\alpha(\Pi_\lambda) = \emptyset$.*

Proof. We note that if $\mu_\alpha(V,W) = (A(x), x \in Q_0)$, then

$$\sum_{x \in Q_0} \text{Tr}(A(x)) = \sum_{a \in Q_1} \text{Tr}(W(a^\star)V(a) - V(a)W(a^\star)) = 0$$

So then it follows that if $(V,W) \in \mu_\alpha^{-1}(\lambda)$, then

$$0 = \sum_{x \in Q_0} \lambda(x)\alpha(x) = \lambda(\alpha)$$

□

Let V be a representation of Q of dimension α. Then the following sequence is exact.

$$0 \to \text{Hom}_Q(V,V) \to \oplus_{x \in Q_0}\text{End}(\mathbb{C}^{\alpha(x)}) \to \oplus_{a \in Q_1}\text{Hom}(V(ta), V(ha)) \to \text{Ext}^1_Q(V,V) \to 0$$

Note that $\text{End}(\mathbb{C}^{\alpha(x)}) \cong \text{Hom}_{\mathbb{C}}(V(x), V(x))$. By dualizing and identifying

$$\text{Rep}_\alpha(Q^\star) = \oplus_{a \in Q_1}\text{Hom}_Q(\mathbb{C}^{h(a)}, \mathbb{C}^{t(a)}) = \oplus_{a \in Q_1}\text{Hom}_Q(V(ta), V(ha))^\star$$

and $\text{End}(\mathbb{C}^{\alpha(x)})$ with $\text{End}(\mathbb{C}^{\alpha(x)})^\star$, we get the exact sequence

$$0 \to Ext^1_Q(V,V)^\star \to \text{Rep}_\alpha(Q^\star) \xrightarrow{\mu_\alpha^V} \oplus_{x \in Q_0}\text{End}(\mathbb{C}^{\alpha(x)}) \xrightarrow{\gamma} \text{Hom}_Q(V,V)^\star \to 0$$

Here we have that the map $\mu_\alpha^V = \mu_\alpha(V, -) : \text{Rep}_\alpha(Q^\star) \to \oplus_{x \in Q_0}\text{End}(\mathbb{C}^{\alpha(x)})$. The last result we need to prove the independence of orientation is the following. By indecomposable summand, we mean an indecomposable representation W such that $V = W \oplus Y$ for some representation Y.

Theorem 4.6. *We have*

$$(\mu_\alpha^V)^{-1}(\lambda) \neq \emptyset$$

if and only if for every indecomposable summand Y of V we have $\lambda(\underline{dim}Y) = 0$.

Proof. If $(\mu_\alpha^V)^{-1}(\lambda)$ is nonempty, then $\gamma(\lambda) = \lambda(\alpha) = 0$ by exactness. Let $p : V \to V$ be projection onto Y. Then $0 = \gamma(\lambda)(p) = \sum_{x \in Q_0} \lambda(x)\text{Tr}(p(x)) = \lambda(\underline{\text{dim}}Y)$. We now show the converse, starting with when V is indecomposable and $\lambda(\alpha) = 0$. We want to show that $\gamma(\lambda) = 0$ since then by exactness, we would have $(\mu_\alpha^V)^{-1}(\lambda) \neq \emptyset$. The endomorphisms of V are spanned by the identity and nilpotent endomorphisms so it is sufficient to check those two cases. We already know that $\gamma(\lambda)(1_V) = \lambda(\alpha) = 0$. Assume $f \in \text{End}_Q(V)$ is nilpotent. Then $\gamma(\lambda)(f) = \sum_{x \in Q_0} \text{Tr}(\lambda(x)f(x)) = 0$ since the trace of a nilpotent matrix is 0. Thus $\gamma(\lambda) = 0$ and $\mu_\alpha^{-1}(\lambda) \neq \emptyset$.

If $V = V_1 \oplus ... \oplus V_r$ with each V_i indecomposable of dimension α_i. Then there exist $(V_i, W_i) \in \text{Rep}_{\alpha_i}(\Pi_\lambda)$ by the previous case. Then $(V, W_1 \oplus ... \oplus W_r) \in \mu_\alpha^{-1}(\lambda)$. □

We can now prove that orientation does not matter. We note that this theorem is the biggest reason why we will need to use indivisible dimension vectors.

Theorem 4.7. *Suppose that α is an indivisible dimension vector. Whether there exists an indecomposable representation of Q of dimension α is independent of the orientation of Q.*

Proof. Let Q and Q' be quivers that differ only by orientation. Then their double quivers $\overline{Q}$ and $\overline{Q'}$ are the same. We first show that we can choose $\lambda \in \mathbb{C}^{Q_0}$ such that for all $\beta \in \mathbb{Z}^{Q_0}$, $\lambda(\beta) = 0$ if and only if β is a rational multiple of α. Let $n = |Q_0|$. Choose $\delta^{(1)}, ..., \delta^{(n-1)} \in \mathbb{Q}^n$ such that $\{c \in \mathbb{Q}^n | \delta^{(1)}(c) = ... = \delta^{(n-1)}(c) = 0\}$ is the $\mathbb{Q}-$span of α. Let $\lambda = t_1\delta^{(1)} + ... + t_{n-1}\delta^{(n-1)}$ where $t_1, ..., t_n \in \mathbb{C}$ are linearly independent over $\mathbb{Q}$. Clearly $\lambda(\beta) = 0$ when $\beta = ta$ for some $t \in \mathbb{Q}$. Now assume $\lambda(\beta) = 0$. Since the t_i are independent, then $\delta^{(i)}(\beta) = 0$ for each i so β is in the $\mathbb{Q}-$span of α. Thus $\beta = t\alpha$.

Let V be an indecomposable representation of Q of dimension α. Then V can be lifted to a representation Z of $\overline{Q}$. Restricting Z to Q' gives a representation V'. It follows by Theorem 4.6 that $\lambda(\underline{\dim} Y) = 0$. By the remark above, then $\underline{\dim} Y$ is a rational multiple of α. Since we assume that α is indivisible, then $\underline{\dim} Y = 0$ or $\underline{\dim} Y = \alpha$. Thus V' is indecomposable. □

4.3. **Proof of weak Kac's Theorem.** We start by establishing some cases when there must exist an indecomposable representation of a certain dimension.

Lemma 4.8. *Let α be an indivisible dimension vector. Suppose that there exists an indecomposable representation V of Q of dimension α. Then either $\alpha = \epsilon_x$ and $V \cong S_x$ or $\sigma_x(\alpha) \in \mathbb{N}^{Q_0}$ and there exists an indecomposable representation of dimension $\sigma_x(\alpha)$.*

Proof. Let Q' be a quiver differing from Q by only orientation where x is a sink. Then we know that there exists a representation V' or Q' of dimension α. The statement then follows directly from Theorem 3.6 since the two cases depend only on whether $C_x^+(V') = 0$ or not. □

Theorem 4.9. *Suppose $\alpha \in \mathbb{N}^{Q_0}$ is indivisible, and $\beta = w(\alpha)$ for $w \in \mathcal{W}$. Then there exists an indecomposable representation of dimension α if and only if there exists an indecomposable representation of dimension β or $-\beta$.*

Proof. Write w as $w = \sigma_{x_r}...\sigma_{x_1}$. We will use induction on r to prove the statement. When $r = 0$, this is clear. By our induction hypothesis, assume that there exists an indecomposable representation of dimension γ where $\gamma = \pm\sigma_{x_{r-1}}...\sigma_{x_1}(\alpha)$ and $\beta = \pm\sigma_{x_r}(\gamma)$. We apply Lemma 4.8. First, suppose that $\sigma_{x_r}(\gamma) \notin \mathbb{N}^{Q_0}$ and $\gamma = \epsilon_{x_r}$. Then $\sigma_{x_r}(\gamma) = -\epsilon_{x_r}$. So the simple representation S_{x_r} is an indecomposable representation of dimension $\pm\beta$. Now suppose that $\sigma_{x_r} \in \mathbb{N}^{Q_0}$. Then directly from Lemma 4.8, there exists an indecomposable representation of dimension $\sigma_{x_r}(\gamma) = \pm\beta$. □

We now prove the results we need about the roots to prove weak Kac's Theorem using the previous two lemmas.

Lemma 4.10. *For every $\alpha \in \Phi_{re}^+$ there exists an indecomposable representation of dimension α and $\Phi_{re} = \Phi_{re}^+ \cup \Phi_{re}^-$.*

Proof. By definition, we know that $\alpha = w(\epsilon_x)$ for some $w \in \mathcal{W}$. From Theorem 4.9, we know that there exists an indecomposable representation with dimension α or $-\alpha$. Since α is assumed to be positive, then it must have dimension α. □

Lemma 4.11. *If $\alpha \in K$, then there exists infinitely many indecomposable representations of dimension α.*

We will use the above result from [1].

Theorem 4.12. *(weak Kac's Theorem) Suppose that α is indivisible. Then there exists an indecomposable representation of dimension α if and only if $\alpha \in \Phi_+$.*

Proof. We have already shown that if $\alpha \in \Phi_{re}^+$, then there exists an indecomposable representation of dimension α in Lemma 4.10. We now show that if $\alpha \in \Phi_{im}^+$ and is indivisible, then there exists an indecomposable representation of dimension α. Let $\beta \in K$ such that $w(\beta) = \alpha$ for $w = \sigma_{x_r} ... \sigma_{x_1} \in \mathcal{W}$. This exists by definition. Note that if β was not indivisible and was equal to $\beta = s\beta'$, then $w(\beta) = w(s\beta') = sw(\beta') = \alpha$. This would contradict α being indivisible. We will now use induction on r. If $r = 0$, then $\beta = \alpha$ and the statement is true. Now assume that statement is true for $\gamma = \sigma_{x_{r-1}} ... \sigma_{x_1}(\beta)$. We apply Lemma 4.8. If $\alpha = \sigma_{x_r}(\gamma) \notin \mathbb{N}^{Q_0}$, then $\gamma = \epsilon_{x_r}$. Then $\alpha \in \Phi_{re} \cap \Phi_{im} = \emptyset$ which is a contradiction. So $\alpha \in \mathbb{N}^{Q_0}$ and there exists an indecomposable representation of dimension α.
The final step to proving this theorem will be to show that if there exists an indecomposable representation of dimension α, then $\alpha \in \Phi_+$. We show this by induction on $|\alpha| = \sum_{x \in Q_0} \alpha(x)$. If $|\alpha| = 1$, then this must be a simple representation. Suppose $|\alpha| > 1$ and there exists an indecomposable representation of dimension α. If there exists $x \in Q_0$ such that $(\alpha, \epsilon_x) > 0$, then

$$|\sigma_x \alpha| = |\alpha| - (\alpha, \epsilon_x) < |\alpha|$$

By our induction hypothesis, then $\sigma_x \alpha \in \Phi_+$. Then $\sigma_x^2 \alpha = \alpha$ implies $\alpha \in \Phi_+$. Suppose that $(\alpha, \epsilon_x) \leq 0$ for all $x \in Q_0$. We also know that the support of α is connected since otherwise we could decompose it. Therefore $\alpha \in K$ so by 4.11 there exists an indecomposable representation of dimension α. □

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