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Introduction

Students of mathematics learn how to complete the rational numbers $\mathbb{Q}$ using the distance metric in order to obtain the real numbers $\mathbb{R}$ (see e.g. [24, Chap. 2]). In lectures on number theory, this idea is used for the construction of p-adic numbers by a modification of the metric (see e.g. [24, Chap. 6]). In functional analysis, this is often used in order to complete metric spaces; e.g. $L_2(a,b)$ is the completion of $C_2(a,b)$.

In commutative algebra, this concept is used in order to associate with a commutative ring R and an ideal $\mathfrak{a}$ of R a commutative ring $\hat{R}^{\mathfrak{a}}$, as explained in most introductory textbooks on commutative algebra. The first typical example is to pass from a polynomial ring $\Bbbk[X_1,\ldots,X_n]$ in n variables over a field $\Bbbk$ to the formal power series ring $\Bbbk[[X_1,\ldots,X_n]]$. This also extends to modules, and we may define the completion $\hat{M}^{\mathfrak{a}}$ of an R-module M with respect to an ideal. It turns out that the behaviour of complete modules is, in many respects, better than that of arbitrary modules. This is not surprising, and the same remark holds for metric spaces and complete metric spaces.

On the other hand, there is a dual concept of torsion with respect to an ideal $\mathfrak{a}$ of a commutative ring R. Namely, an R-module M is called $\mathfrak{a}$-torsion if it equals its $\mathfrak{a}$-torsion submodule $\Gamma_{\mathfrak{a}}(M) = \{m \in M | a^t m = 0 \text{ for some } t \in \mathbb{N} \text{ and all } a \in \mathfrak{a}\}$. The $\mathfrak{a}$-torsion functor $\Gamma_{\mathfrak{a}}$ (also called the $\mathfrak{a}$-section functor) is left exact. By the work of Grothendieck (see [40, 47]), its right-derived functors play an essential rôle in algebraic geometry and later in commutative algebra.

The right-derived functors of the torsion functor – called local cohomology functors – take a central position in research in the field of commutative algebra and related subjects. The term "local cohomology" became an item in the Mathematics Classification Scheme with more than 1300 documents in the database of Mathematical Abstracts. There are several textbooks and monographs about local cohomology (see e.g. [15, 47]).

The section functor $\Gamma_{\mathfrak{a}}(\cdot)$ is given by $\varinjlim \operatorname{Hom}_R(R/\mathfrak{a}^t, \cdot)$. The completion functor $\Lambda^{\mathfrak{a}}(\cdot)$ in commutative algebra is given by $\varprojlim(R/\mathfrak{a}^t \otimes_R \cdot)$. That is, it is formally dual to the section functor. However, it is neither left nor right exact in general. In any textbook about commutative algebra, there are introductions to completions with the basics for Noetherian rings and finitely generated modules. In his book [82], Strooker has shown that adic completion and completeness for non-finitely generated modules also play a rôle in many homological questions. The study of the left-derived functors of the completion functor – called local homology functors – was initiated by Matlis (see [56, 57]).

Among others results, Grothendieck proved that the local cohomology of a module over a Noetherian ring may be computed with a Čech complex. He took advantage of the fact that finite sequences in a Noetherian ring have a very nice property; namely, they are weakly pro-regular. In their fundamental paper [38], Greenlees and May proved that both the local homology and cohomology of a module with respect to any ideal of a commutative ring generated by a pro-regular sequence $\underline{x} = x_1, \ldots, x_k$ may be computed with Koszul complexes (with, on the local homology side, the additional assumption that each element x_i of the sequence $\underline{x}$ also forms a length one pro-regular sequence). Note that the notion of a pro-regular sequence is slightly stronger than that of a weakly pro-regular sequence and is more suitable for induction on the length of the sequence. Note also that finite sequences in a Noetherian ring are always pro-regular. Later, Lipman et al. (see [1, 2]), the first author (see [76]), and Yekutieli et al. (see [64]) also considered complexes of modules and highly generalized Greenlees and May's results, using this time the notion of weakly pro-regular sequences. There is also a duality between local homology and cohomology, already observed in [78] for modules over a Noetherian ring, highly generalized in [1] and [64].

The first aim of this monograph is to systematically study completion and to generalize and extend various completeness criteria, in particular in the general situation of a not necessarily Noetherian ring and not necessarily finitely generated modules. We focus not only on modules but also on complexes of modules. The completeness criteria in [17, 35, 49, 76] concern separated modules. It is natural to wonder what happens for non-separated modules. It turns out that they provide pseudo-completeness criteria. This also yields the notion of homologically complete complexes, a notion introduced and investigated by Yekutieli in [64] and in [87]. In [64], the dual notion of cohomologically torsion complexes is also considered. Now, the above-mentioned completeness criteria provide us with characterizations of homologically complete complexes and dually of cohomologically torsion complexes, leading us to several new results. As far as the authors know, this is the first monograph to provide a thorough presentation of completion and its left-derived functors – the local homology functors. In many cases, the formal duality between completion and torsion yields real dualities between local homology and local cohomology. Because of this, we also provide a rather systematic study of local cohomology for complexes. We start this with the study of Čech homology and cohomology, which coincides with local homology and cohomology when the ideal is generated by a weakly pro-regular sequence. It turns out that most of the results in local homology and local cohomology are valid under only the assumption that the involved ideal is generated by a weakly pro-regular sequence. They are in fact results about Čech homology and cohomology. We provide vanishing and non-vanishing results. We also explore various dualities.

The monograph is divided into three parts. The first part "Modules" consists of three chapters and is mainly devoted to the study of the adic completion functor and its left-derived functors. However, Chap. 1 deals with preliminaries about complexes, needed for the use of derived functors. In this first chapter, we also recall direct and inverse limits and the relations between them and some basic facts about Matlis duality.

Chapter 2 is the heart of the first part. In the first three sections, there are basic results on adic completion. Most of them are known but not always easily available in the literature, or stated and proved with superfluous hypotheses. We complete the picture with some examples and some new statements. A few of these facts hold in full generality, and most of them require that the adic topology is taken with respect to a finitely generated ideal. Then, we generalize a result of Bartijn (see [6]) about the flatness of adic completions of flat modules over a Noetherian ring. The left-derived functors of the completion are studied in rather a great generality. For this, the notion of pseudo-complete modules will be useful. We also introduce some other classes of modules with properties related to adic topologies. In particular, we study the class $\mathcal{C}_{\mathfrak{a}}$ of modules with the property that the left-derived functors of the $\mathfrak{a}$-adic completion vanish in positive degree, while in degree 0 it coincides with the $\mathfrak{a}$-completion functor. This property holds for finitely generated modules over Noetherian rings but also for more general modules in some non-Noetherian settings. There is a generalization of the notion of flatness, called relative flatness (with respect to an ideal). Dually to relative flatness, we also investigate relatively injective modules and torsion modules, where the class of modules $\mathcal{B}_{\mathfrak{a}}$ (a kind of dual of the class $\mathcal{C}_{\mathfrak{a}}$) also plays a rôle. The last section contains several examples. Among them are examples illustrating that relative flatness does not imply flatness and that relatively injective modules are not always injective.

In Chap. 3, there is a generalization of Jensen's completeness criterion (see [49]): "Let $(R, \mathfrak{m})$ denote a Noetherian local ring. Then, a finitely generated R-module M is complete in its $\mathfrak{m}$-adic topology if and only if $\operatorname{Ext}_R^i(F, M) = 0$ for all $i > 0$ and any flat R-module F". As in [76], we generalize this result and a corresponding result due to Frankild and Sather-Wagstaff (see [35]) to obtain a completeness criterion for any separated module (not necessarily finitely generated) and any ideal $\mathfrak{a}$ by the Ext-vanishing of certain "test" modules. For an arbitrary module, this yields a pseudo-completeness criterion. Following Strooker [82], we define the Ext-depth and the Tor-codepth of an R-module M with respect to an ideal $\mathfrak{a}$ by

$$\text{E-dp}(\mathfrak{a}, M) := \inf\{i \in \mathbb{N} | \operatorname{Ext}^i_R(R/\mathfrak{a}, M) \neq 0\}\text{resp.}$$

$$\text{T-codp}(\mathfrak{a}, M) := \inf\{i \in \mathbb{N} | \operatorname{Tor}_i^R(R/\mathfrak{a}, M) \neq 0\}.$$

These invariants became important tools in commutative algebra. The first invariant $\inf\{i \in \mathbb{N} | \operatorname{Ext}^i_R(R/\mathfrak{a}, M) \neq 0\}$ had already been considered and investigated by Foxby, mainly in the case where $\mathfrak{a}$ is the maximal ideal of a Noetherian ring, while the second one, later called width, is implicit in Foxby's definition of the small support (see [31]). The rôle of the Ext-depth and Tor-codepth in local homology and cohomology has been observed in [33] and [79]. In this chapter, we focus our attention on modules of infinite Tor-codepth. It turns out that such modules occur naturally in the completion process and also play a rôle in many criteria, including the completeness criteria in Chap. 3 and some others in Part II. In [82], it was shown in full generality that the Ext-depth of a module may be computed with local cohomology. Dually, we also show how the Tor-codepth of a module may be computed with local homology, i.e. with the left-derived functors of the completion functor, under only the assumption that the involved ideal is finitely generated (for a Noetherian ring, this was also proved in [33]). These notions will be extended to complexes in Part II. (However, the extension to complexes of the above-mentioned results often requires additional hypotheses, like weak pro-regularity or some boundedness conditions.)

In Part II "Complexes", there is an application and extension of local homology and cohomology to complexes, where we are mainly interested in unbounded complexes. This leads to various extensions and generalizations of known results. We shall start our investigations with the study of Čech homology and cohomology of complexes with respect to any finite sequence in a commutative ring. Then, we will apply it to the study of local homology and cohomology with respect to an ideal generated by a weakly pro-regular sequence. This approach has the advantage of emphasizing the rôle of the Čech complex and allowing a realization of many isomorphisms in the derived category by natural quasi-isomorphisms of complexes. However, we first need some preliminaries.

In Chap. 4, we start with a report about homological preliminaries, in particular for unbounded complexes. Concerning their resolutions, we summarize the work of Avramov and Foxby resp. Spaltenstein (see [5, 81]). Moreover, there is a presentation of the construction of the telescope for direct systems of complexes and the microscope for inverse systems of complexes following Greenlees and May (see [38]). Also, we present a study of minimal injective resolutions of unbounded complexes, including the case of not necessarily Noetherian rings.

Chapter 5 is devoted to Koszul complexes and the notions of Ext-depth and Tor-codepth of a complex with respect to an ideal $\mathfrak{a}$ of a commutative ring R. These notions have been introduced and investigated in [82] for R-modules, where among other results it is proved that these can be computed by using a Koszul complex when the ideal $\mathfrak{a}$ is finitely generated (see [82, 6.1.6, 6.1.7]). In the work of Foxby and Iyengar (see [33]), this is shown for unbounded complexes over Noetherian rings, but most of their results hold in a greater generality. Following their ideas, we extend these results to any finitely generated ideals of a commutative ring. Also, we introduce the important class of complexes with infinite Tor-codepth and then add some remarks on complexes with finite Ext-depth and finite Tor-codepth and some useful inequalities.

In Chap. 6, we start with the investigation of the Čech complexes for a finite system of elements in a commutative ring. We investigate Čech homology and cohomology and some related classes, the classes of complexes which coincide with their Čech homology or with their Čech cohomology. We also prove the Ext-depth Tor-codepth sensitivity of the Čech complex as well as some vanishing results and some inequalities with the focus on unbounded complexes. A helpful technical tool is a bounded free resolution of the Čech complex as introduced in [75].

In Chap. 7, we first recall that $\mathrm{L}\Lambda^{\mathfrak{a}}$ and $\mathrm{R}\Gamma_{\mathfrak{a}}$ are well defined in the derived category and fix some notations. As in [38], we describe both functors in terms of telescope and microscope, respectively. We also provide some first vanishing results and results involving the class of complexes of infinite Tor-codepth, valid in great generality. Then, we show under some boundedness conditions that the Ext-depth and Tor-codepth with respect to a finitely generated ideal may again be computed with local cohomology and local homology, respectively. To get rid of the boundedness conditions, we need some more assumptions, namely that the ring is Noetherian (see [33]), more generally that the ideal is generated by a weakly pro-regular sequence, an important technical tool. In the case of an ideal $\mathfrak{a} \subset R$ generated by a weakly pro-regular sequence, the Čech homology and cohomology yield representatives of $\mathrm{L}\Lambda^{\mathfrak{a}}$ and $\mathrm{R}\Gamma_{\mathfrak{a}}$ respectively (see [2, 65]). This will allow us to extend results previously known for modules over a Noetherian ring to unbounded complexes in the setting of an ideal generated by a weakly pro-regular sequence. Local cohomology is an important tool in homological algebra because of the Grothendieck non-vanishing results related to the depth and dimension of a finitely generated module over a Noetherian local ring. In the last section, we also obtain some non-vanishing results for the local cohomology as well as for the local homology of any complex, and some new inequalities (generalizing those obtained in [79] for modules). Moreover, we explore the behaviour of the Ext-depth and Tor-codepth of a complex under the functors $\mathrm{L}\Lambda^{\mathfrak{a}}$ and $\mathrm{R}\Gamma_{\mathfrak{a}}$.

The Koszul complex $K_{\bullet}(\underline{U}-x; X[[\underline{U}]])$, studied in Chap. 8, plays an interesting rôle. Here, $\underline{x} = x_1, \ldots, x_k$ denotes a sequence of elements in a commutative ring R, $\underline{U} = U_1, \ldots, U_k$ a set of variables and X an R-complex, and we form the polynomial ring $R[\underline{U}] = R[U_1, \ldots, U_k]$ and identify the $R[\underline{U}]$-complex $\mathrm{Hom}_R(R[\underline{U}], X)$ with the formal power series complex $X[[\underline{U}]]$. In the case of a weakly pro-regular sequence $\underline{x}$ generating the ideal $\mathfrak{a}$ it turns out that $K_{\bullet}(\underline{U}-x; X[[\underline{U}]])$ is a representative of $\mathrm{L}\Lambda^{\mathfrak{a}}(X)$ in the derived category. That is, $H_i(\underline{U}-x; X[[\underline{U}]]) \cong \Lambda_i^{\mathfrak{a}}(X)$ for all $i \in \mathbb{Z}$. Let M denote an R-module. For $\underline{x}$ an M-weakly pro-regular sequence, it turns out that $H_0(\underline{U}-x; M[[\underline{U}]]) \cong \hat{M}^{\mathfrak{a}}$. It also follows that the sequence $\underline{U}-\underline{x}$ is completely secant on $M[[\underline{U}]]$, in other words that $H_i(\underline{U}-\underline{x}; M[[\underline{U}]]) = 0$ for all $i > 0$.

In Chap. 9, we continue with some applications and supplements. Among them, we describe some composites of RHom, LΛ and RΓ with respect to different ideals $\mathfrak{a}$ and $\mathfrak{b}$. When $\mathfrak{a}=\mathfrak{b}$ some of these results already appeared in [1] and [64]. Here, we derive them from some more general composite results involving Čech complexes. As in [1] and [64], we also provide adjointness results between LΛ and RΓ, derived from the adjointness between telescope and microscope. A further section covers the endomorphisms of certain complexes related to the Čech complex. Another feature is the Mayer–Vietoris sequences for local or Čech cohomology and homology with some unexpected consequences for weakly pro-regular sequences. We again look at the classes of modules $\mathcal{C}_{\mathfrak{a}}$ and $\mathcal{B}_{\mathfrak{a}}$ introduced in Chap. 2, classes of modules with properties related to completion and torsion. In particular, we show how they might be useful for the computation of the local homology or cohomology of complexes. We also study the class of homologically complete complexes and the class of cohomologically torsion complexes. This allows extension to complexes of the completeness criteria for modules described in Chap. 3 as well as their counterpart in local cohomology. (Some of these criteria are also given in [65] and [87].) Some comparisons between these classes are also provided. Also important is change of ring theorems.

Under certain additional assumptions, the dual of the section functor is isomorphic to the completion of the dual. This is the starting point for various duality statements, the theme of Part III.

In Chap. 10, we start with some Čech duality formulas and apply them to Cohen–Macaulay or Gorenstein local rings to obtain some first versions of the Grothendieck Local Duality Theorem. As a by-product, we prove a few new characterizations of Gorenstein local rings. We also obtain new characterizations of $\mathfrak{m}$-torsion and $\mathfrak{m}$-pseudo-complete modules over a Gorenstein local ring.

To obtain a more general version of Grothendieck's Local Duality Theorem, the notion of a dualizing complex, introduced in Chap. 11, plays an essential rôle. By a result of Kawasaki, the existence of a dualizing complex for a Noetherian local ring is equivalent to it being a factor ring of a Gorenstein ring (see [50]). By Cohen's Structure Theorem, a complete local ring is the factor ring of a regular ring. That is, it possesses a dualizing complex. We recover this result without the use of Cohen's Structure Theorem. That is, we show that the Matlis dual of the Čech complex built on a parameter system of a Noetherian local ring is the dualizing complex of its completion. This result also shows an interesting interaction between the notion of a dualizing complex for a Noetherian ring and the notion of Čech complexes. It will be used when we consider a change of rings of the form $R \to \hat{R}^{\mathfrak{a}}$. This will provide us with an explicit construction of a dualizing complex for $\hat{R}^{\mathfrak{a}}$ involving the Čech complex built on a generating set of $\mathfrak{a}$. The last section contains some new properties of dualizing complexes related to the completion functor, a recurrent theme in this monograph.

In Chap. 12, we prove several variations of the Grothendieck Local Duality Theorem and we derive a family of dualities including Matlis duality and dualizing complexes as special cases. Besides some new aspects of duality, we generalize Hartshorne's affine duality concerning local cohomology stated in [44] and provide its counterpart in local homology. There are some extended dualities with applications to complexes with mini-max homology. In the last section, we prove Greenlees' Warwick duality and several extensions.

We end the monograph with an Appendix. It contains an elementary proof of Grothendieck's non-vanishing theorem for local cohomology and its counterpart in local homology. There we also recall the notion of a pro-regular sequence as defined and used by Greenlees and May, a notion stronger than that of weakly pro-regular sequences used in this monograph. Furthermore, we provide some examples showing the necessity of the assumptions for the validity of several of our results. Following Greenlees and May in [38], we also briefly consider the right-derived functor of the completion functor in the setting of unbounded complexes (Greenlees and May only considered modules). Finally, we recall results of Bass and Chase, respectively, characterizing Noetherian and coherent rings. In the last section of the Appendix, we also provide explicit examples of modules showing the necessity of the assumptions for the validity of the results of Bass and Chase resp. and also some further results.

In our presentation, we do not really work in derived categories. We only use its language, as explained in Chap. 4. More precisely, we provide natural quasi-isomorphisms of complexes realizing the corresponding isomorphisms in the derived category. From our point of view, this illustrates in a more natural way the results and the underlying constructions.

The preliminaries for potential readers are courses in commutative algebra and homological algebra. We tried to include most of the non-standard facts on both subjects. Though we are concerned with adic completion, only some basic general topology is required, but topological arguments have to be used with caution; algebraic arguments are safer. One of the main reasons for this need of precaution is the fact that the adic topology of a submodule of a module M might be strictly finer than the one induced by the adic topology of M. In the text, we follow a ring-theoretic framework; i.e. we do not include results in the language of algebraic geometry, as was done by Lipman et al. (see [1]). Our intention was to present a systematic treatment of the subject together with our own contributions for interested researchers or graduate students with a basic knowledge of commutative and homological algebra. This is also a first presentation of some of the results of [1] in the form of a monograph. Moreover, we include some results of the research papers [65] and [64], outlined in the derived category, in our frame of interest.

In our opinion, the present monograph fills a gap in the literature. There are no monographs about completion and their derived functors in contrast to several monographs about local cohomology. Because the completion functor is not left exact, it makes investigations more complicated than in the case of local cohomology. Note, however, that any result in local homology gives rise to a corresponding one in local cohomology, but we cannot go the other way. This is because the dual of the section functor is isomorphic to the completion of the dual under weak pro-regularity conditions (which are always satisfied when the ring is Noetherian). The book could also be used as a second course, continuing commutative algebra resp. homological algebra.

The book results from the co-operation of both authors over several years. It took some time to fix the details for the subject and to work out the necessary prerequisites. As an outcome, the interested reader might find several aspects of commutative and homological algebra not available elsewhere.

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Contents

Part I
Modules

Chapter 1
Preliminaries and Auxiliary Results

Though this first part is mainly devoted to modules, we first fix the complex terminology and notations to be used when some derived functors come into play, recalling some useful well-known facts. Then we present some basic facts on Matlis duality, and on inverse and direct limits. Most readers will not be tempted to look at this first chapter but may prefer to refer back to it for notations and definitions when needed.

Here and in the other chapters R will always denote a commutative ring with non-zero unit. For an excellent source of Commutative Algebra with a view towards homological applications the reader may refer to [46].

1.1 Complexes

1.1.1 *Terminology about Complexes.* An R-ascending complex, or ascending complex for short, is a family of R-modules $\{X^i\}_{i\in\mathbb{Z}}$ and R-homomorphisms $d_X^i : X^i \to X^{i+1}$ for $i \in \mathbb{Z}$ such that $d_x^{i+1} \circ d_X^i = 0$ for all $i \in \mathbb{Z}$. It will be denoted by $X^\bullet$ or simply by X. A morphism $f : X \to Y$ of two ascending R-complexes X and Y consists of a family $\{f^i\}_{i\in\mathbb{Z}}$ of R-homomorphisms $f^i : X^i \to Y^i$ such that $d_Y^i \circ f^i = f^{i+1} \circ d_X^i$ for all $i \in \mathbb{Z}$. The morphism f is an isomorphism if all the f^i are. When two complexes are isomorphic we write $X \cong Y$.

An ascending complex $X^\bullet$ is sometimes considered as a descending complex by putting $X_i = X^{-i}$ and defining the differential $X_i \to X_{i-1}$ by $d_i^X = d_X^{-i}$.

Complexes can be *shifted*. For an ascending complex X and an integer $k \in \mathbb{Z}$ define $X^{[k]}$ to be the complex given by $X^{[k]i} = X^{k+i}$ and the differential $d_{X^{[k]}} = (-1)^k d_X$. For a descending complex X we also define the complex $X_{[k]}$ by $X_{[k]i} = X_{i+k}$ and the differential $d^{X_{[k]}} = (-1)^k d^X$.

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We say that a covariant (resp. contravariant) functor G from the category of R-complexes into another category of complexes *commutes with shift* if we have natural isomorphisms

$$(G(X))^{[1]} \cong G(X^{[1]}) \quad (\text{resp.}(G(X))^{[1]} \cong G(X_{[1]})$$

for all R-complexes X. If G is a functor from the category of R-modules into another category of modules, its natural extension to complexes commutes with shift.

We say that a complex is *right-bounded* if it looks like $\cdots \to * \to 0$. For a descending complex X this means that $X_i = 0$ for $i \ll 0$, and for an ascending one that $X^i = 0$ for $i \gg 0$. Similarly we say that a complex is *left-bounded* if it looks like $0 \to * \to \cdots$. A *bounded complex* is one which is both right and left-bounded. The length of a bounded complex X is defined by $\operatorname{length}(X) = \sup\{i \mid X^i \neq 0\} - \inf\{i \mid X^i \neq 0\}$.

The cohomology (respectively the homology) modules of an ascending (respectively descending) complex X is denoted by $H^i(X) := \operatorname{Ker} d_X^i / \operatorname{Im} d_X^{i-1}$ (respectively $H_i(X) := \operatorname{Ker} d_i^X / \operatorname{Im} d_{i+1}^X$) as usual.

As usual we consider an R-module M as an R-complex concentrated in degree zero with trivial differentials.

1.1.2 Let M denote an R-module (resp. an R-complex) over a commutative ring R. Let I be an arbitrary set. Then we write $M^I = \prod_{i \in I} M_i$ and $M^{(I)} = \sum_{i \in I} M_i$, where $M_i = M$ for all $i \in I$.

Note the following: any set bijection $I \leftrightarrow I'$ induces isomorphisms of R-modules (resp. of R-complexes) $M^{(I)} \cong M^{(I')}$ and $M^I \cong M^{I'}$.

1.1.3 *Quasi-isomorphisms.* A morphism $f : X \to Y$ of complexes is called a quasi-isomorphism if it induces isomorphisms at the (co)homology level. We use the notation $X \xrightarrow{\sim} Y$ or $Y \xleftarrow{\sim} X$ to denote a quasi-isomorphism. As usual we say that two complexes X and Y are *quasi-isomorphic* and we write $X \simeq Y$ if there is a finite sequence of quasi-isomorphisms $X \xrightarrow{\sim} \cdots \xleftarrow{\sim} Y$.

An exact complex, that is, one for which $H_i(X) = 0$ for all $i \in \mathbb{Z}$, is also called *(co)homologically trivial*. Hence the notation $X \simeq 0$ indicates that the complex X is (co)homologically trivial.

1.1.4 *Homotopy.* Two morphisms of R-complexes $f, g : X_\bullet \to Y_\bullet$ are *homotopic* if there is a family of homomorphisms $s_i : X_i \to Y_{i+1}$ such that $d_{i+1}^Y \circ s_i + s_{i-1} \circ d_i^X = f_i - g_i$. When two morphisms are homotopic we write $f \approx g$. In that case $H_i(f) = H_i(g)$ for all $i \in \mathbb{Z}$.

If $f \approx g$, then $\operatorname{Hom}_R(f, Z) \approx \operatorname{Hom}_R(g, Z)$, $\operatorname{Hom}_R(Z, f) \approx \operatorname{Hom}_R(Z, g)$ and $f \otimes_R Z \approx g \otimes_R Z$ for any R-complex Z.

A morphism of complexes $f : X \to Y$ is called a *homotopism* if there is a morphism $f' : Y \to X$ such that $f \circ f' \approx \operatorname{id}_Y$ and $f' \circ f \approx \operatorname{id}_X$. Homotopisms are quasi-isomorphisms. (See [85] or [14] for more details.)

1.1.5 *Tensor product and Hom.* Given two complexes X and Y the homomorphism complex $\mathrm{Hom}_R(X, Y)$ is defined in such a way that

$$\mathrm{Hom}_R(X, Y)^n = \prod_{j-i=n} \mathrm{Hom}_R(X^i, Y^j)$$

with differential $\prod_{j-i=n} \mathrm{Hom}_R(d_X^{i-1}, Y^j) + (-1)^{n+1} \mathrm{Hom}_R(X^i, d_Y^j)$. That is, cycles of degree n are exactly the morphisms $X \to Y^{[n]}$, two cycles of degree n give the same class in $H^n(\mathrm{Hom}_R(X, Y))$ precisely when they are homotopic. In particular, the module of morphisms $X \to Y$ is the module $\mathrm{Ker}(d^0_{\mathrm{Hom}_R(X,Y)})$.

For two R-complexes X, Y the tensor product $X \otimes_R Y$ is defined by $(X \otimes_R Y)^n = \oplus_{i+j=n} X^i \otimes_R Y^j$ with the differential $d_X^i \otimes \mathrm{id}_Y + (-1)^i \mathrm{id}_X \otimes d_Y^j$. Clearly $X \otimes_R Y \cong Y \otimes_R X$. Note that we still have the adjointness formula: for three complexes X, Y, Z there is an isomorphism

$$\mathrm{Hom}_R(X \otimes_R Y, Z) \cong \mathrm{Hom}_R(X, \mathrm{Hom}_R(Y, Z))$$

natural in X, Y and Z. The units and counits are the natural morphisms

$$\mathrm{Hom}_R(Y, X) \otimes_R Y \to X \text{ and } X \to \mathrm{Hom}_R(Y, X \otimes_R Y).$$

See [43, pp. 90, 93] or [85, pp. 58, 62] or [14] for details.

Note also that Hom and $\otimes$ commute with shifts: for all $k \in \mathbb{Z}$ we have natural isomorphisms

$$\mathrm{Hom}_R(X, Y^{[k]}) \cong \mathrm{Hom}_R(X, Y)^{[k]} \cong \mathrm{Hom}_R(X^{[-k]}, Y) \text{ and}$$
$$X^{[k]} \otimes_R Y \cong (X \otimes_R Y)^{[k)]} \cong X \otimes_R Y^{[k]}.$$

To become acquainted with the homomorphism complex we sketch a proof of the following well-known fact.

Lemma 1.1.6 *(a) Let P be a right-bounded R-complex of projective modules and X any exact R-complex. Then the complex* $\mathrm{Hom}_R(P, X)$ *is exact.*

(b) Let I be a left-bounded R-complex of injective modules and X any exact R-complex. Then the complex $\mathrm{Hom}_R(X, I)$ *is exact.*

Proof For the first assertion it is enough to prove that any morphism $f : P \to X^{[n]}$ is homotopic to zero. Up to a shift we may assume that P has the form $\cdots \to P_1 \to P_0 \to 0$ and that $n = 0$. As $f_0(P_0) \subset \mathrm{Ker}(d_0^X) = \mathrm{Im}(d_1^X)$ we have a homomorphism $s_0 : P_0 \to X_1$ such that $f_0 - d_1^X \circ s_0 = 0$. Then $(f_1 - s_0 \circ d_1^P)(P_1) \subset \mathrm{Ker}(d_1^X) = \mathrm{Im}(d_2^X)$ and the map $f_1 - s_0 \circ d_1^P$ factors through X_2, giving us a homomorphism $s_1 : P_1 \to X_2$. We iterate the process and obtain the wanted homotopy.

The proof of the second assertion is similar. □

1.1.7 *Homothety morphisms.* Let X be any complex over a commutative ring R. Multiplication by $r \in R$ provides a natural morphism

$$h : R \to \mathrm{Hom}_R(X, X)$$

also called the homothety morphism. Note that $h(1) = \mathrm{id}_X$ is a cocycle of degree 0 in $\mathrm{Hom}_R(X, X)$, and that id_X induces the identity on $H^i(\mathrm{Hom}_R(X, X))$ for all $i \in \mathbb{Z}$. If $H^0(\mathrm{Hom}_R(X, X)) \cong R$ and $H^i(\mathrm{Hom}_R(X, X)) = 0$ for all $i \neq 0$ then the homothety morphism is a quasi-isomorphism.

1.1.8 *Split-exact or contractible complexes.* We say that a complex X is split-exact or contractible if the endomorphism id_X is homotopic to zero. Such a complex is obviously exact. If the complex X is split-exact so are the complexes

$$\mathrm{Hom}_R(X, Y), \ \mathrm{Hom}_R(Y, X) \ \text{ and } Y \otimes_R X$$

for any other R-complex Y.

Note that any right-bounded exact complex of projective modules is split-exact and that any left-bounded exact complex of injective modules is split-exact. See [14] for more details around this.

The following will be used rather often.

Observations 1.1.9 Let L be a right-bounded R-complex of finitely generated projective R-modules and X another R-complex. Then

(a) $\mathrm{Hom}_R(L, X) \cong \mathrm{Hom}_R(L, R) \otimes_R X$ if X is left-bounded or L is bounded,
(b) $L \otimes_R X \cong \mathrm{Hom}_R(\mathrm{Hom}_R(L, R), X)$ if X is right-bounded or L is bounded.

1.1.10 *The amplitude of a complex.* We say that a complex X is *homologically left-bounded* if $H(X)$, viewed as a complex with zero differential, is left-bounded. That is, if there is an integer r such that $H^i(X) = 0$ for all $i < r$ (equivalently $H_i(X) = 0$ for all $i > r$).

We say that a complex X is *homologically right-bounded* if $H(X)$ is right-bounded. That is, if there is an integer s such that $H^i(X) = 0$ for all $i > s$ (equivalently $H_i(X) = 0$ for all $i < s$). We say that a complex is *homologically bounded* if it is both homologically left-bounded and homologically right-bounded.

We define the *amplitude* of a complex X by

$$\mathrm{amp}(X) = \sup\{i \mid H^i(X) \neq 0\} - \inf\{i \mid H^i(X) \neq 0\}.$$

Note that $0 \leq \mathrm{amp}(X) < \infty$ if and only if X is homologically bounded and not exact.

1.1.11 *Resolutions of modules.* Let $\mathcal{X}$ be a class of modules. By a $\mathcal{X}$-complex we mean a complex with modules in $\mathcal{X}$. A left $\mathcal{X}$-resolution of a module M is a quasi-isomorphism $X \xrightarrow{\sim} M$ where X is a $\mathcal{X}$-complex and $X_i = 0$ for $i < 0$. A right $\mathcal{X}$-resolution of a module M is a quasi-isomorphism $M \xrightarrow{\sim} X$ where X is an $\mathcal{X}$-complex and $X^i = 0$ for $i < 0$.

1.1.12 *Resolutions of complexes with boundedness conditions.* Any homologically left-bounded complex X has a left-bounded injective resolution. More precisely, there is a quasi-isomorphism $X \xrightarrow{\sim} I$ where I is a complex of injective modules such that $I^i = 0$ for all $i < r := \inf\{i \mid H^i(X) \neq 0\}$.

Any homologically right-bounded complex X has a right-bounded free resolution. More precisely, there is a quasi-isomorphism $L \xrightarrow{\sim} X$ where L is a complex of free modules such that $L_i = 0$ for all $i < s := \inf\{i \mid H_i(X) \neq 0\}$. If, moreover, the $H_i(X)$'s are finitely generated and if the ring is Noetherian, then we may choose for L a complex of finitely generated free modules. See [43] for the details.

(Resolutions of unbounded complexes will be considered in Part II.)

1.2 Inverse Limits

Recalls 1.2.1 An *inverse system of R-modules over* $(\mathbb{N}, \leq)$ is a system of R-modules and *transition* homomorphisms $\mathfrak{M} = \{\rho_{i,j} : M_j \to M_i \mid 0 \leq i \leq j \in \mathbb{N}\}$ such that $\rho_{i,i} = \mathrm{id}_{M_i}$ for all i and $\rho_{i,j} \circ \rho_{j,k} = \rho_{i,k}$ for any triple of natural numbers $i \leq j \leq k$. The inverse limit of such a system, denoted by $\varprojlim M_t$ or $\varprojlim \mathfrak{M}$, is defined by a universal property. More precisely the inverse systems over $(\mathbb{N}, \leq)$ may be viewed as contravariant functors from the small category $(\mathbb{N}, \leq)$ to the category of R-modules; so they form an abelian category. There is a natural embedding of the category of R-modules into the category of inverse systems, sending any module M to the constant inverse system for which $M_i = M$ and $\rho_{i,j} = \mathrm{id}_M$. This natural embedding has a right-adjoint, namely the inverse limit functor $\varprojlim$. It follows that the functor $\varprojlim$ is left-exact and commutes with products.

1.2.2 For an inverse system over $\mathbb{N}$ as in 1.2.1 there is an associated homomorphism

$$\psi_{\mathfrak{M}} : \prod_{t \in \mathbb{N}} M_t \to \prod_{t \in \mathbb{N}} M_t \quad \text{defined by} \quad (\psi_{\mathfrak{M}}(w))_t = w_t - \rho_{t,t+1}(w_{t+1}).$$

It is known that $\mathrm{Ker}(\psi_{\mathfrak{M}}) = \varprojlim M_t$. The module $\mathrm{Coker}(\psi_{\mathfrak{M}})$ is also significant, in fact we have $\mathrm{Coker}(\psi_{\mathfrak{M}}) = \varprojlim{}^1 M_t$ (in general $\varprojlim{}^1$ denotes the first right-derived functor of the inverse limit functor (see [49] or [85] for details)). Here we just take this as a notation but we observe the following: given an inverse system over $(\mathbb{N}, \leq)$ of exact sequences

$$0 \to M'_t \to M_t \to M''_t \to 0,$$

by the snake lemma there exists a six-term long exact sequence

$$\begin{aligned} 0 \to \varprojlim M'_t \to \varprojlim M_t \to \varprojlim M''_t \\ \to \varprojlim{}^1 M'_t \to \varprojlim{}^1 M_t \to \varprojlim{}^1 M''_t \to 0. \end{aligned}$$

This justifies the notation $\operatorname{Coker}(\psi_{\mathfrak{M}}) = \varprojlim{}^1 M_t$ and shows again that the functor $\varprojlim$ is left-exact. (Note that $\varprojlim{}^i M_t = 0$ for $i > 1$ and an inverse system $\mathfrak{M}$ over the natural numbers $\mathbb{N}$ (see again [49] for the details). But we do not need this last fact.)

We can restrict to a subset $\mathbb{N}'$ of $\mathbb{N}$ which is *cofinal* in $(\mathbb{N}, \leq)$, that is, a subset such that for all $t \in \mathbb{N}$ there is an $t' \in \mathbb{N}'$ with $t \leq t'$. Then we have $\varprojlim_{t\in\mathbb{N}} M_t = \varprojlim_{t'\in\mathbb{N}'} M_{t'}$ and $\varprojlim{}^1_{t\in\mathbb{N}} M_t = \varprojlim{}^1_{t'\in\mathbb{N}'} M_{t'}$.

Definitions and observations 1.2.3 Let $\mathfrak{M} = \{\rho_{i,j} : M_j \to M_i\}$ be an inverse system of modules over $\mathbb{N}$. If all the homomorphisms $\rho_{t,t+1}$ are surjective we say that the inverse system $\mathfrak{M}$ is *surjective*. In this case the associated homomorphism $\psi_{\mathfrak{M}}$ is surjective, as is easily seen, and $\varprojlim{}^1 M_t = \operatorname{Coker}(\psi_{\mathfrak{M}}) = 0$.

The advantage of this notion is that any inverse system can be embedded into a surjective one (see [49]).

More generally, one says that the inverse system $\mathfrak{M}$ satisfies the *Mittag-Leffler condition* if for all t the decreasing sequence of submodules

$$N_k = \{\operatorname{Im}(\rho_{t,t+k}) \mid k \in \mathbb{N}\}$$

of M_t stabilizes. The associated homomorphism $\psi_{\mathfrak{M}}$ is also surjective in this case.

(Indeed we have an exact sequence of inverse systems $0 \to \mathfrak{M} \to \mathfrak{I} \to \mathfrak{N} \to 0$ where the system $\mathfrak{I}$ is surjective, and it is known that, if $\mathfrak{M}$ satisfies the Mittag-Leffler condition, the inverse limit of this sequence is again exact (see [39, III-0, 13.2.2]). Hence $\operatorname{Coker}(\psi_{\mathfrak{M}}) = 0$ as seen in the six-term long exact sequence.)

Definitions and observation 1.2.4 Following [75] we say that the inverse system $\mathfrak{M} = \{\rho_{i,j} : M_j \to M_i\}$ is *pro-zero* if for all $t \in \mathbb{N}$ there exists a $t' \geq t$ such that the transition homomorphism $\rho_{t',t}$ is zero. This condition implies that the transition map $\psi_{\mathfrak{M}}$ is an isomorphism, and therefore $\varprojlim M_t = 0 = \varprojlim{}^1 M_t$. (Because there is a cofinal subsystem with null transition homomorphisms.)

Let $0 \to \{M_t'\} \to \{M_t\} \to \{M_t''\} \to 0$ denote a short exact sequence of inverse systems of R-modules. Then the middle inverse system is pro-zero if and only if the two outside ones are pro-zero.

Example 1.2.5 An inverse system which is neither surjective nor pro-zero, which does not satisfy the Mittag-Leffler condition, though its associated homomorphism is an isomorphism.

Take for R a formal power series ring $\Bbbk[[x]]$ in one variable x over a field $\Bbbk$. We form an inverse system $\mathfrak{M}$ with $M_t = R$ and $\rho_{t,t+1} : M_{t+1} \to M_t$ multiplication by x for all $t \in \mathbb{N}$. This system is neither surjective nor pro-zero and does not satisfy the Mittag-Leffler condition. But it is rather easy to see that the associated homomorphism $\psi_{\mathfrak{M}}$ is an isomorphism. Indeed, if $w \in \operatorname{Ker}(\psi_{\mathfrak{M}})$, then for all t we have $w_t \in \cap_n x^n R = \{0\}$, hence $w = 0$. On the other hand, given a $w \in \prod_t M_t$, we put $v_t = w_t + x w_{t+1} + \cdots = \sum_{n=t}^{\infty} x^{n-t} w_n$. These v_t give an element $v \in \prod_t M_t$ such that $\psi_{\mathfrak{M}}(v) = w$.

1.2.6 We shall also consider *inverse systems of complexes* over $\mathbb{N}$, briefly described by a system of morphisms of complexes $\mathfrak{P} = \{\rho_{t,t+1} : P_{t+1} \to P_t \mid t \in \mathbb{N}\}$. To such a system there is an associated morphism of complexes

$$\psi_{\mathfrak{P}} : \prod_{t\in\mathbb{N}} P_t \to \prod_{t\in\mathbb{N}} P_t$$

defined degree-wise as in 1.2.1 and we still have

$$\operatorname{Ker}(\psi_{\mathfrak{P}}) = \varprojlim P_t, \quad \operatorname{Coker}(\psi_{\mathfrak{P}}) = \varprojlim{}^1 P_t.$$

If the inverse system $\{P_t\}$ is degree-wise surjective, more generally satisfies degree-wise the Mittag-Leffler condition, then $\psi_{\mathfrak{P}}$ is surjective and $\varprojlim{}^1 P_t = 0$.

1.2.7 *Some constructions.* Given an inverse system $\mathfrak{P}$ of complexes over $\mathbb{N}$ as in 1.2.6 and another complex Y, we can form the inverse system

$$\operatorname{Hom}_R(Y, \mathfrak{P}) := \{\operatorname{Hom}_R(Y, \rho_{t,t+1}) : \operatorname{Hom}_R(Y, P_{t+1}) \to \operatorname{Hom}_R(Y, P_t) \mid t \in \mathbb{N}\}.$$

Then we have

$$\psi_{\operatorname{Hom}_R(Y,\mathfrak{P})} = \operatorname{Hom}_R(Y, \psi_{\mathfrak{P}}) \text{ and } \varprojlim(\operatorname{Hom}_R(Y,\ P_t)) = \operatorname{Hom}_R(Y, \varprojlim P_t)$$

because $\operatorname{Hom}_R(Y, \cdot)$ is left-exact and commutes with products. In other words the functor $\operatorname{Hom}_R(Y, \cdot)$ commutes with $\varprojlim$.

We can also form the inverse system

$$Y \otimes_R \mathfrak{P} := \{Y \otimes_R \rho_{t,t+1} : Y \otimes_R P_{t+1} \to Y \otimes_R P_t \mid t \in \mathbb{N}\}.$$

As a first application of the above and for later use we mention the following.

Lemma 1.2.8 *Let R be any commutative ring and Y be a descending R-complex. Also let $\mathfrak{M} = \{\rho_{t,t+1} : M_{t+1} \to M_t \mid t \in \mathbb{N}\}$ be an inverse system of R-modules satisfying the Mittag-Leffler condition. Then for all $i \in \mathbb{Z}$ there is a natural short exact sequence*

$$0 \to \varprojlim{}^1 H_{i+1}(Y \otimes M_t) \to H_i(\varprojlim(Y \otimes M_t)) \to \varprojlim H_i(Y \otimes M_t) \to 0.$$

Proof First note that the inverse system of complexes $\{Y \otimes M_t\}_{t\in\mathbb{N}}$ satisfies degree-wise the Mittag-Leffler condition. Hence there is a short exact sequence of complexes

$$0 \to \varprojlim(Y \otimes M_t) \longrightarrow \prod_t (Y \otimes M_t) \xrightarrow{\psi} \prod_t (Y \otimes M_t) \to 0,$$

where ψ is the associated morphism. Since homology commutes with direct products the long exact homology sequence induces the statement. □

Lemma 1.2.9 *Let $\{P_t \mid t \in \mathbb{N}\}$ and $\{P'_t \mid t \in \mathbb{N}\}$ denote two inverse systems of R-complexes and assume there is an inverse system of quasi-isomorphisms $P_t \xrightarrow{\sim} P'_t$. Suppose that both systems satisfy degree-wise the Mittag-Leffler condition. Then the induced morphism $\varprojlim P_t \to \varprojlim P'_t$ is also a quasi-isomorphism.*

Proof There is commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \varprojlim P_t \arrow[r] \arrow[d] & \prod P_t \arrow[r, "\psi_{\mathfrak{P}}"] \arrow[d] & \prod P_t \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \varprojlim P'_t \arrow[r] & \prod P'_t \arrow[r, "\psi_{\mathfrak{P}}"] & \prod P'_t \arrow[r] & 0.
\end{tikzcd}$$

In this diagram the rows are exact (see 1.2.6). Moreover, the middle and the right vertical morphisms are quasi-isomorphisms. Hence the left vertical morphism is also a quasi-isomorphism. □

1.2.10 We may also consider inverse systems over any partially ordered set. Then again the inverse limit functor is left-exact and $\mathrm{Hom}_R(X, \varprojlim_i P_i) \cong \varprojlim_i (\mathrm{Hom}_R(X, P_i)$ (this may be seen by adjointness).

In this direction recall the following well-known result. Let$\{M_{i,j}\}$ be an inverse system over $(\mathbb{N} \times \mathbb{N}, \leq \times \leq)$. Then

$$\varprojlim_j(\varprojlim_i\{M_{i,j}\}) = \varprojlim_t\{M_{t,t}\} = \varprojlim_i(\varprojlim_j\{M_{i,j}\}).$$

1.3 Direct Limits

Recalls 1.3.1 A direct system of R-complexes over $\mathbb{N}$ is a system of morphisms of R-complexes $\mathfrak{D} = \{\sigma_{i,j} : D_i \to D_j \mid 0 \leq i \leq j \in \mathbb{N}\}$ such that $\sigma_{i,i} = \mathrm{id}_{D_i}$ for all i and $\sigma_{j,k} \circ \sigma_{i,j} = \sigma_{i,k}$ for any triple of natural numbers $i \leq j \leq k$. Here the direct systems over $(\mathbb{N}, \leq)$ may be viewed as covariant functors from the small category $(\mathbb{N}, \leq)$ to the category of R-complexes. There is again a natural embedding of the category of R-complexes into the category of direct systems, sending a complex X to the constant direct system for which $X_i = X$ and $\sigma_{ij} = \mathrm{id}_X$. This natural embedding has a left-adjoint, namely the direct limit functor $\varinjlim$.

1.3.2 To a direct system of complexes $\mathfrak{D}$ over $\mathbb{N}$ there is associated a morphism of complexes

$$\phi_{\mathfrak{D}} : \bigoplus_{t\in\mathbb{N}} D_t \to \bigoplus_{t\in\mathbb{N}} D_t$$

defined degree-wise by $x \mapsto x - \sigma_{t,t+1}(x)$ for $x \in D_t$. Then it is known that we have the short exact sequence

$$0 \longrightarrow \bigoplus D_t \xrightarrow{\phi_{\mathfrak{D}}} \bigoplus D_t \longrightarrow \varinjlim D_t \longrightarrow 0.$$

It follows that the functor $\varinjlim$ is exact, hence commutes with taking homology.

Moreover, if $D_t \xrightarrow{\sim} D'_t$ is a direct system of quasi-isomorphisms over $\mathbb{N}$, it induces a quasi-isomorphism $\varinjlim D_t \xrightarrow{\sim} \varinjlim D'_t$.

1.3.3 *Some constructions.* Let $\mathfrak{D}$ be a direct system of R-complexes as in 1.3.1 and Y an R-complex.
(a) We can form the direct system

$$\mathfrak{D} \otimes_R Y := \{\sigma_{t,t+1} \otimes_R \mathrm{id}_Y : D_t \otimes_R Y \to D_{t+1} \otimes_R Y \mid t \in \mathbb{N}\}.$$

As tensor products commute with direct sum they also commute with $\varinjlim$. More precisely we have

$$\phi_{\mathfrak{D}\otimes_R Y} = \phi_{\mathfrak{D}} \otimes_R Y \text{ and } \varinjlim(D_t \otimes_R Y) = (\varinjlim D_t) \otimes_R Y.$$

(b) We can also form the inverse system

$$\mathrm{Hom}_R(\mathfrak{D}, Y) := \{\mathrm{Hom}_R(\sigma_{t,t+1}, Y) : \mathrm{Hom}_R(D_{t+1}, Y) \to \mathrm{Hom}_R(D_t, Y) \mid t \in \mathbb{N}\}.$$

For an $s = (s_t) \in \mathrm{Hom}_R(\oplus D_t, Y) = \prod \mathrm{Hom}_R(D_t, Y)$ note that $(s \circ \phi_{\mathfrak{D}})_t = s_t - s_{t+1} \circ \sigma_{t,t+1}$. Hence we have

$$\psi_{\mathrm{Hom}_R(\mathfrak{D},Y)} = \mathrm{Hom}_R(\phi_{\mathfrak{D}}, Y) \text{ and } \varprojlim(\mathrm{Hom}_R(D_t, Y)) \cong \mathrm{Hom}_R(\varinjlim D_t, Y).$$

This reflects the formal duality between inverse and direct limits.

In the case when $Y = I$ is a complex of injective R-modules, so that the functor $\mathrm{Hom}_R(\cdot, I)$ is exact, we also have $\varprojlim{}^1(\mathrm{Hom}_R(D_t, I)) = 0$.

In general the functor $\mathrm{Hom}_R(Z, \cdot)$ does not commute with direct limits, but in particular we have the following.

Lemma 1.3.4 *Let R be any commutative ring and let L be a right-bounded R-complex of finitely generated free modules. Let $\mathfrak{D} = \{D_t \to D_{t+1} \mid t \in \mathbb{N}\}$ denote a direct system of R-complexes with*

$$D_t : 0 \to D_t^0 \to D_t^1 \to \cdots .$$

Then there is a natural isomorphism

$$\varinjlim(\mathrm{Hom}_R(L, D_t)) \cong \mathrm{Hom}_R(L, \varinjlim D_t).$$

Proof We have $\mathrm{Hom}_R(L, D_t) \cong \mathrm{Hom}_R(L, R) \otimes_R D_t$ (see 1.1.9). It follows that

$$\varinjlim(\operatorname{Hom}_R(L, D_t) \cong \varinjlim(\operatorname{Hom}_R(L, R) \otimes_R D_t) \cong \operatorname{Hom}_R(L, R) \otimes_R \varinjlim D_t.$$

As the complex $\varinjlim D_t$ is obviously left-bounded we also have

$$\operatorname{Hom}_R(L, R) \otimes_R \varinjlim D_t \cong \operatorname{Hom}_R(L, \varinjlim D_t)$$

(see 1.1.9 again). □

1.3.5 Direct limits of R-complexes can also be taken over an arbitrary partially ordered set $(I, \prec)$. In this case the direct limit functor commutes with tensor products. It is left-exact and we still have that $\operatorname{Hom}_R(\varinjlim D_i, X) \cong \varprojlim \operatorname{Hom}_R(D_i, X)$. All this follows by adjointness.

If, moreover, I is *right-filtered*, that is, if for all $i, j \in I$ there exists a $k \in I$ such that $i, j \prec k$, then the direct limit functor is exact, hence commutes with taking homology.

A first example of a direct limit is the ring R_x ($x \in R$), that is, the ring of fractions of the form r/x^n, where $r \in R$ and $n \in \mathbb{N}$. Before showing this we recall some basic facts around localization.

Definition 1.3.6 Let S denote a multiplicatively closed subset of a commutative ring R, that is, a subset $S \subset R$ such that $1 \in S$ and $ss' \in S$ for all $s, s' \in S$. The *localization* R_S is a commutative ring and a ring morphism $\theta : R \to R_S$ such that

(a) $\theta(s)$ is invertible for all $s \in S$.
(b) R_S is universal with the property:

$$\begin{tikzcd} R \arrow[rr, "\theta"] \arrow[dr, "f"'] & & R_S \arrow[dl, dashed, "f'"] \\ & R' & \end{tikzcd}$$

if R' is a commutative ring and f is a ring morphism such that $f(s)$ is invertible for all $s \in S$, then there is a unique ring morphism $f' : R_S \to R'$ such that $f' \circ \theta = f$.

As a solution to a universal mapping problem R_S is unique if it exists. It exists in view of the following.

Remark 1.3.7 Let $S \subset R$ denote a multiplicatively closed set in a commutative ring R. For the details of the following we refer to [3], [13], or [52].

(a) We introduce the equivalence relation $\sim$ on $R \times S$

$$(r, s) \sim (r', s') \iff \text{there is an } s'' \in S \text{ such that } s''(rs' - r's) = 0.$$

Denote the equivalence class of (r, s) by r/s and write $R_S = (R \times S)/\sim$. With the induced operations R_S becomes a commutative ring. There is a natural ring morphism $\iota : R \to R_S, r \mapsto r/1$. Recall that $1 \in S$. Moreover, $\iota(s)$ is invertible for all $s \in S$ and clearly $\iota : R \to R_S$ has the universal property of Definition 1.3.6. Whence R_S is the localization of R with respect to S. Then $\mathrm{Ker}(\iota) = \{r \in R \mid sr = 0 \text{ for some } s \in S\}$.

(b) Here is another description of R_S. Let $R[T_s|s \in S]$ denote the polynomial ring in the variables $T_s, s \in S$. Let $I \subset R[T_s|s \in S] =: R[T_S]$ denote the ideal generated by the polynomials $1 - sT_s, s \in S$, and define $\theta : R \to R[T_S]/I$ as the composite $R \to R[T_S] \to R[T_S]/I$. Clearly $R[T_S]/I$ is a commutative ring, θ is a ring morphism and $\theta(s)$ is invertible for all $s \in S$. It is easy to check that $\theta : R \to R[T_S]/I$ has the universal property of Definition 1.3.6. Whence $R_S \cong R[T_S]/I$.

(c) Let $x \in R$ denote an element and let $S = \{1, x, x^2, \dots\}$. Then we write R_x for R_S. Let us consider the direct system $\{\sigma_{n,n+1} : R_n \to R_{n+1} \mid n \in \mathbb{N}\}$, where $R_n = R$ for all $n \in \mathbb{N}$ and $\sigma_{n,n+1}$ is the multiplication by x. The homomorphisms $g_n : R_n = R \to R_x, 1 \mapsto 1/x^n$ induce a homomorphism $g : \varinjlim_n R_n \to R_x$. It is easy to check that this homomorphism g is an isomorphism of R-modules. Whence $\varinjlim_n R_n, \cong R_x$. Moreover, there is a short exact sequence

$$0 \to \oplus_n R_n \xrightarrow{\phi} \oplus_n R_n \to R_x \to 0$$

(see also 1.3.2 for the construction of ϕ). Furthermore, there is a ring isomorphism $R_x \cong R[T](1 - xT)R[T]$, where $R[T]$ denotes the polynomial ring in the variable T over R.

1.4 Ext-Tor Duality and General Matlis Duality

1.4.1 Let M, N, I be modules over a commutative ring R. Assume that I is injective. We recall the Ext-Tor duality

$$\mathrm{Hom}_R(\mathrm{Tor}_i^R(M, N), I)) \cong \mathrm{Ext}_R^i(M, \mathrm{Hom}_R(N, I)).$$

This follows by adjointness and the exactness of $\mathrm{Hom}_R(\cdot, I)$: if $L \xrightarrow{\sim} M$ is a free resolution of M, then $\mathrm{Hom}_R(L \otimes_R N, I) \cong \mathrm{Hom}_R(L, \mathrm{Hom}_R(N, I))$.

Hence if N is flat then $\mathrm{Hom}_R(N, I)$ is injective.

Moreover, if M has a free resolution $L \xrightarrow{\sim} M$ for which there is a natural number k such that L_i is finitely generated when $i \leq k + 1$, then

$$\mathrm{Tor}_i^R(M, \mathrm{Hom}_R(N, I)) \cong \mathrm{Hom}_R(\mathrm{Ext}_R^i(M, N), I)) \text{ for all } i \leq k.$$

This follows because

$$\begin{aligned} d_i^L \otimes_R \mathrm{Hom}_R(N, I) &\cong \mathrm{Hom}_R(\mathrm{Hom}_R(d_i^L, R), \mathrm{Hom}_R(N, I)) \\ &\cong \mathrm{Hom}_R(\mathrm{Hom}_R(d_i^L, N), I) \end{aligned}$$

for $i \leq k+1$ as d_i^L is a homomorphism of finitely generated free modules for $i \leq k+1$ and because I is injective. (See [18] for more details around this.)

We now wonder when the module $\mathrm{Hom}_R(J, I)$ is flat for two injective R-modules J, I. It is clear that any product of injective modules is injective and that any direct sum of flat modules is flat. We may also wonder when any product of flat modules is flat and when any direct sum of injectives is injective (see also the Appendix A.5.4 and A.5.7, for more precise results concerning these questions).

1.4.2 Recall that a commutative ring R is *coherent* when for every finitely generated ideal $\mathfrak{c} \subset R$ there is an exact sequence

$$L_2 \to L_1 \to L_0 \to R/\mathfrak{c} \to 0$$

where L_0, L_1, L_2 are finitely generated free R-modules.

1.4.3 Recall first a Bourbaki flatness criterion [13, Chapitre 1, §2, $n°3$ remarque 1]: an R-module N is flat if and only $\mathrm{Tor}_1^R(R/\mathfrak{c}, N) = 0$ for any finitely generated ideal $\mathfrak{c}$ of R.

It follows that the flat dimension of an R-module M is given by

$$\mathrm{fd}_R(M) = \sup\{i \mid \mathrm{Tor}_i^R(R/\mathfrak{c}, M) \neq 0 \text{ for some finitely generated ideal } \mathfrak{c}\}.$$

Recall 1.4.4 (see [13, Chapter 1, §2, exercise 9]) Let N be a *finitely presented* R-module, that is, an R-module admitting an exact sequence

$$L_1 \to L_0 \to N \to 0$$

where L_0 and L_1 are finitely generated free R-modules. Then the functor $(N \otimes_R \cdot)$ commutes with products.

When the ring is coherent it follows that any product of flat modules is flat (in view of the criterion 1.4.3).

Lemma 1.4.5 *Suppose that a commutative ring R is coherent. Let J and I be two complexes of injective R-modules. Then the complex* $\mathrm{Hom}_R(J, I)$ *is a complex of flat R-modules.*

Proof In view of 1.4.4 we only need to consider the case where I and J are modules. In this case, for all finitely generated ideals $\mathfrak{c}$ of R we have in view of 1.4.1

$$\mathrm{Tor}_1^R(R/\mathfrak{c}, \mathrm{Hom}_R(J, I)) \cong \mathrm{Hom}_R(\mathrm{Ext}_R^1(R/\mathfrak{c}, J), I)) = 0.$$

This implies the flatness of $\mathrm{Hom}_R(J, I)$ in view of 1.4.3. □

As a slight extension of some of the remarks in 1.4.1 the following statement may also be useful.

Lemma 1.4.6 *Let $\mathfrak{a}$ denote a finitely generated ideal of a commutative ring R. Let X denote an arbitrary R complex and I a complex of injective R-modules. Then the natural morphism*

$$R/\mathfrak{a} \otimes_R \operatorname{Hom}_R(X, I) \to \operatorname{Hom}_R(\operatorname{Hom}_R(R/\mathfrak{a}, X), I)$$

is an isomorphism.

Proof At first let X be an R-module and let I be an injective R-module. Then by virtue of 1.4.1 it follows that the natural homomorphism

$$R/\mathfrak{a} \otimes_R \operatorname{Hom}_R(X, I) \to \operatorname{Hom}_R(\operatorname{Hom}_R(R/\mathfrak{a}, X), I)$$

is an isomorphism. Now we extend it to complexes. First note that

$$(R/\mathfrak{a} \otimes_R \operatorname{Hom}_R(X, I))^n = R/\mathfrak{a} \otimes_R (\operatorname{Hom}_R(X, I))^n = R/\mathfrak{a} \otimes_R \prod_{j-i=n} \operatorname{Hom}_R(X^i, I^j).$$

Because $\mathfrak{a}$ is finitely generated the tensor product by $R/\mathfrak{a}$ commutes with direct products (see 1.4.4). Therefore the previous module is isomorphic to

$$\prod_{j-i=n} \operatorname{Hom}_R((\operatorname{Hom}_R(R/\mathfrak{a}, X)^i, Y^j)) = (\operatorname{Hom}_R(\operatorname{Hom}_R(R/\mathfrak{a}, X), Y))^n.$$

By following the corresponding differentials this proves the statement. □

1.4.7 Let us also recall Baer's criterion (see e.g. [14, §1, n°7 Proposition 10]). An R-module M is injective if and only $\operatorname{Ext}_R^1(R/\mathfrak{c}, M) = 0$ for all ideals $\mathfrak{c}$ of R.

It follows that the injective dimension of an R-module M is given by

$$\operatorname{id}_R(M) = \sup\{i \mid \operatorname{Ext}_R^i(R/\mathfrak{c}, M) \neq 0 \text{ for some ideal } \mathfrak{c}\}.$$

When the ring R is Noetherian it also follows that any direct limit or direct sum of injectives modules is injective. This is because the $\operatorname{Ext}_R^i(R/\mathfrak{c}, \cdot)$ commute with direct limits and direct sums when R is Noetherian (to see this take a free resolution $L_\bullet$ of $R/\mathfrak{c}$, where the L_i are free finitely generated, and use 1.1.9).

1.4.8 *General Matlis duality.* Let R be any commutative ring. We write $E_R(M)$ for the injective hull of an R-module M, and simply write E for the direct product of the $E_R(R/\mathfrak{m})$, $\mathfrak{m}$ running through the set of maximal ideals of R. This module E is an injective cogenerator of the category of R-modules. That is, for any element $m \neq 0$ of an R-module M there is an $s \in \operatorname{Hom}_R(M, E)$ such that $s(m) \neq 0$. This gives rise to a *general Matlis duality functor* denoted by $(\cdot)^\vee := \operatorname{Hom}_R(\cdot, E)$. This contravariant functor is faithfully exact and we have

$$\operatorname{Tor}_i^R(M, N)^\vee \cong \operatorname{Ext}_R^i(M, N^\vee)$$

for all R-modules M and N and all $i \geq 0$. Assume in addition that M has a free resolution $L \xrightarrow{\sim} M$ by finitely generated free R-modules. Then there are isomorphisms

$$\mathrm{Tor}_i^R(M, N^\vee) \cong \mathrm{Ext}_R^i(M, N)^\vee$$

for all $i \geq 0$. This is a consequence of 1.4.1.

Note also the natural homomorphism

$$M \to \mathrm{Hom}_R(\mathrm{Hom}_R(M, E), E) = (M^\vee)^\vee$$

obtained by evaluation. It is always injective since E is an injective cogenerator.

1.4.9 Let R be a commutative ring and M an R-module. To each submodule N of the R-module M we associate a submodule of $M^\vee$ defined by $N^o := \{s \in M^\vee \mid s(N) = 0\}$. Similarly to each submodule S of the R-module $M^\vee$ we associate a submodule of M defined by $S^a := \{w \in M \mid S(w) = 0\}$. This gives a correspondence between the submodules of M and those of $M^\vee$ which reverses inclusions and such that $S \subset S^{ao}$, $N \subset N^{oa}$. We note that the last inclusion is in fact an equality: $N^{oa} = N$ for all submodules N of M because E is an injective cogenerator. Then we obtain the following implications.

(a) If $M^\vee$ is Noetherian, then M is Artinian.
(b) If $M^\vee$ is Artinian, then M is Noetherian.

When the ring is Noetherian flat modules and injective modules are interchanged under the general Matlis duality. More generally we have the following well-known result.

Lemma 1.4.10 *Let R be a Noetherian ring and N an R-module. Then*

$$\mathrm{fd}_R(N) = \mathrm{id}_R(N^\vee) \text{ and } \mathrm{id}_R(N) = \mathrm{fd}_R(N^\vee).$$

Proof This is a consequence of 1.4.3, 1.4.7 and the Ext-Tor duality as stated in 1.4.8. □

1.5 Cones and Fibers

1.5.1 Let $h : X \to Y$ denote a morphism of ascending complexes over a commutative ring R. The cone $C(h)$ of this morphism is the graded module defined by $C(h)^i = X^{i+1} \oplus Y^i$ with differential given by the matrices

$$d^i_{C(h)} = \begin{pmatrix} -d_X^{i+1} & 0 \\ -h^{i+1} & d_Y^i \end{pmatrix}.$$

(With the sign convention we follow [43] and [36].)

The fibre $F(h)$ of the morphism $h : X \to Y$ is defined by $F(h) = C(h)^{[-1]}$. Hence $F(h)^i = X^i \oplus Y^{i-1}$ with differential given by the matrix

$$d^i_{F(h)} = \begin{pmatrix} d^i_X & 0 \\ h^i & -d^{i-1}_Y \end{pmatrix}.$$

It is straightforward to check that $C(h)$ and $F(h)$ are indeed ascending complexes. Moreover, we have the following short exact sequences of R-complexes

$$0 \to Y \to C(h) \to X^{[1]} \to 0 \quad \text{and} \quad 0 \to Y^{[-1]} \to F(h) \to X \to 0.$$

Note that $h : X \to Y$ is a quasi-isomorphism if and only if $C(h)$ (respectively $F(h)$) is cohomologically trivial. This follows since the connecting maps in the long exact cohomology sequence associated to the above short exact sequences coincide with $H^i(h)$, $i \in \mathbb{Z}$. In particular, $C(h)$ (respectively $F(h)$) is exact as soon as both X and Y are exact.

The cone and the fiber of a morphism $X \xrightarrow{h} Y$ of descending complexes is defined by lowering the indices: in the descending situation we have $C(h)_i = X_{i-1} \oplus Y_i$ and $F(h)_i = X_i \oplus Y_{i+1}$ for all $i \in \mathbb{Z}$.

1.5.2 *Observation.* Let G be an additive functor of the category of R-modules into another category of modules. This functor extends naturally to complexes and this extension commutes with shift. Now let $h : X \to Y$ be a morphism of complexes. If G is covariant then $C(G(h)) \cong G(C(h))$ and $F(G(h)) \cong G(F(h))$. If G is contravariant then $F(G(h)) \cong G(C(h))$ and $C(G(h)) \cong G(F(h))$.

1.5.3 *A standard homological argument.* Let G be a covariant additive functor from the category of R-modules into another category of modules, and let G_i be its left-derived functors : $G_i(M) := H_i(G(L))$, where $L_\bullet \xrightarrow{\sim} M$ is a free resolution of the R-module M. The functor G_0 is right-exact and $G_0 \cong G$ if and only if G itself is right-exact. Now let $\mathcal{X}_G$ be the class of R-modules N such that $G_i(N) = 0$ for all $i > 0$. A standard homological argument asserts that the $G_i(M)$ can be computed by applying G_0 to a left $\mathcal{X}_G$-resolution $X_\bullet \to M$ of the module M: $G_i(M) = H_i(G_0(X))$ for all $i \geq 0$.

Indeed, let $L_\bullet \xrightarrow{\sim} M$ be a free resolution of M. Then there is a quasi-isomorphism $f : L_\bullet \xrightarrow{\sim} X_\bullet$ and the complex $C(f)$ is a right-bounded exact complex with modules $C(f)_i \in \mathcal{X}_G$. It is rather easy to see that the image by G_0 of such a complex remains exact. Hence we have the quasi-isomorphism $G_0(f) : G(L) = G_0(L) \to G_0(X)$. There are analogous remarks on the right-derived side, and also for contravariant functors.

Cones and fibers will play an essential rôle in Part II. In the spirit of 1.5.2 we also note the following, the proof of which is an easy exercise (because Hom and Tensor commute with shift).

Lemma 1.5.4 *Let X, Y, Z be three R-complexes, ascending or descending, it does not matter, and let $f : X \to Y$ be a morphism of R-complexes. Then we have the following isomorphisms of complexes:*

(a) $C(f) \otimes_R Z \cong C(f \otimes_R Z)$.
(b) $\mathrm{Hom}_R(Z, C(f)) \cong C(\mathrm{Hom}_R(Z, f))$.
(c) $\mathrm{Hom}_R(Z, F(f)) \cong F(\mathrm{Hom}_R(Z, f))$.
(d) $\mathrm{Hom}_R(C(f), Z) \cong F(\mathrm{Hom}_R(f, Z))$.
(e) $\mathrm{Hom}_R(F(f), Z) \cong C(\mathrm{Hom}_R(f, Z))$.

1.5.5 *Functorial properties.* The following also is well known and rather easy to prove.

(a) Any commutative square

$$\begin{tikzcd} X' \arrow[r, "g_s"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g_b"'] & Y \end{tikzcd}$$

induces morphisms $C(g_s, g_b) : C(f') \to C(f)$ and $F(g_s, g_b) : F(f') \to F(f)$. If g_s and g_b are quasi-isomorphisms, so are $C(g_s, g_b)$ and $F(g_s, g_b)$.
(b) Any commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & X' \arrow[r] \arrow[d, "f'"] & X \arrow[r] \arrow[d, "f"] & X'' \arrow[r] \arrow[d, "f''"] & 0 \\ 0 \arrow[r] & Y' \arrow[r] & Y \arrow[r] & Y'' \arrow[r] & 0 \end{tikzcd}$$

induces short exact sequences

$$0 \to C(f') \to C(f) \to C(f'') \to 0 \quad \text{and} \quad 0 \to F(f') \to F(f) \to F(f'') \to 0.$$

We continue with a few more constructions related to cones.

Proposition 1.5.6 (See [14, §2, n°7 Proposition 9] or [32]) *Let R be a commutative ring and*

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

be a zero-sequence of R-complexes and morphisms. Then

(a) *g induces a morphism $C(f) \to Z$ given degree-wise (in ascending notation) by the composite*

$$C(f)^i = X^{i+1} \oplus Y^i \to Y^i \xrightarrow{g^i} Z^i.$$

(b) *f induces a morphism $X \to F(g)$ given degree-wise (in ascending notation) by the composite*

$$X^i \xrightarrow{f^i} Y^i \to Y^i \oplus Z^{i-1} = F(g)^i.$$

(c) *If, moreover, the sequence $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ is exact, then the morphisms g' and f' described in (a) and (b) are quasi-isomorphisms.*

Proof It is rather easy to check that the maps described in (a) and (b) are morphisms of complexes. When the sequence

$$0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$$

is a short exact sequence observe that the morphism described in (a) is surjective with kernel $C(\mathrm{id}_X)$, while the morphism described in (b) is injective with cokernel $F(\mathrm{id}_Z)$. Then deduce the claim in (c) or see [14] or [32] for an outlined proof and all the details. □

Corollary 1.5.7 *Let $0 \to X \xrightarrow{f} Y \to Z \to 0$ be an exact sequence of R-complexes. The R-complex Y is exact if and only if the morphism f induces quasi-isomorphisms*

$$X^{[1]} \xleftarrow{\sim} C(f) \xrightarrow{\sim} Z.$$

1.5.8 *About cylinders.* Let $f : X \to Y$ be a morphism of ascending complexes over a commutative ring R. We take the fiber of f and denote by $p : F(f) \to X$ the natural projection. Hence $p^i : F(f)^i = X^i \oplus Y^{i-1} \to X^i$ is described by the row matrix $\begin{pmatrix} \mathrm{id}_{X^i} & 0 \end{pmatrix}$. Then we define the cylinder of f by

$$\mathrm{Cyl}(f) = C(-p).$$

Hence $\mathrm{Cyl}(f)^i = C(f)^i \oplus X^i = X^{i+1} \oplus Y^i \oplus X^i$ and the differential of $\mathrm{Cyl}(f)$ is given by the matrices

$$d^i_{\mathrm{Cyl}(f)} = \begin{pmatrix} -d_X^{i+1} & 0 & 0 \\ -f^{i+1} & d_Y^i & 0 \\ -\mathrm{id}_X^{i+1} & 0 & d_X^i \end{pmatrix}.$$

There is a morphism of graded modules $h : \mathrm{Cyl}(f) \to Y$ described by the row matrices $h^i = \begin{pmatrix} 0 & \mathrm{id}_Y^i & f^i \end{pmatrix}$. By computation we observe that h is a morphism of complexes and that there is a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r] \arrow[d, equal] & \mathrm{Cyl}(f) \arrow[r] \arrow[d, "h"] & C(f) \arrow[r] \arrow[d] & 0 \\
 & X \arrow[r, "f"'] & Y \arrow[r, "g"'] & Z \arrow[r] & 0
\end{tikzcd}$$

where $Z = \mathrm{Coker}(f)$, where g is the natural surjection and where the vertical morphism on the right is the one described in 1.5.6 (a).

If, moreover, f is injective, that is, if the short sequence $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ is exact, then the vertical morphism on the right $C(f) \to Z$ is a quasi-isomorphism (see 1.5.6) and so is h.

We note that these constructions are functorial. Any commutative square

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "a"'] & Y' \arrow[d, "b"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

induces morphisms $\mathrm{Cyl}(a, b)$ and $C(a, b)$ such that the following diagram

$$\begin{tikzcd} 0 \arrow[r] & X' \arrow[r] \arrow[d, "a"'] & \mathrm{Cyl}(f') \arrow[r] \arrow[d, "\mathrm{Cyl}(a,b)"] & C(f') \arrow[r] \arrow[d, "C(a,b)"] & 0 \\ 0 \arrow[r] & X \arrow[r] & \mathrm{Cyl}(f) \arrow[r] & C(f) \arrow[r] & 0 \end{tikzcd}$$

is commutative. If the morphisms a and b are quasi-isomorphisms, then the morphisms $C(a, b)$ and $\mathrm{Cyl}(a, b)$ are also quasi-isomorphisms. This is a consequence of 1.5.5.

For more information and details about these constructions, see [14, §2, $n°6$ Proposition 7 and §2, $n°7$ Proposition 9] resp. [85, Section 1.5].

1.5.9 *A dual construction: cocylinders.* Let $0 \to Z' \to Y' \xrightarrow{f'} X'$ be an exact sequence of descending complexes over a commutative ring R. We take the cone of f' and denote by $p' : X' \to C(f')$ the natural injection. Then we define the cocylinder of f' by

$$\mathrm{Dyl}(f') = F(-p').$$

Then $\mathrm{Dyl}(f') = X' \oplus Y' \oplus X'_{[1]}$ as a graded module. We note that cylinders and cocylinders are interchanged by the general Matlis duality. In particular, $(\mathrm{Dyl}(f'))^\vee \cong \mathrm{Cyl}(f'^\vee)$. It follows that the morphism of graded modules $h' : Y' \to \mathrm{Dyl}(f')$ described by the column matrices

$$h'_i = \begin{pmatrix} f'_i \\ (\mathrm{id}_Y)_i \\ 0 \end{pmatrix}$$

is a morphism of complexes and that there is a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & Z' \arrow[r] \arrow[d] & Y' \arrow[r, "f'"] \arrow[d, "h'"] & X' \arrow[d, equal] & \\
0 \arrow[r] & F(f') \arrow[r] & \mathrm{Dyl}(f') \arrow[r] & X' \arrow[r] & 0
\end{tikzcd}$$

where the vertical morphism on the left is the one described in 1.5.6 (b).

If, moreover, f' is surjective, that is, if the short sequence $0 \to Z' \to Y' \xrightarrow{f'} X' \to 0$ is exact, then the vertical morphism on the left is a quasi-isomorphism (see 1.5.6) and so is h'.

Here again these constructions are functorial.

Chapter 2
Adic Topology and Completion

In the first three sections of this chapter we summarize the basic results on the subject, include results for which there is no reference in the literature and complete the picture with some new observations. A few of these facts hold in full generality; but most of them require that the adic topology is taken with respect to a finitely generated ideal. The case when the ring is Noetherian is easier to handle, though far from being obvious when dealing with infinitely generated modules. In Sect. 2.4 we provide an extension of Bartijn's result stating that the adic completion of a flat module over a Noetherian ring is flat. Section 2.5 contains the first information on the left-derived functors of the adic completion functor with respect to an ideal $\mathfrak{a}$. Writing them, we have been astonished by how much can be said assuming only the hypothesis that the ideal $\mathfrak{a}$ is finitely generated. As usual there are finer results when the ring is Noetherian, and it is helpful in Part II to see that some of these hold in greater generality and in a more general setting. In Sect. 2.6 we introduce a notion of relative flatness, which is helpful for the study of local homology. Section 2.7 contains some remarks on torsion modules and relatively injective modules.

2.1 Topological Preliminaries

We first recall some known facts on the subject, fix notations and add some remarks and comments.

2.1.1 *Adic topologies.* (a) Let $\mathfrak{a}$ be an ideal of a commutative ring R. Given an R-module M, its submodules $\mathfrak{a}^t M$, $t \in \mathbb{N}$, form a base of open neighbourhoods of the origin for a topology on M, called the $\mathfrak{a}$-adic topology. It is then clear that any morphism between R-modules is continuous for the $\mathfrak{a}$-adic topology.

P. Schenzel and A.-M. Simon, *Completion, Čech and Local Homology and Cohomology*, Springer Monographs in Mathematics,
https://doi.org/10.1007/978-3-319-96517-8_2

(b) Let N be a submodule of M. The closure of N in M for this topology, the $\mathfrak{a}$-closure for short, is $\overline{N}_M^{\mathfrak{a}} := \bigcap_t (N + \mathfrak{a}^t M)$. Thus M is separated in its $\mathfrak{a}$-adic topology, $\mathfrak{a}$-separated for short, exactly when $\bigcap_t \mathfrak{a}^t M = 0$. Moreover, N is $\mathfrak{a}$-closed in M if and only if the quotient module M/N is $\mathfrak{a}$-separated. If $\overline{N}_M^{\mathfrak{a}} = M$ we say that N is $\mathfrak{a}$-dense in M; algebraically this means that $M = N + \mathfrak{a}M$; topologically this also means that the $\mathfrak{a}$-adic topology on the quotient M/N is the coarse one. More generally we say that a morphism $f : M' \to M$ is $\mathfrak{a}$-dense if $\mathrm{Im}(f)$ is $\mathfrak{a}$-dense in M, algebraically this means that $M = \mathrm{Im}(f) + \mathfrak{a}M$.

We shall also say that the R-module M is $\mathfrak{a}$-coarse if its $\mathfrak{a}$-adic topology is the coarse one, algebraically this means that $M = \mathfrak{a}M$.

(c) Note also that the $\mathfrak{a}$-adic topology of a submodule N of M is finer than the one induced by the $\mathfrak{a}$-adic topology of M, so that N is $\mathfrak{a}$-separated as soon as M is. But one has to be careful, it might happen that the $\mathfrak{a}$-adic topology of the submodule N of M is strictly finer than the one induced by the $\mathfrak{a}$-adic topology of M. However, the topology on $\bigcap \mathfrak{a}^t M = \overline{0}_M^{\mathfrak{a}}$ induced by the $\mathfrak{a}$-topology of M is always the coarse one. We shall provide in 2.4.14 an example of a module X over a Noetherian ring for which $0 \neq \overline{0}_X^{\mathfrak{a}}$ is $\mathfrak{a}$-separated (and even $\mathfrak{a}$-complete). Moreover, we shall also see in 2.5.6 that modules Y for which $\bigcap \mathfrak{a}^t Y$ is not $\mathfrak{a}$-coarse when non-zero occur very often in the completion process.

(d) We shall say that two ideals $\mathfrak{a}$ and $\mathfrak{a}'$ are *topologically equivalent* if they define the same adic topology, algebraically this means that $\mathfrak{a}$ contains a power of $\mathfrak{a}'$ and vice versa. Note that finitely generated ideals are topologically equivalent if and only if they have the same radical.

The following easy observation will be used frequently.

Observation 2.1.2 If the R-module X satisfies $X = \mathfrak{a}X$ and if the R-module Y is $\mathfrak{a}$-separated, then $\mathrm{Hom}_R(X, Y) = 0$. This is both topologically and algebraically obvious.

2.1.3 *Adic completions.* (a) The $\mathfrak{a}$-adic topology on M is in fact induced by a uniform structure on M (and even an ultrametric structure in the case when M is $\mathfrak{a}$-separated), so one may form completions. The $\mathfrak{a}$-adic completion functor is denoted by $\Lambda^{\mathfrak{a}}$ or $\hat{(\cdot)}^{\mathfrak{a}}$, or even $\hat{(\cdot)}$ in the case when there is no doubt about the ideal. It is known that $\hat{M}^{\mathfrak{a}} = \varprojlim M/\mathfrak{a}^t M$. If the descending sequence of submodules $\{M'_t\}_{t\geq 1}$ of M form another base of open neighbourhoods of the origin for the $\mathfrak{a}$-adic topology of M, then also $\hat{M}^{\mathfrak{a}} = \varprojlim M/M'_t$. In particular, if $\mathfrak{a}$ and $\mathfrak{a}'$ are topologically equivalent, then $\hat{M}^{\mathfrak{a}} = \varprojlim M/\mathfrak{a}'^t M = \hat{M}^{\mathfrak{a}'}$.

Let $f : M \to N$ denote a homomorphism of R-modules. The inverse limit procedure gives natural maps $\tau_M^{\mathfrak{a}}$ and $\rho_t^{M\mathfrak{a}}$, $t \geq 1$, such that the following diagrams are commutative

$$\begin{tikzcd} M \arrow[rr, "\tau_M^{\mathfrak{a}}"] \arrow[dr, "p_t^{M\mathfrak{a}}"'] & & \hat{M}^{\mathfrak{a}} \arrow[dl, "\rho_t^{M\mathfrak{a}}"] \\ & M/\mathfrak{a}^t M & \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\tau_X^{\mathfrak{a}}"'] & Y \arrow[d, "\tau_Y^{\mathfrak{a}}"] \\ \hat{X}^{\mathfrak{a}} \arrow[r, "\hat{f}^{\mathfrak{a}}"] \arrow[d, "\rho_t^{X\mathfrak{a}}"'] & \hat{Y}^{\mathfrak{a}} \arrow[d, "\rho_t^{Y\mathfrak{a}}"] \\ X/\mathfrak{a}^t X \arrow[r, "R/\mathfrak{a}^t \otimes_R f"] & Y/\mathfrak{a}^t Y \end{tikzcd}$$

where the $p_t^{M\mathfrak{a}}$'s are the natural projections, so that the $\rho_t^{M\mathfrak{a}}$'s are surjective. The superscripts are sometimes dropped or partially dropped from the notation when this is not ambiguous.

(b) Moreover, we note that $\operatorname{Ker}(\tau_M^{\mathfrak{a}}) = \bigcap_t \mathfrak{a}^t M$. Thus M is $\mathfrak{a}$-separated if and only if $\tau_M^{\mathfrak{a}}$ is injective. Note also that M is complete in its $\mathfrak{a}$-adic topology, $\mathfrak{a}$-complete for short, if and only if $\tau_M^{\mathfrak{a}}$ is bijective. When $\tau_M^{\mathfrak{a}}$ is surjective we say that M is $\mathfrak{a}$-*quasi-complete*.

(c) The $\mathfrak{a}$-completion functor is additive, hence commutes with finite direct sums. It follows that a direct summand of an $\mathfrak{a}$-complete module is itself $\mathfrak{a}$-complete.

(d) Recall also that $\hat{R}^{\mathfrak{a}}$ is a commutative ring, even an R-algebra by means of $\tau_R^{\mathfrak{a}}$. For any R-module M the R-module structure on $\hat{M}^{\mathfrak{a}}$ naturally extends to the structure of an $\hat{R}^{\mathfrak{a}}$-module. We thus have a natural map of $\hat{R}^{\mathfrak{a}}$-modules

$$\hat{R}^{\mathfrak{a}} \otimes_R M \to \hat{M}^{\mathfrak{a}} : c \otimes w \mapsto c \cdot \tau_M^{\mathfrak{a}}(w).$$

For these results the reader can also refer to [3] and [58].

Remarks 2.1.4 The completion process, i.e. the inverse limit process, endows $\hat{M}^{\mathfrak{a}}$ with a natural topology for which the $\operatorname{Ker}(\rho_t^{M\mathfrak{a}})$'s form a base of open neighbourhoods of the origin. Of course we have that $\hat{M}^{\mathfrak{a}}$ is complete, hence separated, in this natural topology (the projective limit of the isomorphisms $M/\mathfrak{a}^t M \cong \hat{M}^{\mathfrak{a}}/\operatorname{Ker}(\rho_t^{M\mathfrak{a}})$ is an isomorphism and by construction we have $\bigcap_t \operatorname{Ker}(\rho_t^{M\mathfrak{a}}) = 0$). But it may happen that $\tau_M^{\mathfrak{a}}$ is not $\mathfrak{a}$-dense.

We note that $(\tau_M^{\mathfrak{a}})^{-1}(\operatorname{Ker}(\rho_t^{M\mathfrak{a}})) = \mathfrak{a}^t M$, as follows in view of the commutative triangle of 2.1.3. We also note that $\mathfrak{a}^t \hat{M}^{\mathfrak{a}} \subset \operatorname{Ker}(\rho_t^{M\mathfrak{a}})$. This means that the $\mathfrak{a}$-adic topology on the R-module $\hat{M}^{\mathfrak{a}}$ is finer than its natural one. In general, it can be strictly finer and it can happen that $\hat{M}^{\mathfrak{a}}$ is not $\mathfrak{a}$-complete. However, $\hat{M}^{\mathfrak{a}}$ is always $\mathfrak{a}$-separated since its $\mathfrak{a}$-adic topology is finer than its natural one.

Example 2.1.5 Here is such an example. It originated in [13, Chapitre 3, §2, exercice 12] and is completely treated in Bartijn's thesis [6, I, §3 p. 19]. Let R be a polynomial ring in a countable set of indeterminates $X_1, X_2, \ldots$ over a field $\Bbbk$, and let $\mathfrak{a}$ be the ideal of R generated by these X_i. It is observed that $\hat{R}^{\mathfrak{a}}$ is the ring of formal powers series of the form $\sum_{i=0}^{\infty} P_i$, where $P_i \in R$ is an homogeneous polynomial of degree i, $i \geq 0$. Let $\rho_t : \hat{R}^{\mathfrak{a}} \to R/\mathfrak{a}^t$ be the maps coming from the inverse limit procedure. Bourbaki's reader is invited to check that

$$f_t := \sum_{i=t}^{\infty} X_i^i \in \mathrm{Ker}(\rho_t) \setminus \mathfrak{a}^t \hat{R}^{\mathfrak{a}},$$

which is rather obvious. These $\{f_t\}$ form a Cauchy sequence for the $\mathfrak{a}\hat{R}^{\mathfrak{a}}$-adic topology of $\hat{R}^{\mathfrak{a}}$ as well as for the natural topology. In the natural topology we have $\lim_{t\to\infty} f_t = 0$, but not in the $\mathfrak{a}\hat{R}^{\mathfrak{a}}$-adic topology. The $\mathfrak{a}\hat{R}^{\mathfrak{a}}$-adic Cauchy sequence of the f_t's is not $\mathfrak{a}\hat{R}^{\mathfrak{a}}$-adic convergent, hence $\hat{R}^{\mathfrak{a}}$ is not $\mathfrak{a}$-complete. Note also that the natural map $\tau_R^{\mathfrak{a}}$ is not $\mathfrak{a}$-dense: $\sum_{i=1}^{\infty} X_i^i$ does not belong to the $\mathfrak{a}$-closure of R in $\hat{R}^{\mathfrak{a}}$. But such a bad situation cannot occur when the ideal $\mathfrak{a}$ is finitely generated (see 2.2.2 below).

Remark 2.1.6 The ideal $\mathfrak{a}\hat{R}^{\mathfrak{a}}$ of $\hat{R}^{\mathfrak{a}}$ is contained in the Jacobson radical of $\hat{R}^{\mathfrak{a}}$.

(To see this, it is enough to show that $(1-x)$ is invertible for any $x \in \mathfrak{a}\hat{R}^{\mathfrak{a}}$. This is the case because $\sum_{i=0}^{\infty} x^i \in \hat{R}^{\mathfrak{a}}$ as the limit of a Cauchy sequence in the natural topology and $(1-x)\cdot(\sum_{i=0}^{\infty} x^i) = 1$).

2.1.7 *About purity.* Let R be any commutative ring and let $i : N \hookrightarrow M$ be a submodule of the R-module M. We say that N is pure in M if the map $X \otimes_R i$ is injective for any R-module X. When this is the case $\mathfrak{a}^t M \cap N = \mathfrak{a}^t N$ for all ideals $\mathfrak{a}$ of R, the $\mathfrak{a}$-adic topology of N is induced by the $\mathfrak{a}$-adic topology of M and the short sequence $0 \to \hat{N}^{\mathfrak{a}} \to \hat{M}^{\mathfrak{a}} \to \widehat{M/N}^{\mathfrak{a}} \to 0$ is exact (because it is the inverse limit of short exact sequences of surjective inverse systems).

Assume now that the R-module M is flat. With the Bourbaki flatness criterion recalled in 1.4.3 it is easy to see that N is pure in M if and only if M/N is flat if and only if the map $R/\mathfrak{a} \otimes_R i : N/\mathfrak{a}N \to M/\mathfrak{a}M$ is injective for every ideal $\mathfrak{a}$ of R. Hence a pure submodule of a flat module is flat.

In general the $\mathfrak{a}$-adic completion functor is neither left-exact nor right-exact. An example showing this can be found in [3, Chap. 10, Exercises 1 and 2], another one, in the same spirit, is in 2.4.14, but it is known that it preserves surjections. More generally we have:

Proposition 2.1.8 *see* ([78] *or* [79]) Let $\mathfrak{a}$ be an ideal of a commutative ring R.

(a) Let $f : M \to N$ be a morphism of R-modules. Then $\hat{f}^{\mathfrak{a}}$ is surjective if and only if f is $\mathfrak{a}$-dense (this means that $N = f(M) + \mathfrak{a}N$).
(b) Quotients of $\mathfrak{a}$-quasi-complete modules are $\mathfrak{a}$-quasi-complete.

Proof For the proof of (a) assume first that f is $\mathfrak{a}$-dense. Then $N = f(M) + \mathfrak{a}^t N$ and there is an inverse system of short exact sequences

$$0 \to f^{-1}(\mathfrak{a}^t N)/\mathfrak{a}^t M \to M/\mathfrak{a}^t M \to N/\mathfrak{a}^t N \to 0$$

for all $t \geq 1$. We note that $N = f(M) + \mathfrak{a}N$ also implies that $\mathfrak{a}^t N = f(\mathfrak{a}^t M) + \mathfrak{a}^{t+1} N$ for all $t \geq 1$. Therefore the inverse system on the left is surjective. Then we conclude that $\hat{f}^{\mathfrak{a}}$ is surjective with the six-term long exact sequence in 1.2.2.

If $\hat{f}^{\mathfrak{a}}$ is surjective we look at the commutative rectangles in 2.1.3 with $X = M$ and $Y = N$. Since the maps $\rho_t^{N\mathfrak{a}}$ are surjective we see that $R/\mathfrak{a}^t \otimes_R f$ is surjective too, whence $N = f(M) + \mathfrak{a}^t N$.

Now (b) is an easy consequence of (a). □

2.1.9 *Completion of direct sums.* As mentioned above, tensor products commute with direct sums and inverse limits commute with products, but not with infinite direct sums.

Let $\mathfrak{a}$ be an ideal of a commutative ring R and simply denote by $\hat{(\cdot)}$ the $\mathfrak{a}$-completion functor. Let $\{M_i\}_{i\in I}$ be a family of R-modules. Then we have the commutative square

$$\begin{CD}
\bigoplus_i \hat{M}_i @>>> \prod_i \hat{M}_i \\
@V{\bigoplus_i \rho_t^{M_i}}VV @VV{\prod_i \rho_t^{M_i}}V \\
\bigoplus_i M_i/\mathfrak{a}^t M_i @>>> \prod_i M_i/\mathfrak{a}^t M_i
\end{CD}$$

where the horizontal maps are the natural inclusions and where the $\rho_t^{M_i}$'s are the natural maps as described in 2.1.3. Let $w = (w_i) \in \prod_i \hat{M}_i$ and observe that $\prod_i \rho_t^{M_i}(w) \in \bigoplus_i M_i/\mathfrak{a}^t M_i$ if and only if all but finitely many w_i belong to $\mathrm{Ker}(\rho_t^{M_i})$. Then define the submodule Z of $\prod_i \hat{M}_i$ by

$$Z = \{w \in \prod_i \hat{M}_i \mid \text{ for all } t, \text{ all but finitely many } w_i \in \mathrm{Ker}(\rho_t^{M_i})\}.$$

Thus $\bigoplus_i \hat{M}_i \subset Z \subset \prod_i \hat{M}_i$ and for all t the map $\prod_i \rho_t^{M_i}$ restricts to a map $f_t : Z \to \bigoplus_i M_i/\mathfrak{a}^t M_i$. Now it is easy to see that Z together with these f_t is the inverse limit of the inverse system formed by the $\bigoplus_i M_i/\mathfrak{a}^t M_i$'s, i.e. that Z is the $\mathfrak{a}$-completion of $\bigoplus_i M_i$.

This description is just a generalization of the description of the completion of a free module over a Noetherian ring given in [82, Chap. 2].

Note also that the $\prod_i \mathrm{Ker}(\rho_t^{M_i})$'s form a base of open neighbourhoods of the origin for a topology on $\prod_i \hat{M}_i$ and that Z is the closure of $\bigoplus_i \hat{M}_i$ in $\prod_i \hat{M}_i$ for that topology.

Remarks 2.1.10 (a) Let M and N two R-modules. For any homomorphism $f : M \to \hat{N}^{\mathfrak{a}}$ there is an homomorphism f' making the following triangle commutative

$$\begin{tikzcd}
M \arrow[rr, "f"] \arrow[dr, "\tau_M^{\mathfrak{a}}"'] & & \hat{N}^{\mathfrak{a}} \\
& \hat{M}^{\mathfrak{a}} \arrow[ur, dotted, "f'"'] &
\end{tikzcd}$$

Indeed, recall the homomorphism $\rho_t^{N\mathfrak{a}} : \hat{N}^{\mathfrak{a}} \to N/\mathfrak{a}^t N$ of 2.1.3. The homomorphism $\rho_t^{N\mathfrak{a}} \circ f$ factors through $M/\mathfrak{a}^t M$, inducing a homomorphism $f'_t : M/\mathfrak{a}^t M \to N/\mathfrak{a}^t N$. This holds for all $t \geq 1$ and these f'_t's form an inverse system of homomorphisms, as easily seen. Now the homomorphism $f' := \varprojlim f'_t$ does the job.

(b) Note, however, that the homomorphism f' making the triangle commutative is not necessarily unique in this very general situation. For example, take for R and $\mathfrak{a}$ the ring and ideal of example 2.1.5, and take $M = R$. Here $R/\mathfrak{a} = \Bbbk$ is a field. Because $\tau_R^{\mathfrak{a}}$ is not $\mathfrak{a}$-dense the $\Bbbk$-vector space $\operatorname{Coker}(\tau_R^{\mathfrak{a}})/\mathfrak{a}\operatorname{Coker}(\tau_R^{\mathfrak{a}})$ is not zero. Then take $N = \Bbbk$ so that $\hat{N}^{\mathfrak{a}} = \hat{\Bbbk}^{\mathfrak{a}} \cong \Bbbk$ and take for f the zero homomorphism. Because $\Bbbk$ is a field there is a non-zero homomorphism $\operatorname{Coker}(\tau_R^{\mathfrak{a}}) \to \Bbbk$ inducing a non-zero homomorphism $g : \hat{R}^{\mathfrak{a}} \to \Bbbk$ such that $g \circ \tau_R^{\mathfrak{a}} = 0 = 0 \circ \tau_R^{\mathfrak{a}}$.

(c) The natural map $\tau_{\hat{N}^{\mathfrak{a}}}^{\mathfrak{a}}$ is always split-injective. (To see this put in (a) $M = \hat{N}^{\mathfrak{a}}$ and $f = \mathrm{id}_{\hat{N}^{\mathfrak{a}}}$.)

More important is the following.

Observations 2.1.11 Let $\mathfrak{b}$ be an $\mathfrak{a}$-open ideal of a commutative ring R, i.e. an ideal containing some power $\mathfrak{a}^t$ of $\mathfrak{a}$. We tensorize the triangle in 2.1.3 with $R/\mathfrak{b}$ and observe as in [82] that the homomorphism

$$R/\mathfrak{b} \otimes_R \tau_M^{\mathfrak{a}} : M/\mathfrak{b}M \to \hat{M}^{\mathfrak{a}}/\mathfrak{b}\hat{M}^{\mathfrak{a}} \quad \text{is split-injective.}$$

(Note that by 2.1.3 $\rho_t^{M_{\mathfrak{a}}} \circ \tau_M^{\mathfrak{a}} = p_t^{M_{\mathfrak{a}}}$. Note also that $R/\mathfrak{a}^t \otimes_R p_t^{M\mathfrak{a}} = \mathrm{id}_{M/\mathfrak{a}^t M}$. Therefore $R/\mathfrak{a}^t \otimes_R \tau_M^{\mathfrak{a}}$ is split-injective and so is $R/\mathfrak{b} \otimes_R \tau_M^{\mathfrak{a}}$.)

Passing to the limit, we have that the homomorphism

$$\Lambda^{\mathfrak{a}}(\tau_M^{\mathfrak{a}}) : \Lambda^{\mathfrak{a}}(M) \to \Lambda^{\mathfrak{a}}(\Lambda^{\mathfrak{a}}(M))$$

is split-injective too.

Remark 2.1.12 Suppose that the homomorphisms $R/\mathfrak{a}^t \otimes_R \tau_M^{\mathfrak{a}}$ are isomorphisms. Then it follows that $\mathfrak{a}^t \hat{M}^{\mathfrak{a}} = \operatorname{Ker}(\rho_t^{M\mathfrak{a}})$. Hence the $\mathfrak{a}$-adic topology of $\hat{M}^{\mathfrak{a}}$ coincides with its natural one, and therefore $\hat{M}^{\mathfrak{a}}$ is $\mathfrak{a}$-complete.

2.1.13 *The torsion functors.* Given an ideal $\mathfrak{a}$ of a commutative ring R we also have the $\mathfrak{a}$-torsion functor defined by

$$\Gamma_{\mathfrak{a}}(M) = \{w \in M \mid \mathfrak{a}^t w = 0 \text{ for some } t \in \mathbb{N}\} \cong \varinjlim \operatorname{Hom}_R(R/\mathfrak{a}^t, M)$$

for all R-modules M. This functor is a subfunctor of the identity functor, left-exact and clearly idempotent.

An R-module M such that $M = \Gamma_{\mathfrak{a}}(M)$ is called an $\mathfrak{a}$-*torsion* module.

If the R-module M is $\mathfrak{a}$-torsion, then $\operatorname{Supp}_R M \subseteq V(\mathfrak{a})$. When the ideal $\mathfrak{a}$ is finitely generated then the R-module M is $\mathfrak{a}$-torsion if and only if $\operatorname{Supp}_R M \subseteq V(\mathfrak{a})$.

(To see this note that an R-module M is $\mathfrak{a}$-torsion if and only if each finitely generated submodule of M is $\mathfrak{a}$-torsion. Note also that $\operatorname{Supp}_R M \subseteq V(\mathfrak{a})$ if and only if for each finitely generated submodule M_λ of M, $\operatorname{Supp}_R M_\lambda \subseteq V(\mathfrak{a})$. So we only need to prove the case where M is finitely generated. When a finitely generated module M is $\mathfrak{a}$-torsion it is annihilated by a power of $\mathfrak{a}$ so that $\operatorname{Supp}_R M \subseteq V(\mathfrak{a})$. When the module M is finitely generated we have $\operatorname{Supp}_R M = V(\operatorname{Ann}_R(M))$ so that the condition $\operatorname{Supp}_R M \subseteq V(\mathfrak{a})$ implies $\mathfrak{a} \subseteq \operatorname{Rad}(\operatorname{Ann}_R(M))$. The last condition implies that M is annihilated by a power of $\mathfrak{a}$ if the ideal $\mathfrak{a}$ is finitely generated.)

If the ring R is Noetherian note also that the R-module M is $\mathfrak{a}$-torsion if and only if $\operatorname{Ass}_R(M) \subseteq V(\mathfrak{a})$. (To see this we again may reduce to the case where M is finitely generated.)

2.1.14 *Local cohomology* The right-derived functors of $\Gamma_{\mathfrak{a}}$ are denoted by $H^i_{\mathfrak{a}}$ and usually called the local cohomology functors with respect to $\mathfrak{a}$. As $\varinjlim$ is exact, note that $H^i_{\mathfrak{a}}(M) \cong \varinjlim \operatorname{Ext}^i_R(R/\mathfrak{a}^t, M)$ for all $i \in \mathbb{N}$.

Here is a slight extension of [82, 3.4.9].

Proposition 2.1.15 *Let $\mathfrak{a}$ denote an ideal of a commutative ring R. Suppose that M is an $\mathfrak{a}$-torsion R-module. Then*

(a) M has the unique structure of an $\hat{R}^{\mathfrak{a}}$-module extending its R-module structure.
(b) $\operatorname{Hom}_R(M, X) = \operatorname{Hom}_{\hat{R}^{\mathfrak{a}}}(M, X)$ *for every $\hat{R}^{\mathfrak{a}}$-module X.*

In particular, any R-submodule of M is also an $\hat{R}^{\mathfrak{a}}$-submodule.

Proof For the proof of (a) note that if M is finitely generated there is an integer t such that $\mathfrak{a}^t M = 0$. Hence it has an $\hat{R}^{\mathfrak{a}}$-module structure inherited from the natural map $\rho_t^{R\mathfrak{a}} : \hat{R}^{\mathfrak{a}} \to R/\mathfrak{a}^t$ and the statement is clear. In general M is the union of its finitely generated submodules and it is straightforward to check that the $\hat{R}^{\mathfrak{a}}$-module structures on these patch together.

For the proof of (b) let $f \in \operatorname{Hom}_R(M, X)$, $w \in M$ and $s \in \hat{R}^{\mathfrak{a}}$. We have $\mathfrak{a}^t w = 0 = \mathfrak{a}^t f(w)$ for some $t > 0$. In view of the commutative triangle in 2.1.3 with $M = R$ we can take an element $r \in R$ such $p_t^{R\mathfrak{a}}(r) = \rho_t^{R\mathfrak{a}}(s)$. For such an r note that $sw = rw$ and also $sf(w) = rf(w)$. Hence $f(sw) = f(rw) = rf(w) = sf(w)$. $\square$

See also the forthcoming 2.2.6 and 2.2.13 for more information on $\mathfrak{a}$-torsion modules when the ideal $\mathfrak{a}$ is finitely generated.

Corollary 2.1.16 *For an R-module M the modules $\Gamma_{\mathfrak{a}}(M)$ and $H^i_{\mathfrak{a}}(M)$ are $\mathfrak{a}$-torsion, hence have the structure of an $\hat{R}^{\mathfrak{a}}$-module.*

2.1.17 *The left-derived functors of the $\mathfrak{a}$-completion functor* are simply denoted by $\Lambda^{\mathfrak{a}}_i$ and called the local homology functors, often also denoted by $\mathrm{L}_i \Lambda^{\mathfrak{a}}$.

The first information on these will be given in one of the last sections of this chapter.

2.2 The Case of Finitely Generated Ideals

When an ideal $\mathfrak{a}$ of a commutative ring R is finitely generated, the $\mathfrak{a}$-completion of any R-module M, always complete in its natural topology, is also complete in its $\mathfrak{a}$-adic topology. This important theorem was first proved by Matlis in the case when the ideal $\mathfrak{a}$ is generated by a regular sequence (see [57, Theorem 15]) and in general by Bartijn and Strooker (see [82, Theorem 2.2.5]). They derived it from their investigation of the completion of free modules over a Noetherian ring. Here we provide a more direct proof of it. Then we collect some of its consequences.

First we apply the recalls in 1.2.2 to the $\mathfrak{a}$-completion process.

Remark 2.2.1 We have the inverse system of exact sequences,

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{a}^{t+1}M \arrow[r] \arrow[d, hook] & M \arrow[r] \arrow[d, equal] & M/\mathfrak{a}^{t+1}M \arrow[r] \arrow[d, two heads] & 0 \\ 0 \arrow[r] & \mathfrak{a}^{t}M \arrow[r] & M \arrow[r] & M/\mathfrak{a}^{t}M \arrow[r] & 0, \end{tikzcd}$$

where the right vertical maps are the natural projection and the left vertical maps are the natural inclusions. We note that $\varprojlim\{\mathfrak{a}^t M\} = \bigcap_t \mathfrak{a}^t M$ and, as the projective systems in the middle and on the right are surjective, we obtain a four-term exact sequence

$$0 \longrightarrow \bigcap_n \mathfrak{a}^t M \longrightarrow M \xrightarrow{\tau_M^{\mathfrak{a}}} \hat{M}^{\mathfrak{a}} \longrightarrow \varprojlim{}^1\{\mathfrak{a}^t M\} \longrightarrow 0.$$

The following fact will play a substantial role.

Theorem 2.2.2 *Let $\mathfrak{a}$ be a finitely generated ideal of the commutative ring R and M an R-module. Then the natural maps*

$$R/\mathfrak{b} \otimes_R \tau_M^{\mathfrak{a}} : M/\mathfrak{b}M \to \hat{M}^{\mathfrak{a}}/\mathfrak{b}\hat{M}^{\mathfrak{a}}$$

are isomorphisms for any $\mathfrak{a}$-open ideal $\mathfrak{b}$ of R. In particular, the map $\tau_M^{\mathfrak{a}}$ is $\mathfrak{a}$-dense, $\mathrm{Coker}(\tau_M^{\mathfrak{a}}) = \mathfrak{a}\,\mathrm{Coker}(\tau_M^{\mathfrak{a}})$ *and $\hat{M}^{\mathfrak{a}}$ is $\mathfrak{a}$-complete.*

Proof In view of 2.1.12 we only need prove the first assertion. As we know by 2.1.11 that the maps $R/\mathfrak{b} \otimes_R \tau_M^{\mathfrak{a}}$ are injective when $\mathfrak{b}$ is $\mathfrak{a}$-open, it is enough to prove that they are surjective. As an $\mathfrak{a}$-open ideal $\mathfrak{b}$ contains some power $\mathfrak{a}^t$ of $\mathfrak{a}$, it is also enough to prove that $\mathrm{Coker}(\tau_M^{\mathfrak{a}}) = \mathfrak{a}\mathrm{Coker}(\tau_M^{\mathfrak{a}})$.

By 2.2.1 and 1.2.2 we have $\mathrm{Coker}(\tau_M^{\mathfrak{a}}) = \varprojlim{}^1\{\mathfrak{a}^t M\} = \mathrm{Coker}(\psi)$, where $\psi : \prod_t \mathfrak{a}^t M \to \prod_t \mathfrak{a}^t M$ is the map associated to the inverse system formed with the natural inclusions $\mathfrak{a}^{t+1}M \subset \mathfrak{a}^t M$.

The ideal $\mathfrak{a}$ is finitely generated, $R/\mathfrak{a}$ is finitely presented and the functor $R/\mathfrak{a} \otimes_R \cdot$ commutes with direct products (see 1.4.4). It follows that the map

$$R/\mathfrak{a} \otimes_R \psi : \prod_{t \geq 1} \mathfrak{a}^t M/\mathfrak{a}^{t+1} M \to \prod_{t \geq 1} \mathfrak{a}^t M/\mathfrak{a}^{t+1} M$$

is the map associated to the inverse system formed by the maps

$$\mathfrak{a}^{t+1} M/\mathfrak{a}^{t+2} M \to \mathfrak{a}^t M/\mathfrak{a}^{t+1} M$$

induced by the natural injections $\mathfrak{a}^{t+1} M \subset \mathfrak{a}^t M$. As the previous inverse system is clearly pro-zero, we have that $R/\mathfrak{a} \otimes_R \psi$ is an isomorphism (see 1.2.2). In particular, we have $\operatorname{Coker}(R/\mathfrak{a} \otimes_R \psi) = 0$. With the right-exactness of the tensor product this also means that $R/\mathfrak{a} \otimes_R \operatorname{Coker}(\psi) = 0$. But $\operatorname{Coker}(\psi) = \varprojlim{}^1 \{\mathfrak{a}^t M\} = \operatorname{Coker}(\tau_M^{\mathfrak{a}})$. Therefore $R/\mathfrak{a} \otimes_R \operatorname{Coker}(\tau_M^{\mathfrak{a}}) = 0$. $\square$

We now obtain a universal property for the maps $\tau_M^{\mathfrak{a}}$.

Corollary 2.2.3 *Let $f : M \to N$ be a homomorphism of R-modules and assume that the ideal $\mathfrak{a}$ of R is finitely generated. Then $\hat{f}^{\mathfrak{a}}$ is the unique map $\hat{M}^{\mathfrak{a}} \to \hat{N}^{\mathfrak{a}}$ making the following square commutative*

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\tau_M^{\mathfrak{a}}"'] & N \arrow[d, "\tau_N^{\mathfrak{a}}"] \\ \hat{M}^{\mathfrak{a}} \arrow[r, dotted, "\hat{f}^{\mathfrak{a}}"'] & \hat{N}^{\mathfrak{a}} \end{tikzcd}$$

If $N = C$ is $\mathfrak{a}$-complete, then $\hat{f}^{\mathfrak{a}}$ is also the unique map making the triangle

$$\begin{tikzcd} M \arrow[rr, "f"] \arrow[dr, "\tau_M^{\mathfrak{a}}"'] & & C \\ & \hat{M}^{\mathfrak{a}} \arrow[ur, dotted, "\hat{f}^{\mathfrak{a}}"'] & \end{tikzcd}$$

commutative.

Proof The statements follow because of the following two facts: $\tau_M^{\mathfrak{a}}$ is $\mathfrak{a}$-dense and $\hat{N}^{\mathfrak{a}}$ is $\mathfrak{a}$-separated. (Algebraically this also follows by Theorem 2.2.2, since $R/\mathfrak{a}^t \otimes_R f = R/\mathfrak{a}^t \otimes_R \hat{f}^{\mathfrak{a}}$.) $\square$

Remark 2.2.4 In the situation of 2.2.3 we may take for f the natural map $\tau_M^{\mathfrak{a}} : M \to \hat{M}^{\mathfrak{a}}$. We then observe that $\tau^{\mathfrak{a}}_{\Lambda^{\mathfrak{a}} M}$ and $\Lambda^{\mathfrak{a}}(\tau_M^{\mathfrak{a}})$ coincide and are isomorphisms.

Corollary 2.2.5 *Let $\mathfrak{a}$ be a finitely generated ideal of the commutative ring R. Then*

$$\Lambda^{\mathfrak{a}}(M) \cong \Lambda^{\mathfrak{a}}(\hat{R}^{\mathfrak{a}} \otimes_R M) \cong \Lambda^{\mathfrak{a}\hat{R}^{\mathfrak{a}}}(\hat{R}^{\mathfrak{a}} \otimes_R M) \text{ and}$$
$$\Gamma_{\mathfrak{a}}(M) \cong \Gamma_{\mathfrak{a}}(\operatorname{Hom}_R(\hat{R}^{\mathfrak{a}}, M)) \cong \Gamma_{\mathfrak{a}\hat{R}^{\mathfrak{a}}}(\operatorname{Hom}_R(\hat{R}^{\mathfrak{a}}, M))$$

for any R-module M.

Proof In view of Theorem 2.2.2 we have

$$R/\mathfrak{a}^t \otimes_R M \cong R/\mathfrak{a}^t \otimes_R \hat{R}^{\mathfrak{a}} \otimes_R M \cong \hat{R}^{\mathfrak{a}}/\mathfrak{a}^t \hat{R}^{\mathfrak{a}} \otimes_{\hat{R}^{\mathfrak{a}}} (\hat{R}^{\mathfrak{a}} \otimes_R M).$$

Hence the first statement follows by taking inverse limits.

For the second note also that

$$\operatorname{Hom}_R(R/\mathfrak{a}^t, M) \cong \operatorname{Hom}_R(R/\mathfrak{a}^t, \operatorname{Hom}_R(\hat{R}^{\mathfrak{a}}, M)) \cong \operatorname{Hom}_{\hat{R}^{\mathfrak{a}}}(\hat{R}^{\mathfrak{a}}/\mathfrak{a}^t \hat{R}^{\mathfrak{a}}, \operatorname{Hom}_R(\hat{R}^{\mathfrak{a}}, M)),$$

where the second isomorphism follows by Proposition 2.1.15. Then the statement follows by taking direct limits. □

Corollary 2.2.6 *Let $\mathfrak{a}$ be a finitely generated ideal of a commutative ring R and M an $\mathfrak{a}$-torsion R-module. Then the natural homomorphism $M \to \hat{R}^{\mathfrak{a}} \otimes_R M$ is an isomorphism.*

Proof We already noted in 2.1.15 that the $\mathfrak{a}$-torsion module M has the structure of an $\hat{R}^{\mathfrak{a}}$-module. If M is finitely generated it is annihilated by a power $\mathfrak{a}^t$ of $\mathfrak{a}$, hence $M = R/\mathfrak{a}^t \otimes_R M$. By 2.2.2 we then have

$$\hat{R}^{\mathfrak{a}} \otimes_R M = \hat{R}^{\mathfrak{a}} \otimes_R R/\mathfrak{a}^t \otimes_R M \cong R/\mathfrak{a}^t \otimes_R M = M$$

and the statement is clear. For an arbitrary R-module M note that M is the direct limit of its finitely generated submodules. The tensor product commutes with this direct limit and the result follows. □

2.2.7 *Direct products and completion.* We assume again that the ideal $\mathfrak{a}$ of the commutative ring R is finitely generated. This ensures that the functors $(R/\mathfrak{a}^t \otimes_R \cdot)$, $t \geq 1$, commute with direct products (see 1.4.4). As inverse limits commute with direct products, we see that the $\mathfrak{a}$-adic completion functor also commutes with direct products: $\Lambda^{\mathfrak{a}}(\prod_i M_i) = \prod_i \Lambda^{\mathfrak{a}}(M_i)$. In particular, we have that a direct product of $\mathfrak{a}$-complete R-modules is $\mathfrak{a}$-complete.

2.2.8 *Direct sums and completion.* Let $\{M_i\}_{i \in I}$ be a family of R-modules. When the ideal $\mathfrak{a}$ of R is finitely generated, $\Lambda^{\mathfrak{a}}(\bigoplus_i M_i)$ is isomorphic to the $\mathfrak{a}$-closure of $\bigoplus_i \hat{M}_i^{\mathfrak{a}}$ in $\prod_i \hat{M}_i^{\mathfrak{a}}$.

(In view of 1.4.4 the submodules $\prod_i \mathfrak{a}^t \hat{M}_i^{\mathfrak{a}}$ of $\prod_i \hat{M}_i^{\mathfrak{a}}$ form a base of open neighbourhoods of the origin for the $\mathfrak{a}$-adic topology of $\prod_i \hat{M}_i^{\mathfrak{a}}$. With 2.1.12 and 2.2.2 we have that $\mathfrak{a}^t \hat{M}_i^{\mathfrak{a}} = \operatorname{Ker}(\rho_t^{M_i \mathfrak{a}})$. Now the statement follows from 2.1.9.)

This description is also a generalization of the description of the completion of a free module over a Noetherian ring given in [82, Chap. 2].

2.2.9 *Change of ideals* (see [82, 2.2.6]). Let $\mathfrak{a} \subset \mathfrak{b}$ be two ideals of a commutative ring R. Suppose that $\mathfrak{a}$ is finitely generated. Let M denote an R-module.

(a) By Theorem 2.2.2 we have that $\hat{M}^{\mathfrak{a}}/\mathfrak{b}^t\hat{M}^{\mathfrak{a}} \cong M/\mathfrak{b}^t M$ because $\mathfrak{b}^t$ is an $\mathfrak{a}$-open ideal. Therefore $\Lambda^{\mathfrak{b}}(\Lambda^{\mathfrak{a}}(M)) = \Lambda^{\mathfrak{b}}(M)$. More precisely, we have a commutative triangle

$$\begin{tikzcd} & \hat{M}^{\mathfrak{a}} \arrow[dr, "\tau^{\mathfrak{b}}_{\hat{M}^{\mathfrak{a}}}"] & \\ M \arrow[ur, "\tau^{\mathfrak{a}}_M"] \arrow[rr, "\tau^{\mathfrak{b}}_M"] & & \hat{M}^{\mathfrak{b}}. \end{tikzcd}$$

(b) If M is $\mathfrak{b}$-complete, then M is also $\mathfrak{a}$-complete. Indeed, if $\tau^{\mathfrak{b}}_M$ is an isomorphism, then M appears as a direct summand of $\hat{M}^{\mathfrak{a}}$ which is $\mathfrak{a}$-complete, therefore M is $\mathfrak{a}$-complete.

In the other direction we have the following result in the spirit of [76, 2.4].

Proposition 2.2.10 *Let $\mathfrak{a}_1, \mathfrak{a}_2$ be two finitely generated ideals of a commutative ring R and put $\mathfrak{a} = \mathfrak{a}_1 + \mathfrak{a}_2$. Let M be an R-module which is both $\mathfrak{a}_1$-quasi-complete and $\mathfrak{a}_2$-quasi-complete. Then M is $\mathfrak{a}$-quasi-complete. Moreover, if M is $\mathfrak{a}$-separated, then M is $\mathfrak{a}$-complete.*

Proof Recall the exact sequence

$$0 \to M \xrightarrow{\Delta} M \oplus M \xrightarrow{q} M \to 0,$$

where Δ is the diagonal map and q is defined by $q(m_1, m_2) = m_1 - m_2$. For any integer $t \geq 1$ this sequence gives rise to the following exact sequences

$$0 \to M/(\mathfrak{a}_1^t M \cap \mathfrak{a}_2^t M) \to M/\mathfrak{a}_1^t M \oplus M/\mathfrak{a}_2^t M \to M/(\mathfrak{a}_1^t M + \mathfrak{a}_2^t M) \to 0.$$

These form an inverse system of which we take the inverse limit. We first note that the submodules $(\mathfrak{a}_1^t M + \mathfrak{a}_2^t M)$ of M form a base of open neighbourhoods of the origin for the $\mathfrak{a}$-adic topology of M. As the inverse system on the left is clearly surjective the rows in the following commutative diagram are exact

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d] & M \oplus M \arrow[r] \arrow[d, "\tau^{\mathfrak{a}_1}_M \oplus \tau^{\mathfrak{a}_2}_M"] & M \arrow[r] \arrow[d, "\tau^{\mathfrak{a}}_M"] & 0 \\ 0 \arrow[r] & \varprojlim(M/(\mathfrak{a}_1^t M \cap \mathfrak{a}_2^t M) \arrow[r] & \hat{M}^{\mathfrak{a}_1} \oplus \hat{M}^{\mathfrak{a}_2} \arrow[r] & \hat{M}^{\mathfrak{a}} \arrow[r] & 0 \end{tikzcd}$$

The middle vertical map in the diagram is surjective by hypothesis. Whence it follows that the right vertical map $\tau^{\mathfrak{a}}_M$ is surjective too. This means that M is $\mathfrak{a}$-quasi-complete and even $\mathfrak{a}$-complete when it is $\mathfrak{a}$-separated. □

Proposition 2.2.11 (see [78, 1.3, 1.4]) *Let $\mathfrak{a} \subset R$ denote a finitely generated ideal of a commutative ring R.*

(a) Let C denote an $\mathfrak{a}$-complete R-module. Then every $\mathfrak{a}$-closed submodule $K \subset C$ is $\mathfrak{a}$-complete (though the $\mathfrak{a}$-adic topology of K might be finer than the one induced by the $\mathfrak{a}$-adic topology of C).

(b) Let $X_\bullet : \ldots \to X_{i+1} \to X_i \xrightarrow{d_i} X_{i-1} \to \ldots$ denote a complex of $\mathfrak{a}$-complete modules. Then the homology modules $H_i(X)$ are $\mathfrak{a}$-quasi-complete, and either $H_i(X) = 0$ or $H_i(X) \neq \mathfrak{a}H_i(X)$.

Proof (a): Consider the exact sequence $0 \to K \xrightarrow{i} C \xrightarrow{p} Z \to 0$. We first note that Z is $\mathfrak{a}$-complete: it is $\mathfrak{a}$-separated because K is $\mathfrak{a}$-closed in C, and $\mathfrak{a}$-quasi-complete as a quotient of an $\mathfrak{a}$-complete module (see 2.1.8). This allows us to put $\tau_Z^{\mathfrak{a}} = \mathrm{id}_Z$, $\tau_C^{\mathfrak{a}} = \mathrm{id}_C$ and $p = \hat{p}^{\mathfrak{a}}$. As $p \circ \hat{i}^{\mathfrak{a}} = 0$ the map $\hat{i}^{\mathfrak{a}}$ factors through i. Therefore $\tau_K^{\mathfrak{a}}$ is split-injective. Whence K is $\mathfrak{a}$-complete as a direct summand of the $\mathfrak{a}$-complete module $\hat{K}^{\mathfrak{a}}$.

(b): Put $Z_i = \mathrm{Ker}(d_i)$. As X_{i-1} is $\mathfrak{a}$-complete, hence $\mathfrak{a}$-separated, we have that Z_i is $\mathfrak{a}$-closed in X_i. Thus Z_i is $\mathfrak{a}$-complete by (a). Then observe that $H_i(X) = \mathrm{Coker}(X_{i+1} \to Z_i)$ and use 2.1.8 again. □

More precise information on the homology of a complex of $\mathfrak{a}$-complete modules will be given in the fifth section of this chapter (see 2.5.7).

Corollary 2.2.12 (see [78, 1.5]) *Assume that an ideal $\mathfrak{a}$ of a commutative ring R is finitely generated. Let M be an $\mathfrak{a}$-complete R-module and X any other R-module. Then $\mathrm{Hom}_R(X, M)$ is $\mathfrak{a}$-complete and for any natural number i the following holds: $\mathrm{Ext}_R^i(X, M)$ is $\mathfrak{a}$-quasi-complete and*

$$\mathrm{Ext}_R^i(X, M) \neq 0 \text{ if and only if } \mathrm{Ext}_R^i(X, M) \neq \mathfrak{a}\,\mathrm{Ext}_R^i(X, M).$$

If in addition the ring R is Noetherian and X is finitely generated, then for any i $\mathrm{Tor}_i^R(X, M)$ is $\mathfrak{a}$-quasi-complete and

$$\mathrm{Tor}_i^R(X, M) \neq 0 \text{ if and only if } \mathrm{Tor}_i^R(X, M) \neq \mathfrak{a}\,\mathrm{Tor}_i^R(X, M).$$

Proof Let $L_\bullet$ be a free resolution of X. Then the complex $\mathrm{Hom}_R(L_\bullet, M)$ is a complex of $\mathfrak{a}$-complete modules (products of $\mathfrak{a}$-complete modules are $\mathfrak{a}$-complete (see 2.2.7)). In the case when R is Noetherian and X is finitely generated we take a free resolution $L_\bullet$ of X in which the modules L_i are finitely generated. Then the complex $L_\bullet \otimes_R M$ is also a complex of $\mathfrak{a}$-complete modules (finite direct sums of $\mathfrak{a}$-complete modules are $\mathfrak{a}$-complete). The result then follows by 2.2.11. □

The above proposition and its corollary can sometimes be used as a substitute for Nakayama's lemma.

Observations 2.2.13 Assume that the ideal $\mathfrak{a}$ of the commutative ring R is finitely generated. Then

(a) the $\mathfrak{a}$-torsion functor and the $\mathfrak{a}$-completion functor are related through the general Matlis duality: for any R-module M we have (see 1.4.1 and 1.3.3) that

$$(\Gamma_{\mathfrak{a}}(M))^{\vee} \cong \Lambda^{\mathfrak{a}}(M^{\vee}).$$

(b) It follows that the general Matlis dual of an $\mathfrak{a}$-torsion module is $\mathfrak{a}$-complete.

2.3 Noetherian Rings and Matlis Duality

2.3.1 *Noetherian rings and finitely generated modules.* Concerning adic completions this situation is well understood. We fix a proper ideal $\mathfrak{a}$ of a Noetherian ring R and simply write $\hat{(\cdot)}$ for the $\mathfrak{a}$-adic completion functor.

(a) By Krull's intersection theorem (see e.g. [3, Theorem 10.17]) it follows that R is $\mathfrak{a}$-separated if and only if any element of $1+\mathfrak{a}$ is a non-zero divisor. When $\mathfrak{a}$ is contained in the Jacobson radical of R any finitely generated module M is $\mathfrak{a}$-separated and its submodules are $\mathfrak{a}$-closed in M.

(b) If N is a submodule of the finitely generated module M the $\mathfrak{a}$-adic topology of N is the one induced by the $\mathfrak{a}$-adic topology of M. This is a consequence of the Artin–Rees lemma. It follows that the functor $\hat{(\cdot)}$ is exact on the category of finitely generated modules.

(c) When M is finitely generated the natural map $\hat{R}\otimes_R M \to \hat{M}$ is bijective. More precisely the map $\tau_M^{\mathfrak{a}}$ is the natural map $M \to \hat{R}\otimes_R M$ (because it is so when M is a finitely generated free R-module). It follows that the change of rings $R \to \hat{R}$ is flat. It also yields that finitely generated modules are $\mathfrak{a}$-complete when R is $\mathfrak{a}$-complete.

(d) Let $\mathfrak{c}$ be any proper ideal of R. We take the exact sequence $0 \to \mathfrak{c} \to R \to R/\mathfrak{c} \to 0$ and tensor it by $\hat{R}$, which amounts to taking its $\mathfrak{a}$-adic completion. Then we observe that

$$\mathfrak{c}\hat{R} = \mathfrak{c}\otimes_R \hat{R} = \hat{\mathfrak{c}}, \qquad \widehat{R/\mathfrak{c}} \cong \hat{R}/\mathfrak{c}\hat{R}, \qquad R/\mathfrak{c}\otimes_R \tau_R^{\mathfrak{a}} = \tau_{R/\mathfrak{c}}^{\mathfrak{a}}.$$

If $\mathfrak{c}=\mathfrak{b}$ is $\mathfrak{a}$-open, i.e. if $\mathfrak{b}$ contains some power of $\mathfrak{a}$, then the $\mathfrak{a}$-adic topology on $R/\mathfrak{b}$ is the discrete one and $R/\mathfrak{b} \cong \widehat{R/\mathfrak{b}}$. We recover the fact that the natural map $R/\mathfrak{b}\otimes_R \tau_R^{\mathfrak{a}} : R/\mathfrak{b} \to \hat{R}/\mathfrak{b}\hat{R}$ is bijective, and that $\hat{R}$ is $\mathfrak{a}$-complete.

(e) Assume now that $\mathfrak{c}$ is a proper ideal and that $\mathfrak{a}$ is contained in the Jacobson radical of R. Then $R/(\mathfrak{c}+\mathfrak{a}) \neq 0$ and $0 \neq \widehat{R/\mathfrak{c}} \cong \hat{R}/\mathfrak{c}\hat{R}$ and therefore $\hat{R}$ is R-faithfully flat. Moreover, the map $R/\mathfrak{c}\otimes_R \tau_R^{\mathfrak{a}} = \tau_{R/\mathfrak{c}}^{\mathfrak{a}}$ is injective because $R/\mathfrak{c}$ is $\mathfrak{a}$-separated. It follows that the morphism $\tau_R^{\mathfrak{a}} : R \to \hat{R}^{\mathfrak{a}}$ is pure injective and that the R-module $\operatorname{Coker}(\tau_M^{\mathfrak{a}}) = \hat{R}^{\mathfrak{a}}/R$ is R-flat.

(f) Last but not least, recall that the ring $\hat{R}$ is Noetherian.

All the above is well known and more details can be found in [3], [13] or [58].

2.3.2 *Matlis duality over a Noetherian local ring.* Let $(R, \mathfrak{m}, \Bbbk)$ be a Noetherian local ring with its maximal ideal $\mathfrak{m}$ and residue field $\Bbbk = R/\mathfrak{m}$. Let $E = E_R(\Bbbk)$ be the injective hull of $\Bbbk$. As before write $(\cdot)^\vee = \operatorname{Hom}_R(\cdot, E)$ for the Matlis duality functor. We simply write $\widehat{(\cdot)}$ for the $\mathfrak{m}$-adic completion functor.

(a) Note first that $\operatorname{Ass}_R(E) = \{\mathfrak{m}\}$. Hence the R-module E is $\mathfrak{m}$-torsion and has the structure of an $\hat{R}$-module (see 2.1.15). Moreover, E is also the injective hull of $\Bbbk$ as an $\hat{R}$-module: $E = E_{\hat{R}}(\Bbbk)$ (see [55]). (We summarize the argument. Put $E_1 = E_{\hat{R}}(\Bbbk)$. There is an injection $i : E \hookrightarrow E_1$ of $\hat{R}$-modules because the injection $\Bbbk \hookrightarrow E$ is also a homomorphism of $\hat{R}$-modules (see 2.1.15) and this injection is R-split because E is R-injective. But E_1 is also $\mathfrak{m}$-torsion and $\operatorname{Hom}_R(E_1, E) = \operatorname{Hom}_{\hat{R}}(E_1, E)$ in view of 2.1.15 again. Hence the injection i is also $\hat{R}$-split and $E = E_1$.)

(b) Let M be any R-module. We note that $\operatorname{Ann}_R(M) = \operatorname{Ann}_R(M^\vee)$ by the faithful exactness of the functor $(\cdot)^\vee$. Note also that $M^\vee := \operatorname{Hom}_R(M, E)$ has the structure of an $\hat{R}$-module for any R-module M, and that $M^\vee \cong \operatorname{Hom}_{\hat{R}}(\hat{R} \otimes_R M, E)$.

Now let F be a flat R-module. It is known that $F^\vee = \operatorname{Hom}_R(F, E)$ is R-injective (see 1.4.1). As $\operatorname{Hom}_R(F, E) \cong \operatorname{Hom}_{\hat{R}}(\hat{R} \otimes_R F, E)$ it follows that $F^\vee = \operatorname{Hom}_R(F, E)$ is also $\hat{R}$-injective.

(c) It is easy to see that : $\Bbbk \cong \Bbbk^\vee \cong \Bbbk^{\vee\vee}$. It follows that an R-module M has finite length if and only if $M^\vee$ has. In that case, the natural injection $M \hookrightarrow M^{\vee\vee}$ is an isomorphism.

(d) The endomorphism ring $\operatorname{Hom}_R(E, E) = R^{\vee\vee}$ is isomorphic to $\hat{R}$ and the natural homomorphism $R \to \operatorname{Hom}_R(E, E)$ coincides with $\tau_R^{\mathfrak{a}} : R \to \hat{R}$. (To see this note that $E \cong \varinjlim \operatorname{Hom}_R(R/\mathfrak{m}^t, E)$ because E is $\mathfrak{m}$-torsion. Hence $\operatorname{Hom}_R(E, E) \cong \varprojlim (R/\mathfrak{m}^t)^{\vee\vee} = \varprojlim R/\mathfrak{m}^t = \hat{R}$. The rest is now obvious.)

It follows that $M^{\vee\vee} \cong \hat{R} \otimes_R M \cong \hat{M}$ when the R-module M is finitely generated.

(e) Note that $E = E_R(\Bbbk) = E_{\hat{R}}(\Bbbk)$ is Artinian as an R-module and as an $\hat{R}$-module. To see this, note first that every R-submodule of E is also an $\hat{R}$-submodule (see 2.1.15 again). Hence we may and do assume that $R = \hat{R}$ is $\mathfrak{m}$-complete. In this case, E is Artinian because $E^\vee = R$ is Noetherian (see 1.4.9).

(f) Recall that an R-module N is Artinian if and only if there is an embedding $N \hookrightarrow E^n$ for some natural number n (see e.g. [82, 3.4.11]). (To obtain such an embedding for an Artinian R-module N take in $\cup_n \operatorname{Hom}_R(N, E^n)$ a homomorphism with the property that the kernel is minimal, then observe that such a homomorphism is injective.)

If N is Artinian it follows that $\operatorname{Ass}_R(N) = \{\mathfrak{m}\}$. Therefore N is $\mathfrak{m}$-torsion and N has the structure of an $\hat{R}$-module. The above embedding is also an $\hat{R}$-homomorphism and $N^\vee := \operatorname{Hom}_R(N, E) \cong \operatorname{Hom}_{\hat{R}}(N, E)$ (see 2.1.15 or 2.2.6). Moreover, $N^\vee$ is finitely generated as an $\hat{R}$-module and the natural embedding

$$N \to \operatorname{Hom}_{\hat{R}}(\operatorname{Hom}_{\hat{R}}(N, E), E)$$

is an isomorphism.

(g) The previous result together with 1.4.9 also shows that an R-module M is finitely generated if and only if $M^\vee$ is Artinian.

(h) When the Noetherian local ring R is complete in its $\mathfrak{m}$-adic topology the situation is quite satisfactory.

(1) An R-module M is Noetherian if and only if $M^\vee$ is Artinian.
(2) An R-module M is Artinian if and only if $M^\vee$ is Noetherian.
(3) If the R-module M is either Noetherian or Artinian, then the natural injection $M \hookrightarrow M^{\vee\vee}$ is an isomorphism.

(i) Finally recall that R is Artinian if and only if E is finitely generated. In that case E has finite length.

(If R is Artinian, then R has finite length and so has $E = R^\vee$. In particular, E is finitely generated. Conversely, if $E = R^\vee$ is finitely generated, then R is Artinian (see 1.4.9.)

See Matlis' original paper [55] for more information and details around this, or [82, Chap. 2] for a summary.

For the use of complexes note also the following.

2.3.3 Let $(R, \mathfrak{m}, \Bbbk)$ be a Noetherian local ring and X an R-complex with finitely generated cohomology modules. Then the natural map $X \to X^{\vee\vee}$ induces a quasi-isomorphism

$$X \otimes_R \hat{R}^{\mathfrak{m}} \xrightarrow{\sim} X^{\vee\vee}$$

where $(\cdot)^\vee$ denotes, as before, the Matlis duality functor $\mathrm{Hom}_R(\cdot, E_R(\Bbbk))$.

This is because it holds when $X = M$ is a finitely generated R-module (see 2.3.2 (d)) and because the functor $(\cdot)^{\vee\vee}$ is exact.

For the sake of completeness let us also mention the following statement.

Proposition 2.3.4 (see [90], [25], [10] or [82, 3.4.13]) *Let $(R, \mathfrak{m}, \Bbbk)$ be a complete Noetherian local ring. Let M denote an R-module. The natural homomorphism*

$$M \to M^{\vee\vee}$$

is an isomorphism if and only if there is a finitely generated submodule $N \subseteq M$ such that M/N is Artinian. (Then one says that M is "mini-max".)

2.4 Completions of Flat Modules over a Noetherian Ring

The $\mathfrak{a}$-adic completion of a free module over a Noetherian ring is well documented. Note that an explicit description of it was given in 2.1.9, and another one in 2.2.8.

Among other things Gruson and Raynaud already stated that over a complete Noetherian local ring (complete in its maximal-adic topology) the maximal-adic completion of a free and even of a flat module is flat (see [67, Part II, 2.4.2, 2.4.3]). However their proofs were only sketched. Later Bartijn and Strooker provided a detailed proof of the fact that for any ideal $\mathfrak{a}$ of a Noetherian ring the $\mathfrak{a}$-adic completion of a free module is flat (see [82, Chap. 2]).

More generally, in his thesis J. Bartijn proved that the $\mathfrak{a}$-adic completion of any flat module over a Noetherian ring is flat. Unfortunately his thesis has never been published, but this result was later recovered (see for example [88]). We shall provide here another proof of this rather important result (different from Bartijn's original approach) and at the same time a stronger statement. This in turn will also provide another proof of the more precise result of Bartijn and Strooker about completions of free modules. This requires some preliminaries.

First there is a variant of the Artin–Rees Lemma.

Lemma 2.4.1 *Let $\mathfrak{a}$ be an ideal in a Noetherian ring R and N a finitely generated R-module. Then the inverse system $\{\operatorname{Tor}_k^R(N, R/\mathfrak{a}^t) \mid t \in \mathbb{N}\}$ defined by the surjections $R/\mathfrak{a}^{t+1} \to R/\mathfrak{a}^t$ is pro-zero for all $k \geq 1$.*

Proof Let $0 \to M \to L \to N \to 0$ be a short exact sequence where L is a finitely generated free R-module. Then $\operatorname{Tor}_1^R(N, R/\mathfrak{a}^t) \cong \mathfrak{a}^t L \cap M/\mathfrak{a}^t M$. By the Artin–Rees Lemma there exists an integer c such that $\mathfrak{a}^{t+c} L \cap M \subseteq \mathfrak{a}^t M$ for all $t \geq 1$. This implies that the inverse system $\{\operatorname{Tor}_1^R(N, R/\mathfrak{a}^t)\}$ is pro-zero and proves the statement for $k = 1$. Now the exact sequence induces isomorphisms $\operatorname{Tor}_{k+1}^R(N, R/\mathfrak{a}^t) \cong \operatorname{Tor}_k^R(M, R/\mathfrak{a}^t)$ for $k \geq 1$. Whence an induction completes the proof. $\square$

Lemma 2.4.2 *Let $\mathfrak{a}$ denote an ideal of a Noetherian ring R. Let $\{M_t\}$ denote an inverse system such that the maps $\rho_{t,t+1} : M_{t+1} \to M_t$ are surjective and M_t is $R/\mathfrak{a}^t$-flat for all $t \geq 1$. Let $L_\bullet$ denote a free resolution of a finitely generated R-module N by finitely generated free R-modules. Then $\varprojlim(L_\bullet \otimes M_t)$ is a left resolution of $\varprojlim(N \otimes M_t)$ and $\varprojlim(N \otimes M_t) \cong N \otimes (\varprojlim M_t)$.*

Proof By Lemma 1.2.8 there are short exact sequences

$$0 \to \varprojlim{}^1 \operatorname{Tor}_{k+1}(N, M_t) \to H_k(\varprojlim(L_\bullet \otimes M_t)) \to \varprojlim \operatorname{Tor}_k^R(N, M_t) \to 0$$

for all k. Since M_t is $R/\mathfrak{a}^t$-flat there is an isomorphism

$$\operatorname{Tor}_k^R(N, M_t) \cong \operatorname{Tor}_k^R(N, R/\mathfrak{a}^t) \otimes_{R/\mathfrak{a}^t} M_t$$

for all k. In view of 2.4.1 this implies that the inverse system

$$\{\operatorname{Tor}_k^R(N, M_t) \mid t \in \mathbb{N}\} \text{ is pro-zero for all } k \geq 1.$$

That is, $\varprojlim{}^1 \operatorname{Tor}_k^R(N, M_t) = \varprojlim \operatorname{Tor}_k^R(N, M_t) = 0$ for all $k \geq 1$ and therefore $H_0(\varprojlim(L_\bullet \otimes M_t)) = \varprojlim(N \otimes M_t)$ and $H_k(\varprojlim(L_\bullet \otimes M_t)) = 0$ for all $k \geq 1$.

In order to finish the proof we have to show the last isomorphism. To this end we have the commutative diagram with exact rows

$$\begin{array}{ccccccc}
(L_1 \otimes \varprojlim M_t) & \longrightarrow & (L_0 \otimes \varprojlim M_t) & \longrightarrow & (N \otimes \varprojlim M_t) & \longrightarrow & 0 \\
\| & & \| & & \downarrow & & \\
\varprojlim(L_1 \otimes M_t) & \longrightarrow & \varprojlim(L_0 \otimes M_t) & \longrightarrow & \varprojlim(N \otimes M_t) & \longrightarrow & 0
\end{array}$$

where the first two vertical maps are isomorphisms since L_k, $k = 0, 1$, are finitely generated free R-modules. Whence the last map is an isomorphism too. □

Proposition 2.4.3 *Let R be a Noetherian ring. Let $\{M_t \mid t \in \mathbb{N}\}$ denote an inverse system such that the maps $\rho_{t,t+1} : M_{t+1} \to M_t$ are surjective and M_t is $R/\mathfrak{a}^t$-flat for all $t \geq 1$. Let $0 \to N' \to N \to N'' \to 0$ denote a short exact sequence of finitely generated R-modules. Then the sequence*

$$0 \to \varprojlim(N' \otimes M_t) \to \varprojlim(N \otimes M_t) \to \varprojlim(N'' \otimes M_t) \to 0$$

is exact.

Proof There is a short exact sequence of complexes $0 \to L'_\bullet \to L_\bullet \to L''_\bullet \to 0$, where $L'_\bullet, L_\bullet, L''_\bullet$ are free resolutions of N', N, N'' respectively by finitely generated free R-modules. By tensoring with M_t this induces a short exact sequence of inverse systems of complexes

$$0 \to L'_\bullet \otimes M_t \to L_\bullet \otimes M_t \to L''_\bullet \otimes M_t \to 0.$$

By passing to the inverse limit there is a short exact sequence

$$0 \to \varprojlim(L'_\bullet \otimes M_t) \to \varprojlim(L_\bullet \otimes M_t) \to \varprojlim(L''_\bullet \otimes M_t) \to 0.$$

To this end recall that the induced maps $L'_j \otimes M_{t+1} \to L'_j \otimes M_t$ are surjective for all $j \geq 0$. By virtue of Lemma 2.4.2 the claim now follows by the long exact cohomology sequence. □

We now recover Bartijn's result and a certain extension.

Theorem 2.4.4 *Let $\mathfrak{a}$ be an ideal of a Noetherian ring R and let F be an R-module. Then $\hat{F}^{\mathfrak{a}}$ is R-flat if and only $F/\mathfrak{a}^t F$ is $R/\mathfrak{a}^t$-flat for all $t \geq 1$. In particular, the $\mathfrak{a}$-adic completion of a flat R-module is always R-flat and also $\hat{R}^{\mathfrak{a}}$-flat.*

Proof Let $0 \to N' \to N \to N'' \to 0$ be a short exact sequence of finitely generated R-modules and assume that the module F is such that $F/\mathfrak{a}^t F$ is $R/\mathfrak{a}^t$-flat for all $t \geq 1$. To prove that $\hat{F}^{\mathfrak{a}}$ is R-flat we have to show that the sequence

$$0 \to N' \otimes_R \hat{F}^{\mathfrak{a}} \to N \otimes_R \hat{F}^{\mathfrak{a}} \to N'' \otimes_R \hat{F}^{\mathfrak{a}} \to 0$$

is exact. But this follows by Proposition 2.4.3 and Lemma 2.4.2 by the use of $M_t = F/\mathfrak{a}^t F$.

The other assertions are clear since $F/\mathfrak{a}^t F \cong \hat{F}^{\mathfrak{a}}/\mathfrak{a}^t \hat{F}^{\mathfrak{a}}$ (see 2.2.2). □

The above considerations have some further consequences. For two ideals $\mathfrak{a}$ and $\mathfrak{c}$ of a Noetherian ring R we know that $\widehat{R/\mathfrak{c}}^{\mathfrak{a}} \cong \hat{R}^{\mathfrak{a}}/\mathfrak{c}\hat{R}^{\mathfrak{a}}$ (see 2.3.1 (d)). We now extend this result to a flat module F in place of R.

Proposition 2.4.5 *Let $\mathfrak{a}$ and $\mathfrak{c}$ be two ideals of a Noetherian ring R and let F be a flat R-module. Then*

$$\widehat{F/\mathfrak{c}F}^{\mathfrak{a}} \cong \hat{F}^{\mathfrak{a}}/\mathfrak{c}\hat{F}^{\mathfrak{a}}.$$

Proof We apply 2.4.2 to the finitely generated module $N = R/\mathfrak{c}$ and to the inverse system $\{M_t = F/\mathfrak{a}^t F\}$. We obtain

$$R/\mathfrak{c} \otimes_R \hat{F}^{\mathfrak{a}} \cong \varprojlim (R/\mathfrak{c} \otimes_R F/\mathfrak{a}^t F) = \widehat{F/\mathfrak{c}F}^{\mathfrak{a}},$$

as required. □

The above considerations may be applied to a composite of completion functors.

Proposition 2.4.6 *Let $\mathfrak{a}$ and $\mathfrak{c}$ be two ideals of a Noetherian ring R and let F be a flat R-module. Then $\Lambda^{\mathfrak{c}}(\Lambda^{\mathfrak{a}}(F)) = \Lambda^{(\mathfrak{c}+\mathfrak{a})}(F)$.*

Proof Put $F_{ij} = F/(\mathfrak{a}^i F + \mathfrak{c}^j F)$. With 2.4.5 we first have $\varprojlim_i \{F_{ij}\} = \widehat{F/\mathfrak{c}^j F}^{\mathfrak{a}} \cong \hat{F}^{\mathfrak{a}}/\mathfrak{c}^j \hat{F}^{\mathfrak{a}}$. Then we obtain

$$\Lambda^{(\mathfrak{c}+\mathfrak{a})}(F) = \varprojlim_t \{F_{tt}\} = \varprojlim_j (\varprojlim_i \{F_{ij}\}) = \varprojlim_j \hat{F}^{\mathfrak{a}}/\mathfrak{c}^j \hat{F}^{\mathfrak{a}} = \Lambda^{\mathfrak{c}}(\Lambda^{\mathfrak{a}}(F))$$

by 1.2.10. □

We also recover Bartijn and Strooker's result. First we recall their terminology.

Definition 2.4.7 (see [6], [82]) A free submodule L of an R-module M is called an $\mathfrak{a}$-basic submodule of M if it is pure and $\mathfrak{a}$-dense in M. Note that if the module M has an $\mathfrak{a}$-basic submodule L, then $\hat{M}^{\mathfrak{a}} \cong \hat{L}^{\mathfrak{a}}$.

Theorem 2.4.8 (see [6] or [82, 2.1.9, 2.2.3]) *Let $\mathfrak{a}$ be an ideal of a Noetherian ring R. For simplicity write $\widehat{(\cdot)}$ for the $\mathfrak{a}$-adic completion functor. Let $L = R^{(I)}$ be a free R-module defined as the direct sum of copies of R over an arbitrary index set I. Then*

(a) $\hat{L}$ is the $\mathfrak{a}\hat{R}$-closure of $\hat{R}^{(I)}$ in the complete flat module $\hat{R}^I$, the direct product of $\hat{R}$ over the index set I.
(b) The free $\hat{R}$-module $\hat{R}^{(I)}$ is pure in $\hat{R}^I$ hence in $\hat{L}$.
(c) $\hat{R}^{(I)}$ is an $\mathfrak{a}\hat{R}$-basic submodule of $\hat{L}$.
(d) Moreover, $\hat{L}$ is a pure R-submodule of $\hat{R}^I$.
(e) $\hat{L}$ is R-flat and $\hat{R}$-flat. The $\mathfrak{a}$-adic topology of $\hat{L}$ is induced by the $\mathfrak{a}$-adic topology of $\hat{R}^I$.

Proof The functor $N \otimes_R \cdot$ commutes with direct sums and also with products when N is finitely presented (see 1.4.4). This implies that a product of flat modules over a Noetherian ring is flat, in particular that $\hat{R}^I$ is R-flat. Now the first assertion follows by 2.2.5 and 2.2.8. The second assertion is rather obvious in view of 2.1.7 and the third one follows from the second and 2.2.2.

Because $\hat{R}^I$ is R-flat, for the proof of the purity of $\hat{L}$ in $\hat{R}^I$ one has to prove that the injection $\hat{L} \hookrightarrow \hat{R}^I$ induces an injection $\hat{L}/\mathfrak{c}\hat{L} \hookrightarrow \hat{R}^I/\mathfrak{c}\hat{R}^I$ for any ideal $\mathfrak{c}$ of R. But this follows by 2.4.5 and the first assertion applied to the ring $R/\mathfrak{c}$. Namely, $\hat{L}/\mathfrak{c}\hat{L} \cong \widehat{L/\mathfrak{c}L}$ is the $\mathfrak{a}R/\mathfrak{c}$-closure of $\widehat{R/\mathfrak{c}}^{(I)}$ in $\widehat{R/\mathfrak{c}}^{I}$ and $\widehat{R/\mathfrak{c}}^{I} \cong (\hat{R}/\mathfrak{c}\hat{R})^I \cong \hat{R}^I/\mathfrak{c}\hat{R}^I$.

The last assertion is a direct consequence of the one in (d) in view of the recalls in 2.1.7. □

Remark 2.4.9 In the situation of 2.4.8 (with the same notations) assume moreover that the ideal $\mathfrak{a}$ is contained in the Jacobson radical of R. Then $L = R^{(I)}$ is pure in $\hat{R}^{(I)}$ (by 2.3.1 (e)), hence also in $\hat{L}$, and the module $\operatorname{Coker}(\tau_L^{\mathfrak{a}}) = \hat{L}/L$ is R-flat.

For the sake of completeness we recall the following.

Proposition 2.4.10 (see [78, Propositions 2.1 and 3.2]) *Let R denote a Noetherian ring. Then the flat dimension and the injective dimension of an $\mathfrak{a}$-complete module M are given by*

$$\begin{aligned} \operatorname{fd}_R(M) &= \sup\{i \in \mathbb{N} \mid \operatorname{Tor}_i^R(R/\mathfrak{p}, M) \neq 0 \textit{ for some prime ideal } \mathfrak{p} \supset \mathfrak{a}\}, \\ \operatorname{id}_R(M) &= \sup\{i \in \mathbb{N} \mid \operatorname{Ext}_R^i(R/\mathfrak{p}, M) \neq 0 \textit{ for some prime ideal } \mathfrak{p} \supset \mathfrak{a}\}. \end{aligned}$$

Proof Recall that a finitely generated module N over a Noetherian ring has a composition series, i.e. a finite sequence of submodules

$$0 \subset N_n \subset \cdots \subset N_i \subset \cdots \subset N_0 = N$$

whose successive quotients are of the form $R/\mathfrak{q}$ with $\mathfrak{q} \in \operatorname{Supp}_R N$ (see [13, Chapitre 4, §1, $n°4$ Theorems 1 and 2].

First we look at the flat dimension. We only need to check $\operatorname{Tor}_i^R(N, M)$ where N is finitely generated (see 1.4.3). If $\operatorname{Tor}_i^R(N, M) \neq 0$ and N finitely generated, it follows that $\operatorname{Tor}_i^R(R/\mathfrak{q}, M) \neq 0$ for some $\mathfrak{q} \in \operatorname{Supp}_R N$. If $\mathfrak{a} \not\subseteq \mathfrak{q}$, choose $x \in \mathfrak{a} \setminus \mathfrak{q}$ and look at the long exact sequence of the Tor associated to the exact sequence $0 \to R/\mathfrak{q} \xrightarrow{x} R/\mathfrak{q} \to R/(\mathfrak{q} + xR) \to 0$. When M is $\mathfrak{a}$-complete we have (see 2.2.12) that multiplication by x is not surjective on $\operatorname{Tor}_i^R(R/\mathfrak{q}, M)$. It follows that $\operatorname{Tor}_i^R(R/(\mathfrak{q} + xR), M) \neq 0$. Then we obtain a prime ideal $\mathfrak{q}_1 \supset (\mathfrak{q} + xR)$ with $\operatorname{Tor}_i^R(R/\mathfrak{q}_1, M) \neq 0$. If $\mathfrak{a} \not\subseteq \mathfrak{q}_1$ we continue the process until we reach an ideal $\mathfrak{p} \supset \mathfrak{a}$ with $\operatorname{Tor}_i^R(R/\mathfrak{p}, M) \neq 0$.

The proof of the statement about the injective dimension is similar. □

Remark 2.4.11 The above proposition 2.4.10 together with the previously known 2.4.8 also gives a second proof of Bartijn's result on flatness. Let $L_\bullet$ be a free resolution of the R-flat module F. It is easy to see that $\hat{L}^{\mathfrak{a}}_\bullet$ is a resolution of $\hat{F}^{\mathfrak{a}}$, and even a flat one in view of 2.4.8. For a prime ideal $\mathfrak{p}$ containing $\mathfrak{a}$ we have that $R/\mathfrak{p} \otimes_R \tau^{\mathfrak{a}}_{L_\bullet} : L_\bullet/\mathfrak{p}L_\bullet \to \hat{L}^{\mathfrak{a}}_\bullet/\mathfrak{p}\hat{L}^{\mathfrak{a}}_\bullet$ is an isomorphism of complexes (see 2.2.2). Hence $\operatorname{Tor}_i^R(R/\mathfrak{p}, \hat{F}^{\mathfrak{a}}) \cong \operatorname{Tor}_i^R(R/\mathfrak{p}, F) = 0$ for all $i \geq 0$ and $\hat{F}^{\mathfrak{a}}$ is flat in view of 2.4.10.

Here is another consequence of Lemma 2.4.1. In the following we denote by $M^{(I)}$ the direct sum of copies of M over an arbitrary index set I.

Proposition 2.4.12 *Let $\mathfrak{a}$ be an ideal of a Noetherian ring R, N a finitely generated R-module and $L_\bullet$ a resolution of N by finitely generated free R-modules. Let I be an arbitrary index set. Then $\Lambda^{\mathfrak{a}}(L_\bullet^{(I)})$ is a flat resolution of $\Lambda^{\mathfrak{a}}(N^{(I)})$.*

Proof We already know that the complex $\Lambda^{\mathfrak{a}}(L_\bullet^{(I)})$ is a complex of flat R-modules. Whence it is enough to prove the following:

for any short exact sequence $0 \to M_1 \to L \to M \to 0$, where L is finitely generated and free, the induced sequence

$$0 \to \Lambda^{\mathfrak{a}}(M_1^{(I)}) \to \Lambda^{\mathfrak{a}}(L^{(I)}) \to \Lambda^{\mathfrak{a}}(M^{(I)}) \to 0$$

is exact.

To this end we have two inverse systems of short exact sequences

$$0 \to \operatorname{Tor}_1^R(R/\mathfrak{a}^t, M^{(I)}) \to M_1^{(I)}/\mathfrak{a}^t M_1^{(I)} \to M_1^{(I)}/\mathfrak{a}^t L^{(I)} \cap M_1^{(I)} \to 0,$$
$$0 \to M_1^{(I)}/\mathfrak{a}^t L^{(I)} \cap M_1^{(I)} \to L^{(I)}/\mathfrak{a}^t L^{(I)} \to M^{(I)}/\mathfrak{a}^t M^{(I)} \to 0$$

of which we take the inverse limit. As $\operatorname{Tor}^R(R/\mathfrak{a}^t, \cdot)$ commutes with direct sums we obtain (see Lemma 2.4.1) that the inverse system $\{\operatorname{Tor}_1^R(R/\mathfrak{a}^t, M^{(I)})\}$ is pro-zero and therefore

$$\varprojlim \operatorname{Tor}_1^R(R/\mathfrak{a}^t, M^{(I)}) = 0 = \varprojlim{}^1 \operatorname{Tor}_1^R(R/\mathfrak{a}^t, M^{(I)}).$$

The other inverse systems are surjective and the result follows by 1.2.3 and the six-term long exact sequences (see 1.2.2). □

We now obtain the following, related to a statement of Raynaud and Gruson (see [67, Part II, 2.4.2]).

Corollary 2.4.13 *Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be a short exact sequence of finitely generated modules over a Noetherian ring R. Then the sequence*

$$0 \to \Lambda^{\mathfrak{a}}(M_1^{(I)}) \to \Lambda^{\mathfrak{a}}(M_2^{(I)}) \to \Lambda^{\mathfrak{a}}(M_3^{(I)}) \to 0$$

is exact.

We end this section with the example announced in 2.1.1. It was first given in [80, Example 2.5] to show that the class of $\mathfrak{a}$-complete modules is not closed under extension. It also shows several other strange things which can happen when dealing with non-finitely generated modules.

Example 2.4.14 An $\mathfrak{a}$-quasi-complete module X over a Noetherian ring such that $0 \neq \bigcap_t \mathfrak{a}^t X$ is $\mathfrak{a}$-complete. Let $(R, \mathfrak{m})$ denote a Noetherian local ring that is $\mathfrak{m}$-complete and of positive depth and take $\mathfrak{a} = \mathfrak{m}$. Let M be the $\mathfrak{m}$-completion of the free module $R^{(\mathbb{N})}$ as described in 2.1.9 and write $P = R^{\mathbb{N}}$. Thus M is the $\mathfrak{m}$-closure of $R^{(\mathbb{N})}$ in P (see 2.2.8) and P/M is $\mathfrak{m}$-complete.

Now let y be a regular element of R and let $f : P \to P$ be defined by $(f(w))_i = y^i w_i$ for all $i \in \mathbb{N}$. Clearly $f(P) \subset M$, and we shall see that the module $X = M/f(M)$ is the required example.

As the map f is injective but not surjective we have strict inclusions $f(M) \subsetneq f(P) \subsetneq M$. We observe that $f(P)$ is contained in the $\mathfrak{m}$-closure of $f(M)$ in M: let $w \in P$, then

$$f(w) = (w_0, \ldots, y^t w_n, 0, \ldots) + y^{t+1}(0, \ldots, 0, w_{t+1}, yw_{t+2}, \ldots)$$

and $f(w) \in f(M) + \mathfrak{m}^{t+1}M$ for all $t \geq 1$. Hence $f(M)$ is not $\mathfrak{m}$-closed in M and the $\mathfrak{m}$-quasi-complete module $X = M/f(M)$ is not $\mathfrak{m}$-separated.

We have more. First $f(P)$ is $\mathfrak{m}$-closed in P because

$$P/f(P) = \prod_{i \in \mathbb{N}} R/y^i R$$

is $\mathfrak{m}$-complete as a product of $\mathfrak{m}$-complete modules. But M is a pure submodule of P (see 2.4.8). Hence the $\mathfrak{m}$-adic topology on M is the one induced by the $\mathfrak{m}$-adic topology of P. It follows that $f(P)$ is also $\mathfrak{m}$-closed in M. Hence $M/f(P)$ is $\mathfrak{m}$-complete and $f(P)$ is the $\mathfrak{m}$-closure of $f(M)$ in M. Now look at the exact sequence

$$0 \to f(P)/f(M) \to M/f(M) \to M/f(P) \to 0.$$

The above means that the submodule $f(P)/f(M)$ of the module $M/f(M) = X$ is the $\mathfrak{m}$-closure of the origin in $M/f(M) = X$, that is, $f(P)/f(M) = \bigcap_n \mathfrak{m}^n X$ and $\hat{X}^{\mathfrak{m}} = M/f(P)$. But note that $f(P)/f(M) \cong P/M$ is $\mathfrak{m}$-complete!

By looking at the $\mathfrak{m}$-completion of the exact sequence $0 \to M \xrightarrow{f} M \to M/f(M) \to 0$ we also see that the $\mathfrak{m}$-completion functor is neither right-exact nor left-exact.

Later this example will also serve for other purposes.

2.5 The Left-Derived Functors of Completion

The left-derived functors of the $\mathfrak{a}$-adic completion functor, also called the local homology functors, were first investigated by Matlis in [56], mainly in the case when the ideal $\mathfrak{a}$ is generated by a regular sequence, later in [38, 78, 79] and in a more general setting in [1, 75]. Here we denote them by $\Lambda_i^{\mathfrak{a}}$ and recall and generalize the first basic facts on the subject. Some Ext-vanishing results involving $\Lambda_0^{\mathfrak{a}}$ will be given in the next chapter. More information will be given in Part II.

Remarks 2.5.1 Let $\mathfrak{a}$ *be an arbitrary ideal of a commutative ring* R.

(a) Let M be an R-module with the free resolution $L_\bullet$. There is a commutative diagram with exact rows

$$\begin{tikzcd}
L_1 \arrow[r, "d_1"] \arrow[d, "\tau^{\mathfrak{a}}_{L_1}"] & L_0 \arrow[r, "\varepsilon"] \arrow[d, "\tau^{\mathfrak{a}}_{L_0}"] & M \arrow[r] \arrow[d, dotted, "\eta^{\mathfrak{a}}_M"] & 0 \\
\hat{L}_1^{\mathfrak{a}} \arrow[r, "\hat{d}_1^{\mathfrak{a}}"] & \hat{L}_0^{\mathfrak{a}} \arrow[r] \arrow[d, equal] & \Lambda_0^{\mathfrak{a}}(M) \arrow[r] \arrow[d, dotted, "\gamma^{\mathfrak{a}}_M"] & 0 \\
 & \hat{L}_0^{\mathfrak{a}} \arrow[r, "\hat{\varepsilon}^{\mathfrak{a}}"] & \hat{M}^{\mathfrak{a}} \arrow[r] & 0.
\end{tikzcd}$$

Indeed the top row is exact by assumption and the middle row is exact by definition. This yields the existence of $\eta_M^{\mathfrak{a}}$. We have the existence of $\gamma_M^{\mathfrak{a}}$ because $\hat{\varepsilon}^{\mathfrak{a}} \circ \hat{d}_1^{\mathfrak{a}} = 0$, and $\hat{\varepsilon}^{\mathfrak{a}}$ is surjective in view of 2.1.8. It follows that $\gamma_M^{\mathfrak{a}}$ is also surjective. Because of

$$\gamma_M^{\mathfrak{a}} \circ \eta_M^{\mathfrak{a}} \circ \varepsilon = \hat{\varepsilon}^{\mathfrak{a}} \circ \tau_{L_0}^{\mathfrak{a}} = \tau_M^{\mathfrak{a}} \circ \varepsilon$$

we get $\gamma_M^{\mathfrak{a}} \circ \eta_M^{\mathfrak{a}} = \tau_M^{\mathfrak{a}}$ and the commutative triangle

$$\begin{tikzcd}
 & \Lambda_0^{\mathfrak{a}}(M) \arrow[dr, "\gamma^{\mathfrak{a}}_M"] & \\
M \arrow[ur, "\eta^{\mathfrak{a}}_M"] \arrow[rr, "\tau^{\mathfrak{a}}_M"] & & \hat{M}^{\mathfrak{a}}.
\end{tikzcd}$$

(b) As in [78, 5.1] we note that

$$\Lambda_0^{\mathfrak{a}}(M) = 0 \quad \Leftrightarrow \quad M = \mathfrak{a}M \quad \Leftrightarrow \quad \hat{M}^{\mathfrak{a}} = 0,$$

as it is another consequence of 2.1.8 and 2.1.11.

(c) The functor $\Lambda_0^{\mathfrak{a}}$ is additive, hence commutes with finite direct sums. If $M = M_1 \oplus M_2$ and if $\eta_M^{\mathfrak{a}}$ is an isomorphism, then so are $\eta_{M_1}^{\mathfrak{a}}$ and $\eta_{M_2}^{\mathfrak{a}}$.

(d) If L is finitely generated and free, then $\hat{L}^{\mathfrak{a}} \cong \hat{R}^{\mathfrak{a}} \otimes_R L$. If the R-module M is finitely presented, that is, if there is an exact sequence $L_1 \to L_0 \to M \to 0$ where L_1 and L_0 are free finitely generated, it follows that $\Lambda_0^{\mathfrak{a}}(M) \cong \hat{R}^{\mathfrak{a}} \otimes_R M$.

The homomorphism $\eta_M^{\mathfrak{a}}$ leads to the definition of pseudo-completeness.

Definition 2.5.2 When the natural homomorphism $\eta_M^{\mathfrak{a}} : M \to \Lambda_0^{\mathfrak{a}}(M)$ is an isomorphism, we say that the R-module M is $\mathfrak{a}$-*pseudo-complete*. In that case we have $M = \mathfrak{a}M$ if and only if $M = 0$.

(Caution: our terminology here is slightly different from the one used by Greenlees and May in [38]. Modules we called $\mathfrak{a}$-pseudo-complete are called $\mathfrak{a}$-complete in [38]. Here we prefer to call $\mathfrak{a}$-complete a module which coincides with its $\mathfrak{a}$-adic completion.)

In view of the right-exactness of $\Lambda_0^{\mathfrak{a}}$ the following is rather obvious.

Lemma 2.5.3 *The class of* $\mathfrak{a}$*-pseudo-complete modules has the following property. Given an exact sequence* $M_1 \to M_2 \to M_3 \to 0$*, if* M_1 *and* M_2 *are* $\mathfrak{a}$*-pseudo-complete, then so is* M_3.

Remarks 2.5.4 Assume now that the ideal $\mathfrak{a}$ *of the commutative ring* R *is finitely generated.*

(a) The homomorphism $R/\mathfrak{b} \otimes_R \eta_M^{\mathfrak{a}}$ is bijective for any $\mathfrak{a}$-open ideal $\mathfrak{b}$ and any R-module M. To this end, tensor the commutative diagram in 2.5.1 by $R/\mathfrak{b}$ and remember 2.2.2. In particular, we have

$$\widehat{\Lambda_0^{\mathfrak{a}}(M)}^{\mathfrak{a}} \cong \hat{M}^{\mathfrak{a}} \text{ and more precisely } \gamma_M^{\mathfrak{a}} = \tau_{\Lambda_0^{\mathfrak{a}}(M)}^{\mathfrak{a}}.$$

(b) It follows that the surjective homomorphism $\gamma_M^{\mathfrak{a}} = \tau_{\Lambda_0^{\mathfrak{a}}(M)}^{\mathfrak{a}}$ is an isomorphism if and only if $\Lambda_0^{\mathfrak{a}}(M)$ is $\mathfrak{a}$-complete.

(c) Any $\mathfrak{a}$-pseudo-complete module N is also $\mathfrak{a}$-quasi-complete as a quotient of an $\mathfrak{a}$-complete module (recall 2.1.8). Note that the converse does not hold. For example, a module of the form $\mathrm{Coker}(\tau_M^{\mathfrak{a}})$ is $\mathfrak{a}$-quasi-complete and $\mathrm{Coker}(\tau_M^{\mathfrak{a}}) = \mathfrak{a}\,\mathrm{Coker}(\tau_M^{\mathfrak{a}})$ (see 2.2.2). So $\mathrm{Coker}(\tau_M^{\mathfrak{a}})$ is not $\mathfrak{a}$-pseudo-complete when it is not zero (see 2.5.2).

(d) The modules $\Lambda_i^{\mathfrak{a}}(M)$ are all $\mathfrak{a}$-quasi-complete and $\Lambda_i^{\mathfrak{a}}(M) \neq \mathfrak{a}\Lambda_i^{\mathfrak{a}}(M)$ when $\Lambda_i^{\mathfrak{a}}(M) \neq 0$. This follows by 2.2.11.

When the ideal $\mathfrak{a}$ is finitely generated we shall see that $\Lambda^{\mathfrak{a}}(M)$ and the $\Lambda_i^{\mathfrak{a}}(M)$'s are also $\mathfrak{a}$-pseudo-complete. This needs some preparation.

2.5.5 *A family of exact sequences.* Let us first investigate the functors $\Lambda_i^{\mathfrak{a}}$ in the general case, that is, in the case when $\mathfrak{a}$ is an arbitrary ideal of the commutative ring R, and let us describe the kernel of the natural surjective homomorphism $\gamma_M^{\mathfrak{a}} : \Lambda_0^{\mathfrak{a}}(M) \to \hat{M}^{\mathfrak{a}}$. To this end let $L_\bullet$ denote a free resolution of the R-module M and let $\mathfrak{a}_t$ be a descending sequence of ideals which form a base of open neighbourhoods of the origin for the $\mathfrak{a}$-adic topology of R (we may take $\mathfrak{a}_t = \mathfrak{a}^t$). By virtue of Lemma 1.2.8 there are natural short exact sequences

$$0 \to \varprojlim{}^1 \operatorname{Tor}_{i+1}^R(R/\mathfrak{a}_t, M) \to H_i(\varprojlim(R/\mathfrak{a}_t \otimes_R L_\bullet))$$
$$\to \varprojlim \operatorname{Tor}_i^R(R/\mathfrak{a}_t, M) \to 0$$

for all $i \geq 0$. We note that $\Lambda_i^{\mathfrak{a}}(M) = H_i(\varprojlim(R/\mathfrak{a}_t \otimes_R L_\bullet))$ by definition, so these exact sequences coincide with the ones obtained in [38, Proposition 1.1] in a slightly different way.

For $i = 0$ we have a commutative diagram with exact rows

$$\begin{tikzcd}
 & 0 \arrow[r] & M \arrow[r] \arrow[d, "\eta_M^{\mathfrak{a}}"'] & M \arrow[r] \arrow[d, "\tau_M^{\mathfrak{a}}"] & 0 \\
0 \arrow[r] & \varprojlim{}^1 \operatorname{Tor}_1^R(R/\mathfrak{a}_t, M) \arrow[r] & \Lambda_0^{\mathfrak{a}}(M) \arrow[r] & \hat{M}^{\mathfrak{a}} \arrow[r] & 0.
\end{tikzcd}$$

With a little reflection we observe that the surjection $\Lambda_0^{\mathfrak{a}}(M) \to \hat{M}^{\mathfrak{a}}$ on the bottom row is the map $\gamma_M^{\mathfrak{a}}$ obtained in 2.5.1. Whence

$$\operatorname{Ker}(\gamma_M^{\mathfrak{a}}) \cong \varprojlim{}^1 \operatorname{Tor}_1^R(R/\mathfrak{a}_t, M).$$

In general, $\operatorname{Ker}(\eta_M^{\mathfrak{a}}) \subseteq \operatorname{Ker}(\tau_M^{\mathfrak{a}})$. When the inverse system $\{\operatorname{Tor}_1^R(R/\mathfrak{a}_t, M)\}$ satisfies the Mittag-Leffler condition, so that $\varprojlim{}^1 \operatorname{Tor}_1^R(R/\mathfrak{a}_t, M) = 0$ (see 1.2.3), then $\operatorname{Ker}(\eta_M^{\mathfrak{a}}) = \operatorname{Ker}(\tau_M^{\mathfrak{a}})$.

Lemma 2.5.6 *Let $\mathfrak{a}$ be a finitely generated ideal of a commutative ring R. Let M denote an R-module. Then:*

(a) $\bigcap_t \mathfrak{a}^t \Lambda_0^{\mathfrak{a}}(M) = \operatorname{Ker}(\tau_{\Lambda_0^{\mathfrak{a}}(M)}^{\mathfrak{a}}) \cong \varprojlim{}^1 \operatorname{Tor}_1^R(R/\mathfrak{a}^t, M)$ *is $\mathfrak{a}$-quasi-complete.*
(b) *If* $\operatorname{Ker}(\tau_{\Lambda_0^{\mathfrak{a}}(M)}^{\mathfrak{a}}) \neq 0$, *then* $\operatorname{Ker}(\tau_{\Lambda_0^{\mathfrak{a}}(M)}^{\mathfrak{a}}) \neq \mathfrak{a}\operatorname{Ker}(\tau_{\Lambda_0^{\mathfrak{a}}(M)}^{\mathfrak{a}})$.

Proof By (2.5.4 (a)) we have $\gamma_M^{\mathfrak{a}} = \tau_{\Lambda_0^{\mathfrak{a}}(M)}^{\mathfrak{a}}$ and therefore

$$\bigcap_t \mathfrak{a}^t \Lambda_0^{\mathfrak{a}}(M) = \operatorname{Ker}(\tau_{\Lambda_0^{\mathfrak{a}}(M)}^{\mathfrak{a}}) \cong \varprojlim{}^1 \operatorname{Tor}_1^R(R/\mathfrak{a}^t, M),$$

(see 2.5.5). But $\varprojlim{}^1 \operatorname{Tor}_1^R(R/\mathfrak{a}^t, M) = \operatorname{Coker}(\psi)$, where

$$\psi : \prod_{t \geq 1} \operatorname{Tor}_1^R(R/\mathfrak{a}^t, M \to \prod_{t \geq 1} \operatorname{Tor}_1^R(R/\mathfrak{a}^t, M))$$

is the map associated to the inverse system $\{\operatorname{Tor}_1^R(R/\mathfrak{a}^t, M)\}_{t \in \mathbb{N}}$ (see 1.2.2). The module $\prod_{t \geq 1} \operatorname{Tor}_1^R(R/\mathfrak{a}^t, M)$ is $\mathfrak{a}$-complete as a product of $\mathfrak{a}$-complete modules (see 2.2.7) and (a) follows (see 2.1.8).

Then (b) follows by 2.2.11. □

The above investigations allow a large generalization of some known facts. In the following proposition the first and the third assertion were known when $\mathfrak{a}$ is an ideal of a Noetherian ring (see the more precise statements 2.5.15 and 2.5.16 below), more generally, when the finitely generated ideal $\mathfrak{a}$ satisfies some extra conditions (see [38, Theorem 4.1]). (The whole statement in [38, Theorem 4.1] will be further generalized in Part II.) Note also that in the following we have the $\Lambda_0^{\mathfrak{a}}$ counterpart of Theorem 2.2.2.

Proposition 2.5.7 *Let $\mathfrak{a}$ be a finitely generated ideal of a commutative ring R. Then:*

(a) Any $\mathfrak{a}$-complete R-module C is $\mathfrak{a}$-pseudo-complete, that is, the natural homomorphism $\eta_C^{\mathfrak{a}} : C \to \Lambda_0^{\mathfrak{a}}(C)$ is an isomorphism.
(b) More generally, for any R-complex X of $\mathfrak{a}$-complete modules, the homology modules $H_i(X)$ are $\mathfrak{a}$-pseudo-complete.
(c) In particular, for all R-modules M all the local homology modules $\Lambda_i^{\mathfrak{a}}(M)$ are $\mathfrak{a}$-pseudo-complete and the functor $\Lambda_0^{\mathfrak{a}}(\cdot)$ is idempotent.

Proof (a): We look at the commutative triangle of 2.5.1. When C is $\mathfrak{a}$-complete the homomorphism $\tau_C^{\mathfrak{a}}$ is bijective. Therefore the homomorphism $\gamma_C^{\mathfrak{a}} = \tau_{\Lambda_0^{\mathfrak{a}}(C)}^{\mathfrak{a}}$ is split surjective and $\Lambda_0^{\mathfrak{a}}(C) \cong C \oplus \operatorname{Ker}(\tau_{\Lambda_0^{\mathfrak{a}}(C)}^{\mathfrak{a}})$. But $\widehat{\Lambda_0^{\mathfrak{a}}(C)}^{\mathfrak{a}} \cong \hat{C}^{\mathfrak{a}}$ (see (2.5.4 (a)), and the $\mathfrak{a}$-adic completion commutes with finite direct sums. It follows that the $\mathfrak{a}$-completion of $\operatorname{Ker}(\tau_{\Lambda_0^{\mathfrak{a}}(C)}^{\mathfrak{a}})$ vanishes so that

$$\operatorname{Ker}(\tau_{\Lambda_0^{\mathfrak{a}}(C)}^{\mathfrak{a}}) = \mathfrak{a}\operatorname{Ker}(\tau_{\Lambda_0^{\mathfrak{a}}(C)}^{\mathfrak{a}}),$$

(see 2.5.1 (b)). By 2.5.6 we now obtain $\operatorname{Ker}(\tau_{\Lambda_0^{\mathfrak{a}}(C)}^{\mathfrak{a}}) = 0$. Thus $\tau_{\Lambda_0^{\mathfrak{a}}(C)}^{\mathfrak{a}}$ is an isomorphism as well as $\eta_C^{\mathfrak{a}}$.

(b): Write $B_i = \operatorname{Im}(d_{i+1}^X)$ and $Z_i = \operatorname{Ker}(d_i^X)$. Note that Z_i is $\mathfrak{a}$-complete as an $\mathfrak{a}$-closed submodule of an $\mathfrak{a}$-complete module (see 2.2.11). We have short exact sequences

$$0 \to Z_i \to X_i \to B_{i-1} \to 0 \quad \text{and} \quad 0 \to B_i \to Z_i \to H_i(X) \to 0$$

to which we apply the right-exact functor $\Lambda_0^{\mathfrak{a}}$. From the sequence on the left and (a) we first obtain that B_{i-1} is $\mathfrak{a}$-pseudo-complete (see 2.5.3) and this holds for all $i \in \mathbb{Z}$. From the sequence on the right we then obtain the statement. (c): This is a direct consequence of (b). □

Corollary 2.5.8 *Let $\mathfrak{a}$ be a finitely generated ideal of a commutative ring R. Let M denote an R-module. Then M is $\mathfrak{a}$-complete if and only if M is $\mathfrak{a}$-separated and $\mathfrak{a}$-pseudo-complete.*

Proof If the homomorphism $\eta_M^{\mathfrak{a}} : M \to \Lambda_0^{\mathfrak{a}}(M)$ is an isomorphism then M is $\mathfrak{a}$-quasi-complete (see 2.5.4 (c)). If, moreover, M is $\mathfrak{a}$-separated, it is also $\mathfrak{a}$-complete. The reverse implication follows by 2.5.7. □

For the sake of completeness let us also mention the following.

Corollary 2.5.9 *Let $\mathfrak{a}$ be a finitely generated ideal of a commutative ring R and M an R-module.*

(a) Assume M is $\mathfrak{a}$-pseudo-complete. Then any $\mathfrak{a}$-closed submodule N of M is also $\mathfrak{a}$-pseudo-complete.
(b) Let $\mathfrak{b} \supset \mathfrak{a}$ be another ideal. If M is $\mathfrak{b}$-pseudo-complete, then M is also $\mathfrak{a}$-pseudo-complete.

Proof (a): We consider the exact sequence $0 \to N \xrightarrow{i} M \to M/N \to 0$. Note first that M/N is $\mathfrak{a}$-separated because N is $\mathfrak{a}$-closed in M. As the $\mathfrak{a}$-pseudo-complete module M is also $\mathfrak{a}$-quasi-complete (see 2.5.4) we have that M/N is also $\mathfrak{a}$-quasi-complete (see 2.1.8). Hence M/N is $\mathfrak{a}$-complete. We now argue as in 2.2.11. By 2.5.7 we may put $\eta^{\mathfrak{a}}_{M/N} = \mathrm{id}_{M/N}$ and $\eta^{\mathfrak{a}}_M = \mathrm{id}_M$. We observe that $\Lambda^{\mathfrak{a}}_0(i)$ factors through i, so that N appears as a direct summand of the $\mathfrak{a}$-pseudo-complete module $\Lambda^{\mathfrak{a}}_0(N)$. It follows that N is $\mathfrak{a}$-pseudo-complete.

(b): We argue as in 2.2.9. The homomorphism $\eta^{\mathfrak{b}}_M$ factors through $\eta^{\mathfrak{a}}_M$ (because $\tau^{\mathfrak{b}}_L$ factors through $\tau^{\mathfrak{a}}_L$ for any free resolution of L of M). If $\eta^{\mathfrak{b}}_M$ is an isomorphism then $\eta^{\mathfrak{a}}_M$ is split-injective and M is $\mathfrak{a}$-pseudo-complete as a direct summand of the $\mathfrak{a}$-pseudo complete module $\Lambda^{\mathfrak{a}}_0(M)$. ☐

Example 2.5.10 An $\mathfrak{a}$-pseudo-complete module which is not $\mathfrak{a}$-separated. It is time to provide such an example. But we have already one. Look again at the module $X = M/f(M)$ of example 2.4.14. This X is not $\mathfrak{a}$-separated and is inserted in the short exact sequence

$$0 \to M \xrightarrow{f} M \to M/f(M) \to 0,$$

where M is $\mathfrak{a}$-complete hence also $\mathfrak{a}$-pseudo-complete. It follows that $X = M/f(M)$ is $\mathfrak{a}$-pseudo-complete.

Remark 2.5.11 Here is another consequence of the considerations in 2.5.5. Let $\mathfrak{a}$ be a finitely generated ideal of a coherent ring R and M an R-module. Then

$$(\Gamma_{\mathfrak{a}}(M))^{\vee} \cong \Lambda^{\mathfrak{a}}(M^{\vee}) \cong \Lambda^{\mathfrak{a}}_0(M^{\vee}).$$

The first isomorphism was already noticed in 2.2.13. For the second we note that $\mathrm{Ext}^1_R(R/\mathfrak{a}^t, M)^{\vee} \cong \mathrm{Tor}^R_1(R/\mathfrak{a}^t, M^{\vee})$ (see 1.4.1) and

$$\varprojlim{}^1 \mathrm{Tor}^R_1(R/\mathfrak{a}^t, M^{\vee}) = 0$$

by the last remark in 1.3.3. In view of this the claim follows from the short exact sequence in 2.5.5.

2.5.12 *The classes $\mathcal{C}_{\mathfrak{a}}$ and $\mathcal{C}^+_{\mathfrak{a}}$.* As in [78] we denote by $\mathcal{C}_{\mathfrak{a}}$ the class of R-modules M such that

$$\gamma_M^{\mathfrak{a}} : \Lambda_0^{\mathfrak{a}}(M) \to \hat{M}^{\mathfrak{a}} \text{ is an isomorphism and } \Lambda_i^{\mathfrak{a}}(M) = 0 \text{ for all } i > 0.$$

Let $\{\mathfrak{a}_t\}$ be a decreasing sequence of ideals which form a base of open neighbourhoods of the origin for the $\mathfrak{a}$-adic topology of R. It was already observed in [56, Corollary 4.5] that modules M such that $\operatorname{Tor}_i^R(R/\mathfrak{a}_t, M) = 0$ for all t and $i > 0$ belong to the class $\mathcal{C}_{\mathfrak{a}}$. (This is true because the completion of a short exact sequence $0 \to X \to Y \to M \to 0$ remains exact when $\operatorname{Tor}_1^R(R/\mathfrak{a}_t, M) = 0$ for all t.) In particular, flat modules belong to $\mathcal{C}_{\mathfrak{a}}$.

We also write $\mathcal{C}_{\mathfrak{a}}^+$ for the larger class of modules M such that $\Lambda_i^{\mathfrak{a}}(M) = 0$ for all $i > 0$. A standard homological argument shows that the $\Lambda_i^{\mathfrak{a}}(M)$ can be computed by applying the functor $\Lambda_0^{\mathfrak{a}}$ to a left resolution $Y_\bullet$ of M with modules Y_i in $\mathcal{C}_{\mathfrak{a}}^+$ (see 1.5.3), that is, $\Lambda_i^{\mathfrak{a}}(M) = H_i(\Lambda_0^{\mathfrak{a}}(Y_\bullet))$, or by applying the $\mathfrak{a}$-adic completion functor $\Lambda^{\mathfrak{a}}$ to a left resolution $X_\bullet$ of M with modules X_i in $\mathcal{C}_{\mathfrak{a}}$, in particular with flat modules, that is, $\Lambda_i^{\mathfrak{a}}(M) = H_i(\hat{X}_\bullet^{\mathfrak{a}})$.

This last remark, already observed by Matlis in [56], emphasizes the importance of the class $\mathcal{C}_{\mathfrak{a}}$ when studying the $\mathfrak{a}$-adic completion functor, as well as the rôle of flat modules.

Let us have a closer look at the class $\mathcal{C}_{\mathfrak{a}}$. The following lemma was already noticed in [78, 5.3, 5.4]. Although it was stated for a Noetherian ring there, it holds in greater generality.

Lemma 2.5.13 *Assume that an ideal $\mathfrak{a}$ of a commutative ring R is finitely generated.*

(a) Let $0 \to X \to Y \to Z \to 0$ be a short exact sequence of R-modules. If $Y, Z \in \mathcal{C}_{\mathfrak{a}}$, then also $X \in \mathcal{C}_{\mathfrak{a}}$ and the $\mathfrak{a}$-adic completion of the sequence is exact.

(b) Let $X_\bullet : \ldots \to X_1 \xrightarrow{d_1} X_0 \to 0$ be a complex of R-modules. Suppose that X_i and $H_i(X_\bullet)$ belong to $\mathcal{C}_{\mathfrak{a}}$ for all $i \geq 0$. Then $\widehat{H_i(X_\bullet)}^{\mathfrak{a}} = H_i(\hat{X_\bullet}^{\mathfrak{a}})$.

Proof (a): Applying the functor $\Lambda_0^{\mathfrak{a}}$ to the exact sequence we obtain $\Lambda_i^{\mathfrak{a}}(X) = 0$ for all $i \geq 1$ and a short exact sequence

$$0 \to \Lambda_0^{\mathfrak{a}}(X) \to \hat{Y}^{\mathfrak{a}} \to \hat{Z}^{\mathfrak{a}} \to 0.$$

Thus $\Lambda_0^{\mathfrak{a}}(X)$ is $\mathfrak{a}$-complete as a closed submodule of the $\mathfrak{a}$-complete module $\hat{Y}^{\mathfrak{a}}$ (see 2.2.11). Hence the homomorphism $\gamma_X^{\mathfrak{a}} = \tau^{\mathfrak{a}}_{\Lambda_0^{\mathfrak{a}}(X)}$ is an isomorphism, (see 2.5.4 (b)) and the claims follow.

(b): We put $B_i = \operatorname{Im}(d_{i+1}^X)$, $Z_i = \operatorname{Ker}(d_i^X)$, $Z_0 = X_0$ and split our data into short exact sequences

$$0 \to B_i \to Z_i \to H_i(X_\bullet) \to 0 \quad \text{and} \quad 0 \to Z_{i+1} \to X_{i+1} \to B_i \to 0.$$

By (a) and the assumptions we first have that $B_0 \in \mathcal{C}_{\mathfrak{a}}$, then we have that $Z_i, B_i \in \mathcal{C}_{\mathfrak{a}}$ for all $i \geq 0$ by induction on i. By (a) and the assumptions we again have that the above exact sequences remain exact under $\mathfrak{a}$-completion. The claim in (b) follows. □

We now give a first picture of the situation when the ring is Noetherian.

Remark 2.5.14 When the ring R is Noetherian any finitely generated module belongs to the class $\mathcal{C}_{\mathfrak{a}}$ introduced in 2.5.12. More generally any direct sum of copies of a finitely generated module also belongs to the class $\mathcal{C}_{\mathfrak{a}}$. This follows from 2.4.12 and has nice consequences concerning the local homology of modules over a Noetherian ring, already exploited in [79]. In particular, it provides the following change of rings theorem (see [79, 3.3]). Let $X_1, \dots X_k$ be indeterminates and let φ be the homomorphism

$$\varphi : R[X_1, \dots, X_k] \to R,\ X_i \mapsto x_i.$$

Write $\mathfrak{A}$ for the ideal of $R[X_1, \dots, X_k]$ generated by the X_i's and $\mathfrak{a}$ for the ideal of R generated by the x_i's. As free R-modules, viewed as $R[X_1, \dots, X_k]$-modules, belong to the class $\mathcal{C}_{\mathfrak{A}}$, we have for all R-modules M that $\Lambda_i^{\mathfrak{a}}(M) \cong \Lambda_i^{\mathfrak{A}}(M)$. This reduces the study of the $\Lambda_i^{\mathfrak{a}}(M)$ to the case where the Noetherian ring is $\mathfrak{a}$-separated and where the ideal $\mathfrak{a}$ is generated by a regular sequence. But note already that we shall obtain in Part II a similar change of rings theorem valid in greater generality and in the more general setting of complexes. That is why we postpone some of its consequences. It is also why the properties of $\mathfrak{a}$-complete and $\mathfrak{a}$-pseudo-complete modules over a Noetherian ring are very significant.

Proposition 2.5.15 (see [78, 2.5, 5.2]) *Let R be a Noetherian ring and $\mathfrak{a}$ an ideal of R.*

(a) Let C be an $\mathfrak{a}$-complete module. Then C has a left resolution by $\mathfrak{a}$-complete flat modules,

$$\Lambda_i^{\mathfrak{a}}(C) = 0 \textit{ for all } i \geq 1, \quad \tau_C^{\mathfrak{a}} = \mathrm{id}_C = \eta_C^{\mathfrak{a}}$$

and C belongs to the class $\mathcal{C}_{\mathfrak{a}}$ introduced in 2.5.12.

(b) For any R-module M the module $\Lambda_0^{\mathfrak{a}}(M)$ has a left resolution with $\mathfrak{a}$-complete flat modules, hence $\Lambda_i^{\mathfrak{a}}(\Lambda_0^{\mathfrak{a}}(M)) = 0$ for all $i \geq 1$ and $\eta^{\mathfrak{a}}_{\Lambda_0^{\mathfrak{a}}(M)} = \mathrm{id}_{\Lambda_0^{\mathfrak{a}}(M)}$.

Proof (a): Let L be a free module surjecting to C. Since C is $\mathfrak{a}$-complete and by 2.1.8 we have an exact sequence $0 \to Z \to \hat{L}^{\mathfrak{a}} \to C \to 0$ and Z is $\mathfrak{a}$-closed in $\hat{L}^{\mathfrak{a}}$. It follows that Z is $\mathfrak{a}$-complete (see 2.2.11). The wanted resolution is now obtained by iteration, recall also 2.4.8: $\hat{L}^{\mathfrak{a}}$ is flat when L is free. The other statements in (a) follow from the remarks in 2.5.12.

(b): Let $L_1 \xrightarrow{d_1} L_0 \to M \to 0$ be a free presentation of M. We have the exact sequence

$$\hat{L}_1^{\mathfrak{a}} \xrightarrow{\hat{d}_1^{\mathfrak{a}}} \hat{L}_0^{\mathfrak{a}} \to \Lambda_0^{\mathfrak{a}}(M) \to 0$$

and $\mathrm{Ker}(\hat{d}_1^{\mathfrak{a}})$ is $\mathfrak{a}$-complete as a closed submodule of an $\mathfrak{a}$-complete module. So the wanted resolution of $\Lambda_0^{\mathfrak{a}}(M)$ is obtained as above by iteration and the last claims in (b) follow again from the remarks in 2.5.12. $\square$

Corollary 2.5.16 *Let $\mathfrak{a}$ be an ideal of a Noetherian ring. Then any $\mathfrak{a}$-pseudo-complete module belongs to the class $\mathcal{C}_{\mathfrak{a}}^{+}$ introduced in 2.5.12. In particular, $\Lambda_i^{\mathfrak{a}}(M) \in \mathcal{C}_{\mathfrak{a}}^{+}$ for all R-modules M and all $i \geq 0$.*

Proof We have seen in 2.5.15 that $\Lambda_0^{\mathfrak{a}}(M) \in \mathcal{C}_{\mathfrak{a}}^{+}$, hence $\mathfrak{a}$-pseudo-complete modules are in $\mathcal{C}_{\mathfrak{a}}^{+}$. As the $\Lambda_i^{\mathfrak{a}}(M)$'s are $\mathfrak{a}$-pseudo-complete in view of 2.5.7 they are also in $\mathcal{C}_{\mathfrak{a}}^{+}$. □

By the long exact homology sequence of the $\Lambda_i^{\mathfrak{a}}$'s we also obtain the following corollary.

Corollary 2.5.17 *Let $\mathfrak{a}$ be an ideal of a Noetherian ring R. Then the class of $\mathfrak{a}$-pseudo-complete modules has the following property: Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be a short exact sequence of R-modules. If two of the three modules M_1, M_2, M_3 are $\mathfrak{a}$-pseudo-complete, so is the third.*

The next corollary provides a refinement of 2.5.7 (b) for the case of a Noetherian ring.

Corollary 2.5.18 *Let $\mathfrak{a}$ be an ideal of a Noetherian ring R. Let X be a complex of $\mathfrak{a}$-pseudo-complete modules. Then the modules $B_i := \mathrm{Im}(d_{i+1})$, $Z_i := \mathrm{Ker}(d_i)$ and $H_i(X) = Z_i/B_i$ are $\mathfrak{a}$-pseudo-complete for all $i \in \mathbb{Z}$.*

Proof With the exact sequence $X_i \to X_{i-1} \to X_{i-1}/B_{i-1} \to 0$ we first have that X_{i-1}/B_{i-1} is $\mathfrak{a}$-pseudo-complete by 2.5.3, and this holds for all $i \in \mathbb{Z}$. By 2.5.17 we now obtain successively that $B_{i-1} \cong X_i/Z_i$, Z_i and $H_i(X) = Z_i/B_i$ are $\mathfrak{a}$-pseudo-complete. □

Except for the fact that $\mathfrak{a}$-complete and $\mathfrak{a}$-pseudo complete modules have an $\mathfrak{a}$-complete flat resolutions, we shall see in Part II that the above proposition and its corollaries also hold for ideals other than those of a Noetherian ring (see 7.5.13 which generalizes part of 2.5.15). The ideals we have in mind are those generated by a weakly pro-regular sequence, a notion studied by the first author in [75] and highly exploited in Part II. One trick will be to use the Čech complex built on the weakly pro-regular sequence. Another trick will be that the study of the local homology with respect to such an ideal may be reduced to the case where the ring is Noetherian.

Proposition 2.5.19 *Assume that the ring R is Noetherian. Then the functors $\Lambda_i^{\mathfrak{a}}$ commute with products: $\Lambda_i^{\mathfrak{a}}(\prod_\lambda M_\lambda) \cong \prod_\lambda \Lambda_i^{\mathfrak{a}}(M_\lambda)$ for all $i \in \mathbb{N}$.*

Proof Over a Noetherian ring products of flat modules are flat and $\mathfrak{a}$-completions commute with products (see 2.2.7). As taking homology also commutes with products the conclusion follows. □

Remark 2.5.20 When the commutative ring R is Noetherian the local homology and cohomology are related through the general Matlis duality: for any R-module M we have

$$(H_{\mathfrak{a}}^i(M))^\vee \cong \Lambda_i^{\mathfrak{a}}(M^\vee) \text{ and } (H_{\mathfrak{a}}^0(M))^\vee \cong \Lambda^{\mathfrak{a}}(M^\vee)$$

(see [78]). In particular, $\Lambda_0^{\mathfrak{a}}(M^\vee)$ is $\mathfrak{a}$-complete and the surjective homomorphism $\gamma_{M^\vee}^{\mathfrak{a}} = \tau_{\Lambda_0^{\mathfrak{a}}(M^\vee)}^{\mathfrak{a}}$ is an isomorphism (see 2.5.4). But this also holds in greater generality and in a more general setting, as we shall see in Part II (see 9.2.5).

We may compose adic-completions and Λ_0 functors with respect to two ideals. In this direction we have the following, which also contains a counterpart of 2.2.10.

Proposition 2.5.21 *Let $\mathfrak{a}$, $\mathfrak{b}$ denote two ideals of a Noetherian ring R. Let M be an R-module. Then*

(a) $\Lambda_0^{\mathfrak{b}}(\Lambda_0^{\mathfrak{a}}(M)) \cong \Lambda_0^{(\mathfrak{b}+\mathfrak{a})}(M)$.
(b) *M is $(\mathfrak{a}+\mathfrak{b})$-pseudo-complete if and only if M is both $\mathfrak{a}$-pseudo-complete and $\mathfrak{b}$-pseudo-complete.*

Proof (a): Let $F_1 \xrightarrow{d} F_0 \to M \to 0$ be an exact sequence where F_1 and F_0 are flat. Then $\Lambda_0^{\mathfrak{a}}(M) \cong \operatorname{Coker}(\hat{d}^{\mathfrak{a}})$ (see 2.5.12). This holds for any ideal, in particular for both $\mathfrak{a}$ and $\mathfrak{a}+\mathfrak{b}$. We have seen in 2.4.6 that $\Lambda^{\mathfrak{b}}(\Lambda^{\mathfrak{a}}(F)) = \Lambda^{(\mathfrak{b}+\mathfrak{a})}(F)$ for all flat R-modules, and we know by 2.4.4 that completions of flat modules are flat. All this together gives the claim.

(b): This is a direct consequence of (a) and 2.5.9. □

Observations 2.5.22 Let $\mathfrak{a}$ be an ideal of a Noetherian ring R. Simply write $\hat{(\cdot)}$ for the $\mathfrak{a}$-adic completion functor. We know that $\hat{R}$ is R-flat and that $R/\mathfrak{a}^t \cong \hat{R}/\mathfrak{a}^t\hat{R}$ has the structure of an $\hat{R}$-module. Let M be an R-module. Then there are isomorphisms

$$\operatorname{Tor}_i^R(R/\mathfrak{a}^t, M) \cong \operatorname{Tor}_i^{\hat{R}}(\hat{R}/\mathfrak{a}^t\hat{R}, \hat{R}\otimes_R M)) \text{ and } \Lambda_i^{\mathfrak{a}}(M) \cong \Lambda_i^{\mathfrak{a}\hat{R}}(\hat{R}\otimes_R M)$$

for all $i \in \mathbb{N}$. To this end note the following: a free resolution $L_\bullet$ of M, tensorized with $\hat{R}$, gives a free resolution of the $\hat{R}$-module $\hat{R}\otimes_R M$ and

$$L_\bullet/\mathfrak{a}^t L_\bullet \cong \hat{R}/\mathfrak{a}^t\hat{R}\otimes_{\hat{R}} (\hat{R}\otimes_R L_\bullet).$$

Whence $\varprojlim L_\bullet/\mathfrak{a}^t L_\bullet \cong \varprojlim(\hat{R}/\hat{\mathfrak{a}}^t \otimes_{\hat{R}} (\hat{R}\otimes_R L_\bullet))$.

Let M, N denote two R-modules and assume that M has the structure of an $\hat{R}$-module (this holds if M is $\mathfrak{a}$-complete). Then $\operatorname{Ext}_R^i(N, M)$ has the structure of an $\hat{R}$-module. More precisely, and because $\hat{R}$ is R-flat, there is a canonical isomorphism

$$\operatorname{Ext}_R^i(N, M) \cong \operatorname{Ext}_{\hat{R}}^i(N\otimes_R \hat{R}, M)$$

for all $i \in \mathbb{N}$.

In particular, $\operatorname{Ext}_R^i(R/\mathfrak{a}^t, M) \cong \operatorname{Ext}_{\hat{R}}^i(\hat{R}/\mathfrak{a}^t\hat{R}, M)$ for all $i \in \mathbb{N}$.

2.6 Relative Flatness and Completion in the General Case

In this section $\mathfrak{a}$ is an arbitrary ideal of a commutative ring R. When dealing with $\mathfrak{a}$-completions it turns out that a notion of relative flatness is often sufficient. This notion is also important by itself as well as for further investigations.

Definition 2.6.1 Let $\mathfrak{a}$ be an ideal of a commutative ring R. An R-module M is called relatively-$\mathfrak{a}$-flat if $\operatorname{Tor}_i^R(R/\mathfrak{b}, M) = 0$ for all $i \geq 1$ and every $\mathfrak{a}$-open ideal $\mathfrak{b}$, that is to say for all ideals $\mathfrak{b}$ containing some power $\mathfrak{a}^t$ of $\mathfrak{a}$.

A flat R-module is of course relatively-$\mathfrak{a}$-flat for any ideal $\mathfrak{a}$ of R. Moreover, relatively-$\mathfrak{a}$-flat modules belong to the class $\mathcal{C}_{\mathfrak{a}}$ introduced in 2.5.12. Note that relatively-$\mathfrak{a}$-flat modules occur naturally in the completion process, as shown by the following examples.

Examples 2.6.2 We define $R_s = R/(\cap_t \mathfrak{a}^t)$ and $S_{\mathfrak{a}}(R) = \operatorname{Coker}(\tau_R^{\mathfrak{a}})$. Assume that R is Noetherian. Then the R-modules

$$R_s,\ \bar{0}_R^{\mathfrak{a}} = \cap_t \mathfrak{a}^t \ \text{ and } \ S_{\mathfrak{a}}(R)$$

are relatively-$\mathfrak{a}$-flat.

Indeed, first observe that $\hat{R}_s^{\mathfrak{a}} = \hat{R}^{\mathfrak{a}}$. Then take a free resolution L of R_s by means of finitely generated free modules. Then $\hat{R}^{\mathfrak{a}} \otimes_R L$ is a flat resolution of $\hat{R}^{\mathfrak{a}}$ because $\hat{R}^{\mathfrak{a}}$ is R-flat. Again since $\hat{R}^{\mathfrak{a}}$ is R-flat it follows that $H_i(R/\mathfrak{b} \otimes_R \hat{R}^{\mathfrak{a}} \otimes_R L) = 0$ for all $i \geq 1$ and every ideal $\mathfrak{b}$ of R. If the ideal $\mathfrak{b}$ is $\mathfrak{a}$-open we have $R/\mathfrak{b} \cong \hat{R}^{\mathfrak{a}}/\mathfrak{b}\hat{R}^{\mathfrak{a}}$ and therefore $R/\mathfrak{b} \otimes_R \hat{R}^{\mathfrak{a}} \otimes_R L \cong R/\mathfrak{b} \otimes_R L$. Whence the claim for R_s follows.

Now we consider the short exact sequences $0 \to \cap_t \mathfrak{a}^t \to R \to R_s \to 0$ and $0 \to R_s \to \hat{R}^{\mathfrak{a}} \to S_{\mathfrak{a}}(R) \to 0$. The claims concerning $\cap_t \mathfrak{a}^t$ and $S_{\mathfrak{a}}(R)$ follow from the associated long exact sequence of the $\operatorname{Tor}^i(R/\mathfrak{b}, \cdot)$, where $\mathfrak{b}$ is $\mathfrak{a}$-open, the relative-$\mathfrak{a}$-flatness of R_s and the isomorphism $R_s/\mathfrak{b}R_s = R/\mathfrak{b} \cong \hat{R}^{\mathfrak{a}}/\mathfrak{b}\hat{R}^{\mathfrak{a}}$.

Note that $S_{\mathfrak{a}}(R)$ is not flat in general. For example, take for $\mathfrak{a}$ an ideal of a Noetherian domain which is not contained in the Jacobson radical. Then R is not $\mathfrak{a}$-complete (see 2.1.6) and there is a proper ideal $\mathfrak{c}$ such that $\mathfrak{a} + \mathfrak{c} = R$. For such a $\mathfrak{c}$ the $\mathfrak{a}$-adic topology on $R/\mathfrak{c}$ is the coarse one and $R/\mathfrak{c} \otimes_R \hat{R}^{\mathfrak{a}} \cong \widehat{R/\mathfrak{c}}^{\mathfrak{a}} = 0$. It follows that $\operatorname{Tor}_1^R(R/\mathfrak{c}, S_{\mathfrak{a}}(R)) \cong R/\mathfrak{c} \neq 0$, as shown by the exact sequence $0 \to R \to \hat{R}^{\mathfrak{a}} \to S_{\mathfrak{a}}(R) \to 0$ tensored by $R/\mathfrak{c}$.

Lemma 2.6.3 *Let $\mathfrak{a}$ be an ideal of a commutative ring R and let $\mathfrak{a}_t$, $t \in \mathbb{N}$, be a descending sequence of ideals which form a base of open neighbourhoods of the origin for the $\mathfrak{a}$-adic topology of R. Let M denote an R-module.*

(a) The following conditions are equivalent.

- *(i) M is relatively-$\mathfrak{a}$-flat,*
- *(ii) $M/\mathfrak{a}_t M$ is $R/\mathfrak{a}_t$-flat for all t and* $\operatorname{Tor}_i^R(R/\mathfrak{a}_t, M) = 0$ *for all t and all $i \geq 1$.*

(b) *If the conditions in (a) are satisfied then $M/\mathfrak{b}M$ is $R/\mathfrak{b}$-flat for all $\mathfrak{a}$-open ideals $\mathfrak{b}$. If in addition the ring R is Noetherian, then $\hat{M}^{\mathfrak{a}}$ is R-flat and $\hat{R}^{\mathfrak{a}}$-flat.*

Proof Let L be a free resolution of M.

(a): First assume that M is relatively-$\mathfrak{a}$-flat. For a fixed t let $\mathfrak{c}$ be any ideal containing $\mathfrak{a}_t$. Then $L/\mathfrak{a}_t L$ is a free $R/\mathfrak{a}_t$-resolution of $M/\mathfrak{a}_t M$ by the definition of relative-$\mathfrak{a}$-flatness. Because $\mathfrak{a}_t \subset \mathfrak{c}$ we have

$$R/\mathfrak{c} \otimes_{R/\mathfrak{a}_t} L/\mathfrak{a}_t L \cong R/\mathfrak{c} \otimes_R L.$$

Because $\mathfrak{c}$ is $\mathfrak{a}$-open we also have that the complex $R/\mathfrak{c} \otimes_R L$ is exact in degree ≥ 1. Condition (ii) now follows by the flatness criterion recalled in 1.4.3.

Assume now the conditions in (ii) and let $\mathfrak{b}$ be any $\mathfrak{a}$-open ideal. This $\mathfrak{b}$ contains some $\mathfrak{a}_t$ and we have by assumption that $L/\mathfrak{a}_t L$ is a free $R/\mathfrak{a}_t$-resolution of the flat $R/\mathfrak{a}_t$-module $M/\mathfrak{a}_t M$. Hence $L/\mathfrak{b}L \cong L/\mathfrak{a}_t L \otimes_{R/\mathfrak{a}_t} R/\mathfrak{b}$ is exact in positive degree.

(b): This follows from (a) and 2.4.4. (It may also be deduced from 2.4.8 and 2.4.10.) □

The following result seems to be more or less known, at least partially. In the following we need it in a substantial way.

Proposition 2.6.4 *Let $\mathfrak{a}$ be an ideal of a commutative ring R and let F be a relatively-$\mathfrak{a}$-flat R-module such that $F/\mathfrak{a}F$ is $R/\mathfrak{a}$-free.*

Then the module $F/\mathfrak{a}^t F$ is $R/\mathfrak{a}^t$-free for all $t \geq 1$ and $\hat{F}^{\mathfrak{a}}$ is the $\mathfrak{a}$-adic completion of a free module.

More precisely, let $p_t : F \to F/\mathfrak{a}^t F$ denote the natural projections and let $(w_i, i \in I)$ be a family of elements of F such that $(p_1(w_i), i \in I)$ forms a base of the free $R/\mathfrak{a}$-module $F/\mathfrak{a}F$. Let also L be a free R-module with base $(e_i, i \in I)$ and let $h : L \to F$ be the homomorphism defined by $h(e_i) = w_i$.

Then h is an $\mathfrak{a}$-dense homomorphism such that $R/\mathfrak{a}^t \otimes_R h : L/\mathfrak{a}^t L \to F/\mathfrak{a}^t F$ is an isomorphism for all $t \geq 1$. Therefore $\hat{h}^{\mathfrak{a}} : \hat{L}^{\mathfrak{a}} \to \hat{F}^{\mathfrak{a}}$ is also an isomorphism.

Proof By the construction we have $F = h(L) + \mathfrak{a}F$ and by iteration $F = h(L) + \mathfrak{a}^t F$. Therefore the maps $R/\mathfrak{a}^t \otimes_R h : L/\mathfrak{a}^t L \to F/\mathfrak{a}^t F$ are surjective. They induce exact sequences

$$\mathcal{E}_t : 0 \to K_t \longrightarrow L/\mathfrak{a}^t L \longrightarrow F/\mathfrak{a}^t F \to 0$$

of $R/\mathfrak{a}^t$-modules. Whence it is enough to show that $K_t = 0$ for all $t \geq 1$.

By 2.6.3 we know that $F/\mathfrak{a}^t F$ is $R/\mathfrak{a}^t$-flat. Hence the tensored sequences $R/\mathfrak{a} \otimes_{R/\mathfrak{a}^t} \mathcal{E}_t$ are exact. It follows that

$$R/\mathfrak{a} \otimes_{R/\mathfrak{a}^t} \mathcal{E}_t = \mathcal{E}_1 : 0 \to K_1 \to L/\mathfrak{a}L \to F/\mathfrak{a}F \to 0$$

and therefore $R/\mathfrak{a} \otimes_{R/\mathfrak{a}^t} K_t = K_1$. As $K_1 = 0$ by construction it yields that $K_t = \mathfrak{a}K_t$ and $K_t = \mathfrak{a}^t K_t$ by iteration. But $\mathfrak{a}^t K_t = 0$ because K_t is an $R/\mathfrak{a}^t$-module. Whence $K_t = 0$ for all $t \geq 1$. Finally we obtain that $\hat{h}^{\mathfrak{a}} : \hat{L}^{\mathfrak{a}} \to \hat{F}^{\mathfrak{a}}$ is an isomorphism. □

Let $0 \to H \to G \to F \to 0$ be a short exact sequence of R-modules. When F is relatively $\mathfrak{a}$-flat we have already noticed that the $\mathfrak{a}$-adic completion of this sequence remains exact. Moreover, when $F/\mathfrak{a}F$ is free as an $R/\mathfrak{a}$-module we can say more.

Theorem 2.6.5 *Let $\mathfrak{a}$ be an ideal of a commutative ring R and let*

$$\mathcal{S} : 0 \to H \to G \xrightarrow{f} F \to 0$$

be a short exact sequence of R-modules, where F is a relatively-$\mathfrak{a}$-flat module such that $F/\mathfrak{a}F$ is $R/\mathfrak{a}$-free. Then the $\mathfrak{a}$-adic completion of the short sequence $\mathcal{S}$ is split-exact.

Proof The short sequence $\hat{\mathcal{S}}^{\mathfrak{a}}$ is exact because F is relatively-$\mathfrak{a}$-flat. Let $h : L \to F$ be the homomorphism described in 2.6.4. We lift it and obtain a homomorphism $s : L \to G$ such that $h = f \circ s$. By $\mathfrak{a}$-completions we have that $\hat{h}^{\mathfrak{a}}$ is an isomorphism. It follows that $\hat{f}^{\mathfrak{a}}$ is split-surjective. □

Let us have a closer look at the homomorphism h described in 2.6.4. It is not always injective and the w_i of 2.6.4 are not necessarily linearly independent in F. For example, take a Noetherian ring R and an ideal $\mathfrak{a}$ such that R is not $\mathfrak{a}$-separated, then take for F the relatively-$\mathfrak{a}$-flat module $R_s := R/\bigcap_t \mathfrak{a}^t$. Some cases are better, we mention the following.

Proposition 2.6.6 *We take the notations and hypotheses of Proposition 2.6.4 and assume moreover that R is $\mathfrak{a}$-separated.*

Then h is injective and $\operatorname{Tor}_i^R(R/\mathfrak{b}, \operatorname{Coker}(h)) = 0$ *for all $\mathfrak{a}$-open ideals $\mathfrak{b}$ and all i. Hence* $\operatorname{Coker}(h)$ *is relatively $\mathfrak{a}$-flat and* $\operatorname{Coker}(h) = \mathfrak{a}\operatorname{Coker}(h)$.

Proof If R is $\mathfrak{a}$-separated, then the map $\tau_L^{\mathfrak{a}}$ is injective. Therefore h is also injective because of $\tau_F^{\mathfrak{a}} \circ h = \hat{h}^{\mathfrak{a}} \circ \tau_L^{\mathfrak{a}}$. Then $\operatorname{Coker}(h)$ has the required property because the homomorphisms $R/\mathfrak{a}^t \otimes_R h$ are isomorphisms. □

For the next good case we need another flatness criterion.

2.6.7 *A Bourbaki flatness criterion.* (see [13, Chap. 1, §2, Proposition 13, Corollary 1]) To present the criterion we use matrix notations and denote by $A^{m\times n}$ the set of $m \times n$ matrices with entries in A.

A module M over a commutative ring R is flat if and only if it satisfies the following condition:

For all $b \in R^{1\times n}$, $w \in M^{n\times 1}$ and all relations $bw = 0$, there are $m \in \mathbb{N}$, $v \in M^{m\times 1}$ and $c \in R^{n\times m}$ such that $w = cv$ and $bc = 0$.

Proposition 2.6.8 (see [79, 2.1]) *We take the notations and hypotheses of Proposition 2.6.4. Assume moreover that F is flat and that the ideal $\mathfrak{a}$ is contained in the Jacobson radical of R.*

Then h is pure injective and $h(L)$ is an $\mathfrak{a}$-basic submodule of F. Hence $\operatorname{Coker}(h)$ *is flat and* $\operatorname{Coker}(h) = \mathfrak{a}\operatorname{Coker}(h)$.

Proof First we prove the freeness of the w_i in F. We use matrix notations as above. If we have a relation $\sum_{i=1}^{n} b_i w_i = 0$ with $b_i \in R$, then we have $v_1, \ldots v_m \in F$ and a matrix $c \in R^{n\times m}$ such that $w = cv$ and $bc = 0$. The matrix c is right-invertible modulo the ideal $\mathfrak{a}$ because the images of the w_i in $F/\mathfrak{a}F$ form a base of the free $R/\mathfrak{a}$-module $F/\mathfrak{a}F$. But the ideal $\mathfrak{a}$ is contained in the Jacobson radical of R by assumption. It follows that the matrix c is also right-invertible in R. Then $b = 0$ because $bc = 0$. Therefore the w_i's are free and the map h is injective.

To prove the purity of L in F it is enough to prove that the maps $R/\mathfrak{c} \otimes_R h$ are injective for all ideals $\mathfrak{c}$ of R (see 2.1.7). But this follows from the first part of the proof applied to the ring $R/\mathfrak{c}$, the flat $R/\mathfrak{c}$-module $F/\mathfrak{c}F$, the morphism $R/\mathfrak{c} \otimes_R h$ and the ideal $(\mathfrak{a} + \mathfrak{c})/\mathfrak{c}$ of $R/\mathfrak{c}$. Note that $(\mathfrak{a} + \mathfrak{c})/\mathfrak{c}$ is contained in the Jacobson radical of $R/\mathfrak{c}$. □

In the above proposition the case where $\mathfrak{a} = \mathfrak{m}$ is the maximal ideal of a local ring was already observed in [7, Proposition 3.3].

The following may have some technical interest.

Corollary 2.6.9 *Let $\mathfrak{a}$ be an ideal of a commutative ring R. Let M denote an R-module satisfying $M = \mathfrak{a}M$. Assume that either*

(a) R is $\mathfrak{a}$-separated or
(b) $\mathfrak{a}$ is contained in the Jacobson radical of R.

Then there is a short exact sequence $0 \to M' \to T \to M \to 0$ such that $T = \mathfrak{a}T$ and T is relatively $\mathfrak{a}$-flat. Moreover, in case (b) we have such a sequence where T is flat.

Proof Choose a short exact sequence $0 \to N \xrightarrow{u} L_0 \to M \to 0$ where L_0 is a free R-module. Because $M/\mathfrak{a}M = 0$ the map $R/\mathfrak{a} \otimes_R u$ is surjective. Take a family of elements $(v_i \in N; i \in I)$ such that their images in $L_0/\mathfrak{a}L_0$ form a base of this $R/\mathfrak{a}$-free module. Now let L be a free module of base $(e_i, i \in I)$ and define $h_1 : L \to N$ by $h_1(e_i) = v_i$. Then there is a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, "\mathrm{id}_L"] \arrow[d, "h_1"] & L \arrow[r] \arrow[d, "h"] & 0 & \\ 0 \arrow[r] & N \arrow[r] & L_0 \arrow[r] & M \arrow[r] & 0. \end{tikzcd}$$

We define $T = \mathrm{Coker}(h)$ and $M' = \mathrm{Coker}(h_1)$. Then the snake lemma provides the short exact sequence. The module T has the required properties by 2.6.6 and 2.6.8 applied to L_0. □

When $\mathfrak{a}$ is an ideal of a Noetherian ring we have seen that the $\mathfrak{a}$-completion of a flat module is flat. This does not hold in general (see examples 2.8.7 and 2.8.4). But relative-$\mathfrak{a}$-flatness is preserved provided the ideal $\mathfrak{a}$ satisfies some mild conditions.

2.6.10 *The property C.* Let $\mathfrak{a}$ be an ideal of a commutative ring R. We say that the pair $(R, \mathfrak{a})$ has property C if each $\mathfrak{a}$-complete R-module M belongs to the class $\mathcal{C}_{\mathfrak{a}}$ introduced in 2.5.12, that is, if $\Lambda_i^{\mathfrak{a}}(M) = 0$ for all $i > 0$) and if the natural morphism $\gamma_M^{\mathfrak{a}} : \Lambda_0^{\mathfrak{a}}(M) \to \hat{M}^{\mathfrak{a}}$ is an isomorphism.

Recall that the pair $(R, \mathfrak{a})$ always has property C when R is Noetherian (see 2.5.15). More generally we shall see later in 7.5.13 that this is also the case when $\mathfrak{a}$ is finitely generated and generated by a so-called weakly pro-regular sequence. These are not the only cases. For a rather trivial example take for $\mathfrak{a}$ a nilpotent ideal. Then the $\mathfrak{a}$-adic topology of any R-module is the discrete one. Hence every R-module is $\mathfrak{a}$-complete and belongs to the class $\mathcal{C}_{\mathfrak{a}}$. Note that a nilpotent ideal of a non-Noetherian ring is not necessarily finitely generated.

Proposition 2.6.11 *Let R be a commutative ring, $\mathfrak{a}$ a finitely generated ideal and let M denote a relatively-$\mathfrak{a}$-flat R-module. Assume that the pair $(R, \mathfrak{a})$ has the property C introduced in 2.6.10. Then $\hat{M}^{\mathfrak{a}}$ is relatively-$\mathfrak{a}$-flat too.*

Proof We choose a free R-module L with a surjective homomorphism $f : L \to \operatorname{Coker}(\tau_M^{\mathfrak{a}})$. Then there is a surjective homomorphism $g : L \to \hat{M}^{\mathfrak{a}}$ such that f factors through g. This gives us an exact sequence

$$\mathcal{E} : 0 \to N \xrightarrow{u} M \oplus L \xrightarrow{p} \hat{M}^{\mathfrak{a}} \to 0 \quad \text{where} \quad p = (\tau_M^{\mathfrak{a}}, g)$$

and a module N associated to M. The assumption on M implies $M \in \mathcal{C}_{\mathfrak{a}}$. We also have $\hat{M}^{\mathfrak{a}} \in \mathcal{C}_{\mathfrak{a}}$ with the assumption on the pair $(R, \mathfrak{a})$. Because $L \in \mathcal{C}_{\mathfrak{a}}$ the short exact sequence $\mathcal{E}$ yields that $N \in \mathcal{C}_{\mathfrak{a}}$ and that the sequence $\mathcal{E}$ remains exact by $\mathfrak{a}$-completion (see 2.5.13). On the other hand the map $\hat{p}^{\mathfrak{a}}$ is split-surjective because the map $\widehat{\tau_M^{\mathfrak{a}}}^{\mathfrak{a}}$ is an isomorphism (see 2.2.2). Hence the short exact sequence $\hat{\mathcal{E}}^{\mathfrak{a}}$ is split-exact. We first obtain that the morphism $\hat{N}^{\mathfrak{a}} \to \hat{L}^{\mathfrak{a}}$ stemming from u is an isomorphism.

Let $\mathfrak{b}$ denote any $\mathfrak{a}$-open ideal. By tensoring with $R/\mathfrak{b}$ we obtain an isomorphism $N/\mathfrak{b}N \cong L/\mathfrak{b}L$ as follows by 2.2.2 again. Then the long exact sequence associated to the short exact sequence $\mathcal{E}$ by applying the tensor product with $R/\mathfrak{b}$ yields an exact sequence:

$$0 \to \operatorname{Tor}_1^R(R/\mathfrak{b}, \hat{M}^{\mathfrak{a}}) \to N/\mathfrak{b}N \to M/\mathfrak{b}M \oplus L/\mathfrak{b}L \to M/\mathfrak{b}M \to 0.$$

By the previous investigation the map $N/\mathfrak{b}N \to M/\mathfrak{b}M \oplus L/\mathfrak{b}L$ is injective. This first proves the vanishing $\operatorname{Tor}_1^R(R/\mathfrak{b}, \hat{M}^{\mathfrak{a}}) = 0$.

Now we proceed by induction. For any relatively-$\mathfrak{a}$-flat R-module M and for any $\mathfrak{a}$-open ideal $\mathfrak{b}$ we assume by induction that $\operatorname{Tor}_i^R(R/\mathfrak{b}, \hat{M}^{\mathfrak{a}}) = 0$ for $1 \le i \le t$. We need to prove that $\operatorname{Tor}_{t+1}^R(R/\mathfrak{b}, \hat{M}^{\mathfrak{a}}) = 0$. But

$$\operatorname{Tor}_{t+1}^R(R/\mathfrak{b}, \hat{M}^{\mathfrak{a}}) \cong \operatorname{Tor}_t^R(R/\mathfrak{b}, N)$$

as follows by the short exact sequence $\mathcal{E}$ above. Hence we only need to prove that $\operatorname{Tor}_t^R(R/\mathfrak{b}, N) = 0$. To this end we take a free resolution F of this R-module N associated to M. We construct short exact sequences

$$\mathcal{S}_i : 0 \to N_{i+1} \to F_i \to N_i \to 0 \text{ for all } i \geq 0,$$

with F_i a free module starting with $N = N_0$. As $N \in \mathcal{C}_\mathfrak{a}$ it follows that $N_i \in \mathcal{C}_\mathfrak{a}$ for all $i \geq 0$. Whence the short exact sequences $\mathcal{S}_i$ remain exact after $\mathfrak{a}$-completion (see 2.5.13 again). We first look at the short exact sequence

$$\widehat{\mathcal{S}_0}^{\mathfrak{a}} : 0 \to \hat{N_1}^{\mathfrak{a}} \to \hat{F_0}^{\mathfrak{a}} \to \hat{N}^{\mathfrak{a}} \to 0.$$

By the first part of the proof we have $\operatorname{Tor}_1^R(R/\mathfrak{b}, \hat{N}^\mathfrak{a}) \cong \operatorname{Tor}_1^R(R/\mathfrak{b}, \hat{L}^\mathfrak{a}) = 0$. Hence the short sequence $R/\mathfrak{b} \otimes_R \widehat{\mathcal{S}_0}^{\mathfrak{a}}$ is exact. But there are isomorphisms of short sequences $R/\mathfrak{b} \otimes_R \widehat{\mathcal{S}_i}^{\mathfrak{a}} \cong R/\mathfrak{b} \otimes_R \mathcal{S}_i$. It follows that $\operatorname{Tor}_1^R(R/\mathfrak{b}, N) = 0$, which settles the case $t = 1$.

Assume now $t \geq 2$. Because of $\hat{N}^\mathfrak{a} \cong \hat{L}^\mathfrak{a}$ and the inductive hypothesis we have

$$\operatorname{Tor}_i^R(R/\mathfrak{b}, \hat{N}^\mathfrak{a}) = 0 = \operatorname{Tor}_i^R(R/\mathfrak{b}, \hat{F}_j^\mathfrak{a}) \text{ for all } 1 \leq i \leq t \text{ and all } j.$$

Playing with the short exact sequences $\widehat{\mathcal{S}_i}^{\mathfrak{a}}$ we obtain

$$\operatorname{Tor}_1^R(R/\mathfrak{b}, \hat{N}_{t-1}^\mathfrak{a}) = \operatorname{Tor}_t^R(R/\mathfrak{b}, \hat{N}^\mathfrak{a}) = 0.$$

Thus the short sequence $R/\mathfrak{b} \otimes_R \widehat{\mathcal{S}_{t-1}}^{\mathfrak{a}}$ is exact and so is the short sequence $R/\mathfrak{b} \otimes_R \mathcal{S}_{t-1}$. It follows that $\operatorname{Tor}_1^R(R/\mathfrak{b}, N_{t-1}) = 0$. As $\operatorname{Tor}_t^R(R/\mathfrak{b}, N) = \operatorname{Tor}_1^R((R/\mathfrak{b}, N_{t-1})$ for $t \geq 2$ the inductive step is completed. ☐

The same statement with a different proof appeared in [88, Theorem 4.3] under the slightly stronger hypothesis that the ideal $\mathfrak{a}$ is generated by a weakly pro-regular sequence. In that case, see also the forthcoming 9.5.7 for a companion result.

2.7 Relatively Injective and Torsion Modules

Here again $\mathfrak{a}$ is an arbitrary ideal of a commutative ring R. We investigate a notion dual to the notion of relative flatness. It might present some interest in the study of local cohomology.

Definition and observation 2.7.1 Let $\mathfrak{a}$ be an ideal of a commutative ring R. We say that an R-module M is *relatively-$\mathfrak{a}$-injective* if $\operatorname{Ext}_R^i(R/\mathfrak{b}, M) = 0$ for all $\mathfrak{a}$-open ideal $\mathfrak{b}$ of R and all $i \geq 1$. For such a module we note that $H_\mathfrak{a}^i(M) = 0$ for all $i \geq 1$ since $H_\mathfrak{a}^i(M) = \varinjlim \operatorname{Ext}_R^i(R/\mathfrak{a}^t, M)$.

Lemma 2.7.2 *Let $\mathfrak{a}$ be an ideal of a commutative ring R. Let M denote an R-module and $\mathfrak{b}$ an $\mathfrak{a}$-open ideal.*

(a) If M is relatively-$\mathfrak{a}$-injective then $\operatorname{Hom}_R(R/\mathfrak{b}, M)$ *is $R/\mathfrak{b}$-injective.*

(b) M is relatively-$\mathfrak{a}$-injective if and only for all $t \geq 1$ $\operatorname{Hom}_R(R/\mathfrak{a}^t, M)$ *is $R/\mathfrak{a}^t$-injective and* $\operatorname{Ext}^i_R(R/\mathfrak{a}^t, M) = 0$ *for all $i \geq 1$.*

Proof Let I be a injective resolution of M. We prove (a) at first. For any ideal $\mathfrak{c}$ containing $\mathfrak{b}$ the complex $\operatorname{Hom}_R(R/\mathfrak{c}, I)$ is exact in degree $i > 0$. In particular, the complex $\operatorname{Hom}_R(R/\mathfrak{b}, I)$ provides an injective $R/\mathfrak{b}$-resolution of $\operatorname{Hom}_R(R/\mathfrak{b}, M)$. As

$$\operatorname{Hom}_R(R/\mathfrak{c}, I) = \operatorname{Hom}_{R/\mathfrak{b}}(R/\mathfrak{c}, \operatorname{Hom}_R(R/\mathfrak{b}, I))$$

the conclusion follows by the Baer criterion (see 1.4.7).

The only if part of (b) is a direct consequence of (a). The proof of the if part, similar to the proof of the corresponding statement in 2.6.3, is left to the reader. □

Let us first investigate the above notion for Noetherian rings. We need some preparation, some remarks on $\mathfrak{a}$-torsion modules, analogous to those we made on $\mathfrak{a}$-complete modules.

Lemma 2.7.3 *Let $\mathfrak{a}$ be an ideal of a commutative ring R. Let M denote an $\mathfrak{a}$-torsion R-module and N an arbitrary R-module.*

(a) Any direct sum of $\mathfrak{a}$-torsion R-modules is $\mathfrak{a}$-torsion.
(b) $\operatorname{Tor}^R_i(N, M)$ *is $\mathfrak{a}$-torsion for all $i \geq 0$.*
(c) Suppose in addition that R is Noetherian and that N is finitely generated. Then $\operatorname{Ext}^i_R(N, M)$ *is $\mathfrak{a}$-torsion for all $i \geq 0$.*

Proof The statement in (a) is obvious. For (b) let L be a free resolution of N. Then $L \otimes_R M$ is a complex of $\mathfrak{a}$-torsion modules. Hence $H_i(L \otimes_R M) = \operatorname{Tor}^R_i(N, M)$ is $\mathfrak{a}$-torsion as a sub-quotient of an $\mathfrak{a}$-torsion module. The proof of (c) is similar, taking for L a resolution by means of finitely generated modules. □

Here is a counterpart of Proposition 2.4.10 on the $\mathfrak{a}$-torsion side.

Proposition 2.7.4 *Let R denote a Noetherian ring and M an $\mathfrak{a}$-torsion R-module. Then the injective and the flat dimension of M are given by*

$$\operatorname{id}_R(M) = \sup\{i \in \mathbb{N} \mid \operatorname{Ext}^i_R(R/\mathfrak{p}, M) \neq 0 \textit{ for some prime ideal } \mathfrak{p} \supset \mathfrak{a}\},$$
$$\operatorname{fd}_R(M) = \sup\{i \in \mathbb{N} \mid \operatorname{Tor}^R_i(R/\mathfrak{p}, M) \neq 0 \textit{ for some prime ideal } \mathfrak{p} \supset \mathfrak{a}\}.$$

Proof As in the proof of 2.4.10 we shall use the fact that a finitely generated module N over a Noetherian ring has a composition series

$$0 \subset N_n \subset \cdots \subset N_i \subset \cdots \subset N_0 = N$$

where the successive quotients are of the form $R/\mathfrak{q}$ with $\mathfrak{q} \in \operatorname{Supp}_R N$. For the first equality we only need check the $\operatorname{Ext}^i_R(N, M)$ where N is finitely generated (see

1.4.7). If $\operatorname{Ext}^i_R(N, M) \neq 0$, N finitely generated, it follows that $\operatorname{Ext}^i_R(R/\mathfrak{q}, M) \neq 0$ for some $\mathfrak{q} \in \operatorname{Supp}_R N$. If $\mathfrak{a} \not\subseteq \mathfrak{q}$, take $x \in \mathfrak{a} \setminus \mathfrak{q}$ and look at the long exact sequence of the $\operatorname{Ext}_R(\cdot, M)$ associated to the exact sequence

$$0 \to R/\mathfrak{q} \xrightarrow{x} R/\mathfrak{q} \to R/(\mathfrak{q} + xR) \to 0.$$

We note that multiplication by x is not injective on $\operatorname{Ext}^i_R(R/\mathfrak{q}, M)$ because this module is $\mathfrak{a}$-torsion. It follows that $\operatorname{Ext}^i_R(R/(\mathfrak{q} + xR), M) \neq 0$. Then we obtain a prime ideal $\mathfrak{q}_1 \supset (\mathfrak{q} + xR)$ with $\operatorname{Ext}^i_R(R/\mathfrak{q}_1, M) \neq 0$. If $\mathfrak{a} \not\subseteq \mathfrak{q}_1$ we continue the process until we reach an ideal $\mathfrak{p} \supset \mathfrak{a}$ with $\operatorname{Ext}^i_R(R/\mathfrak{p}, M) \neq 0$.

The proof of the statement about the flat dimension is similar. □

Corollary 2.7.5 *Let R denote a Noetherian ring and M a relatively-$\mathfrak{a}$-injective R-module. Then*

(a) $\Gamma_{\mathfrak{a}}(M)$ is R-injective.
(b) Moreover, $\Gamma_{\mathfrak{a}}(M)$ has the structure of an $\hat{R}^{\mathfrak{a}}$-module and $\Gamma_{\mathfrak{a}}(M)$ is also $\hat{R}^{\mathfrak{a}}$-injective.

Proof Let I be an injective resolution of M. It is known that $\Gamma_{\mathfrak{a}}(I)$ is a complex of injective modules. Because $H^i_{\mathfrak{a}}(M) = 0$ for $i \geq 1$ (see 2.7.1) we have that $\Gamma_{\mathfrak{a}}(I)$ is an injective resolution of $\Gamma_{\mathfrak{a}}(M)$. For any prime ideal $\mathfrak{p}$ containing $\mathfrak{a}$ we have $\operatorname{Hom}_R(R/\mathfrak{p}, I) = \operatorname{Hom}_R(R/\mathfrak{p}, \Gamma_{\mathfrak{a}}(I))$. As $\operatorname{Hom}_R(R/\mathfrak{p}, I)$ is exact in degree $i \geq 1$ by assumption the statement in (a) follows by 2.7.4.

The module $\Gamma_{\mathfrak{a}}(M)$ has the structure of an $\hat{R}^{\mathfrak{a}}$-module because it is $\mathfrak{a}$-torsion (see 2.1.15). We have $\operatorname{Ext}^i_{\hat{R}}(\hat{R}/\mathfrak{a}^t\hat{R}, \Gamma_{\mathfrak{a}}(M)) \cong \operatorname{Ext}^i_R(R/\mathfrak{a}^t, \Gamma_{\mathfrak{a}}(M)) = 0$ for all $i > 0$ because $\Gamma_{\mathfrak{a}}(M)$ is R-injective (see also 2.5.22). Note also that $\operatorname{Hom}_{\hat{R}}(\hat{R}/\mathfrak{a}^t\hat{R}, \Gamma_{\mathfrak{a}}(M)) \cong \operatorname{Hom}_R(R/\mathfrak{a}^t, \Gamma_{\mathfrak{a}}(M))$ is $R/\mathfrak{a}^t$-injective. Since $R/\mathfrak{a}^t \cong \hat{R}^{\mathfrak{a}}/\mathfrak{a}^t\hat{R}^{\mathfrak{a}}$ it follows that $\Gamma_{\mathfrak{a}}(M)$ viewed as an $\hat{R}^{\mathfrak{a}}$-module is relatively $\mathfrak{a}\hat{R}^{\mathfrak{a}}$-injective (see 2.7.2). Now the second statement follows from the first. □

2.7.6 *The class $\mathcal{B}_{\mathfrak{a}}$.* Let $\mathfrak{a}$ be an ideal of a commutative ring R. We denote by $\mathcal{B}_{\mathfrak{a}}$ the class of R-modules M such that $H^i_{\mathfrak{a}}(M) = 0$ for all $i \geq 1$. By a standard cohomological argument as in 1.5.3 we note that the local cohomology of an R-module M may be computed with a right resolution $M \xrightarrow{\sim} J$, where the modules J^i are in $\mathcal{B}_{\mathfrak{a}}$ and $J^i = 0$ for $i < 0$: for such a resolution we have $H^i(\Gamma_{\mathfrak{a}}(J)) \cong H^i_{\mathfrak{a}}(M)$. Note also that relatively-$\mathfrak{a}$-injective modules are in $\mathcal{B}_{\mathfrak{a}}$.

Remarks 2.7.7 Let $\mathfrak{a}$ be an ideal of a Noetherian ring and M an $\mathfrak{a}$-torsion R-module. Then M has an injective resolution I where all the I^i are $\mathfrak{a}$-torsion. (For any injection $M \hookrightarrow J$ of M into an injective R-module J we have $M = \Gamma_{\mathfrak{a}}(M) \hookrightarrow \Gamma_{\mathfrak{a}}(J)$ and it is known that $\Gamma_{\mathfrak{a}}(J)$ is injective. As $\Gamma_{\mathfrak{a}}(J)/M$ is also $\mathfrak{a}$-torsion the resolution is obtained by iteration.) It follows that $H^i_{\mathfrak{a}}(M) = 0$ for all $i \geq 1$ and therefore $M \in \mathcal{B}_{\mathfrak{a}}$.

We are now prepared for a statement analogous to 2.6.11 on the $\mathfrak{a}$-torsion side. When $\mathfrak{a}$ is an ideal of a Noetherian ring it is known that $\Gamma_{\mathfrak{a}}(I)$ is injective when

I is an injective R-module. This does not hold in general (see example 2.8.8). But the property of being relatively-$\mathfrak{a}$-injective is preserved provided the ideal $\mathfrak{a}$ satisfies some mild conditions.

2.7.8 *The property B.* Let $\mathfrak{a}$ be an ideal of a commutative ring R. We say that the pair $(R, \mathfrak{a})$ has property B if each $\mathfrak{a}$-torsion R-module M belongs to the class $\mathcal{B}_{\mathfrak{a}}$ introduced in 2.7.6, that is, if $H^i_{\mathfrak{a}}(M) = 0$ for all $i > 0$.

Recall that the pair $(R, \mathfrak{a})$ always has property B when R is Noetherian (see 2.7.7). More generally we shall see later in 7.4.7 that this is also the case when the ideal $\mathfrak{a}$ is generated by a weakly pro-regular sequence.

The following is well known, we state it in the form we need and provide a quick proof. (It can also be quickly deduced from [82, 5.3.15] or 3.4.3 or from the more general 7.6.2.)

Lemma 2.7.9 *Let $\mathfrak{a}$ be an ideal of a commutative ring R. Let N denote an R-module. If $H^i_{\mathfrak{a}}(N) = 0$ for all $i \geq 0$, then also $\mathrm{Ext}^i_R(R/\mathfrak{b}, N) = 0$ for all $i \geq 0$ and any $\mathfrak{a}$-open ideal $\mathfrak{b}$.*

Proof Let $\mathfrak{b}$ be any $\mathfrak{a}$-open ideal. If $H^0_{\mathfrak{a}}(N) = 0$ then $\mathrm{Hom}_R(R/\mathfrak{b}, N) = 0$. Assume now $H^i_{\mathfrak{a}}(N) = 0$ for all $i \geq 0$. We proceed by induction, assume that $\mathrm{Ext}^i_R(R/\mathfrak{b}, N) = 0$ for all $i \leq r$ and consider the exact sequence

$$0 \to N \to I \to N_1 \to 0$$

where I is an injective hull of N. We first note that $H^0_{\mathfrak{a}}(I) = 0$. Hence $H^i_{\mathfrak{a}}(N_1) = 0$ for all $i \geq 0$. As $\mathrm{Ext}^{r+1}_R(R/\mathfrak{b}, N) \cong \mathrm{Ext}^r_R(R/\mathfrak{b}, N_1)$ the inductive step is complete. $\square$

We are ready for the counterpart of 2.6.11. As usual the injective situation is easier to handle.

Proposition 2.7.10 *Let R a commutative ring, $\mathfrak{a}$ an ideal and let M denote an R-module. Assume that the pair $(R, \mathfrak{a})$ has the property B introduced in 2.7.8. Assume also that M is relatively-$\mathfrak{a}$-injective. Then $\Gamma_{\mathfrak{a}}(M)$ is relatively-$\mathfrak{a}$-injective.*

Proof Put $N = M/\Gamma_{\mathfrak{a}}(M)$. We consider the short exact sequence

$$0 \to \Gamma_{\mathfrak{a}}(M) \to M \to N \to 0$$

and the associated long exact sequence in local cohomology. By the hypothesis on the pair $(R, \mathfrak{a})$ we have that $H^i_{\mathfrak{a}}(\Gamma_{\mathfrak{a}}(M)) = 0$ for all $i \geq 1$. By the hypothesis on M we also have $H^i_{\mathfrak{a}}(M) = \varinjlim \mathrm{Ext}^i_R(R/\mathfrak{a}^t, M) = 0$ for all $i \geq 1$. It follows that $H^i_{\mathfrak{a}}(N) = 0$ for all i. Now let $\mathfrak{b}$ be any $\mathfrak{a}$-open ideal. In view of 2.7.9 the conclusion follows by the long exact sequence of the $\mathrm{Ext}^i_R(R/\mathfrak{b}, \cdot)$ associated to the above short exact sequence. $\square$

In the particular case when the ideal $\mathfrak{a}$ is generated by a weakly pro-regular sequence the above result will be refined in 9.5.8.

2.8 Some Examples

In this section we present some rings the $\mathfrak{a}$-completions of which are not flat (though relatively-$\mathfrak{a}$-flat). We also present an injective module the $\mathfrak{a}$-torsion part of which is not injective.

2.8.1 For the first example we recall some facts about 1-Kähler differentials. Namely let A be an R-algebra. Let $f : A \otimes_R A \to A, a \otimes b \mapsto ab$, the multiplication homomorphism and $I = \ker f$. Consider $A \otimes_R A$ as an R-module via left multiplication. Then the A-module $\Omega^1_{A/R} := I/I^2$ is called the module of (relative) Kähler differentials. For these and related subjects we refer to [51, §1].

Lemma 2.8.2 *Let $\Bbbk$ denote a field of characteristic zero and $A = \Bbbk[[x]]$ the formal power series ring in one variable. Let $Q = \Bbbk((x))$ denote the quotient field of A. Then* $\dim_Q \Omega^1_{Q/\Bbbk} = \infty$.

Proof By [51, 5.4 Corollary] and the fact that $\operatorname{trdeg}(Q, \Bbbk) = \infty$ it follows that $\dim_Q \Omega^1_{Q/\Bbbk} = \infty$. □

The previous result has an important corollary.

Corollary 2.8.3 *Let $A = \Bbbk[[x]]$ be as above and $B = A \otimes_\Bbbk A$ with the natural surjective map $B \xrightarrow{f} A \to 0$ and* $\ker f = I$. *Then I/I^2 is not finitely generated as an A-module. Moreover, $I \subset B$ is not a finitely generated ideal of B and therefore B is not a Noetherian ring.*

Proof Let Q denote the quotient field of A. There are isomorphisms

$$I/I^2 \otimes_A Q \cong \Omega^1_{A/\Bbbk} \otimes_A Q \cong \Omega^1_{Q/\Bbbk}.$$

Since $\dim_Q \Omega^1_{Q/\Bbbk} = \infty$ the A-module I/I^2 is of infinite rank and therefore not finitely generated. This implies that $I \subset B$ is not a finitely generated ideal. Note that if I is a finitely generated ideal of B there is a surjection $B^k \to I \to 0$ for a finite number k. This implies a surjection $A^k \to I/I^2 \to 0$, which contradicts the fact that I/I^2 is not finitely generated over A. □

With these investigations we are prepared for the following interesting example due to A. Yekutieli (see [88]).

Example 2.8.4 (*see* [88]) Let $\Bbbk$ denote a field of characteristic zero and $A = \Bbbk[[t]]$ the formal power series ring in one variable. Let $B = \Bbbk[[x]] \otimes_\Bbbk \Bbbk[[y]]$ and let $\mathfrak{a} = (x, y)B$. Then

(a) The ring B is not Noetherian.
(b) We have $\hat{B}^{\mathfrak{a}} \cong \Bbbk[[x, y]]$.
(c) $\hat{B}^{\mathfrak{a}}$ is not B-flat.
(d) $\hat{B}^{\mathfrak{a}}$ is relatively-$\mathfrak{a}$-flat.

Proof First there is a surjective homomorphism

$$B \xrightarrow{g} A \to 0 \text{ by } x \mapsto t, y \mapsto t.$$

Then B is isomorphic to the ring B in Corollary 2.8.3. Under this isomorphism $\ker g = I$. Since I is not finitely generated it follows that B is not Noetherian and (a) follows.

For the proof of (b) we recall that $B/(x^t, y^t)B \cong \Bbbk[x, y]/(x^t, y^t)$, which proves the claim (by passing to the inverse limit).

For the proof of (c) suppose that $\hat{B} := \hat{B}^{\mathfrak{a}}$ is B-flat. Then we have the short exact sequence

$$0 \to I \otimes_B \hat{B} \to \hat{B} \to \hat{B} \otimes_B A \to 0.$$

Therefore $I \otimes_B \hat{B} \subset \hat{B} \cong \Bbbk[[x, y]]$ is a finitely generated $\hat{B}$-module. Moreover, we have the following commutative diagram

$$\begin{tikzcd} B \arrow[rr, "g"] \arrow[dr, "\tau"'] & & A \\ & \hat{B} \cong \Bbbk[[x, y]], \arrow[ur, "h"'] & \end{tikzcd}$$

where the surjection $h : \hat{B} \to A$ is given by $x \mapsto t, y \mapsto t$. Via the homomorphism $h : \hat{B} \to A$ we may consider the A-module I/I^2 as a $\hat{B}$-module. The surjections $I \to I/I^2$ and $\hat{B} \to A$ induce surjections

$$I \otimes_B \hat{B} \to I/I^2 \otimes_B \hat{B} \to I/I^2 \otimes_B A = I/I^2.$$

But $I \otimes_B \hat{B}$ is a finitely generated $\hat{B}$-module by assumption. Whence I/I^2 is finitely as a $\hat{B}$-module and therefore also as an A-module. But this contradicts 2.8.3, and so $\hat{B}$ is not B-flat.

For the proof of (d) note that the sequences (x^t, y^t) are regular on both B and $\hat{B}$. It follows that $\operatorname{Tor}_i^B(B/(x^t, y^t)B, \hat{B}) = 0$ for all $i \geq 1$. As $\hat{B}/(x^t, y^t)\hat{B} \cong B/(x^t, y^t)B$ we obtain this last statement by 2.6.3. $\square$

A much simpler example with similar properties as those of 2.8.4 is the following 2.8.7. It works in any characteristic. In order to present it we first recall the idealization process, also needed in the subsequent sections.

2.8.5 *Trivial extensions.* (see [61, p. 2]) Let R denote a commutative ring. Let M be an R-module. Then we define $S = R \ltimes M$, the trivial extension of R by M. This is the R-module $R \oplus M$ with an internal multiplication given by $(r, m) \cdot (s, n) = (rs, rn + sm)$. Note that the map $R \to S, r \mapsto (r, 0)$ is injective and that $R \to S$ is a ring extension. For an R-submodule $N \subset M$ it follows that $0 \ltimes N$ is an ideal of S. Moreover, $0 \ltimes M$ is nilpotent of degree 2 and $R \ltimes M/0 \ltimes M \cong R$.

Remarks 2.8.6 Let $S = R \ltimes M$ be as in 2.8.5.

(a) Denote by $\mathfrak{a}$ the ideal $0 \ltimes M$ of S. If $M \neq 0$, then $R \cong S/\mathfrak{a}$ viewed as an S-module is not flat and $\mathfrak{a}$ viewed as an S-module is not injective.
Indeed, there is the short exact sequence $0 \to \mathfrak{a} \to S \to S/\mathfrak{a} \to 0$. With the associated long exact sequence of the $\operatorname{Tor}^S(S/\mathfrak{a}, \cdot)$ we note that $\operatorname{Tor}_1^S(S/\mathfrak{a}, S/\mathfrak{a}) \cong \mathfrak{a}/\mathfrak{a}^2 \cong \mathfrak{a} \neq 0$. Whence $S/\mathfrak{a}$ is not S-flat. Then we consider the associated long exact sequence of the $\operatorname{Ext}_R(\cdot, \mathfrak{a})$

$$0 \to \operatorname{Hom}_S(S/\mathfrak{a}, \mathfrak{a}) \to \mathfrak{a} \xrightarrow{f} \operatorname{Hom}_S(\mathfrak{a}, \mathfrak{a}) \to \operatorname{Ext}_S^1(S/\mathfrak{a}, \mathfrak{a}) \to 0.$$

Since $\mathfrak{a}^2 = 0$ we note that $\operatorname{Hom}_S(S/\mathfrak{a}, \mathfrak{a}) \cong \mathfrak{a}$ and that the homomorphism f is zero. Because $0 \neq \operatorname{Hom}_S(\mathfrak{a}, \mathfrak{a})$ it follows that $\operatorname{Ext}_S^1(S/\mathfrak{a}, \mathfrak{a})$ does not vanish. Whence $\mathfrak{a}$ is not S-injective.

(b) The ideals of S are of the form $\mathfrak{c} \ltimes (N + \mathfrak{c}M)$, for $\mathfrak{c}$ an ideal of R and N an R-submodule of N. (This is because $(1, v) \cdot (r, w) = (r, rv + w)$ for all $r \in R$ and $w, v \in M$.) For an ideal $\mathfrak{a} = \mathfrak{c} \ltimes (N + \mathfrak{c}M)$ it follows that $S/\mathfrak{a} \cong R/\mathfrak{c} \ltimes M/(N + \mathfrak{c}M)$, which is the idealization of $R/\mathfrak{c}$ by the $R/\mathfrak{c}$-module $(M/(N + \mathfrak{c}M)$.

(c) Moreover, $\operatorname{Spec}(S) = \{\mathfrak{p} \ltimes M \mid \mathfrak{p} \in \operatorname{Spec}(R)\}$. If R has a unique maximal ideal $\mathfrak{m}$, then S has a unique maximal ideal $\mathfrak{m} \ltimes M$.

(d) S is Noetherian if and only if R and M are both Noetherian. In that case $\dim S = \dim R$.

We are ready for our example.

Example 2.8.7 Let R be a complete Noetherian local ring of positive depth and let $x \in R$ be a regular element. Then R is also xR-complete (see 2.2.9). We form the trivial extension $S = R \ltimes R_x$ and take completion with respect to the ideal $\mathfrak{c} := xS$ of S. Clearly the ring S is not Noetherian. Observe that

$$\hat{S}^{\mathfrak{c}} \cong \varprojlim S/x^t S \cong \varprojlim (R/x^t R \oplus R_x/x^t R_x) \cong R.$$

Therefore $\hat{S}^{\mathfrak{c}}$ is a Noetherian ring and $\hat{S}^{\mathfrak{c}} \cong S/(0 \ltimes R_x)$ is not S-flat in view of 2.8.6.

Note, however, that $\hat{S}^{\mathfrak{c}}$ is relatively-$\mathfrak{c}$-flat. This follows by the criterion in 2.6.3. Indeed, multiplication by x is injective on both S and $\hat{S}^{\mathfrak{c}}$, hence $\operatorname{Tor}_i^S(S/\mathfrak{c}^t, \hat{S}^{\mathfrak{c}}) = 0$ for all $i \geq 1$ and clearly $\hat{S}^{\mathfrak{c}}/\mathfrak{c}^t \hat{S}^{\mathfrak{c}} \cong S/\mathfrak{c}^t$ is $S/\mathfrak{c}^t$-flat.

Here is an example of a commutative ring S, an injective S-module I and an ideal $\mathfrak{c}$ of S such that $\Gamma_{\mathfrak{c}}(I)$ is not injective.

Example 2.8.8 Let $R = \Bbbk[[x]]$ denote the power series ring in one variable over the field $\Bbbk$. Let $E = E_R(\Bbbk)$ denote the injective hull of the residue field. Then define $S = R \ltimes E$. There is an isomorphism of S-modules $\operatorname{Hom}_R(S, E) \cong S$, in particular S is self-injective (see A.4.6). We consider the ideal $\mathfrak{c} := xS$ of S and note $\Gamma_{xS}(S) = 0 \ltimes E$. Hence $\Gamma_{xS}(S)$ is not injective as an S-module in view of 2.8.6.

A more detailed study of the question of when the $\mathfrak{a}$-torsion submodule $\Gamma_{\mathfrak{a}}(I)$ of an injective R-module I is again R-injective was done by Quy and Rohrer (see [66]). They also provide different counterexamples to this question.

Chapter 3
Ext-Tor Vanishing and Completeness Criteria

As a classical result, Jensen proved that, if $\mathfrak{a} = \mathfrak{m}$ is the maximal ideal of a Noetherian local ring R, then a finitely generated R-module M is $\mathfrak{m}$-complete if and only if $\operatorname{Ext}^1_R(F, M) = 0$ for all countably generated flat modules F if and only if $\operatorname{Ext}^i_R(F, M) = 0$ for all $i \geq 1$ and any flat module F (see [48] and [49, Proposition 8.2]).

More recently, Frankild and Sather–Wagstaff obtained another completeness criterion relative to an ideal $\mathfrak{a}$ contained in the Jacobson radical of a Noetherian ring R. They proved that a finitely generated R-module M is $\mathfrak{a}$-complete if and only if $\operatorname{Ext}^i_R(\hat{R}^{\mathfrak{a}}, M) = 0$ for all $i \geq 1$ (see [35, Theorem A]).

Non-finitely generated modules have also been considered. For instance, Buchweitz and Flenner proved that, if $\mathfrak{a}$ is a maximal ideal of a commutative ring R and M is an $\mathfrak{a}$-complete R-module, then $\operatorname{Ext}^i_R(F, M) = 0$ for all $i \geq 1$ and any flat modules (see [17, Theorem 2.3]).

For an arbitrary ideal $\mathfrak{a}$ of a commutative ring R and an R-module M it was proved (see [76, Theorem 4.4]) that $\operatorname{Ext}^i_R(F, \hat{M}^{\mathfrak{a}}) = 0$ for all $i \geq 0$ as soon as F is a flat module with $F = \mathfrak{a}F$. The same paper provides another completeness criterion: if $\mathfrak{a} = (x_1, \ldots, x_k)$ is an ideal of a Noetherian ring R then an $\mathfrak{a}$-separated R-module M is $\mathfrak{a}$-complete if and only $\operatorname{Ext}^1_R(\oplus_{i=0}^k R_{x_i}, M) = 0$ (see [76, Theorem 1.1]).

In this chapter we continue with some further investigations, we revisit and generalize the above-mentioned $\mathfrak{a}$-adic completeness criteria. We observe that the flat modules F occurring in these results all have the property that $F/\mathfrak{a}F$ is free as an $R/\mathfrak{a}$-module. That is why modules M such that $M/\mathfrak{a}M$ is free (possibly zero) will retain our attention.

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3.1 Completeness and Pseudo-completeness Criteria

In this section we start with an elaboration and extension of results in [49], [17] and [76]. We also present new and different proofs. First we have a generalization of both [17, 2.3] and [76, Theorem 4.4].

Theorem 3.1.1 *Let $\mathfrak{a}$ denote an ideal of a commutative ring R. Let F be a relatively-$\mathfrak{a}$-flat module such that $F/\mathfrak{a}F$ is $R/\mathfrak{a}$-free.*

(a) $\operatorname{Ext}^i_R(F, \hat{M}^{\mathfrak{a}}) = 0$ *for all $i \geq 1$ and any R-module M.*
(b) Suppose that the R-module M is $\mathfrak{a}$-complete. Then $\operatorname{Ext}^i(F, M) = 0$ *for all $i \geq 1$.*
(c) Suppose that the R-module M is $\mathfrak{a}$-pseudo-complete. Then $\operatorname{Ext}^i(F, M) = 0$ *for all $i \geq 1$.*

Proof First we choose a free R-module L with a surjection $L \to F$. So there is a short exact sequence

$$\mathcal{S} : 0 \to F_1 \xrightarrow{i} L \to F \to 0.$$

We recall that the short sequence $\hat{\mathcal{S}}^{\mathfrak{a}}$ is split-exact (see 2.6.5). Moreover, we observe that the short sequence $R/\mathfrak{a} \otimes_R \mathcal{S}$ also is split-exact (it is exact because F is relatively $\mathfrak{a}$-flat and it is split because $F/\mathfrak{a}F$ is $R/\mathfrak{a}$-free).

(a): Let $h : F_1 \to \hat{M}^{\mathfrak{a}}$ be an arbitrary homomorphism. Then we have a homomorphism $h' : \hat{F}_1^{\mathfrak{a}} \to \hat{M}^{\mathfrak{a}}$ such that $h = h' \circ \tau^{\mathfrak{a}}_{F_1}$ (see 2.1.10). The homomorphism h' factors through $\hat{i}^{\mathfrak{a}}$ because $\hat{i}^{\mathfrak{a}}$ is split-injective. Hence there is a homomorphism $g : \hat{L}^{\mathfrak{a}} \to \hat{M}^{\mathfrak{a}}$ such that $g \circ \hat{i}^{\mathfrak{a}} = h'$. By view of the commutative diagram

$$\begin{tikzcd}
F_1 \arrow[r, "i"] \arrow[d, "\tau^{\mathfrak{a}}_{F_1}"] & L \arrow[d, "\tau^{\mathfrak{a}}_{L}"] \\
\hat{F_1}^{\mathfrak{a}} \arrow[r, "\hat{i}^{\mathfrak{a}}"] & \hat{L}^{\mathfrak{a}}
\end{tikzcd}$$

we have that $h = h' \circ \tau^{\mathfrak{a}}_{F_1} = g \circ \hat{i}^{\mathfrak{a}} \circ \tau^{\mathfrak{a}}_{F_1} = g \circ \tau^{\mathfrak{a}}_{L} \circ i$. Whence h factors through i and therefore $\operatorname{Ext}^1_R(F, \hat{M}^{\mathfrak{a}}) = 0$.

Next we observe that the module F_1 is also relatively-$\mathfrak{a}$-flat. As the sequence $R/\mathfrak{a} \otimes_R \mathcal{S}$ is split-exact we get that $F_1/\mathfrak{a}F_1$ is $R/\mathfrak{a}$-projective. Moreover we also have that the relatively-$\mathfrak{a}$-flat R-module $F_1 \oplus F$ has the property that $(F_1 \oplus F)/\mathfrak{a}(F_1 \oplus F) \cong L/\mathfrak{a}L$ is $R/\mathfrak{a}$-free. By the first part of the proof it follows now that $\operatorname{Ext}^1_R(F_1 \oplus F, \hat{M}^{\mathfrak{a}}) = 0$, and therefore $0 = \operatorname{Ext}^1_R(F_1, \hat{M}^{\mathfrak{a}}) = \operatorname{Ext}^2_R(F, \hat{M}^{\mathfrak{a}})$. The result follows by induction.

(b): When the R-module M is $\mathfrak{a}$-complete we have $M \cong \hat{M}^{\mathfrak{a}}$. Hence (b) is a particular case of (a).

(c): Assume now that the R-module M is $\mathfrak{a}$-pseudo-complete, i.e. that the natural map $\eta^{\mathfrak{a}}_M : M \to \Lambda^{\mathfrak{a}}_0(M)$ is an isomorphism. This allows us to put $\eta^{\mathfrak{a}}_M = \mathrm{id}_M$. First observe that we may also put $\Lambda^{\mathfrak{a}}_0(\mathcal{S}) = \hat{\mathcal{S}}^{\mathfrak{a}}$ and $\eta^{\mathfrak{a}}_{\mathcal{S}} = \tau^{\mathfrak{a}}_{\mathcal{S}}$ because the modules in the short exact sequence $\mathcal{S}$ are relatively-$\mathfrak{a}$-flat, hence belong to the class $\mathcal{C}_{\mathfrak{a}}$ introduced in 2.5.12. Then the statement is proved in the same way as in (a). Let $h : F_1 \to M$ be a homomorphism. Then $\Lambda_0^{\mathfrak{a}}(h) : \Lambda_0^{\mathfrak{a}}(F_1) = \hat{F}_1^{\mathfrak{a}} \to \Lambda_0^{\mathfrak{a}}(M) = M$ factors through $\Lambda_0^{\mathfrak{a}}(i) = \hat{i}^{\mathfrak{a}}$ because $\hat{i}^{\mathfrak{a}}$ is split-injective. That is, there is a homomorphism $g : \hat{L}^{\mathfrak{a}} \to M$ such that $g \circ \Lambda_0^{\mathfrak{a}}(i) = \Lambda_0^{\mathfrak{a}}(h)$. In view of the commutative diagrams

$$\begin{tikzcd} F_1 \arrow[r, "h"] \arrow[d, "\eta^{\mathfrak{a}}_{F_1}"] & M \arrow[d, "\eta^{\mathfrak{a}}_M"] \\ \Lambda_0^{\mathfrak{a}}(F_1) \arrow[r, "\Lambda_0^{\mathfrak{a}}(h)"] & \Lambda_0^{\mathfrak{a}}(M), \end{tikzcd} \qquad \begin{tikzcd} F_1 \arrow[r, "i"] \arrow[d, "\eta^{\mathfrak{a}}_{F_1}"] & L \arrow[d, "\eta^{\mathfrak{a}} L"] \\ \Lambda_0^{\mathfrak{a}}(F_1) \arrow[r, "\Lambda_0^{\mathfrak{a}}(i)"] & \Lambda_0^{\mathfrak{a}}(L). \end{tikzcd}$$

we have $h = \Lambda_0^{\mathfrak{a}}(h) \circ \eta^{\mathfrak{a}}_{F_1}$ and $h = g \circ \Lambda_0^{\mathfrak{a}}(i) \circ \eta^{\mathfrak{a}}_{F_1} = g \circ \eta^{\mathfrak{a}}_L \circ i$. Whence h factors through i and $\mathrm{Ext}^1_R(F, M) = 0$. We now finish the proof as in (a) by induction. $\square$

In relation to the statements in (a) and (b) note the following: (a) is more general than (b) because a module of the form $\hat{M}^{\mathfrak{a}}$ is not necessarily $\mathfrak{a}$-complete when the ideal $\mathfrak{a}$ is not finitely generated.

Remark 3.1.2 Some particular cases of the above will attract more attention. Let $\mathfrak{a}$ be an ideal of a Noetherian ring R. As above, let $S_{\mathfrak{a}}(R) = \mathrm{Coker}(\tau_R^{\mathfrak{a}})$ and let M be an $\mathfrak{a}$-complete R-module. Then

(a) $\mathrm{Ext}^i_R(\hat{R}^{\mathfrak{a}}, M) = 0$ for all $i \geq 1$ and
(b) $\mathrm{Ext}^i_R(S_{\mathfrak{a}}(R), M) = 0$ for all $i \geq 0$.

This is true because $\hat{R}^{\mathfrak{a}}$ is R-flat and because $\hat{R}^{\mathfrak{a}}/\mathfrak{a}\hat{R}^{\mathfrak{a}} \cong R/\mathfrak{a}$ is $R/\mathfrak{a}$-free. Moreover, $S_{\mathfrak{a}}(R)$ is relatively $\mathfrak{a}$-flat (see 2.6.2) and $S_{\mathfrak{a}}(R)/\mathfrak{a}S_{\mathfrak{a}}(R) = 0$, so that we also have $\mathrm{Hom}_R(S_{\mathfrak{a}}(R), M) = 0$ by 2.1.2.

Hence the natural homomorphism $\mathrm{Hom}_R(\hat{R}^{\mathfrak{a}}, M) \to \mathrm{Hom}_R(R, M) = M$ is an isomorphism when M is $\mathfrak{a}$-complete. (To see this apply the functor $\mathrm{Hom}_R(\cdot, M)$ to the exact sequence $0 \to R_s \to \hat{R}^{\mathfrak{a}} \to S_{\mathfrak{a}}(R) \to 0$, where $R_s = R/\bigcap_t \mathfrak{a}^t$, and observe that $\mathrm{Hom}_R(R_s, M) \cong \mathrm{Hom}_R(R, M)$.)

The $\mathfrak{a}$-completeness criteria we have in mind concern the case where the ideal $\mathfrak{a}$ is finitely generated. We first investigate the case where the ideal $\mathfrak{a}$ is principal, say generated by a single element $x \in R$.

Recall 3.1.3 Let x be an element of a commutative ring R. Then R_x, the ring of fractions of the form $\frac{r}{x^n}$, $r \in R$, is a countably generated flat R-module of projective dimension ≤ 1. It is coarse in its xR-adic topology since $R_x = xR_x$. It can be viewed as a direct limit: put $R_t = R$, $t \geq 0$. Then $R_x = \varinjlim\{R_{t+1} \xrightarrow{x} R_t\}$. Its projective resolution is given by the exact sequence

$$0 \to L \xrightarrow{d} L \to R_x \to 0,$$

where L is a free R-module with countable base $(e_0, e_1, e_2, \ldots)$. The map d is defined by $d(e_i) = e_i - xe_{i+1}$. Whence the morphisms of R_x into an arbitrary R-module M are in bijection with the sequences $(w_0, w_1, \ldots, w_i, \ldots) \in \prod M$ such that $w_i = xw_{i+1}$. In view of this the following result is rather obvious.

Lemma 3.1.4 *Let T be an R-module such that $T = xT \neq 0$. Then* $\operatorname{Hom}_R(R_x, T) \neq 0$.

In the following we prove a partial converse of Theorem 3.1.1 in the particular case when $F = R_x$. It fills a gap in the proof of [76, Theorem 1.1].

Proposition 3.1.5 *Let x be an element of a commutative ring R. Let M be an R-module, separated in its xR-adic topology. Then M is xR-complete if and only if* $\operatorname{Ext}^1_R(R_x, M) = 0$.

Proof The only if part is a particular case of 3.1.1. For the reverse implication take the exact sequence

$$0 \to M \to \hat{M}^{xR} \to \operatorname{Coker}(\tau_M^{xR}) \to 0$$

and the associated long exact cohomology sequence of the $\operatorname{Ext}^i_R(R_x, \cdot)$.

By 2.1.2 we have that $\operatorname{Hom}_R(R_x, M) = 0 = \operatorname{Hom}_R(R_x, \hat{M}^{xR})$. Therefore the long exact cohomology sequence provides an injection

$$0 \to \operatorname{Hom}_R(R_x, \operatorname{Coker}(\tau_M^{xR})) \to \operatorname{Ext}^1_R(R_x, M).$$

Thus the assumption $\operatorname{Ext}^1_R(R_x, M) = 0$ implies $\operatorname{Hom}_R(R_x, \operatorname{Coker}(\tau_M^{xR})) = 0$. On the other hand we have

$$\operatorname{Coker}(\tau_M^{xR}) = x\operatorname{Coker}(\tau_M^{xR})$$

(see 2.2.2). By virtue of 3.1.4 it follows that $\operatorname{Coker}(\tau_M^{xR}) = 0$. That is, $M \cong \hat{M}^{xR}$ is xR-complete. □

We can now state [76, Theorem 1.1] without the assumption that the ring R is Noetherian. Note that the following together with 3.1.1 also provides a generalization of Jensen's criterion in [49, Proposition 8.2].

Theorem 3.1.6 *Let $\mathfrak{a} = (x_1, \ldots, x_k)R$ be a finitely generated ideal of a commutative ring R. Let M be an $\mathfrak{a}$-separated R-module. Then M is $\mathfrak{a}$-complete if and only if* $\operatorname{Ext}^1_R(\oplus_{i=1}^k R_{x_i}, M) = 0$.

Proof The "only if" part is a consequence of 3.1.1. For the converse first note that for all $x \in \mathfrak{a}$ the xR-adic topology on M is finer than the $\mathfrak{a}$-adic topology. Thus M is also xR-separated. By 3.1.5 we obtain that M is complete in its x_iR-adic topology for all $i = 1, \ldots, k$. Now the result follows by an iterative use of 2.2.10. □

Remark 3.1.7 In the situation of 3.1.6 and because M is $\mathfrak{a}$-separated we also have $\operatorname{Hom}_R(\oplus_{i=1}^k R_{x_i}, M) = 0$ in view of 2.1.2. But we cannot replace in 3.1.6 the condition that M is $\mathfrak{a}$-separated by the weaker condition $\operatorname{Hom}_R(\oplus_{i=1}^k R_{x_i}, M) = 0$ without harm. Indeed, recall the module X of Example 2.4.14. This X is not $\mathfrak{a}$-separated and is inserted in a short exact sequence $0 \to C_1 \to X \to C_2 \to 0$, where C_1 and C_2 are $\mathfrak{a}$-complete. With the associated long exact cohomology sequence of the $\operatorname{Ext}^i_R(R_x, \cdot)$, from 2.1.2 and 3.1.1 it follows that $\operatorname{Ext}^i_R(R_x, X) = 0$ for all $x \in \mathfrak{a}$ and all $i \geq 0$. But note that this module X, which is not $\mathfrak{a}$-complete, is however $\mathfrak{a}$-pseudo-complete, as observed in 2.5.10. This example is a particular case of the following general result.

Proposition 3.1.8 *Let $\mathfrak{a}$ denote a finitely generated ideal of a commutative ring R. Let M be an $\mathfrak{a}$-pseudo-complete R-module. Then for all relatively-$\mathfrak{a}$-flat R-modules such that $F = \mathfrak{a}F$ we have $\operatorname{Ext}^i_R(F, M) = 0$ for all $i \geq 0$.*

Proof Let $L \xrightarrow{\sim} M$ denote a free resolution of M. It induces an exact sequence $0 \to Z \to \hat{L}_1^{\mathfrak{a}} \to \hat{L}_0^{\mathfrak{a}} \to M \to 0$, where the R-module Z is $\mathfrak{a}$-complete as an $\mathfrak{a}$-closed submodule of an $\mathfrak{a}$-complete module (see 2.2.11). We split the sequence into short exact sequences

$$0 \to Z \to \hat{L}_1^{\mathfrak{a}} \to M_1 \to 0 \text{ and } 0 \to M_1 \to \hat{L}_0^{\mathfrak{a}} \to M \to 0.$$

Applying the associated long exact cohomology sequence of the $\operatorname{Ext}^i_R(F, \cdot)$ to the first sequence we first obtain $\operatorname{Ext}^i_R(F, M_1) = 0$ for all $i \geq 0$ (see 2.1.2 and 3.1.1). We then obtain the claim with the second sequence. $\square$

Here is some more precise information in the case when the ideal $\mathfrak{a}$ is principal.

Observations 3.1.9 Let x denote an element of a commutative ring R and let M be an R-module.

(a) Multiplication by x gives us an inverse system of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & 0 :_M x^{t+1} \arrow[r] \arrow[d, "x"] & M \arrow[r, "x^{t+1}"] \arrow[d, "x"] & x^{t+1}M \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & 0 :_M x^t \arrow[r] & M \arrow[r, "x^t"] & x^t M \arrow[r] & 0
\end{tikzcd}$$

where the right vertical maps are the natural inclusions. By comparing the homomorphism $\psi_{\{M,x\}}$ associated to the inverse system $\{M, x\}$ in the middle with the free resolution of R_x given in 3.1.3 we observe that $\psi_{\{M,x\}} = \operatorname{Hom}_R(d, M)$, so that

$$\varprojlim\{M, x\} \cong \operatorname{Hom}_R(R_x, M) \text{ and } \varprojlim{}^1\{M, x\} \cong \operatorname{Ext}^1_R(R_x, M)$$

(see 1.2.2). By passing to the limit we obtain a six-term long exact sequence

$$0 \to \varprojlim\{0 :_M x^t, x\} \to \operatorname{Hom}_R(R_x, M) \to \cap_{n\geq 1} x^t M$$
$$\to \varprojlim{}^1\{0 :_M x^t, x\} \to \operatorname{Ext}^1(R_x, M) \to \varprojlim{}^1\{x^t M\} \to 0.$$

(b) The following implications hold.

(1) If $\operatorname{Ext}^1_R(R_x, M) = 0$, then M is xR-quasi-complete.
(As $\operatorname{Coker}(\tau_M^{xR}) = \varprojlim{}^1\{x^t M\}$ (see 2.2.1), this is obtained with the last part of the six-term long exact sequence.)
(2) If M is xR-separated then $\operatorname{Hom}_R(R_x, M) = 0$.
(This is a particular case of 2.1.2.)
(3) If M is xR-complete then $\operatorname{Hom}_R(R_x, M) = \operatorname{Ext}^1_R(R_x, M) = 0$.
(This follows from $(b)(2)$ and 3.1.1.)

Note that this provides a second proof of 3.1.5.

(c) The modules $0 :_M x^t$ are xR-complete and so is their product. In view of the description of $\varprojlim$ and $\varprojlim{}^1$ given in 1.2.2 it follows that both modules $\varprojlim\{0 :_M x^t, x\}$ and $\varprojlim{}^1\{0 :_M x^t, x\}$ are xR-pseudo-complete (see 2.5.7).

With certain additional assumptions on M we can prove some reverse implications in (3.1.9 (b)).

Proposition 3.1.10 (see [76, 3.3]) *Let x denote an element of a commutative ring R. Let M be an R-module and suppose that the ascending sequence $0 :_M x^t$ of submodules of M becomes stationary. Then*

$$\operatorname{Hom}_R(R_x, M) \cong \cap_{n\geq 1} x^t M \text{ and } \operatorname{Ext}^1(R_x, M) \cong \varprojlim{}^1\{x^t M\}.$$

Moreover, we have the following equivalences.

(1) $\operatorname{Ext}^1_R(R_x, M) = 0$ if and only if M is xR-quasi-complete.
(2) M is xR-separated if and only if $\operatorname{Hom}_R(R_x, M) = 0$.
(3) M is xR-complete if and only if $\operatorname{Hom}_R(R_x, M) = \operatorname{Ext}^1_R(R_x, M) = 0$.

Proof The assumption means that the inverse system $\{0 :_M x^t, x\}$ is pro-zero, so that $\varprojlim\{0 :_M x^t, x\} = \varprojlim{}^1\{0 :_M x^t, x\} = 0$ (see 1.2.4). We then obtain the isomorphisms with the six-term exact sequence in 3.1.9. By the observations in (3.1.9 (b)) we only need to prove the if part of the equivalences. The second isomorphism proves the if part of (1) ($\operatorname{Coker}(\tau_M^{xR} \cong \varprojlim{}^1\{x^t M\}$ (see 2.2.1)). The if part of (2) is given by the first isomorphism and the equivalence (3) follows. □

We end this section with a converse of 3.1.8 in the case when the ring is Noetherian.

Theorem 3.1.11 *Let $\mathfrak{a} = (x_1, \ldots, x_k) R$ be an ideal of a Noetherian ring R and let M be an R-module. The following conditions are equivalent:*

(i) *M is $\mathfrak{a}$-pseudo-complete.*
(ii) *For every relatively-$\mathfrak{a}$-flat module F with $F = \mathfrak{a}F$ we have*

$$\operatorname{Ext}^i_R(F, M) = 0 \textit{ for all } i \geq 0,$$

(iii) $\operatorname{Hom}_R(\oplus_{i=1}^k R_{x_i}, M) = 0 = \operatorname{Ext}^1_R(\oplus_{i=1}^k R_{x_i}, M)$.

Proof In view of 3.1.8 it is enough to prove (iii) $\Rightarrow$ (i). At first we consider the case where the ideal $\mathfrak{a}$ is principal, say generated by a single element x. In that case, the condition $\operatorname{Ext}^1(R_x, M) = 0$ implies that M is xR-quasi-complete (see (3.1.9 (b)(1))). Whence we have the short exact sequence

$$0 \to \cap_{t \geq 1} x^t M \to M \to \hat{M}^{xR} \to 0.$$

With both conditions $\operatorname{Hom}_R(R_x, M) = \operatorname{Ext}^1(R_x, M) = 0$ and in view of the six-term long exact sequence in 3.1.9 we have that $\bigcap_t x^t M \cong \varprojlim{}^1 \{0 :_M x^t, x\}$. But the latter module is xR-pseudo-complete (see 3.1.9 (c)), and so is the module $\hat{M}^{xR}$ (see 2.5.7). Now the claim for a principal ideal follows by 2.5.17 applied to our short exact sequence.

The general case is obtained with an iterative use of 2.5.21. $\square$

The above theorem, which also emphasizes the importance of the separateness condition in 3.1.6, will be highly generalized in Part II (see 9.6.8).

3.2 Modules of Infinite Co-depth

In the following we need both notions of Ext-depth and Tor-codepth for a module.

3.2.1 *Ext-depth and Tor-codepth.* Let $\mathfrak{a}$ be an ideal in a commutative ring R. As in [82] we define the Ext-depth and the Tor-codepth of an R-module M with respect to the ideal $\mathfrak{a}$ as follows:

$$\begin{aligned} \text{E-dp}(\mathfrak{a}, M) &= \operatorname{ext}^-_R(R/\mathfrak{a}, M) := \inf\{i \mid \operatorname{Ext}^i_R(R/\mathfrak{a}, M) \neq 0\} \\ \text{T-codp}(\mathfrak{a}, M) &= \operatorname{tor}^R_-(R/\mathfrak{a}, M) := \inf\{i \mid \operatorname{Tor}^R_i(R/\mathfrak{a}, M) \neq 0\}. \end{aligned}$$

By convention $0 \leq \text{E-dp}(\mathfrak{a}, M), \text{T-codp}(\mathfrak{a}, M) \leq \infty$. In particular, the vanishing of $\operatorname{Ext}^i_R(R/\mathfrak{a}, M)$ for all $i \geq 0$ (resp. $\operatorname{Tor}^R_i(R/\mathfrak{a}, M) = 0$ for all $i \geq 0$) says by definition that $\text{E-dp}(\mathfrak{a}, M) = \infty$ (resp. $\text{T-codp}(\mathfrak{a}, M) = \infty$).

For two ideals $\mathfrak{a} \subset \mathfrak{b}$, all $t \geq 1$ and any R-module M we have the comparison formulas:

$$\begin{aligned} \text{E-dp}(\mathfrak{a}^t, M) &= \text{E-dp}(\mathfrak{a}, M) \leq \text{E-dp}(\mathfrak{b}, M) \text{ and} \\ \text{T-codp}(\mathfrak{a}^t, M) &= \text{T-codp}(\mathfrak{a}, M) \leq \text{T-codp}(\mathfrak{b}, M). \end{aligned}$$

For the assertion on the Ext-depth, see [82, 5.3.11 and 5.3.16]. The statement on the Tor-codepth is shown in [79, 1.6]. It follows by the use of the Ext-Tor duality. (The reader may also compare these numbers by using formal acyclicity lemmas as stated in [78, 6.1] or [82, 1.1.1, 1.1.2], or refer to Part II (see 5.1.6 and 5.1.7), where complex versions are available.)

Definitions and observation 3.2.2 *The class* $\mathcal{T}_{\mathfrak{a}}$. We denote by $\mathcal{T}_{\mathfrak{a}}$ the class of R-modules M with $\operatorname{T-codp}(\mathfrak{a}, M) = \infty$. That is,

$$M \in \mathcal{T}_{\mathfrak{a}} \text{ if and only if } \operatorname{Tor}_i^R(R/\mathfrak{a}, M) = 0 \text{ for all } i \in \mathbb{Z}.$$

By the comparison formulas in 3.2.1 we have that a module T belongs to $\mathcal{T}_{\mathfrak{a}}$ if and only if $T = \mathfrak{a}T$ and T is relatively $\mathfrak{a}$-flat in the sense of 2.6.1. We note that this class $\mathcal{T}_{\mathfrak{a}}$ is contained in the class $\mathcal{C}_{\mathfrak{a}}$ introduced in 2.5.12, that is,

$$T \in \mathcal{T}_{\mathfrak{a}} \text{ implies } \hat{T}^{\mathfrak{a}} = 0 = \Lambda_i^{\mathfrak{a}}(T) \text{ for all } i \geq 0.$$

Moreover, if the ring R is Noetherian then $T \in \mathcal{T}_{\mathfrak{a}}$ implies $\hat{R}^{\mathfrak{a}} \otimes_R T \in \mathcal{T}_{\mathfrak{a}\hat{R}^{\mathfrak{a}}}$ (see 2.5.22).

Modules in $\mathcal{T}_{\mathfrak{a}}$ play an essential rôle in the study of completion. For example, the module $\oplus_{i=1}^k R_{x_i}$ appearing in criteria 3.1.6 and 3.1.11 is in the class $\mathcal{T}_{\mathfrak{a}}$ for $\mathfrak{a} = \underline{x}R$. Moreover, modules in $\mathcal{T}_{\mathfrak{a}}$ appear naturally in the completion process. By 2.6.2 and 2.2.2 we have that $\operatorname{Coker}(\tau_R^{\mathfrak{a}}) \in \mathcal{T}_{\mathfrak{a}}$ provided the ring is Noetherian. More generally, we have the following.

Proposition 3.2.3 *Let* $\mathfrak{a}$ *be an ideal of a Noetherian ring* R. *Let* M *be an* R*-module and put* $S_{\mathfrak{a}}(M) := \operatorname{Coker}(\tau_M^{\mathfrak{a}})$. *Then* $M/\cap\mathfrak{a}^t M \in \mathcal{C}_{\mathfrak{a}}$ *if and only if* $S_{\mathfrak{a}}(M) \in \mathcal{T}_{\mathfrak{a}}$.

Proof Put $M_s := M/\cap_t \mathfrak{a}^t M$. Then $\hat{M}^{\mathfrak{a}} = \hat{M}_s^{\mathfrak{a}}$ and $S_{\mathfrak{a}}(M) = S_{\mathfrak{a}}(M_s)$ as is easily seen. Moreover, we have the short exact sequence

$$0 \to M_s \xrightarrow{\tau_{M_s}^{\mathfrak{a}}} \hat{M}_s^{\mathfrak{a}} \to S_{\mathfrak{a}}(M_s) \to 0.$$

Let L denote a free resolution of M_s and assume first that $M_s \in \mathcal{C}_{\mathfrak{a}}$. Then $\hat{L}^{\mathfrak{a}}$ is an R-resolution of $\hat{M}_s^{\mathfrak{a}}$ and even a flat resolution by 2.4.4. But the homomorphism

$$\tau_L \otimes_R R/\mathfrak{a} : L \otimes_R R/\mathfrak{a} \to \hat{L}^{\mathfrak{a}} \otimes_R R/\mathfrak{a}$$

is an isomorphism of complexes. Hence the induced homomorphisms

$$\operatorname{Tor}_i^R(M_s, R/\mathfrak{a}) \to \operatorname{Tor}_i^R(\hat{M}_s^{\mathfrak{a}}, R/\mathfrak{a})$$

are isomorphisms for all $i \geq 0$. By the long exact sequence of the Tor functors it follows that $\operatorname{Tor}_i^R(S_{\mathfrak{a}}(M_s), R/\mathfrak{a}) = 0$ for all $i \geq 0$ and so $S_{\mathfrak{a}}(M) = S_{\mathfrak{a}}(M_s) \in \mathcal{T}_{\mathfrak{a}}$.

Conversely, if $S_{\mathfrak{a}}(M) = S_{\mathfrak{a}}(M_s) \in \mathcal{T}_{\mathfrak{a}}$, then $M_s \in \mathcal{C}_{\mathfrak{a}}$ as a direct consequence of 2.5.13 and 2.5.15. □

Corollary 3.2.4 *When the ring R is Noetherian and the R-module M is finitely generated we know that $M/\cap \mathfrak{a}^t M \in \mathcal{C}_{\mathfrak{a}}$. Therefore $S_{\mathfrak{a}}(M) := \operatorname{Coker}(\tau_M^{\mathfrak{a}}) \in \mathcal{T}_{\mathfrak{a}}$.*

Here is a characterization of modules in $\mathcal{T}_{\mathfrak{a}}$. It may be compared with the completeness criterion in 3.1.6.

Proposition 3.2.5 *Let $\mathfrak{a}$ be a finitely generated ideal of the commutative ring R and T an R-module. The following conditions are equivalent:*

(i) $T \in \mathcal{T}_{\mathfrak{a}}$.
(ii) $\operatorname{Ext}^i_R(T, M) = 0$ *for any $\mathfrak{a}$-pseudo-complete module M and all $i \geq 0$.*
(iii) $\operatorname{Ext}^i_R(T, M) = 0$ *for any $\mathfrak{a}$-complete module M and all $i \geq 0$.*

Proof We already know that (i) implies (ii) (see 3.1.8). Moreover, (ii) implies (iii) because $\mathfrak{a}$-complete modules are $\mathfrak{a}$-pseudo-complete (see 2.5.7). To finish the proof we use the general Matlis duality as introduced in 1.4.8. When condition (iii) is satisfied it follows that $\operatorname{Ext}^i_R(T, (R/\mathfrak{a})^\vee) = 0$ for all $i \geq 0$ ($(R/\mathfrak{a})^\vee$ is $\mathfrak{a}$-complete because it is annihilated by $\mathfrak{a}$). We conclude with the Ext-Tor duality. □

The importance of these notions is underlined by the following formula of Auslander–Buchsbaum type. (It is a direct consequence of the Ext-depth Tor-codepth sensitivity of the Koszul complex as stated for modules in [82] and extended to complexes in 5.3.3 together with the self-duality of the Koszul complex.)

Theorem 3.2.6 (see [82, 6.1.8], or Part II (5.3.5 and 5.3.9) for a complex version) *Let $\mathfrak{a} = (x_1, \ldots, x_k)R$ be a finitely generated ideal of a commutative ring R. Let M denote an R-module. Then the following two conditions are equivalent:*

(i) $\operatorname{E-dp}(\mathfrak{a}, M) < \infty$.
(ii) $\operatorname{T-codp}(\mathfrak{a}, M) < \infty$.

If one of these conditions is satisfied, then $\operatorname{E-dp}(\mathfrak{a}, M) + \operatorname{T-codp}(\mathfrak{a}, M) \leq k$.

Proposition 3.2.7 *Let $\mathfrak{a}$ be an ideal of the commutative ring R. Let M, N be two R-modules with $M \xrightarrow{\sim} I$ a minimal injective resolution of M. Then*

(a) $\operatorname{E-dp}(\mathfrak{a}, M) = \infty$ *if and only if $\Gamma_{\mathfrak{a}}(I) = 0$.*
(b) If N is $\mathfrak{a}$-torsion and if $\operatorname{E-dp}(\mathfrak{a}, M) = \infty$, *then* $\operatorname{Ext}^i_R(N, M) = 0$ *for all $i \geq 0$.*

Proof (a): If $\Gamma_{\mathfrak{a}}(I) = 0$ then $\operatorname{E-dp}(\mathfrak{a}, M) = \infty$ because

$$\operatorname{Hom}_R(R/\mathfrak{a}^t, I) \cong \operatorname{Hom}_R(R/\mathfrak{a}^t, \Gamma_{\mathfrak{a}}(I))$$

for all $t \geq 1$.

Conversely, assume that $\operatorname{E-dp}(\mathfrak{a}, M) = \infty$. Then $\Gamma_{\mathfrak{a}}(M) = 0$ and also $\Gamma_{\mathfrak{a}}(I^0) = 0$ because I^0 is an injective hull of M. Now consider the short exact sequence $0 \to$

$M \to I^0 \to M_1 \to 0$. We have $\mathrm{Hom}_R(R/\mathfrak{a}, M_1) = 0$ because $\mathrm{Hom}_R(R/\mathfrak{a}, I^0) = 0 = \mathrm{Ext}^1_R(R/\mathfrak{a}, M)$. For $i > 0$ we also have $\mathrm{Ext}^i_R(R/\mathfrak{a}, M_1) \cong \mathrm{Ext}^{i+1}_R(R/\mathfrak{a}, M) = 0$. Hence $\text{E-dp}(\mathfrak{a}, M_1) = \infty$ and we conclude by induction.

(b): When N is $\mathfrak{a}$-torsion we have $\mathrm{Hom}_R(N, I) \cong \mathrm{Hom}_R(N, \Gamma_{\mathfrak{a}}(I))$. So the claim in (b) follows from the assertion in (a). □

We can now fill a gap in the statement of [76, Theorem 3.2] (this had some implicit assumptions).

Corollary 3.2.8 *Let $\mathfrak{a}$ be an ideal of a Noetherian ring R. Let $M \in \mathcal{C}_{\mathfrak{a}}$ be an R-module which is $\mathfrak{a}$-separated and write as before $S_{\mathfrak{a}}(M) := \mathrm{Coker}(\tau^{\mathfrak{a}}_M)$. Let N be another R-module such that $\mathrm{Supp}_R(N) \subseteq V(\mathfrak{a})$. Then*

(a) $\mathrm{Ext}^i_R(N, S_{\mathfrak{a}}(M) = 0$ *for all* $i \geq 0$ *and*
(b) *the natural maps* $\mathrm{Ext}^i_R(N, M) \to \mathrm{Ext}^i_R(N, \hat{M}^{\mathfrak{a}})$ *are all isomorphisms.*

Proof We have $S_{\mathfrak{a}}(M) \in \mathcal{T}_{\mathfrak{a}}$ (see 3.2.3) hence $\text{E-dp}(\mathfrak{a}, S_{\mathfrak{a}}(M)) = \infty$ (see 3.2.6). But the module N is $\mathfrak{a}$-torsion (see 2.1.13). Hence the first assertion is a direct consequence of 3.2.7. The second one is then obtained from the long exact cohomology sequence associated to the short exact sequence $0 \to M \to \hat{M}^{\mathfrak{a}} \to S_{\mathfrak{a}}(M) \to 0$. □

Now we prepare for another property of modules in $\mathcal{T}_{\mathfrak{a}}$. The following lemma is well-known in homological algebra. For the sake of completeness we include a proof.

Lemma 3.2.9 *Let $L : \ldots \to L_1 \to L_0 \to 0$ denote a complex of free R-modules. Let $\mathfrak{a} \subset R$ be an ideal and assume that the complex $R/\mathfrak{a} \otimes_R L$ is exact. Then*

$$\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{Hom}_R(L, X))$$

is exact for every R-complex X.

Proof By adjointness we have the isomorphism of complexes

$$\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{Hom}_R(L, X)) \cong \mathrm{Hom}_R(R/\mathfrak{a} \otimes_R L, X).$$

By assumption the complex $R/\mathfrak{a} \otimes_R L$ is a right-bounded exact $R/\mathfrak{a}$-complex of free $R/\mathfrak{a}$-modules. Hence it is split-exact and the conclusion follows (see 1.1.8). □

Proposition 3.2.10 *Let $\mathfrak{a}$ be an ideal of a commutative ring. Let T, M denote two R-modules such that $T \in \mathcal{T}_{\mathfrak{a}}$ and $\mathrm{Ext}^i_R(T, M) = 0$ for all $i \neq 1$. Then:*

(a) $\text{E-dp}(\mathfrak{a}, \mathrm{Ext}^1_R(T, M)) = \infty$.
(b) $\mathrm{Ext}^1_R(T, M) \in \mathcal{T}_{\mathfrak{a}}$ *provided $\mathfrak{a}$ is finitely generated.*

Proof Let L be a free resolution of T and let J be an injective resolution of M. Now consider $I = \mathrm{Hom}_R(L, J)$. Then I is a complex of injective modules with $H^i(I) = \mathrm{Ext}^i_R(T, M)$.

Because $\operatorname{Hom}_R(T, M) = 0$ it follows that the map $I^0 \to I^1$ is injective, hence split-injective. Let $I'^1 = I^1/I^0$, then we obtain a split short exact sequence

$$0 \to \operatorname{Hom}_R(R/\mathfrak{a}, I^0) \to \operatorname{Hom}_R(R/\mathfrak{a}, I^1) \to \operatorname{Hom}_R(R/\mathfrak{a}, I'^1) \to 0$$

and an augmented injective resolution of $\operatorname{Ext}^1_R(T, M)$

$$0 \to \operatorname{Ext}^1_R(T, M) \to I'^1 \to I^2 \to \dots$$

since $\operatorname{Ext}^i_R(T, M) = 0$ for $i \neq 1$.

Moreover, the assumption $\text{T-codp}(\mathfrak{a}, T) = \infty$ means that the complex $L/\mathfrak{a}L$ is exact. By 3.2.9 we obtain that the complex $\operatorname{Hom}_R(R/\mathfrak{a}, I)$ is also exact. Putting all this together we obtain that the complex

$$0 \to \operatorname{Hom}_R(R/\mathfrak{a}, I'^1) \to \operatorname{Hom}_R(R/\mathfrak{a}, I^2) \to \dots$$

is exact. That is, $\text{E-dp}(\mathfrak{a}, \operatorname{Ext}^1_R(T, M)) = \infty$. The second assertion follows by 3.2.6. $\square$

The Tor-codepth is related to Foxby's notion of small support introduced in [31].

3.2.11 *Small support*. Let M be a module over a Noetherian ring R. The small support of M is defined by

$$\operatorname{supp}_R M = \{\mathfrak{p} \in \operatorname{Spec} R \mid \text{T-codp}(\mathfrak{p}R_\mathfrak{p}, M_\mathfrak{p}) < \infty\}.$$

Clearly $\operatorname{supp}_R M \subseteq \operatorname{Supp}_R M$, where Supp denotes the ordinary support. Moreover, both sets have the same minimal elements (see [82, Lemma 7.2.7]). Let us also recall that

$$\operatorname{supp}_R M \cap \operatorname{supp}_R N = \emptyset \text{ if and only if } \operatorname{Tor}^R_i(N, M) = 0 \text{ for all } i \geq 0$$

(see [31] or [82, 7.2.8]). It follows that

$$\text{T-codp}(\mathfrak{a}, M) = \infty \text{ if and only if } \operatorname{supp}_R M \cap V(\mathfrak{a}) = \emptyset.$$

When M is finitely generated we have $\operatorname{supp}_R(M) = \operatorname{Supp}_R(M)$ as is easily seen.

3.3 When is a Finitely Generated Module Complete?

In this section we are concerned with completeness criteria for finitely generated modules over a Noetherian ring. Here is a first remark.

Remark 3.3.1 Let M be a finitely generated module over a Noetherian ring R and assume that the ideal $\mathfrak{a}$ is contained in the Jacobson radical. Then M is $\mathfrak{a}$-separated. If M is not $\mathfrak{a}$-complete, then $\hat{M}^{\mathfrak{a}}$ is not finitely generated over R.

Indeed, if $\hat{M}^{\mathfrak{a}}$ is finitely generated over R, then the R-module $\hat{M}^{\mathfrak{a}}/M$ is also finitely generated. Hence by Nakayama's lemma it vanishes because we always have $\hat{M}^{\mathfrak{a}}/M = \mathfrak{a}(\hat{M}^{\mathfrak{a}}/M)$ (see 2.2.2).

The case $M = R$ of the above was observed by Frankild and Sather–Wagstaff (see [35, Theorem B]).

Here is a first criterion for completeness in terms of Tor and supports.

Proposition 3.3.2 *Let M be a finitely generated module over a Noetherian ring R. Assume that R is $\mathfrak{a}$-separated and put $S_{\mathfrak{a}}(R) = \operatorname{Coker}(\tau_R^{\mathfrak{a}}) = \hat{R}^{\mathfrak{a}}/R$. Then the following conditions are equivalent:*

(i) *M is $\mathfrak{a}$-complete.*
(ii) $S_{\mathfrak{a}}(R) \otimes_R M = \operatorname{Tor}_1^R(S_{\mathfrak{a}}(R), M) = 0$.
(iii) $\operatorname{Tor}_i^R(S_{\mathfrak{a}}(R), M) = 0$ *for all* $i \geq 0$.
(iv) $\operatorname{supp}_R(S_{\mathfrak{a}}(R)) \cap \operatorname{Supp}_R M = \emptyset$.

Proof We tensor the short exact sequence $0 \to R \xrightarrow{\tau_R^{\mathfrak{a}}} \hat{R}^{\mathfrak{a}} \longrightarrow S_{\mathfrak{a}}(R) \to 0$ by M. Then we obtain the exact sequence

$$0 \to \operatorname{Tor}_1^R(S_{\mathfrak{a}}(R), M) \to M \to \hat{R}^{\mathfrak{a}} \otimes_R M \to S_{\mathfrak{a}}(R) \otimes_R M \to 0.$$

Because M is finitely generated we have $\tau_R^{\mathfrak{a}} \otimes_R M = \tau_M^{\mathfrak{a}}$ and $\hat{R}^{\mathfrak{a}} \otimes_R M = \hat{M}^{\mathfrak{a}}$. It follows that $\tau_M^{\mathfrak{a}}$ is an isomorphism, that is, M is $\mathfrak{a}$-complete, if and only if the conditions in (ii) are satisfied.

The equivalence between (ii) and (iii) is clear since the flat dimension of $S_{\mathfrak{a}}(R)$ is at most one.

The equivalence between (iii) and (iv) is clear in view of the recalls in 3.2.11. □

Question 3.3.3 When the ring R is Noetherian we know that $S_{\mathfrak{a}}(R) \in \mathcal{T}_{\mathfrak{a}}$, that is, $\operatorname{Tor}_i^R(R/\mathfrak{a}, S_{\mathfrak{a}}(R)) = 0$ for all $i \geq 0$ (see 3.2.4). It follows that $\operatorname{supp}(S_{\mathfrak{a}}(R)) \subseteq \operatorname{Spec}(R) \setminus V(\mathfrak{a})$ by 3.2.11. We now raise the following question. When do we have $\operatorname{supp}(S_{\mathfrak{a}}(R)) = \operatorname{Spec}(R) \setminus V(\mathfrak{a})$?

Here is another criterion in terms of Ext. Note the similarity with 3.3.2.

Proposition 3.3.4 *Assume that R is an $\mathfrak{a}$-separated Noetherian ring. Put again $S_{\mathfrak{a}}(R) = Coker(\tau_{\mathfrak{a}}^R)$. For a finitely generated R-module the following conditions are equivalent:*

(i) *M is $\mathfrak{a}$-complete.*
(ii) $\operatorname{Ext}_R^i(S_{\mathfrak{a}}(R), M) = 0$ *for all* $i \geq 0$.
(iii) $\operatorname{Hom}_R(S_{\mathfrak{a}}(R), M) = \operatorname{Ext}_R^1(S_{\mathfrak{a}}(R), M) = 0$.

Proof We have the following exact sequence

$$\begin{aligned}0 &\to \operatorname{Hom}_R(S_{\mathfrak{a}}(R), M) \to \operatorname{Hom}_R(\hat{R}^{\mathfrak{a}}, M) \to M \\ &\to \operatorname{Ext}^1_R(S_{\mathfrak{a}}(R), M) \to \operatorname{Ext}^1_R(\hat{R}^{\mathfrak{a}}, M) \to 0\end{aligned}$$

induced by the short exact sequence $0 \to R \to \hat{R}^{\mathfrak{a}} \to S_{\mathfrak{a}}(R) \to 0$.

First we show (iii) $\Rightarrow$ (i). The conditions in (iii) implies that

$$\operatorname{Hom}_R(\hat{R}^{\mathfrak{a}}, M) \cong M.$$

Hence M has the structure of an $\hat{R}^{\mathfrak{a}}$-module extending its R-module structure. With this structure M is also a finitely generated $\hat{R}^{\mathfrak{a}}$-module, hence $\mathfrak{a}\hat{R}^{\mathfrak{a}}$- and $\mathfrak{a}$-complete.

The implication (i) $\Rightarrow$ (ii) was already observed in 3.1.2, while the implication (ii) $\Rightarrow$ (iii) is obvious. □

Now we are prepared for a slight generalization of the Frankild–Sather–Wagstaff criterion. It originally only concerned the case where $\mathfrak{a}$ is contained in the Jacobson radical. For the result we have in mind we need an estimate of the projective dimension of a flat module.

Proposition 3.3.5 (see [67, Part II, Corollary 3.2.7]) *The projective dimension of a flat module over a Noetherian ring R is at most* $\dim R$.

The following corollary was observed by Frankild and Sather–Wagstaff in [35].

Corollary 3.3.6 *Let $\mathfrak{a}$ be an ideal of a Noetherian ring R and let M an R-module. Then* $\operatorname{Ext}^i_R(\hat{R}^{\mathfrak{a}}, M) = 0$ *for all* $i > \dim R/\operatorname{Ann}_R(M)$.

Proof Put $B = R/\operatorname{Ann}_R(M)$ and take a projective resolution P of $\hat{R}^{\mathfrak{a}}$. As $\hat{R}^{\mathfrak{a}}$ is R-flat, we have that $B \otimes_R P$ is a B-projective resolution of $B \otimes_R \hat{R}^{\mathfrak{a}} \cong \hat{B}^{\mathfrak{a}}$. Now we compute the Ext's. Because $\operatorname{Hom}_R(P, M) \cong \operatorname{Hom}_B(B \otimes_R P, M)$ we get isomorphisms $\operatorname{Ext}^i_R(\hat{R}^{\mathfrak{a}}, M) \cong \operatorname{Ext}^i_B(\hat{B}^{\mathfrak{a}}, M)$. As the ring $\hat{B}^{\mathfrak{a}}$ is B-flat the result follows by 3.3.5. □

Here is the slight generalization of [35, Theorem A].

Theorem 3.3.7 *Let R be a Noetherian ring and let M be a finitely generated R-module. Suppose that both R and M are $\mathfrak{a}$-separated. Then the following conditions are equivalent:*

(i) *M is $\mathfrak{a}$-complete.*
(ii) $\operatorname{Ext}^i_R(\hat{R}^{\mathfrak{a}}, M) = 0$ *for all* $i \geq 1$.
(iii) $\operatorname{Ext}^i_R(\hat{R}^{\mathfrak{a}}, M) = 0$ *for all i such that* $1 \leq i \leq \dim M$.

Proof The implication (i) $\Rightarrow$ (ii) is a consequence of Theorem 3.1.1 as already observed in 3.1.2. By 3.3.6 we have that (ii) $\Leftrightarrow$ (iii). For the proof of the implication (ii) $\Rightarrow$ (i) take the short exact sequence

$$0 \to R \to \hat{R}^{\mathfrak{a}} \to S_{\mathfrak{a}}(R) \to 0$$

and derive the long exact sequence of the $\mathrm{Ext}^i_R(\cdot, M)$ associated to it. Note that $\mathrm{Hom}_R(S_{\mathfrak{a}}(R), M) = 0$ because $S_{\mathfrak{a}}(R) = \mathfrak{a}S_{\mathfrak{a}}(R)$ and M is $\mathfrak{a}$-separated (see 2.1.12). Therefore we obtain a short exact sequence

$$\mathcal{E} : 0 \to \mathrm{Hom}_R(\hat{R}^{\mathfrak{a}}, M) \to M \to \mathrm{Ext}^1_R(S_{\mathfrak{a}}(R), M) \to 0$$

and the vanishing $\mathrm{Ext}^i_R(S_{\mathfrak{a}}(R), M) = 0$ for all $i \neq 1$.

We now recall that $S_{\mathfrak{a}}(R) \in \mathcal{T}_{\mathfrak{a}}$ (see 3.2.4). Whence, with the help of 3.2.10, we obtain $\text{T-codp}(\mathfrak{a}, \mathrm{Ext}^1_R(S_{\mathfrak{a}}(R), M)) = \infty$, in particular we have $\mathrm{Ext}^1_R(S_{\mathfrak{a}}(R), M) = \mathfrak{a}\,\mathrm{Ext}^1_R(S_{\mathfrak{a}}(R), M)$.

On the other hand note that the R-module $\mathrm{Hom}_R(\hat{R}^{\mathfrak{a}}, M)$ is $\mathfrak{a}$-complete because it is an $\hat{R}^{\mathfrak{a}}$-module, finitely generated over R (as a submodule of M), hence also finitely generated over $\hat{R}^{\mathfrak{a}}$.

We now take the completion of the exact sequence $\mathcal{E}$, which is a sequence of finitely generated R-modules, and obtain a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_R(\hat{R}^{\mathfrak{a}}, M) \arrow[r] \arrow[d, equal] & M \arrow[r] \arrow[d, "\tau^{\mathfrak{a}}_M"'] & \mathrm{Ext}^1_R(S_{\mathfrak{a}}, M) \arrow[r] & 0 \\
0 \arrow[r] & \mathrm{Hom}_R(\hat{R}^{\mathfrak{a}}, M) \arrow[r] & \hat{M}^{\mathfrak{a}} \arrow[rr] & & 0.
\end{tikzcd}$$

This shows that the natural map $\tau^{\mathfrak{a}}_M$ is split-surjective, hence bijective since M is $\mathfrak{a}$-separated. □

Remarks 3.3.8 (a) Let M be an R-module. Then there is a natural homomorphism of R-modules

$$\mathrm{Hom}_R(\hat{R}^{\mathfrak{a}}, M) \to M, \ f \mapsto f(1).$$

If this homomorphism is injective its image has the structure of an $\hat{R}^{\mathfrak{a}}$-module extending its R-module structure. It is rather easy to see that this image is the largest submodule of M having this property. Moreover, when R is Noetherian and M is finitely generated, this image is also the largest $\mathfrak{a}$-complete submodule of M.

(b) In 3.3.2, 3.3.4 and 3.3.7 the hypothesis that M is finitely generated cannot be dropped without harm. Consider the case where R itself is $\mathfrak{a}$-complete.

3.4 Ext-Depth and Tor-Codepth with Local (Co-)Homology

When the ideal $\mathfrak{a}$ is finitely generated the Tor-codepth of a module can be computed with local homology. This will be a consequence of the following result.

Proposition 3.4.1 *Let $\mathfrak{a}$ denote a finitely generated ideal of a commutative ring R. Let L be a right-bounded complex of projective modules. Then we have the equality*

$$\inf\{i \mid H_i(\hat{L}^{\mathfrak{a}}) \neq 0\} = \inf\{i \mid H_i(R/\mathfrak{a} \otimes_R L) \neq 0\}.$$

Proof Without loss of generality we may assume – up to a shift – that L has the form $\cdots \to L_1 \xrightarrow{d_1} L_0 \to 0$. We write $\hat{(\cdot)}$ for the $\mathfrak{a}$-completion functor. First note that $H_0(R/\mathfrak{a} \otimes_R L) \neq 0$ if and only if d_1 is not $\mathfrak{a}$-dense by definition and d_1 is not $\mathfrak{a}$-dense if and only if $\hat{d}_1$ is not surjective (see 2.1.8). Hence both "inf"s vanish simultaneously.

Assume now there is a natural number $n > 0$ such that $H_i(\hat{L}) = 0$ for all $i < n$. Then $\hat{d}_1$ is surjective and the natural map $\tau^{\mathfrak{a}}_{L_0} : L_0 \to \hat{L}_0$ factors through $\hat{L}_1$: there is a homomorphism $s_0 : L_0 \to \hat{L}_1$ such that $\tau^{\mathfrak{a}}_{L_0} = \hat{d}_1 \circ s_0$. Taking $\mathfrak{a}$-completion we obtain $\hat{d}_1 \circ \hat{s}_0 = \mathrm{id}_{\hat{L}_0}$. Thus $\hat{d}_1$ is split-surjective and $\mathrm{Ker}(\hat{d}_1) = \mathrm{Im}(\mathrm{id}_{\hat{L}_1} - \hat{s}_0 \circ \hat{d}_1)$. If $n > 1$, then the induced homomorphism $\hat{L}_2 \to \mathrm{Ker}(\hat{d}_1)$ is surjective and the homomorphism $(\mathrm{id}_{\hat{L}_1} - \hat{s}_0 \circ \hat{d}_1) \circ \tau^{\mathfrak{a}}_{L_1}$ (its image is contained in $\mathrm{Ker}(\hat{d}_1)$) factors through $\hat{L}_2$: there is a homomorphism $s_1 : L_1 \to \hat{L}_2$ such that $(\mathrm{id}_{\hat{L}_1} - \hat{s}_0 \circ \hat{d}_1) \circ \tau^{\mathfrak{a}}_{L_1} = \hat{d}_2 \circ s_1$, thus such that $\mathrm{id}_{\hat{L}_1} = \hat{s}_0 \circ \hat{d}_1 + \hat{d}_2 \circ \hat{s}_1$.

We iterate the process and obtain a partial homotopy, that is, a sequence of maps $\hat{s}_i : \hat{L}_i \to \widehat{L_{i+1}}$, $0 \leq i < n$, such that $\mathrm{id}_{\hat{L}_i} = \widehat{s_{i-1}} \circ \hat{d}_i + \widehat{d_{i+1}} \circ \hat{s}_i$:

$$\begin{tikzcd}
L_n \arrow[r, "d_n"] \arrow[d, "\tau^{\mathfrak{a}}_{L_n}"'] & L_{n-1} \arrow[r] \arrow[d, "\tau^{\mathfrak{a}}_{L_{n-1}}"] \arrow[dl, dashed, "s_{n-1}"'] & \dots \arrow[r] & L_1 \arrow[r, "d_1"] \arrow[d, "\tau^{\mathfrak{a}}_{L_1}"'] & L_0 \arrow[r] \arrow[d, "\tau^{\mathfrak{a}}_{L_0}"] \arrow[dl, dashed, "s_0"'] & 0 \\
\widehat{L_n} \arrow[r, "\widehat{d_n}"] \arrow[d, "\mathrm{id}_{\widehat{L_n}}"'] & \widehat{L_{n-1}} \arrow[r] \arrow[d, "\mathrm{id}_{\widehat{L_{n-1}}}"] \arrow[dl, dashed, "\widehat{s_{n-1}}"'] & \dots \arrow[r] & \widehat{L_1} \arrow[r, "\widehat{d_1}"] \arrow[d, "\mathrm{id}_{\widehat{L_1}}"'] & \widehat{L_0} \arrow[r] \arrow[d, "\mathrm{id}_{\widehat{L_0}}"] \arrow[dl, dashed, "\widehat{s_0}"'] & 0 \\
\widehat{L_n} \arrow[r, "\widehat{d_n}"'] & \widehat{L_{n-1}} \arrow[r] & \dots \arrow[r] & \widehat{L_1} \arrow[r, "\widehat{d_1}"'] & \widehat{L_0} \arrow[r] & 0
\end{tikzcd}$$

By tensoring these data with $R/\mathfrak{a}$ we obtain $H_i(R/\mathfrak{a} \otimes_R L) = 0$ for all $i < n$. This proves the inequality

$$\inf\{i \mid H_i(\hat{L}^{\mathfrak{a}}) \neq 0\} \leq \inf\{i \mid H_i(R/\mathfrak{a} \otimes_R L) \neq 0\}.$$

Moreover, note also that the complex $\hat{L}$ together with the sequence of the $\hat{s}_i$ induces split short exact sequences

$$0 \to Z_i \to \hat{L}_i \to Z_{i-1} \to 0 \ \text{ for all } \ i = 1, \dots, n,$$

where $Z_i = \mathrm{Ker}(\hat{d}_i)$ for $1 \leq i \leq n$ and $Z_0 = \hat{L}_0$. By applying $(R/\mathfrak{a} \otimes_R \cdot)$ and since $R/\mathfrak{a} \otimes_R \hat{L}_i \cong R/\mathfrak{a} \otimes_R L_i$ it follows that $R/\mathfrak{a} \otimes_R Z_n \cong \mathrm{Ker}(R/\mathfrak{a} \otimes_R d_n)$.

Assume now $n = \inf\{i \mid H_i(\hat{L}^{\mathfrak{a}}) \neq 0\} < \infty$. It remains to show that $H_n(R/\mathfrak{a} \otimes_R L) \neq 0$. As $H_n(\hat{L}) \neq 0$ the induced homomorphism $\widehat{L_{n+1}} \to Z_n$ is not surjective. As it is a homomorphism between $\mathfrak{a}$-complete modules (Z_n is $\mathfrak{a}$-complete by 2.2.11) it is not $\mathfrak{a}$-dense. It follows that the induced homomorphism $R/\mathfrak{a} \otimes_R \widehat{L_{n+1}} \to R/\mathfrak{a} \otimes Z_n$ is not surjective. But $R/\mathfrak{a} \otimes_R \widehat{L_{n+1}} \cong R/\mathfrak{a} \otimes_R L_{n+1}$ and $R/\mathfrak{a} \otimes_R Z_n \cong \operatorname{Ker}(R/\mathfrak{a} \otimes_R d_n)$ as seen above. The conclusion follows. □

Corollary 3.4.2 *Let $\mathfrak{a}$ denote a finitely generated ideal of a commutative ring R. Let M be an R-module with one of its free resolutions L. Then:*

(a) $\lambda^{\mathfrak{a}}_{-}(M) := \inf\{i \mid \Lambda^{\mathfrak{a}}_i(M) \neq 0\} = \text{T-codp}(\mathfrak{a}, M)$.
(b) $\text{T-codp}(\mathfrak{a}, M) = \infty$ *if and only if* $\hat{L}^{\mathfrak{a}}$ *is split-exact.*

Proof By the definition of the Tor-codepth the statement in (a) is a direct consequence of Proposition 3.4.1. For the equivalence in (b) note that it is shown in the proof of 3.4.1 that $\hat{L}^{\mathfrak{a}}$ is split-exact as soon as it is exact. □

The Ext-depth counterpart of 3.4.1 and 3.4.2 is easier and does not require any finiteness hypothesis on the ideal $\mathfrak{a}$.

Proposition 3.4.3 [82, 5.3.15] *For any ideal $\mathfrak{a}$ of a commutative ring R and any R-module M we have*

$$\text{E-dp}(\mathfrak{a}, M) = \inf\{i \mid H^i_{\mathfrak{a}}(M) \neq 0\} := h^-_{\mathfrak{a}}(M).$$

The above proposition is also a consequence of the following more general result, helpful in Part II.

Proposition 3.4.4 *Let $\mathfrak{a}$ be an ideal of a commutative ring R. Let I denote a left-bounded complex of injective modules. Then*

$$\inf\{i \mid H^i(\Gamma_{\mathfrak{a}}(I)) \neq 0\} = \inf\{i \mid H^i(\operatorname{Hom}_R(R/\mathfrak{a}^t, I)) \neq 0 \textit{ for some } t \in \mathbb{N}\}.$$

Proof Up to a shift we may assume that I has the following form $0 \to I^0 \xrightarrow{d^0} I^1 \to \ldots$. As we have the injections

$$\operatorname{Hom}_R(R/\mathfrak{a}, I) \hookrightarrow \operatorname{Hom}_R(R/\mathfrak{a}^t, I) \hookrightarrow \Gamma_{\mathfrak{a}}(I) \cong \varinjlim \operatorname{Hom}_R(R/\mathfrak{a}^t, I) \hookrightarrow I$$

we easily see that both "inf"s vanish simultaneously.

Assume now there is a natural number $n > 0$ such that $H^i(\Gamma_{\mathfrak{a}}(I)) = 0$ for all $i < n$. Therefore the homomorphism $\Gamma_{\mathfrak{a}}(d^0)$ is injective and the homomorphism $\Gamma_{\mathfrak{a}}(I^0) \hookrightarrow I^0$ extends to a homomorphism $\Gamma_{\mathfrak{a}}(I^1) \to I^0$, the image of which is in $\Gamma_{\mathfrak{a}}(I^0)$. Hence $\Gamma_{\mathfrak{a}}(d^0)$ is split-injective and we have a homomorphism $s^1 : \Gamma_{\mathfrak{a}}(I^1) \to \Gamma_{\mathfrak{a}}(I^0)$ such that $s^1 \circ \Gamma_{\mathfrak{a}}(d^0) = \mathrm{id}_{\Gamma_{\mathfrak{a}}(I^0)}$.

As in 3.4.1 we iterate the process and obtain a partial homotopy, that is, a sequence of maps $s^i : \Gamma_{\mathfrak{a}}(I^i) \to \Gamma_{\mathfrak{a}}(I^{i-1})$ for $i = 1, \ldots, n$, such that $\mathrm{id}_{\Gamma_{\mathfrak{a}}(I^i)} = s^{i+1} \circ \Gamma_{\mathfrak{a}}(d^i) + \Gamma_{\mathfrak{a}}(d^{i-1}) \circ s^i$ for $1 \leq i < n$:

$$\begin{tikzcd}
0 \arrow[r] & \Gamma_{\mathfrak{a}}(I^0) \arrow[r, "\Gamma_{\mathfrak{a}}(d^0)"] \arrow[d, "\mathrm{id}_{\Gamma_{\mathfrak{a}}(I^0)}"'] & \Gamma_{\mathfrak{a}}(I^1) \arrow[r] \arrow[d, "\mathrm{id}_{\Gamma_{\mathfrak{a}}(I^1)}"] \arrow[dl, dashed, "s^1"'] & \ldots \arrow[r] & \Gamma_{\mathfrak{a}}(I^{n-1}) \arrow[r] \arrow[d, "\mathrm{id}_{\Gamma_{\mathfrak{a}}(I^{n-1})}"'] & \Gamma_{\mathfrak{a}}(I^n) \arrow[d, "\mathrm{id}_{\Gamma_{\mathfrak{a}}(I^n)}"] \arrow[dl, dashed, "s^n"'] \\
0 \arrow[r] & \Gamma_{\mathfrak{a}}(I^0) \arrow[r, "\Gamma_{\mathfrak{a}}(d^0)"'] \arrow[d, hook] & \Gamma_{\mathfrak{a}}(I^1) \arrow[r] \arrow[d, hook] \arrow[dl, dashed] & \ldots \arrow[r] & \Gamma_{\mathfrak{a}}(I^{n-1}) \arrow[r] \arrow[d, hook] & \Gamma_{\mathfrak{a}}(I^n) \arrow[d, hook] \arrow[dl, dashed] \\
0 \arrow[r] & I^0 \arrow[r, "d^0"] & I^1 \arrow[r] & \ldots \arrow[r] & I^{n-1} \arrow[r, "d_{n-1}"] & I^n
\end{tikzcd}$$

We apply $\operatorname{Hom}_R(R/\mathfrak{a}^t, \cdot)$ to the data and obtain $H^i(\operatorname{Hom}_R(R/\mathfrak{a}^t, I) = 0$ for all $i < n$ and all t. This gives the inequality

$$\inf\{i \mid H^i(\Gamma_{\mathfrak{a}}(I)) \neq 0\} \leq \inf\{i \mid H^i(\operatorname{Hom}_R(R/\mathfrak{a}^t, I)) \neq 0 \text{ for some } t \geq 1\}.$$

Assume now $H^n(\Gamma_{\mathfrak{a}}(I)) \neq 0$. As $H^n(\Gamma_{\mathfrak{a}}(I)) = \varinjlim(H^n(\operatorname{Hom}_R(R/\mathfrak{a}^t, I)))$ (because $\varinjlim$ is exact and commutes with taking cohomology) we also have $H^n(\operatorname{Hom}_R(R/\mathfrak{a}^t, I)) \neq 0$ for some t. This finishes the proof. □

Combining the results of this section with 3.2.6 we obtain.

Corollary 3.4.5 *For a finitely generated ideal $\mathfrak{a}$ of a commutative ring R and an R-module M the following conditions are equivalent:*

(i) $M \in \mathfrak{T}_{\mathfrak{a}}$, *that is* $\text{T-codp}(\mathfrak{a}, M) = \infty$,
(ii) $\Lambda_i^{\mathfrak{a}}(M) = 0$ *for all* $i \geq 0$,
(iii) $\text{E-dp}(\mathfrak{a}, M) = \infty$,
(iv) $H_{\mathfrak{a}}^i(M) = 0$ *for all* $i \geq 0$.

Part II
Complexes

Chapter 4
Homological Preliminaries

In this chapter we summarize some homological preliminaries needed for our investigations in the sequel. In particular, we investigate unbounded complexes. To this end some additional considerations for their resolutions are necessary, that is, we report and summarize part of the work of Avramov and Foxby resp. Spaltenstein (see [5] and [81]) not available in this form elsewhere. After recalling some results about double complexes we start with the extension to complexes of results on the microscope and telescope introduced by Greenlees and May in [38]. This is used in order to get certain resolutions of unbounded complexes, as suggested by Avramov and Foxby. We also discuss minimal injective resolutions for unbounded complexes.

4.1 Double Complexes and Truncations

4.1.1 *Double complexes.* A double complex of R-modules is a triple

$$K^{\bullet,\bullet} = (\oplus K^{i,j}, {}'d^{i,j}, {}''d^{i,j}),$$

where $K^{i,j}$, $(i, j) \in \mathbb{Z}^2$, is a collection of R-modules and where ${}'d^{i,j}, {}''d^{i,j}$ are homomorphisms

$${}'d^{i,j} : K^{i,j} \to K^{i+1,j}, \quad {}''d^{i,j} : K^{i,j} \to K^{i,j+1}$$

satisfying

$${}'d^2 = 0, \quad {}''d^2 = 0, \quad {}'d \circ {}''d + {}''d \circ {}'d = 0.$$

A typical example of a double complex occurs as the tensor product of two R-complexes X, Y: put $K^{i,j} = X^i \otimes Y^j$, ${}'d^{i,j} = d_X^i \otimes_R \mathrm{id}_{Y_j}$, ${}''d^{i,j} = (-1)^i \mathrm{id}_{X_i} \otimes_R d_Y^j$.

P. Schenzel and A.-M. Simon, *Completion, Čech and Local Homology and Cohomology*, Springer Monographs in Mathematics,
https://doi.org/10.1007/978-3-319-96517-8_4

As in the case of tensor products one might associate a single complex $(K^\bullet, d)$ to a double complex in the following way: define $K^n = \oplus_{i+j=n} K^{i,j}$ and $d = {}'d + {}''d$. When some boundedness conditions are satisfied the complex $\mathrm{Hom}_R(X, Y)$ may also be viewed as the single complex associated to the double complex $K^{i,j} = \mathrm{Hom}_R(X_i, Y^j)$ with differentials induced up to a sign by d_X and d^Y.

4.1.2 *Spectral sequences.* More generally, let $(K^{i,j}, {}'d, {}''d)$ denote a double complex with the associated single complex $K^\bullet$ as above. There are two filtrations defined by

$$'F_p = \oplus_{r \geq p} K^{r,j} \quad \text{and} \quad ''F_q = \oplus_{s \geq q} K^{i,s}$$

and associated spectral sequences defined at the first stage by

$$'E_1^{p,q} = {}''H^q(K^{p,\bullet}) \quad \text{and} \quad ''E_1^{p,q} = {}'H^q(K^{\bullet,p}).$$

We denote by $''H^q(K)$ the row complex

$$\cdots \to {}''H^q(K^{p,\bullet}) \to {}''H^q(K^{p+1,\bullet}) \to \cdots$$

with differential induced by $'d$, similarly by $'H^q(K)$ the column complex

$$\cdots \to {}'H^q(K^{\bullet,p}) \to {}'H^q(K^{\bullet,p+1}) \to \cdots$$

with differential induced by $''d$. With these notations we have

$$'E_2^{p,q} = {}'H^p({}''H^q(K)) \quad \text{and} \quad ''E_2^{p,q} = {}''H^p({}'H^q(K)).$$

The first (resp. the second) filtration induces a filtration on the cohomology modules $H^i(K^\bullet)$. If that filtration is finite we say that the first (resp. the second) spectral sequence converges. In that case the spectral sequence provides information on the filtered modules $H^i(K^\bullet)$. If only a finite number of rows $K^{\bullet,q}$ are non-zero then the first spectral sequence converges : $'E_2^{p,q} \Rightarrow H^{p+q}(K^\bullet)$. If only a finite number of columns $K^{p,\bullet}$ are non-zero then the second spectral sequence converges. If the complex is in the first quadrant both spectral sequences converge.

For the details we refer to [14, §2, Exerc. 17] or the corresponding chapters in [18], [36] or [85]. Another presentation (with full details) is given by McCleary (see [59]).

Spectral sequences do not always converge. But in some cases there are other tricks to compute the cohomology of a double complex. Here is a well-known example.

Lemma 4.1.3 *Assume that the double complex $(K^{i,j}, {}'d, {}''d)$ is in the lower half plane, that is, $K^{i,j} = 0$ for all $j \geq 0$.*

If the row complexes $K^{\bullet,j}$ are exact for all $j \leq 0$, then the single complex $K^\bullet$ is exact.

Proof Write G_t for the single complex associated to the sub-double complex $\oplus_{j \geq -t} K^{i,j}$, $t \geq 0$. We have short exact sequences

$$0 \to G_t \to G_{t+1} \to K^{\bullet,-t-1} \to 0$$

and the morphism $G_t \to G_{t+1}$ is a quasi-isomorphism as the row complex $K^{\bullet,-t-1}$ is exact. But the complex $G_0 = K^{\bullet,0}$ is exact. Hence all the complexes G_t are exact and so is their direct limit. Observe now that $\varinjlim_t G_t$ is the single complex associated to our double complex.

Of course, in the above lemma the role of rows and columns may be interchanged.

Corollary 4.1.4 (see [14, §4, $n°3$, Lemme 1]) *Let F be a right-bounded R-complex of flat R-modules. Then for any exact R-complex X the complex $X \otimes_R F$ is exact.*

4.1.5 *Soft truncations.* Let X be an R-complex and write as usual $Z^i(X) = \mathrm{Ker}(d^i_X)$ and $B^i(X) = \mathrm{Im}(d^{i-1}_X)$ (resp. $Z_i(X) = \mathrm{Ker}(d^X_i)$ and $B_i(X) = \mathrm{Im}(d^X_{i+1})$).

(a) The right soft truncation of X at spot t is the subcomplex

$$\tau_{t]}(X) : \cdots \to X^{t-2} \to X^{t-1} \to Z^t(X) \to 0.$$

For $t \geq 0$ these subcomplexes form a direct system with $\varinjlim\{\tau_{t]}(X) \to \tau_{t+1]}(X)\} = X$ as easily seen. In the case when X is homologically right-bounded let $s \geq \sup\{i \mid H^i(X) \neq 0\}$, then we have the quasi-isomorphism $\tau_{s]}(X) \xrightarrow{\sim} X$.

(b) The left soft truncation of X at spot t is the quotient complex

$$\tau_{[t}(X) : 0 \to X_t/B_t(X) \to X_{t-1} \to X_{t-2} \to \cdots .$$

For $t \geq 0$ these quotient complexes form an inverse system, degree-wise surjective, with $\varprojlim\{\tau_{[t+1}(X) \to \tau_{[t}(X)\} = X$, as is easily seen. In the case when X is homologically left-bounded let $s \geq \sup\{i \mid H_i(X) \neq 0\}$, then we have the quasi-isomorphism $X \xrightarrow{\sim} \tau_{[s}(X)$.

(c) If $\mathrm{amp}(X) = 0$, say we have an s such that $H_i(X) = 0$ for all $i \neq s$, we write M for the module $H_s(X)$ and view M as a complex concentrated in degree zero. With these notations we have the quasi-isomorphisms

$$M_{[-s]} \cong \tau_{[s}(\tau_{s]}(X)) \xleftarrow{\sim} \tau_{s]}(X) \xrightarrow{\sim} X.$$

(d) More generally, any homologically bounded complex is quasi-isomorphic to a bounded one.

4.2 The Microscope

In this and the next section we shall report on two constructions, known in topology and invented in homological algebra by Greenlees and May (see [38]). They will be a helpful tool in the sequel.

Definition 4.2.1 Let $\mathfrak{P} = \{\rho_{t,t+1} : P_{t+1} \to P_t \mid t \in \mathbb{N}\}$ denote an inverse system of R-complexes. There is a morphism associated to $\mathfrak{P}$

$$\psi_{\mathfrak{P}} : \prod P_t \to \prod P_t,\ (x_t) \mapsto (x_t - \rho_{t+1}(x_{t+1})) \text{ for } (x_t) \in \prod P_t.$$

In accordance with [38] we define the microscope of $\mathfrak{P}$ as the fibre of $\psi_{\mathfrak{P}}$:

$$\operatorname{Mic}(\mathfrak{P}) = F(\psi_{\mathfrak{P}}).$$

We note that Mic can be viewed as a functor from the category of inverse systems of R-complexes to the category of R-complexes.

Example 4.2.2 Assume $\mathfrak{P}$ is an inverse system of R-modules, say $P_t = M_t$ and $\mathfrak{P} = \mathfrak{M}$. Then $\operatorname{Mic}(\mathfrak{M})$ is a length one complex. Viewing it as a descending complex we have $H_0(\operatorname{Mic}(\mathfrak{M})) \cong \varprojlim M_t$ and $H_{-1}(\operatorname{Mic}(\mathfrak{M})) \cong \varprojlim{}^1 M_t$.

Clearly the microscope is related to inverse limits. Let us make the relation more precise.

Lemma 4.2.3 *Let $\mathfrak{P} = \{\rho_{t,t+1} : P_{t+1} \to P_t \mid t \in \mathbb{N}\}$ be an inverse system of R-complexes.*

(a) There is a natural morphism of complexes $\varprojlim P_t \to \operatorname{Mic}(\mathfrak{P})$. It is a quasi-isomorphism provided the inverse system $\{P_t\}$ satisfies degree-wise the Mittag-Leffler condition.

(b) By viewing the P_t as descending complexes we have functorial short exact sequences

$$0 \to \varprojlim{}^1 H_{i+1}(P_t) \to H_i(\operatorname{Mic}(\mathfrak{P})) \to \varprojlim H_i(P_t) \to 0$$

for all $i \in \mathbb{Z}$.

(c) If the inverse systems $\{H_i(P_t) \mid t \in \mathbb{N}\}$ are pro-zero for all $i \in \mathbb{Z}$, then $\operatorname{Mic}(\mathfrak{P})$ is exact.

Proof By the definition of the inverse limit there is an exact sequence of complexes

$$0 \to \varprojlim P_t \to \prod P_t \xrightarrow{\psi_{\mathfrak{P}}} \prod P_t.$$

By Proposition 1.5.6 and the definition of $\operatorname{Mic}(\mathfrak{P})$ there is a natural morphism $\varprojlim P_t \to \operatorname{Mic}(\mathfrak{P})$, and this morphism is a quasi-isomorphism provided the map

$\psi_{\mathfrak{P}}$ is surjective. This condition is fulfilled provided the inverse system $\{P_t\}$ satisfies degree-wise the Mittag-Leffler condition (see 1.2.6). This proves (a).

For the proof of (b) note that we have a short exact sequence of complexes

$$0 \to \left(\prod P_t\right)_{[+1]} \to \operatorname{Mic}(\mathfrak{P}) \to \prod P_t \to 0$$

(see the definition of the fiber in 1.5.1). Then we consider the induced long exact homology sequence and break it up into short exact sequences. By taking into account that direct products commute with taking homology the result now follows.

The statement in (c) is an immediate consequence of (b) (see 1.2.4). $\square$

Example 4.2.4 (a) Let X be an R-complex and let $\mathfrak{P}$ be the constant inverse system over X, that is, $P_t = X$ and $\rho_{t,t+1} = \mathrm{id}_X$. We have an isomorphism and a quasi-isomorphism $X \cong \varprojlim \mathfrak{P} \xrightarrow{\sim} \operatorname{Mic}(\mathfrak{P})$.

(b) Let $\mathfrak{a}$ be an ideal of a commutative ring R, M an R-module and consider the inverse system $\{M/\mathfrak{a}^t M\}$, where the transition homomorphisms $M/\mathfrak{a}^{t+1}M \to M/\mathfrak{a}^t M$ are the natural ones. There is a quasi-isomorphism

$$\hat{M}^{\mathfrak{a}} \xrightarrow{\sim} \operatorname{Mic}(\{M/\mathfrak{a}^t M\})$$

(see 4.2.3 (a)). Note also that $\operatorname{Mic}(\{M/\mathfrak{a}^t M\})$, viewed as a descending complex, is a complex concentrated in degree 0 and -1.

The microscope construction behaves well with respect to short exact sequences of inverse systems and quasi-isomorphisms. This is in contrast to the inverse limit of them.

Lemma 4.2.5 *Let $\mathfrak{P}, \mathfrak{P}', \mathfrak{P}''$ be inverse systems of R-complexes over $\mathbb{N}$.*

(a) Let $0 \to \mathfrak{P}' \to \mathfrak{P} \to \mathfrak{P}'' \to 0$ be a short exact sequence of inverse systems. Then the induced sequence

$$0 \to \operatorname{Mic}(\mathfrak{P}') \to \operatorname{Mic}(\mathfrak{P}) \to \operatorname{Mic}(\mathfrak{P}'') \to 0$$

is an exact sequence of complexes.

(b) Let $\mathfrak{P}' \to \mathfrak{P}$ be a morphism of inverse systems such that the morphisms $P'_t \to P_t$ are quasi-isomorphisms for all $t \in \mathbb{N}$. Then the induced morphism $\operatorname{Mic}(\mathfrak{P}') \to \operatorname{Mic}(\mathfrak{P})$ is a quasi-isomorphism too.

(c) If the R-complexes P_t are exact for all $t \in \mathbb{N}$, then the complex $\operatorname{Mic}(\mathfrak{P})$ is exact.

Proof Products commute with taking homology, in particular a product of short exact sequences is a short exact sequence, a product of quasi-isomorphisms is a quasi-isomorphism and a product of exact complexes is exact. Hence the statements are rather obvious by construction, as follows by 1.5.5. $\square$

Proposition 4.2.6 *Let $\mathfrak{P}$ be an inverse system of R-complexes over $\mathbb{N}$ and let X be an R-complex. Then*

(a) *there is the inverse system* $\operatorname{Hom}_R(X, \mathfrak{P}) = \{\operatorname{Hom}_R(X, P_t)\}$ *and*
(b) *a natural isomorphism* $\operatorname{Mic}(\operatorname{Hom}_R(X, \mathfrak{P})) \cong \operatorname{Hom}_R(X, \operatorname{Mic}(\mathfrak{P}))$.

Proof Recall that $\operatorname{Hom}_R(X, \cdot)$ commutes with direct products, so that

$$\operatorname{Hom}_R(X, \psi_{\mathfrak{P}}) = \psi_{\operatorname{Hom}_R(X,\mathfrak{P})}$$

(see 1.2.7). Then conclude by 1.5.4. □

4.2.7 Let $\mathfrak{P}$ be an inverse system of R-complexes. For any R-complex X we can form the inverse system $\mathfrak{P} \otimes_R X := \{P_t \otimes_R X\}_{t\in\mathbb{N}}$. Doing so we obtain a functor $\operatorname{Mic}(\mathfrak{P} \otimes_R \cdot)$ of the category of R-complexes into itself. This new functor also sometimes has nice properties.

Proposition 4.2.8 *Let $\mathfrak{P}$ be an inverse system of R-complexes over $\mathbb{N}$ and assume that the complexes P_t, $t \in \mathbb{N}$, are right-bounded complexes of flat R-modules. Then the functor* $\operatorname{Mic}(\mathfrak{P} \otimes_R \cdot)$ *is exact and preserves quasi-isomorphisms. In particular, the complex* $\operatorname{Mic}(\mathfrak{P} \otimes_R X)$ *is exact as soon as the complex X is exact.*

Proof Let $0 \to X \to Y \to Z \to 0$ be an exact sequence of R-complexes. By the assumption on the P_t's we have an inverse system of exact sequences

$$0 \to P_t \otimes_R X \to P_t \otimes_R Y \to P_t \otimes_R Z \to 0.$$

It now follows (see 4.2.5 (a)) that the sequence

$$0 \to \operatorname{Mic}(\mathfrak{P} \otimes_R X) \to \operatorname{Mic}(\mathfrak{P} \otimes_R Y) \to \operatorname{Mic}(\mathfrak{P} \otimes_R Z) \to 0$$

is exact. By the assumption on the P_t's again it follows that the complexes $P_t \otimes_R X$ are exact as soon as X is exact (see 4.1.4) so that the functor $P_t \otimes_R \cdot$ preserves quasi-isomorphisms (by 1.5.4). The second and the last statement now follow by 4.2.5 (b) and 4.2.5 (c). □

Here is another particular case.

Proposition 4.2.9 *Let $\mathfrak{P} = \{P_t\}$ be an inverse system of R-complexes over $\mathbb{N}$ and let M be a finitely presented R-module. Then* $\operatorname{Mic}(\mathfrak{P} \otimes_R M) \cong \operatorname{Mic}(\mathfrak{P}) \otimes_R M$.

Proof By definition $\operatorname{Mic}(\mathfrak{P} \otimes_R M) = F(\psi_{\mathfrak{P}\otimes_R M})$, the fiber of the morphism

$$\left(\prod_t P_t \otimes_R M\right) \overset{\psi_{\mathfrak{P}\otimes_R M}}{\to} \left(\prod_t P_t \otimes_R M\right).$$

As M is finitely presented, the functor $(\cdot \otimes_R M)$ commutes with products (see 1.4.4). It follows that $(\prod_t P_t \otimes_R M) \cong (\prod_t P_t) \otimes_R M$, that $\psi_{\mathfrak{P}\otimes_R M} \cong \psi_{\mathfrak{P}} \otimes_R M$ and finally that $F(\psi_{\mathfrak{P}\otimes_R M}) \cong F(\psi_{\mathfrak{P}}) \otimes_R M$ in view of 1.5.4 and the definition of the fiber. The claim follows. □

4.3 The Telescope

We present another construction, also due to Greenlees and May (see [38]), and mention its main properties. Moreover, we add some mixed isomorphisms.

Definition 4.3.1 Let $\mathfrak{D} = \{\sigma_{t,t+1} : D_t \to D_{t+1} \mid t \in \mathbb{N}\}$ denote a direct system over $\mathbb{N}$ of R-complexes D_t. There is a morphism associated to $\mathfrak{D}$

$$\phi_{\mathfrak{D}} : \oplus D_t \to \oplus D_t,\ x \mapsto x - \sigma_{t,t+1}(x) \ \text{ for } x \in D_t.$$

Then in accordance with [38] we define the telescope of the direct system $\mathfrak{D}$ as the cone of $\phi_{\mathfrak{D}}$: that is

$$\mathrm{Tel}(\mathfrak{D}) = C(\phi_{\mathfrak{D}}).$$

Note that this construction is functorial, any morphism $\mathfrak{D} \to \mathfrak{D}'$ of direct systems over $\mathbb{N}$ induces a morphism $\mathrm{Tel}(\mathfrak{D}) \to \mathrm{Tel}(\mathfrak{D}')$ of complexes. The following shows the relation between telescopes and direct limits.

Lemma 4.3.2 *For any direct system $\mathfrak{D} = \{D_t\}$ of R-complexes over $\mathbb{N}$ there is a natural quasi-isomorphism* $\mathrm{Tel}(\mathfrak{D}) \xrightarrow{\sim} \varinjlim D_t$. *Hence there are natural isomorphisms* $H^i(\mathrm{Tel}(\mathfrak{D})) \cong \varinjlim H^i(D_t)$ *for all $i \in \mathbb{Z}$.*

Proof By the definition of the direct limit there is a short exact sequence of complexes

$$0 \to \oplus D_t \xrightarrow{\phi_{\mathfrak{D}}} \oplus D_t \to \varinjlim D_t \to 0.$$

The quasi-isomorphism is now obtained by Lemma 1.5.6 and induces the isomorphisms at the homology level. □

Example 4.3.3 (a) Let x be an element of a commutative ring R and $\mathfrak{D}$ be the direct system for which $D_t = R$ and $\sigma_{t,t+1}$ is the multiplication by x. Then $\mathrm{Tel}(\mathfrak{D})$ is a length one complex and we have $\mathrm{Tel}(\mathfrak{D}) \xrightarrow{\sim} R_x$, where R_x is the ring of fractions of the form r/x^t. In fact, $\mathrm{Tel}(\mathfrak{D})$ is the free resolution of R_x described in 3.1.3.

(b) Let X be an R-complex and $\mathfrak{D}$ be the constant direct system over X, that is, the one for which $D_t = X$ and $\sigma_{t,t+1} = \mathrm{id}_X$. We have a quasi-isomorphism and an isomorphism $\mathrm{Tel}(\mathfrak{D}) \xrightarrow{\sim} \varinjlim \mathfrak{D} \cong X$.

(c) Let $\mathfrak{a}$ be an ideal of a commutative ring R, M an R-module and consider the direct system $\{\mathrm{Hom}_R(R/\mathfrak{a}^t, M)\}$, where the transition homomorphisms are induced by the natural homomorphisms $R/\mathfrak{a}^{t+1} \to R/\mathfrak{a}^t$. There is a quasi-isomorphism

$$\mathrm{Tel}(\{\mathrm{Hom}_R(R/\mathfrak{a}^t, M)\}) \xrightarrow{\sim} \Gamma_{\mathfrak{a}}(M)$$

(see 4.3.2). Note also that $\mathrm{Tel}(\{\mathrm{Hom}_R(R/\mathfrak{a}^t, M)\})$, viewed as an ascending complex, is a complex concentrated in degree 0 and -1.

We now mention some useful properties of the general telescope.

Lemma 4.3.4 *Let $\mathfrak{D}, \mathfrak{D}', \mathfrak{D}''$ denote direct systems of R-complexes over $\mathbb{N}$. Then:*

(a) Let $0 \to \mathfrak{D}' \to \mathfrak{D} \to \mathfrak{D}'' \to 0$ be a short exact sequence of direct systems over $\mathbb{N}$. Then the induced sequence

$$0 \to \mathrm{Tel}(\mathfrak{D}') \to \mathrm{Tel}(\mathfrak{D}) \to \mathrm{Tel}(\mathfrak{D}'') \to 0$$

is a short exact sequence of complexes.

(b) Let $\mathfrak{D}' \to \mathfrak{D}$ be a morphism of direct systems of R-complexes over $\mathbb{N}$ such that the morphisms $D_t' \to D_t$ are quasi-isomorphisms for all $t \in \mathbb{N}$. Then the induced map $\mathrm{Tel}(\mathfrak{D}') \xrightarrow{\sim} \mathrm{Tel}(\mathfrak{D})$ is a quasi-isomorphism.

(c) Let $\mathfrak{D} = \{\mathfrak{D}_t\}_{t \in \mathbb{N}}$ be a direct system of R-complexes. If all the complexes D_t, $t \in \mathbb{N}$, are exact, then the complex $\mathrm{Tel}(\mathfrak{D})$ is exact.

Proof A direct sum of short exact sequences is a short exact sequence, a direct sum of quasi-isomorphisms is a quasi-isomorphism and a direct sum of exact complexes is exact. Hence the statements are rather obvious by construction (see 1.5.5).

Proposition 4.3.5 *Let $\mathfrak{D} = \{D_t\}$ be a direct system of R-complexes over $\mathbb{N}$. For any R-complex X we have the direct system $\mathfrak{D} \otimes_R X := \{D_t \otimes_R X\}$ and a natural isomorphism*

$$\mathrm{Tel}(\mathfrak{D} \otimes_R X) \cong \mathrm{Tel}(\mathfrak{D}) \otimes_R X.$$

Proof As tensor products commute with direct sums we have that $\phi_{\mathfrak{D} \otimes_R X} = \phi_{\mathfrak{D}} \otimes_R X$ (see 1.3.3). The result follows by Lemma 1.5.4. □

Next we note a duality between the telescope and the microscope, which reflects the formal duality between direct and inverse limits.

Proposition 4.3.6 *Let $\mathfrak{D} = \{D_t\}$ be a direct system of R-complexes over $\mathbb{N}$ and let X be any R-complex. Then $\mathrm{Hom}_R(\mathfrak{D}, X) := \{\mathrm{Hom}_R(D_t, X)\}$ is an inverse system of complexes and there is an isomorphism*

$$\mathrm{Hom}_R(\mathrm{Tel}(\mathfrak{D}), X)) \cong \mathrm{Mic}(\mathrm{Hom}_R(\mathfrak{D}, X))$$

natural in both $\mathfrak{D}$ and X. In particular, we have $(\mathrm{Tel}(\mathfrak{D}))^\vee \cong \mathrm{Mic}(\mathfrak{D}^\vee)$, where $(\cdot)^\vee$ denotes the general Matlis duality introduced in 1.4.1.

Proof First we have $\mathrm{Hom}_R(C(\phi_{\mathfrak{D}}), X) \cong F(\mathrm{Hom}_R(\phi_{\mathfrak{D}}, X))$ (see 1.5.4) and also $\mathrm{Hom}_R(\phi_{\mathfrak{D}}, X) = \psi_{\mathrm{Hom}_R(\mathfrak{D}, X)}$ (see 1.3.3 (b)). The conclusion follows. □

Putting 4.3.6 and 4.3.5 together we obtain an adjointness formula.

Proposition 4.3.7 *Let $\mathfrak{D} = \{D_t\}$ be a direct system of R-complexes over $\mathbb{N}$. Let X, Y be two R-complexes. There is a natural isomorphism*

$$\mathrm{Hom}_R(\mathrm{Tel}(\mathfrak{D} \otimes_R X), Y) \cong \mathrm{Hom}_R(X, \mathrm{Mic}(\mathrm{Hom}_R(\mathfrak{D}, Y)).$$

Proof By virtue of 4.3.5 and by adjointness we have the following isomorphism

$$\operatorname{Hom}_R(\operatorname{Tel}(\mathfrak{D} \otimes_R X), Y) \cong \operatorname{Hom}_R(X, \operatorname{Hom}_R(\operatorname{Tel}(\mathfrak{D}), Y)).$$

Because of 4.3.6 we get $\operatorname{Hom}_R(\operatorname{Tel}(\mathfrak{D}), Y)) \cong \operatorname{Mic}(\operatorname{Hom}_R(\mathfrak{D}, Y))$. This finally proves the claim. □

4.3.8 Let $\mathfrak{P}$ be an inverse system of R-complexes over $\mathbb{N}$. For any R-complex X we can also form the direct system $\operatorname{Hom}_R(\mathfrak{P}, X) := \{\operatorname{Hom}_R(P_t, X) \to \operatorname{Hom}_R(P_{t+1}, X) \mid t \in \mathbb{N}\}$. Doing so we obtain a functor $\operatorname{Tel}(\operatorname{Hom}_R(\mathfrak{P}, \cdot))$ of the category of R-complexes into itself. This new functor also sometimes has nice properties.

Lemma 4.3.9 *Let $\mathfrak{P} = \{P_t\}$ be an inverse system of R-complexes over $\mathbb{N}$ and assume that the complexes P_t, $t \in \mathbb{N}$, are right-bounded complexes of projective R-modules. Then the functor* $\operatorname{Tel}(\operatorname{Hom}_R(\mathfrak{P}, \cdot))$ *is exact and preserves quasi-isomorphisms. In particular, the complex* $\operatorname{Tel}(\operatorname{Hom}_R(\mathfrak{P}, X))$ *is exact as soon as the R-complex X is exact.*

Proof Let $0 \to X \to Y \to Z \to 0$ be an exact sequence of R-complexes. By the assumption on the P_t's we have a direct system of exact sequences

$$0 \to \operatorname{Hom}_R(P_t, X) \to \operatorname{Hom}_R(P_t, Y) \to \operatorname{Hom}_R(P_t, Z) \to 0.$$

It now follows by (4.3.4 (a)) that the sequence

$$0 \to \operatorname{Tel}(\operatorname{Hom}_R(\mathfrak{P}, X)) \to \operatorname{Tel}(\operatorname{Hom}_R(\mathfrak{P}, Y)) \to \operatorname{Tel}(\operatorname{Hom}_R(\mathfrak{P}, Z)) \to 0$$

is exact. By the assumption on the P_t's again we know that the complexes $\operatorname{Hom}_R(P_t, X)$ are exact as soon as X is exact (see 1.1.6) Whence the functors $\operatorname{Hom}_R(P_t, \cdot)$ preserve quasi-isomorphisms (in view of 1.5.4). The second and the last statement now follow (see 4.3.4 (b)) and (4.3.4 (c)). □

4.4 Special Resolutions and Their Uses

4.4.1 Let R be a commutative ring. As usual we say that an R-complex X is flat (resp. projective, injective) if all the X^i are.

If the R-complex X is flat (resp. projective, injective), then the functor $X \otimes_R \cdot$ (resp. $\operatorname{Hom}_R(X, \cdot)$, $\operatorname{Hom}_R(\cdot, X)$) is exact on the category of R-complexes: it sends short exact sequences of R-complexes to short exact sequences.

The following notions and results are crucial for the homological algebra of unbounded-complexes.

Definition 4.4.2 Following Spaltenstein [81] we say that an R-complex X is K-flat (resp. K-projective, K-injective) if the functor $X \otimes_R \cdot$ (resp. $\operatorname{Hom}_R(X, \cdot)$,

$\mathrm{Hom}_R(\cdot, X)$) preserves quasi-isomorphisms. In view of 1.5.4 this is equivalent to saying that this functor preserves exactness of R-complexes, which means that for any exact R-complex Z the complex $X \otimes_R Z$ (resp. $\mathrm{Hom}_R(X, Z)$, $\mathrm{Hom}_R(Z, X)$) is still exact.

See [81] and [5] for more details around these notions and their use. Here we shall only recall some facts we want to have at hand, including some proofs or sketches of proofs to become acquainted with them.

Remark 4.4.3 Let F, F', I, P, P', X denote R-complexes.

(a) If F is K-flat and I is K-injective, then $\mathrm{Hom}_R(F, I)$ is K-injective. (This is obtained by adjointness:

$$\mathrm{Hom}_R(X, \mathrm{Hom}_R(F, I)) \cong \mathrm{Hom}_R(X \otimes_R F, I)$$

is exact when X is exact.)

(b) If P and P' are K-projective, then $P \otimes_R P'$ is K-projective. (This is also obtained by adjointness.)

(c) If F and F' are K-flat, then $F \otimes_R F'$ is K-flat. (This is obtained by associativity.)

(d) A K-projective R-complex P is K-flat. (This can be viewed with the general Matlis duality functor, which is faithfully exact: $(P \otimes_R X)^\vee \cong \mathrm{Hom}_R(P, X^\vee)$ is exact when X is exact.)

(e) Recall the classical results. A right-bounded complex F of flat R-modules is K-flat, this follows from 4.1.4. A right-bounded complex P of projective R-modules is K-projective and a left-bounded complex I of injective R-modules is K-injective, this follows from 1.1.6.

(f) If the R-complex X is K-projective (resp. K-injective, K-flat), then so is any shifted complex $X^{[k]}$.

Lemma 4.4.4 *Any contractible R-complex is K-flat, K-projective and K-injective. Conversely, an exact R-complex which is K-projective (resp. K-injective) is contractible.*

Proof The first assertion is a direct consequence of 1.1.8.

If the R-complex P is K-projective and if $P \xrightarrow{\sim} 0$ we have

$$\mathrm{Hom}_R(P, P) \xrightarrow{\sim} \mathrm{Hom}_R(P, 0) = 0.$$

Hence $\mathrm{Hom}_R(P, P)$ is exact. In particular id_P, which is a cycle of degree zero in $\mathrm{Hom}_R(P, P)$, is homotopic to zero. The injective side of the second assertion is proved in a similar way. $\square$

Proposition 4.4.5 (see [81, Proposition 1.4]) *An R-complex P is K-projective if and only if it has the following property: for any morphism $f : P \to Y$ and quasi-isomorphism $g : X \xrightarrow{\sim} Y$ of complexes there is a morphism $f' : P \to X$, unique up to homotopy, such that $g \circ f'$ is homotopic to f:*

$$\begin{tikzcd} & P \arrow[dl, dotted, "f'"'] \arrow[d, "f"] \\ X \arrow[r, "g"] & Y \end{tikzcd}$$

Proof A morphism $f : P \to Y$ is a cycle of degree zero in $\mathrm{Hom}_R(P, Y)$. If P is K-projective and $g : X \xrightarrow{\sim} Y$ is a quasi-isomorphism we have the quasi-isomorphism $\mathrm{Hom}_R(P, g) : \mathrm{Hom}_R(P, X) \xrightarrow{\sim} \mathrm{Hom}_R(P, Y)$. Hence we have a cycle of degree zero in $\mathrm{Hom}_R(P, X)$, unique modulo $H^0(\mathrm{Hom}_R(P, X))$, mapping onto f. This means that we have a morphism $f' : P \to X$, unique up to homotopy, such that $g \circ f'$ is homotopic to f. Conversely, if $g : X \xrightarrow{\sim} Y$ is a quasi-isomorphism and if P has the property of the statement we apply it to the quasi-isomorphism $g^{[n]} : X^{[n]} \xrightarrow{\sim} Y^{[n]}$, $n \in \mathbb{Z}$, to conclude that $\mathrm{Hom}_R(P, g)$ is a quasi-isomorphism. □

Corollary 4.4.6 (see [81, Proposition 1.4]) *Consider the following diagram*

$$\begin{tikzcd} P \arrow[r, "\sim", "h"'] & X \arrow[d, "f"] \\ P' \arrow[r, "\sim", "h'"'] & X' \end{tikzcd}$$

of a morphism f and quasi-isomorphisms h and h' of R-complexes, where P and P' are K-projective. Then there is a morphism $f' : P \to P'$, unique up to homotopy, such that $f \circ h$ is homotopic to $h' \circ f'$.

Remark 4.4.7 By reversing arrows in 4.4.5 and 4.4.6 we obtain a characterization and a property of K-injective R-complexes (see [81, 1.5]).

Corollary 4.4.8 (see [5, 1.4 P, 1.4.I]) *If two K-projective (resp. K-injective) R-complexes X and X' are quasi-isomorphic, then there is a quasi-isomorphism $X \xrightarrow{\sim} X'$.*

Proposition 4.4.9 *Let $f : X \to Y$ be a morphism of R-complexes. If X and Y are K-flat (resp. K-projective, K-injective), then so are the cone $C(f)$ and the fibre $F(f)$.*

Proof This is a rather direct consequence of 1.5.4. Let Z be an exact complex. If X and Y are K-flat, then

$$X \otimes_R Z \simeq 0 \simeq Y \otimes_R Z$$

and $C(f) \otimes_R Z \cong C(f \otimes_R Z)$ is exact.

If X and Y are K-projective, then

$$\mathrm{Hom}_R(X, Z) \simeq 0 \simeq \mathrm{Hom}_R(Y, Z)$$

and $\mathrm{Hom}_R(C(f), Z) \cong F(\mathrm{Hom}_R(f, Z)$ is exact.

If X and Y are K-injective, then

$$\mathrm{Hom}_R(Z, X) \simeq 0 \simeq \mathrm{Hom}_R(Z, Y)$$

and $\mathrm{Hom}_R(Z, C(f)) \cong C(\mathrm{Hom}_R(Z, f)$ is exact. □

Moreover, Spaltenstein proved the following existence theorem.

Theorem 4.4.10 (see [81]) *For any R-complex X there are quasi-isomorphisms $P \xrightarrow{\sim} X$ and $X \xrightarrow{\sim} J$, where the R-complex P is K-projective and the R-complex J is K-injective.*

The quasi-isomorphism $P \xrightarrow{\sim} X$ is called a K-projective resolution of X and the quasi-isomorphism $X \xrightarrow{\sim} J$ is called a K-injective resolution of X.

Proof See [81] or the following more precise Theorem 4.4.20. □

As an application we have the following:

Corollary 4.4.11 (see [5]) *Let Z be any R-complex.*

(a) Any quasi-isomorphism $g : F \xrightarrow{\sim} F'$ between K-flat R-complexes induces a quasi-isomorphism

$$Z \otimes_R g : Z \otimes_R F \xrightarrow{\sim} Z \otimes_R F'.$$

(b) Any quasi-isomorphism $f : X \xrightarrow{\sim} X'$ between K-injective (resp. K-projective) R-complexes induces quasi-isomorphisms

(b1) $Z \otimes_R f : Z \otimes_R X \to Z \otimes_R X'$,
(b2) $\mathrm{Hom}_R(f, Z) : \mathrm{Hom}_R(X', Z) \to \mathrm{Hom}_R(X, Z)$, *and*
(b3) $\mathrm{Hom}_R(Z, f) : \mathrm{Hom}_R(Z, X) \to \mathrm{Hom}_R(Z, X')$.

Proof For the proof of (a) we take a K-projective resolution $L \xrightarrow{\sim} Z$. The data in (a) provide a commutative square

$$\begin{array}{ccc} L \otimes_R F & \longrightarrow & L \otimes_R F' \\ \downarrow & & \downarrow \\ Z \otimes_R F & \longrightarrow & Z \otimes_R F'. \end{array}$$

In this square the vertical morphisms are quasi-isomorphisms because of the assumptions on F, F'. Moreover, as L is K-projective hence also K-flat, we have that the top horizontal morphism also is a quasi-isomorphism. The claim in (a) follows.

Now we prove (b). The cone $C(f)$ is K-injective (resp. K-projective) (see 4.4.9) and exact. Hence it is contractible (see 4.4.4). It follows that the complexes $Z \otimes_R C(f)$, $\mathrm{Hom}_R(C(f), Z)$ and $\mathrm{Hom}_R(Z, C(f))$ are also contractible (see 1.1.8), hence exact. As $Z \otimes_R C(f) \cong C(Z \otimes_R f)$, $\operatorname{Hom}_R(C(f), Z) \cong F(\operatorname{Hom}_R(f, Z)$ and $\operatorname{Hom}_R(Z, C(f)) \cong C(\operatorname{Hom}_R(Z, f)$ (see 1.5.4) the conclusion follows by the remarks in 1.5.1. $\square$

4.4.12 *Towards the homological algebra of unbounded complexes.* Let G be an additive covariant functor from the category of R-complexes into another category of complexes and assume that G commutes with shift. As a morphism in the category of R-complexes becomes an isomorphism in the derived category exactly when it is a quasi-isomorphism and in view of the preceding we have that the left-derived functor $\mathrm{L}\, G$ (respectively the right-derived functor $\mathrm{R}\, G$) is well defined in the derived category as soon as G preserves quasi-isomorphisms between K-projective R-complexes (respectively between K-injective R-complexes). In that case, given a complex X and one of its K-projective resolutions $P \to X$ (respectively one of its K-injective resolutions $X \to I$), then $\mathrm{L}\, G(X)$ is represented by $G(P)$ (respectively $\mathrm{R}G(X)$ is represented by $G(I)$).

In these notes we do not really work in the derived category. We mainly work in the category of R-complexes but use $\mathrm{L}\, G$ and $\mathrm{R}\, G$ as convenient notations, noting that $\mathrm{L}\, G(X)$ and $\mathrm{R}\, G(X)$, viewed as R-complexes, are well defined up to a quasi-isomorphism as soon as they exist. We also refer to a short explanation of the derived category in [36, III, §1, 6] *entitled "What is the derived category".*

More precisely, if X' is another R-complex quasi-isomorphic to X, if $P' \xrightarrow{\sim} X'$ is a K-projective resolution (respectively $X' \xrightarrow{\sim} I'$ is a K-injective resolution), then we have a quasi-isomorphism $P \xrightarrow{\sim} P'$ (respectively $I \xrightarrow{\sim} I'$) (in view of 4.4.8). Hence we also have a quasi-isomorphism $G(P) \xrightarrow{\sim} G(P')$ (respectively a quasi-isomorphism $G(I) \xrightarrow{\sim} G(I')$) with the hypotheses on G.

More down to earth: we note that $H_i(G(P))$ and $H^i(G(I))$ depend only on X and not on the choice of the resolutions P and I, so that without ambiguity we can define $G_i(X) = H_i(G(P))$ and $G^i(X) = H^i(G(I))$. By doing so we obtain functors G_i and G^i from the category of R-complexes to the category of R-modules called respectively the i^{th} left-derived and the i^{th} right-derived functor of G. These functors G_i, G^i have the following property: if $X \simeq X'$, then $G^i(X) \cong G^i(X')$ and $G_i(X) \cong G_i(X')$. Note also that $G^i(X)$ and $G_i(X)$ are functorial in X (in view of 4.4.6 and 4.4.7).

When the functor G is contravariant the existence of $\mathrm{L}\, G$ and $\mathrm{R}\, G$, of G_i and G^i, is discussed in the same way, but exchanging the roles of K-projective and K-injective resolutions.

Finally, note that, if G is an additive functor from the category of R-modules into another category of modules, the classical notion of left-derived and right-derived functors coincides with the one described above.

In the following isomorphisms in the derived category will be denoted by $\simeq$.

As a first application of the above, we now discuss RHom and $\otimes_R^{\mathrm{L}}$.

4.4.13 We first note that RHom and $\otimes_R^{\mathrm{L}}$ are well defined in the derived category. This follows by 4.4.11 and the remarks in 4.4.12. By 1.1.5 note also that the functors

$\mathrm{Hom}_R(X, \cdot)$, $\mathrm{Hom}_R(\cdot, Y)$ and $(X \otimes_R \cdot)$ commute with shifts. In particular, if $F \xrightarrow{\sim} X$ and $F' \xrightarrow{\sim} Y$ are K-flat or K-projective resolutions, then $X \otimes_R^{\mathrm{L}} Y$ is represented by any of the quasi-isomorphic complexes

$$X \otimes_R F' \xleftarrow{\sim} F \otimes_R F' \xrightarrow{\sim} F \otimes_R Y.$$

If $P \xrightarrow{\sim} X$ is a K-projective resolution and $Y \xrightarrow{\sim} I$ is a K-injective resolution, then $\mathrm{RHom}_R(X, Y)$ is represented by any of the quasi-isomorphic complexes

$$\mathrm{Hom}_R(X, I) \xrightarrow{\sim} \mathrm{Hom}_R(P, I) \xleftarrow{\sim} \mathrm{Hom}_R(P, Y).$$

In accordance with [5] we now define

$$\mathrm{Ext}^i_R(X, Y) := H^i(\mathrm{RHom}_R(X, Y)) \text{ and } \mathrm{Tor}_i^R(X, Y) := H_i(X \otimes_R^{\mathrm{L}} Y)$$

for all $i \in \mathbb{Z}$.

Moreover, we again have the Hom-Tensor adjointness formula in the derived category: for three complexes X, Y, Z we have a natural isomorphism

$$\mathrm{RHom}_R(X \otimes_R^{\mathrm{L}} Y, Z) \simeq \mathrm{RHom}_R(X, \mathrm{RHom}_R(Y, Z))$$

in the derived category. This follows from the classical adjointness formula: we may replace Y by one of its K-flat resolutions $F \xrightarrow{\sim} Y$ and Z by one of its K-injective resolutions $Z \xrightarrow{\sim} J$, noting with 4.4.3 that $\mathrm{Hom}_R(F, J)$ is K-injective. The unit and counit are represented by the natural morphisms

$$\mathrm{Hom}_R(P, X) \otimes_R P \to X \text{ and } X \to \mathrm{Hom}_R(P, X \otimes_R P),$$

where $P \xrightarrow{\sim} Y$ is a K-projective resolution of Y.

It follows that we again have an Ext-Tor duality formula for complexes: for all $i \in \mathbb{Z}$ we have natural isomorphisms

$$(X \otimes_R^{\mathrm{L}} Y)^\vee \simeq \mathrm{RHom}_R(X, Y^\vee) \text{ and } \mathrm{Tor}_i^R(X, Y)^\vee \cong \mathrm{Ext}^i_R(X, Y^\vee),$$

where $(\cdot)^\vee = \mathrm{Hom}_R(\cdot, E)$ for an injective cogenerator E.

Moreover, if Y is left-bounded and if X has a right-bounded free resolution $L \xrightarrow{\sim} X$ with the L_i's finitely generated, then

$$X \otimes_R^{\mathrm{L}} Y^\vee \simeq \mathrm{RHom}_R(X, Y)^\vee \text{ and } \mathrm{Tor}_i^R(X, Y^\vee) \cong \mathrm{Ext}^i_R(X, Y)^\vee.$$

This last fact follows by the observations in 1.1.9.

Remark 4.4.14 Note that a K-flat R-complex is not necessarily flat (any contractible complex is K-flat). Note also that a flat R-complex is not necessarily K-flat. A typical example is the following.

Take $R = \mathbb{Z}/4\mathbb{Z}$ and for X the $\mathbb{Z}/4\mathbb{Z}$-complex defined by $X_i = \mathbb{Z}/4\mathbb{Z}$ and d_i^X is the multiplication by 2. This complex X is flat and exact: $X \xrightarrow{\sim} 0$, but $X \otimes_{\mathbb{Z}/4\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$ is not exact. Hence X is not K-flat in view of 4.4.11.

The same remarks hold on the projective and injective side (by considering also the general Matlis dual complex $X^\vee$). This example also shows that there are exact complexes which are neither K-flat, nor K-projective, nor K-injective.

Remark 4.4.15 If $X \to Y$ is a homotopism and if X is K-projective (resp. K-injective, K-flat), then so is Y.

The proofs follows from the recall in 1.1.4 and are left to the reader. But note that a complex quasi-isomorphic to a K-projective (resp. K-injective, K-flat) complex is not necessarily K-projective (resp. K-injective, K-flat). (To see this note that any exact complex is quasi-isomorphic to a contractible one, then remember 4.4.4.)

Though the existence of K-projective and K-injective resolutions is enough to develop a (co)homological theory of unbounded complexes, we sometimes need better resolutions.

Definition 4.4.16 Following Avramov and Foxby [5] we say that an R-complex X is DG-flat (resp. DG-projective, DG-injective) if it is both flat and K-flat (resp. projective and K-projective, injective and K-injective).

Remark 4.4.17 With the above definitions and some of the remarks in 4.4.3 recall the following.

(a) Let $f : X \to Y$ be a morphism of complexes. If X and Y are DG-projective (resp. DG-injective, DG-flat), then so are the complexes $C(f)$ and $F(f)$.

(b) If F is DG-flat and J is DG-injective, then $\mathrm{Hom}_R(F, J)$ is DG-injective.

(c) If P and P' are DG-projective, then $P \otimes_R P'$ is DG-projective.

(d) If F and F' are DG-flat, then $F \otimes_R F'$ is DG-flat.

(e) Direct sums of DG-projective R-complexes are DG-projective.

(f) Direct products of DG-injective R-complexes are DG-injective.

To prove the existence of DG-projective resolutions and DG-injective resolutions we first recall that any right-bounded R-complex X has a right-bounded projective resolution $P \xrightarrow{\sim} X$ and that such a P is DG-projective (see 4.4.3). Dually we also recall that any left-bounded R-complex X has a left-bounded injective resolution $X \xrightarrow{\sim} I$ and that such an I is DG-injective. Then we recall Spaltenstein's construction.

Lemma 4.4.18 (see [81]) *Let* $0 \to X \xrightarrow{f} Y \to Z \to 0$ *be a short exact sequence of right-bounded ascending* R*-complexes and let* $u_X : P \xrightarrow{\sim} X$ *be a right-bounded projective resolution. Then there is a right-bounded projective resolution* $u_Y : Q \xrightarrow{\sim} Y$ *and a commutative diagram with exact rows*

$$\begin{tikzcd}
0 \arrow[r] & P \arrow[r] \arrow[d, "u_X"'] & Q \arrow[d, "u_Y"] \\
0 \arrow[r] & X \arrow[r, "f"'] & Y
\end{tikzcd}$$

Proof We write $f' = u_X \circ f$, then we take the fiber of f' and a right-bounded projective resolution $v : S \xrightarrow{\sim} F(f')$. We consider the two commutative squares

$$\begin{tikzcd}
P \arrow[r, "f'"] \arrow[d, "u_x"'] & Y \arrow[d, equal] \\
X \arrow[r, "f"] & Y
\end{tikzcd} \quad \text{and} \quad \begin{tikzcd}
S \arrow[r, "w'"] \arrow[d, "v"'] & P \arrow[d, equal] \\
F(f') \arrow[r, "w"] & P
\end{tikzcd}$$

where $w : F(f') \to P$ is the natural projection. We recall that $\mathrm{Cyl}(f') = C(-w)$ and obtain (see the observations in 1.5.8) a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & P \arrow[r] \arrow[d, equal] & C(-w') \arrow[r] \arrow[d] & S^{[1]} \arrow[r] \arrow[d, "v^{[1]}"] & 0 \\
0 \arrow[r] & P \arrow[r] \arrow[d, "u_X"] & \mathrm{Cyl}(f') \arrow[r] \arrow[d] & C(f') \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & X \arrow[r] \arrow[d, equal] & \mathrm{Cyl}(f) \arrow[r] \arrow[d, "h"] & C(f) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & X \arrow[r, "f"] & Y \arrow[r] & Z \arrow[r] & 0.
\end{tikzcd}$$

In this diagram the vertical morphisms on the right and on the left are quasi-isomorphisms and so are the ones on the middle. Finally, we observe that $C(-w')$ is a right-bounded complex of projective R-modules and take $Q = C(-w')$.

Lemma 4.4.19 (see [81]) *Let* $0 \to X \to Y \xrightarrow{f} Z \to 0$ *be a short exact sequence of left-bounded descending* R*-complexes and let* $v_Z : Z \xrightarrow{\sim} I$ *be a left-bounded injective resolution. Then there is a left-bounded injective resolution* $v_Y : Y \xrightarrow{\sim} J$ *and a commutative diagram with exact rows*

$$\begin{tikzcd}
Y \arrow[r, "f"] \arrow[d, "v_Y"'] & Z \arrow[r] \arrow[d, "v_Z"] & 0 \\
J \arrow[r] & I \arrow[r] & 0
\end{tikzcd}$$

Proof The proof is similar to that of 4.4.18, using 1.5.9 in place of 1.5.8.

Theorem 4.4.20 (see [5, 1.6]) *Let X denote an R-complex.*

(a) X has a DG-projective resolution: there exists a quasi-isomorphism $P \xrightarrow{\sim} X$ where P is a DG-projective R-complex, hence also DG-flat.
(b) X has a DG-injective resolution: there exists a quasi-isomorphism $X \xrightarrow{\sim} I$ where I is a DG-injective R-complex.

Proof (a) We start with a right-bounded projective resolution of the right soft truncation $\tau_{0]}(X)$ defined in 4.1.5, then construct, step by step and with the help of 4.4.18 a direct system of quasi-isomorphisms

$$\begin{tikzcd} P_{t+1} \arrow[r, "\sim"] & \tau_{t+1]}(X) \\ P_t \arrow[r, "\sim"] \arrow[u] & \tau_{t]}(X) \arrow[u] \end{tikzcd}$$

for all $t \geq 0$, where the P_t are right-bounded R-complexes of projective R-modules, hence DG-projective. Then we take direct limits and telescopes. This gives quasi-isomorphisms

$$\mathrm{Tel}(\mathfrak{P}) \xrightarrow{\sim} \varinjlim P_t \xrightarrow{\sim} \varinjlim X_{t]} = X,$$

where $\mathfrak{P}$ denotes the direct system $\{P_t \to P_{t+1}\}$ (see 1.3.2, 4.1.5 and 4.3.2). By definition $\mathrm{Tel}(\mathfrak{P}) = C(\phi_{\mathfrak{P}})$, where $\phi_{\mathfrak{P}} : \oplus P_t \to \oplus P_t$ is a morphism between DG-projective complexes (see 4.4.17 (e)). It follows that $\mathrm{Tel}(\mathfrak{P})$ is DG-projective (see 4.4.17 (a)) and the above composite gives the resolution wanted in (a).

(b) We start with a left-bounded injective resolution of the left soft truncation $\tau_{[0}(X)$. With the help of 4.4.19 we construct, step by step, an inverse system of quasi-isomorphisms

$$\begin{tikzcd} \tau_{[t+1}(X) \arrow[r, "\sim"] \arrow[d] & J_{t+1} \arrow[d] \\ \tau_{[t}(X) \arrow[r, "\sim"] & J_t \end{tikzcd}$$

for all $t \geq 0$, where the J_t are left-bounded R-complexes of injective modules, hence DG-injective (see [81, 3.3]). Then we take microscopes and inverse limits. As the inverse system $\{\tau_{[t}(X)\}$ is degree-wise surjective there are quasi-isomorphisms

$$X = \varprojlim \tau_{[t}(X) \xrightarrow{\sim} \mathrm{Mic}(\{\tau_{[t}(X)\}) \xrightarrow{\sim} \mathrm{Mic}(\{J_t\})$$

(see 4.1.5, 4.2.3 and (4.2.5 (b)). It remains to check that $\mathrm{Mic}(\{J_t\})$ is DG-injective. But this is clear, by definition $\mathrm{Mic}(\{J_t\}) = F(\psi_{\{J_t\}})$ and $\psi_{\{J_t\}} : \prod J_t \to \prod J_t$ is a morphism between DG-injective complexes (see 4.4.17 (f)). Hence $\mathrm{Mic}(\{J_t\})$ is DG-injective by 4.4.17 (a). $\square$

4.4.21 *Change of rings*

(A) Let $\varphi : R \to B$ be a homomorphism of commutative rings, let X be an R-complex and Y a B-complex. The B-complex Y may be viewed as an R-complex via φ. There are two B-complexes associated to the R-complex X, namely

$$X_{(\varphi)} := B \otimes_R X \text{ and } X^{(\varphi)} := \operatorname{Hom}_R(B, X).$$

When there is no doubt about the homomorphism φ we also write

$$X_{(\varphi)} := X_B \text{ and } X^{(\varphi)} := X^B.$$

In this situation we have natural isomorphisms

$$\operatorname{Hom}_R(X, Y) \cong \operatorname{Hom}_B(X_B, Y),\ \operatorname{Hom}_R(Y, X) \cong \operatorname{Hom}_B(Y, X^B)$$
$$\text{and } X \otimes_R Y \cong X_B \otimes_B Y.$$

We now recall the following consequences of the above.

(B) Let M be an R-module. Then:

(b1) If the R-module M is injective, then the B-module M^B is injective.

(b2) If the R-module M is flat, then the B-module M_B is flat.

(b3) If the R-module M is free (resp. projective), then the B-module M_B is free (resp. projective).

(C) Let X be an R-complex. Then the following statements are true:

(c1) If the R-complex X is K-injective (resp. DG-injective), then the B-complex X^B is K-injective (resp. DG-injective).

(c2) If the R-complex X is K-flat (resp. DG-flat), then the B-complex X_B is K-flat (resp. DG-flat).

(c3) If the R-complex X is K-projective (resp. DG-projective), then the B-complex X_B is K-projective (resp. DG-projective).

(D) Assume now that the homomorphism $\varphi : R \to B$ is flat.

(d1) If the R-complex X is K-injective (resp. DG-injective), then the complex $X^B = \operatorname{Hom}_R(B, X)$, viewed as an R-complex, is also K-injective (resp. DG-injective).

(d2) Let X, X' be two R-complexes. Then

$$\operatorname{Tor}_i^R(X, X') \otimes_R B \cong \operatorname{Tor}_i^B(X_B, X'_B)$$

for all $i \in \mathbb{Z}$. In particular, Tor localizes.

(If $F \xrightarrow{\sim} X$ is a K-flat resolution of X, then $F_B \xrightarrow{\sim} X_B$ is a K-flat resolution of X_B and $(F \otimes_R X') \otimes_R B \cong F_B \otimes_R X'_B$. We take homology to obtain the claim.)

(d3) Assume that the R-complex X has a right-bounded free resolution $L \xrightarrow{\sim} X$ such that all the L_i are finitely generated. Assume also that the R-complex X' is left-bounded, then

$$\operatorname{Ext}_R^i(X, X') \otimes_R B \cong \operatorname{Ext}_B^i(X_B, X'_B)$$

for all $i \in \mathbb{Z}$. In particular, $\mathrm{Ext}_R(X, X')$ localizes in this case.

(From our hypotheses we have $\mathrm{Hom}_R(L, X') \cong \mathrm{Hom}_R(L, R) \otimes_R X'$ (see 1.1.9) so that $\mathrm{Hom}_R(L, X') \otimes_R B \cong \mathrm{Hom}_R(L, R) \otimes_R (X' \otimes_R B) \cong \mathrm{Hom}_R(L, X'_B) \cong \mathrm{Hom}_B(L_B, X'_B)$. We take cohomology and obtain the result.)

(d4) Note that the hypotheses on X in (d3) are satisfied when X is homologically right-bounded with finitely generated homology modules and when the ring R is Noetherian. In particular, if R is Noetherian, if X is a homologically right-bounded with finitely generated homology modules and if X' is a homologically left-bounded R-complex, then

$$\mathrm{Ext}^i_R(X, X') \otimes \hat{R}^{\mathfrak{a}} \cong \mathrm{Ext}^i_{\hat{R}^{\mathfrak{a}}}(X \otimes \hat{R}^{\mathfrak{a}}, X' \otimes \hat{R}^{\mathfrak{a}})$$

for all ideals $\mathfrak{a}$ of R and all R-complexes X. This is because $\hat{R}^{\mathfrak{a}}$ is R-flat.

The content of the previous subsection is the basis for our use of homological algebra, in particular of unbounded complexes. For further information on the concept of derived category and its use in the homological theory of complexes, the reader may consult [85, Chapter 10] and the books [36], [53] or [87].

4.5 Minimal Injective Resolutions for Unbounded Complexes

In this section we revisit the notion of minimal injective resolutions, particularly helpful when the ring is Noetherian, though available over any commutative ring. Minimal injective resolutions of a module over a Noetherian ring are well understood, see Bass' fundamental paper [9]. One of the features here is that we also consider minimal injective resolutions of unbounded complexes.

4.5.1 Let $f : M \to N$ be a homomorphism of modules over the commutative ring R and let M' be a submodule of M. We say that M' is a *common direct summand of M and N under f* if there is a commutative square

$$\begin{tikzcd} M' \arrow[r, "u"] \arrow[d, "i"'] & N' \\ M \arrow[r, "f"] & N \arrow[u, "p"'] \end{tikzcd}$$

where u is an isomorphism. In that case, i is split-injective, p is split-surjective, and f induces a homomorphism $M/M' \to N/N'$.

Definition 4.5.2 *Minimal injective complexes.* Let I be a complex of injective modules over a commutative ring R. We say that this injective complex is minimal at spot j if I^j is an essential extension of $\mathrm{Ker}(d_I^j)$, in other words if I^j is the injective hull of $\mathrm{Ker}(d_I^j)$. We say that I is minimal if for all $j \in \mathbb{Z}$ it is minimal at spot j.

4.5.3 *Cancellation trick.* Let I denote any injective complex and let us look at spot j for some $j \in \mathbb{Z}$. Let $E_R(\mathrm{Ker}(d_I^j))$ be the injective hull of $\mathrm{Ker}(d_I^j)$. The injection $\mathrm{Ker}(d_I^j) \hookrightarrow I^j$ extends to an injection $E_R(\mathrm{Ker}(d_I^j)) \hookrightarrow I^j$, hence $E_R(\mathrm{Ker}(d_I^j))$ is a direct summand of I^j and we may choose an injective module I'^j such that $I^j = E_R(\mathrm{Ker}(d_I^j)) \oplus I'^j$. The restriction of d_I^j to I'^j is clearly injective, hence I'^j appears as a common direct summand of I^j and I^{j+1} under d_I^j. We cancel this common direct summand. Doing so we obtain an injective complex $\bar{I}$ with $\bar{I}^j = E_R(\mathrm{Ker}(d_I^j))$ and a degree-wise surjective morphism $p : I \to \bar{I}$. We observe that $\bar{I}$ is minimal at spot j because p maps $\mathrm{Ker}(d_I^j)$ into $\mathrm{Ker}(d_{\bar{I}}^j)$.

Lemma 4.5.4 *(a) Let I denote any injective complex over a commutative ring R. This I is minimal at spot j if and only if I^j and I^{j+1} do not have any non-trivial common direct summands under d_I^j.*

(b) Let $0 \to I' \to I \to I'' \to 0$ be a short exact sequence of injective complexes over a commutative ring R. If I is minimal at some spot $j \in \mathbb{Z}$, then so are I' and I''.

Proof The first assertion is a direct consequence of the cancellation trick described in 4.5.3. The second is a direct consequence of the definition of minimality. □

We do not know if the following, well-known when I is left-bounded and R is Noetherian, has already been noticed in general.

Proposition 4.5.5 *Let I denote an injective complex over a commutative ring R. There is a short exact sequence of complexes, degree-wise split-exact*

$$0 \to I_{(e)} \to I \xrightarrow{p} J \to 0,$$

where p is a quasi-isomorphism, J is a minimal injective complex and $I_{(e)}$ is an exact complex of injective modules.

If, moreover, I is DG-injective, then we have such a sequence where J is also DG-injective and where $I_{(e)}$ is contractible.

Proof We first look at even spots. We may write $I^{2j} = E_R(\mathrm{Ker}(d_I^{2j})) \oplus I'^{2j}$ for some choice of the I'^{2j} and we note that I'^{2j} is a common direct summand of I^{2j} and I^{2j+1} under d^{2j} (see 4.5.3). Then we cancel these common direct summands and obtain a short exact sequence, degree-wise split-exact,

$$0 \to I' \to I \xrightarrow{f} \bar{I} \to 0,$$

where I' is the complex

$$\cdots \xrightarrow{0} I'^{2j} = I'^{2j} \xrightarrow{0} I'^{2j+2} = I'^{2j+2} \xrightarrow{0} \cdots.$$

We note that I' is contractible, hence exact and DG-injective, that f is a quasi-isomorphism and that the injective complex $\bar{I}$ is minimal at spots $2j$: $\bar{I}^{2j} =$

$E_R(\mathrm{Ker}(d_I^{2j})) = E_R(\mathrm{Ker}(d_{\bar{I}}^{2j}))$ for all $j \in \mathbb{Z}$. If, moreover, I is DG-injective, then so is $\bar{I}$ because I' is contractible.

Then we take care of the odd spots of the complex $\bar{I}$ in the same way as above, we obtain a short exact sequence, degree-wise split-exact,

$$0 \to I'' \to \bar{I} \xrightarrow{g} J \to 0,$$

where the injective complex I'' is contractible and where J is minimal at odd spots. As the complex $\bar{I}$ is minimal at even spot, so is the complex J (see 4.5.4). Hence J is minimal.

We put $p = g \circ f$ and obtain the claim. Note that if I is DG-injective, so are $\bar{I}$ and J, hence also $I_{(e)}$, so that the exact DG-injective complex $I_{(e)}$ is contractible. □

Corollary 4.5.6 *Every R-complex X has a minimal DG-injective resolution.*

If, moreover, X is homologically left-bounded, then X has a left-bounded minimal DG-injective resolution I with

$$\inf\{j \mid I^j \neq 0\} = \inf\{j \mid H^j(X) \neq 0\}.$$

Minimal injective complexes have nice properties as we shall investigate in the following.

Proposition 4.5.7 *Let $\mathfrak{m}$ be any maximal ideal of a commutative ring R and let I denote a minimal injective R-complex.*

Then the complex $\mathrm{Hom}_R(\Bbbk(\mathfrak{m}), I)$ has zero differentials.

Proof Note that $\Bbbk(\mathfrak{m}) = R/\mathfrak{m}$ is a simple R-module. If $f : \Bbbk(\mathfrak{m}) \to I^j$ is some non-zero homomorphism it is injective by the simplicity of $\Bbbk(\mathfrak{m})$ and factors through $E_R(R/\mathfrak{m})$, giving a homomorphism $f' : E_R(R/\mathfrak{m}) \to I^j$ which is again injective because $E_R(R/\mathfrak{m})$ is an essential extension of $\Bbbk(\mathfrak{m})$. If $d^j \circ f \neq 0$ then $d^j \circ f$ is injective because $\Bbbk(\mathfrak{m})$ is simple. It follows that $d^j \circ f'$ is injective too. Hence $E_R(R/\mathfrak{m})$ appears as a common direct summand of I^j and I^{j+1} under d^j, contradicting the minimality of I. □

We shall need more information on injective modules.

Recalls 4.5.8 (a) Let R be a commutative ring. As usual we denote by $E_R(M)$ or simply $E(M)$ the injective hull of an R-module M. For a prime ideal $\mathfrak{p}$ of R and an element $x \notin \mathfrak{p}$ we note that multiplication by x acts as an automorphism on $E_R(R/\mathfrak{p})$ so that $E_R(R/\mathfrak{p})$ has the structure of an $R_\mathfrak{p}$-module. With its $R_\mathfrak{p}$-module structure we note that $E_R(R/\mathfrak{p})$ is also the injective hull of $\Bbbk(\mathfrak{p}) := R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$, the residue field of the local ring $R_\mathfrak{p}$: $E_R(R/\mathfrak{p}) = E_{R_\mathfrak{p}}(\Bbbk(\mathfrak{p}))$. That is because for any $R_\mathfrak{p}$-complex X we have $X \otimes_R R_\mathfrak{p} \cong X$ and $\mathrm{Hom}_{R_\mathfrak{p}}(X, E_R(R/\mathfrak{p})) \cong \mathrm{Hom}_R(X, E_R(R/\mathfrak{p}))$. As $R/\mathfrak{p}$ is a domain note also that the injective modules $E_R(R/\mathfrak{p})$ are indecomposable as R-modules hence also as $R_\mathfrak{p}$-modules.

(b) Assume now that R is Noetherian. Then $\operatorname{Ass}_R(E_R(R/\mathfrak{p})) = \{\mathfrak{p}\}$, hence $E_R(R/\mathfrak{p})$ and $\Bbbk(\mathfrak{p})$ are $\mathfrak{p}$-torsion modules. We also have that the $E_R(R/\mathfrak{p})$'s, $\mathfrak{p} \in \operatorname{Spec}(R)$, are the only indecomposable injective modules. Moreover, every injective R-module may be written as a direct sum of modules of the form $E_R(R/\mathfrak{p})$, $\mathfrak{p} \in \operatorname{Spec}(R)$, in a unique way, up to the order of the direct summands. See Matlis' original paper [55] for more details.

Remark 4.5.9 Let J denote an injective module over a Noetherian ring R and let $\mathfrak{q}$ be a prime ideal. Then the R-modules $\Gamma_\mathfrak{q}(J)$ and $J_\mathfrak{q}$ are also injective. More precisely we may write

$$J \cong \oplus_{\mathfrak{p}\in \operatorname{Spec}\ R}(E_R(R/\mathfrak{p}))^{(c(\mathfrak{p},J))},$$

where the $c(\mathfrak{p}, J)$'s are cardinal numbers and where $(E_R(R/\mathfrak{p}))^{(c(\mathfrak{p},J))}$ denote the direct sum of $c(\mathfrak{p}, J)$ copies of $E_R(R/\mathfrak{p})$. It follows that

$$\Gamma_\mathfrak{q}(J) \cong \oplus_{\mathfrak{p}\in V(\mathfrak{q})}(E_R(R/\mathfrak{p}))^{(c(\mathfrak{p},J))} \text{ and}$$
$$J_\mathfrak{q} \cong \oplus_{\mathfrak{p}\subseteq q}(E_R(R/\mathfrak{p}))^{(c(\mathfrak{p},J))}.$$

This shows that the R-modules $\Gamma_\mathfrak{q}(J)$ and $J_\mathfrak{q}$ are both injective. It also shows that the natural homomorphism $J \to J_\mathfrak{q}$ is split-surjective. Then note that $J_\mathfrak{q}$ viewed as an $R_\mathfrak{q}$-module is also injective.

Now let $\mathfrak{q} \in \operatorname{Spec}(R)$. Then it yields that

$$\operatorname{Hom}_R(\Bbbk(\mathfrak{q}), J_\mathfrak{q}) \cong \operatorname{Hom}_{R_\mathfrak{q}}(\Bbbk(\mathfrak{q}), J_\mathfrak{q}) \cong \operatorname{Hom}_{R_\mathfrak{q}}(\Bbbk(\mathfrak{q}), \Gamma_\mathfrak{q}(J_\mathfrak{q}))$$

because the $R_\mathfrak{q}$-module $\Bbbk(\mathfrak{q})$ is $\mathfrak{q}$-torsion. Note also that

$$\Gamma_\mathfrak{q}(J_\mathfrak{q}) \cong (E_R(R/\mathfrak{q}))^{(c(\mathfrak{q},J))}.$$

Proposition 4.5.10 *Let I denote an injective complex over a Noetherian ring R. If I is minimal then the complexes $I_\mathfrak{q}$ and $\Gamma_\mathfrak{q}(I)$ are also minimal injective for every prime ideal $\mathfrak{q}$ of R. Moreover, $I_\mathfrak{q}$ viewed as an $R_\mathfrak{q}$-complex is also minimal injective.*

Proof From the remarks in 4.5.9 we note that $\Gamma_\mathfrak{q}(I)$ is an injective subcomplex of I and that $I_\mathfrak{q}$ is an injective complex quotient of I. Hence the statements follow by 4.5.4. □

We continue the investigation of the minimal injective resolution of an R-complex in the case when the ring is Noetherian. In that case minimal injective complexes have the following characterization.

Proposition 4.5.11 *Let I be an injective complex over a Noetherian ring R. The following conditions are equivalent:*

(i) I is minimal,
(ii) the complex $\operatorname{Hom}_R(\Bbbk(\mathfrak{p}), I_\mathfrak{p})$ has zero differentials for all $\mathfrak{p} \in \operatorname{Spec}(R)$.

Proof First note that $\operatorname{Hom}_R(\Bbbk(\mathfrak{p}), I_\mathfrak{p}) \cong \operatorname{Hom}_{R_\mathfrak{p}}(\Bbbk(\mathfrak{p}), I_\mathfrak{p})$. We have seen in 4.5.10 that the $R_\mathfrak{p}$-complex $I_\mathfrak{p}$ is minimal injective when I is minimal. Hence the implication (i) $\Rightarrow$ (ii) follows by 4.5.7. On the other hand, if I is not minimal, then for some $j \in \mathbb{Z}$ the modules I^j and I^{j+1} have a non-zero common direct summand under d^j, hence also a non-zero common direct summand of the form $E_R(R/\mathfrak{p})$ for some $\mathfrak{p} \in \operatorname{Spec}(R)$. It follows that condition (ii) is not satisfied. $\square$

We need some more information on the functors Hom and Ext. We first investigate their behaviour under a ring homomorphism $R \to B$. Then for an R-complex X the complex $X \otimes_R B$ has the structure of a B-complex by ring extensions. For two R-complexes X and Y there is a natural morphism

$$\operatorname{Hom}_R(Y, X) \otimes_R B \to \operatorname{Hom}_B(Y \otimes_R B, X \otimes_R B)$$

of B-complexes. We investigate when it is an isomorphism or a quasi-isomorphism.

Lemma 4.5.12 *Let X be a complex over a Noetherian ring R. Let Y be a second R-complex, degree-wise finitely generated. Let $R \to B$ denote a flat ring homomorphism. Then the natural morphism of B-complexes*

$$\operatorname{Hom}_R(Y, X) \otimes_R B \to \operatorname{Hom}_B(Y \otimes_R B, X \otimes_R B)$$

is an isomorphism provided one of the following conditions is satisfied:

(a) Y is right-bounded and X is left-bounded,
(b) Y is bounded,
(c) X is bounded.

Proof The statement is known when $Y = N$ and $X = M$ are R-modules. In general, $\operatorname{Hom}_R(Y, X)^n = \prod_{i+j=n} \operatorname{Hom}_R(Y_i, X^j)$ is a finite product of the $\operatorname{Hom}_R(Y_i, X^j)$'s when one of the conditions (a), (b) and (c) is satisfied. Because the functor $\cdot \otimes_R B$ commutes with finite products the statement follows. $\square$

For a multiplicatively closed set T in a commutative ring R, $(1 \in T)$, we denote by R_T or $T^{-1}R$ the ring of fractions r/s, $r \in R$, $s \in T$. Then for an R-complex X we denote by X_T or $T^{-1}X$ the R_T-complex $X \otimes_R R_T$.

Proposition 4.5.13 *Let R denote a Noetherian ring and let T be a multiplicatively closed subset in R. Let X be a homologically left-bounded R-complex with one of its left-bounded injective resolution $X \xrightarrow{\sim} I$. Let Y be any homologically right-bounded R-complex and assume that the homology modules $H_i(Y)$ are finitely generated.*

(a) The natural morphism

$$\operatorname{Hom}_R(Y, I) \otimes_R R_T \to \operatorname{Hom}_{R_T}(Y \otimes_R R_T, I \otimes_R R_T)$$

is a quasi-isomorphism.

(b) For all $i \in \mathbb{Z}$ we have

$$\operatorname{Ext}^i_R(Y, X) \otimes_R R_T \cong \operatorname{Ext}^i_{R_T}(Y_T, X_T).$$

In particular, we have

$$(\operatorname{Ext}^i_R(R/\mathfrak{p}, X))_{\mathfrak{p}} \cong \operatorname{Ext}^i_{R_{\mathfrak{p}}}(\Bbbk(\mathfrak{p}), X_{\mathfrak{p}})$$

for all prime ideals $\mathfrak{p}$ of R.

Proof First note that $X \otimes_R R_T \xrightarrow{\sim} I \otimes_R R_T$ is an R_T-injective resolution of $X \otimes_R R_T$ because R_T is R-flat. The complex Y has a right-bounded free resolution $L \xrightarrow{\sim} Y$ where all the L_i's are finitely generated (see 1.1.12). This also induces a quasi-isomorphism $L \otimes_R R_T \xrightarrow{\sim} Y \otimes_R R_T$ and a free resolution of the R_T-complex $Y \otimes_R R_T$. There is a commutative square

$$\begin{tikzcd}
\operatorname{Hom}_R(Y, I) \otimes_R R_T \arrow[r] \arrow[d] & \operatorname{Hom}_{R_T}(Y \otimes_R R_T, I \otimes_R R_T) \arrow[d] \\
\operatorname{Hom}_R(L, I) \otimes_R R_T \arrow[r] & \operatorname{Hom}_{R_T}(L \otimes_R R_T, I \otimes_R R_T)
\end{tikzcd}$$

In this square the vertical morphisms are quasi-isomorphisms as easily seen. As the bottom horizontal morphism is an isomorphism by 4.5.12 the statement in (a) follows. We then take cohomology to finish the proof. (Note, however, that the isomorphisms in (b) are also a particular case of (4.4.21 (d3).) □

We now investigate some finiteness conditions in cohomology.

Lemma 4.5.14 *Let X and Y be two complexes over a Noetherian ring R with finitely generated cohomology modules. Assume that one of the following conditions is satisfied:*

(a) Y is homologically right-bounded and X is homologically left-bounded.
(b) X is a bounded complex of injective modules.

Then the R-modules $\operatorname{Ext}^i_R(Y, X)$ are finitely generated.

Proof Assume first that condition (a) is satisfied. Then as above the complex X has a left-bounded injective resolution $X \xrightarrow{\sim} I$ and the complex Y has a right-bounded free resolution $L \xrightarrow{\sim} Y$ where all the L_i's are finitely generated (see 1.1.12). We recall that $\operatorname{Ext}^i_R(Y, X) \cong H^i(\operatorname{Hom}_R(L, I))$. For the cohomology of $\operatorname{Hom}_R(L, I)$ we have the following spectral sequence

$$E_2^{p,q} = H^p(\operatorname{Hom}_R(L, H^q(I))) \Longrightarrow H^{p+q}(\operatorname{Hom}_R(L, I)).$$

By the assumption the E_2-terms are finitely generated. Therefore all the subsequent stages are finitely generated, and the $E_\infty^{p,q}$ are also finitely generated R-modules for all p, q. That is, the R-module $H^{p+q}(\mathrm{Hom}_R(L, I))$ admits a finite filtration by finitely generated R-modules. Whence it is finitely generated as an R-module.

Assume now that condition (b) is satisfied. Then $H^i(\mathrm{Hom}_R, Y, X) = \mathrm{Ext}^i_R(Y, X)$. As X is bounded note also that $H^i(\mathrm{Hom}_R(Y, X))$ depends only on a soft truncation $\cdots \to Y_n \to \mathrm{Ker}(d^Y_{n-1}) \to 0$ of Y for $n \gg 0$. Hence we may and do assume that Y is right-bounded. In that case (b) follows from (a).

Proposition 4.5.15 *Let X be a homologically left-bounded complex over a Noetherian ring R and assume that the cohomology R-modules $H^i(X)$ are finitely generated. Then the $\Bbbk(\mathfrak{p})$-vector spaces $\mathrm{Ext}^i_{R_\mathfrak{p}}(\Bbbk(\mathfrak{p}), X_\mathfrak{p})$ have finite dimension for any prime ideal $\mathfrak{p}$ of R.*

Proof We have $(\mathrm{Ext}^i_R(R/\mathfrak{p}, X))_\mathfrak{p} \cong \mathrm{Ext}^i_{R_\mathfrak{p}}(\Bbbk(\mathfrak{p}), X_\mathfrak{p})$ by virtue of 4.5.13. As the R-modules $\mathrm{Ext}^i_R(R/\mathfrak{p}, X)$ are finitely generated by 4.5.14 the statement follows. □

4.5.16 *Bass numbers.* Let R be a Noetherian ring. The Bass numbers of an R-module M were introduced in [9]. For an unbounded complex X, for $j \in \mathbb{Z}$ and $\mathfrak{p} \in \mathrm{Spec}(R)$ we define

$$\mu_j(\mathfrak{p}, X) = \dim_{\Bbbk(\mathfrak{p})}(\mathrm{Ext}^j_R(R/\mathfrak{p}, X) \otimes_R R_\mathfrak{p})$$

(note that $\mathrm{Ext}^j_R(R/\mathfrak{p}, X) \otimes_R R_\mathfrak{p} \cong \mathrm{Ext}^j_R(R/\mathfrak{p}, X) \otimes_R R/\mathfrak{p} \otimes_R R_\mathfrak{p}$ is a $\Bbbk(\mathfrak{p})$-vector space). In general, these $\mu_j(\mathfrak{p}, X)$ are only cardinal numbers.

When X is homologically left-bounded then

$$\mu_j(\mathfrak{p}, X) = \dim_{\Bbbk(\mathfrak{p})} \mathrm{Ext}^j_{R_\mathfrak{p}}(\Bbbk(\mathfrak{p}), X_\mathfrak{p}),$$

in view of 4.5.13.

When X is homologically left-bounded and, moreover, when the $H^i(X)$ are finitely generated then the $\mu_j(\mathfrak{p}, X)$ are natural numbers by 4.5.15.

Remark 4.5.17 Let X be any complex over a Noetherian ring R and let $X \xrightarrow{\sim} I$ be one of its DG-injective resolutions. Then

$$\mathrm{Ext}^j_R(R/\mathfrak{p}, X) \otimes_R R_\mathfrak{p} \cong H^j(\mathrm{Hom}_R(R/\mathfrak{p}, I) \otimes_R R_\mathfrak{p})$$

because $R_\mathfrak{p}$ is R-flat. Note also that

$$\mathrm{Hom}_R(R/\mathfrak{p}, I) \otimes_R R_\mathfrak{p} \cong \mathrm{Hom}_{R_\mathfrak{p}}(\Bbbk(\mathfrak{p}), I_\mathfrak{p})$$

in view of 4.5.12 and therefore

$$\mu_j(\mathfrak{p}, X) = \dim_{\Bbbk(\mathfrak{p})} H^j(\mathrm{Hom}_{R_\mathfrak{p}}(\Bbbk(\mathfrak{p}), I_\mathfrak{p})).$$

Note however that the injective complex $I_\mathfrak{p}$ might be not DG-injective in general (see [20]). In general, there are no interpretations of $H^j(\mathrm{Hom}_{R_\mathfrak{p}}(\Bbbk(\mathfrak{p}), I_\mathfrak{p}))$ in terms of Ext. This explains the way we define the Bass numbers of an arbitrary complex.

We have seen in 4.5.6 that any R-complex X has a minimal DG-injective resolution. When the ring R is Noetherian we shall prove in the following that such a resolution is unique up to isomorphism and at the same time that our definition of the Bass numbers coincides with the one originally given by Bass in the case when $X = M$ is an R-module.

Theorem 4.5.18 *Let R be a Noetherian ring and X be any R-complex. Then X has a minimal DG-injective resolution $X \xrightarrow{\sim} I$, unique up to isomorphism, where*

$$I^j \cong \oplus_{\mathfrak{p}\in \operatorname{Spec}(R)} E_R(R/\mathfrak{p}))^{(\mu_j(\mathfrak{p},X))}.$$

Proof Let $X \xrightarrow{\sim} I$ be one of the minimal DG-injective resolutions of X (such a resolution exists, see 4.5.6). As in 4.5.9 we may write

$$I^j \cong \oplus_{\mathfrak{p}\in \operatorname{Spec}(R)} E_R(R/\mathfrak{p}))^{(c(\mathfrak{p},I^j))}$$

where the $c(\mathfrak{p}, I^j)$ are cardinal numbers and where $E_R(R/\mathfrak{p}))^{(c(\mathfrak{p},I^j))}$ denotes the direct sum of $c(\mathfrak{p}, I^j)$ copies of $E_R(R/\mathfrak{p})$. We have to prove that $c(\mathfrak{p}, I^j) = \mu_j(\mathfrak{p}, X)$.

We have seen in 4.5.17 that $\mu_j(\mathfrak{p}, X) = \dim_{\Bbbk(\mathfrak{p})} H^j(\operatorname{Hom}_{R_\mathfrak{p}}(\Bbbk(\mathfrak{p}), I_\mathfrak{p}))$. Because the complexes $I_\mathfrak{p}$ are minimal injective (see 4.5.10) we know by 4.5.11 that the complexes $\operatorname{Hom}_{R_\mathfrak{p}}(\Bbbk(\mathfrak{p}), I_\mathfrak{p})$ have zero differentials. Hence

$$\mu_j(\mathfrak{p}, X) = \dim_{\Bbbk(\mathfrak{p})} \operatorname{Hom}_{R_\mathfrak{p}}(\Bbbk(\mathfrak{p}), I^j_\mathfrak{p}).$$

We have that

$$\begin{aligned}\operatorname{Hom}_{R_\mathfrak{p}}(\Bbbk(\mathfrak{p}), I^j_\mathfrak{p}) &\cong \operatorname{Hom}_{R_\mathfrak{p}}(\Bbbk(\mathfrak{p}), E_R(R/\mathfrak{p})^{(c(\mathfrak{p},I^j))}) \\ &\cong \operatorname{Hom}_{R_\mathfrak{p}}(\Bbbk(\mathfrak{p}), \Bbbk(\mathfrak{p})^{(c(\mathfrak{p},I^j))}),\end{aligned}$$

for the first isomorphism see the remarks at the end of 4.5.9, the last one is rather obvious. The equalities $c(\mathfrak{p}, I^j) = \mu_j(\mathfrak{p}, X)$ follow.

Now let $X \xrightarrow{\sim} J$ be another minimal DG-injective resolution of X. There are quasi-isomorphisms $I \xrightarrow{\sim} J \xrightarrow{\sim} I$, the composite of them is homotopic to the identity (see 4.4.8 and also 4.4.7). It follows that the composite morphism

$$\operatorname{Hom}_{R_\mathfrak{p}}(\Bbbk(\mathfrak{p}), I_\mathfrak{p}) \to \operatorname{Hom}_{R_\mathfrak{p}}(\Bbbk(\mathfrak{p}), J_\mathfrak{p}) \to \operatorname{Hom}_{R_\mathfrak{p}}(\Bbbk(\mathfrak{p}), I_\mathfrak{p})$$

is also homotopic to the identity. But these last complexes have zero differentials. Hence this last composite is the identity and this holds for all $\mathfrak{p} \in \operatorname{Spec}(R)$. In view of the above description of $\operatorname{Hom}_{R_\mathfrak{p}}(\Bbbk(\mathfrak{p}), I_\mathfrak{p})$ it follows that the quasi-isomorphism $I \xrightarrow{\sim} J$ is an isomorphism. $\square$

For the sake of completeness and later use note also the following.

Proposition 4.5.19 *Let X be a homologically left-bounded complex over a Noetherian ring R and assume that the R-modules $H^i(X)$ are finitely generated. Let $\mathfrak{p} \subset \mathfrak{q}$ be two prime ideals of R such that* $\operatorname{height}(\mathfrak{p}/\mathfrak{q}) = 1$. *If $\mu_j(\mathfrak{p}, X) \neq 0$ then $\mu_{j+1}(\mathfrak{q}, X) \neq 0$.*

Proof This was proved by Bass in the case when $X = M$ is an R-module (see [9]). We only note here that Bass' proof extends to the present case. (This is true because in our situation the $R_{\mathfrak{q}}$-modules $\operatorname{Ext}^i_R(R/\mathfrak{p}, X) \otimes_R R_{\mathfrak{q}}$ are finitely generated (see 4.5.14.)) □

In the case when $X = M$ is an R-module note also the following consequences of the above considerations.

Recalls 4.5.20 Let M be a module over a Noetherian Ring R. Then

(a) $\mu_0(\mathfrak{p}, M) \neq 0$ if and only if $\mathfrak{p} \in \operatorname{Ass}_R(M)$,
(b) $\operatorname{id}_R M = \operatorname{length}(I)$, where I is a minimal injective resolution of M,
(c) if $\underline{x} = x_1, \dots, x_k$ is a sequence regular on both R and M then

$$M/\underline{x}M \cong \operatorname{Ext}^k_R(R/\underline{x}R, M) \text{ and } \operatorname{id}_R M = \operatorname{id}_{R/\underline{x}R}(M/\underline{x}M) + k,$$

(d) if M is finitely generated then $\dim_R M \leq \operatorname{id}_R M$.

Moreover it was conjectured by H. Bass (see [9]) that a Noetherian local ring possessing a non-zero finitely generated module of finite injective dimension is a Cohen–Macaulay ring. This conjecture became part of Hochster's homological conjectures (see [45]), among them the intersection conjecture. In fact Hochster solved a number of these homological conjectures in the case of equi-characteristic Noetherian local rings. Since P. Roberts proved the intersection conjecture in general (see [70]) the Bass conjecture also became true (see also the forthcoming more general Theorem 10.1.7 and its Corollary 10.1.8 for more details about this).

4.6 Ext and Tor with Inverse and Direct Limits

Let R denote a commutative ring. We have seen that the functor $\operatorname{Hom}_R(\cdot, Y)$ transforms direct limits into inverse limits and that the functor $\operatorname{Hom}_R(X, \cdot)$ commutes with inverse limits (see 1.3.3 and 1.2.7). This has some consequences on the functors Ext, which we now investigate.

Lemma 4.6.1 *Let $\mathfrak{X} = \{\sigma_{ij} : X_i \to X_j \mid 0 \leq i \leq j \in \mathbb{N}\}$ be a direct system of complexes over a commutative ring R and let I be a DG-injective complex. Then $\operatorname{Hom}_R(\mathfrak{X}, I) = \{\operatorname{Hom}_R(X_t, I)\}$ forms an inverse system and we have an isomorphism and a quasi-isomorphism*

$$\varprojlim(\operatorname{Hom}_R(X_t, I)) \cong \operatorname{Hom}_R(\varinjlim X_t, I) \xrightarrow{\sim} \operatorname{Mic}(\operatorname{Hom}_R(\mathfrak{X}, I)).$$

Moreover, there are short exact sequences

$$0 \to \varprojlim{}^1 H^{i-1}(\mathrm{Hom}_R(X_t, I)) \to H^i(\mathrm{Hom}_R(\varinjlim X_t, I))$$
$$\to \varprojlim H^i(\mathrm{Hom}_R(X_t, I)) \to 0$$

for all $i \in \mathbb{Z}$.

Proof With the morphism $\phi_{\mathfrak{X}}$ as defined in 1.3.2 there is a short exact sequence

$$0 \to \oplus_t X_t \xrightarrow{\phi_{\mathfrak{X}}} \oplus_t X_t \to \varinjlim X_t \to 0.$$

By applying the functor $\mathrm{Hom}_R(\cdot, I)$ it induces a short exact sequence of complexes

$$0 \to \mathrm{Hom}_R(\varinjlim X_t, I) \to \prod \mathrm{Hom}_R(X_t, I) \xrightarrow{\psi_{\mathrm{Hom}_R(\mathfrak{X}, I)}} \prod \mathrm{Hom}_R(D_t, I) \to 0,$$

recall that $\mathrm{Hom}_R(\phi_{\mathfrak{X}}, I) = \psi_{\mathrm{Hom}_R(\mathfrak{X}, I)}$ (see 1.3.3). This gives the isomorphism of the statement. The quasi-isomorphism follows by the definition of Mic and 1.5.6. Then the short exact sequences follow by 4.2.3 applied to the inverse system $\mathrm{Hom}_R(\mathfrak{X}, I)$. □

In a certain sense our first result gives an answer to part of [71, Remark Section 7.2].

Proposition 4.6.2 *Let* R *denote a commutative ring and let* $\mathfrak{X} = \{\sigma_{ij} : X_i \to X_j \mid 0 \le i \le j \in \mathbb{N}\}$ *be a direct system of* R*-complexes. Write* $X = \varinjlim X_t$*. Then there are short exact sequences*

$$0 \to \varprojlim{}^1 \mathrm{Ext}_R^{i-1}(X_t, Y) \to \mathrm{Ext}_R^i(X, Y) \to \varprojlim \mathrm{Ext}_R^i(X_t, Y) \to 0$$

for all $i \in \mathbb{N}$ *and any* R*-complex* Y.

Proof Let $Y \xrightarrow{\sim} I$ denote a DG-injective resolution of Y. Then the result follows by the short exact sequences in 4.6.1 and the definition of the Ext-modules. □

Another result in this direction is related to inverse systems of complexes.

Lemma 4.6.3 *Let* $\mathfrak{Y} = \{\rho_{t,t+1} : Y_{t+1} \to Y_t \mid t \in \mathbb{N}\}$ *denote an inverse system of complexes over a commutative ring* R *and assume that it satisfies degree-wise the Mittag-Leffler condition. Let* P *be a* DG*-projective complex. Then* $\mathrm{Hom}_R(P, \mathfrak{Y}) = \{\mathrm{Hom}_R(P, Y_t)\}$ *forms an inverse system and there is an isomorphism and a quasi-isomorphism*

$$\varprojlim \mathrm{Hom}_R(P, Y_t) \cong \mathrm{Hom}_R(P, \varprojlim Y_t) \xrightarrow{\sim} \mathrm{Mic}(\mathrm{Hom}_R(P, \mathfrak{Y})).$$

Moreover, there are short exact sequences

$$\begin{aligned} 0 \to \varprojlim{}^1 H^{i-1}(\mathrm{Hom}_R(P, Y_t)) &\to H^i(\mathrm{Hom}_R(P, \varprojlim Y_t)) \\ &\to \varprojlim H^i(\mathrm{Hom}_R(P, Y_t)) \to 0 \end{aligned}$$

for all $i \in \mathbb{Z}$.

Proof By the hypothesis on $\mathfrak{Y}$ we have $\varprojlim{}^1 Y_t = 0$ and a short exact sequence

$$0 \to \varprojlim Y_t \to \prod Y_t \xrightarrow{\psi_{\mathfrak{Y}}} \prod Y_t \to 0,$$

where $\psi_{\mathfrak{Y}}$ is the morphism defined in 1.2.6. Since P is DG-projective it induces the short exact sequence

$$0 \to \mathrm{Hom}_R(P, \varprojlim Y_t) \to \prod \mathrm{Hom}_R(P, Y_t) \xrightarrow{\psi} \prod \mathrm{Hom}_R(P, Y_t) \to 0,$$

where $\psi = \mathrm{Hom}_R(P, \psi_{\mathfrak{Y}}) = \psi_{\mathrm{Hom}_R(P,\mathfrak{Y})}$ (see 1.2.7). This gives the isomorphism of the statement. The quasi-isomorphism follows by the definition of Mic and 1.5.6. Then the short exact sequences follow by 4.2.3 applied to the inverse system $\mathrm{Hom}_R(P, \mathfrak{Y})$. ☐

As before we get a result related to [71, Remark Section 7.2].

Proposition 4.6.4 *Let R denote a commutative ring and let $\mathfrak{Y} = \{\rho_{t,t+1} : Y_{t+1} \to Y_t \mid t \in \mathbb{N}\}$ be an inverse system of R-complexes satisfying degree-wise the Mittag-Leffler condition. Write $Y = \varprojlim Y_t$. Then there is a short exact sequence*

$$0 \to \varprojlim{}^1 \mathrm{Ext}_R^{i-1}(X, Y_t) \to \mathrm{Ext}_R^i(X, Y) \to \varprojlim \mathrm{Ext}_R^i(X, Y_t) \to 0$$

for all $i \in \mathbb{N}$ and any R-complex X.

Proof Let $P \xrightarrow{\sim} X$ be a DG-projective resolution of the R-complex X. Then the result follows by 4.6.3 and the definition of the Ext-modules. ☐

We now investigate the behaviour of tensor products and Tor with inverse limits.

Lemma 4.6.5 *Let R denote a commutative ring. Let $\mathfrak{Y} = \{\rho_{t,t+1} : Y_{t+1} \to Y_t \mid t \in \mathbb{N}\}$ denote an inverse system of R-complexes that satisfies degree-wise the Mittag-Leffler condition. Let L denote a right-bounded complex of finitely generated free R-modules. Then there is an isomorphism $\varprojlim(L \otimes_R Y_t) \cong (L \otimes_R \varprojlim Y_t)$.*

Proof First we note that $L \otimes_R \mathfrak{Y} = \{L \otimes_R Y_t\}$ form an inverse system that satisfies degree-wise the Mittag-Leffler condition. By 1.2.3 there is a short exact sequence of complexes

$$0 \to \varprojlim Y_t \to \prod_t Y_t \xrightarrow{\psi_{\mathfrak{Y}}} \prod_t Y_t \to 0.$$

Since L is a complex of free R-modules it induces by tensoring with L a short exact sequence of complexes such that the following diagram with exact rows is commutative

$$\begin{tikzcd}
0 \arrow[r] & L \otimes_R \varprojlim Y_t \arrow[r] \arrow[d] & L \otimes_R (\prod Y_t) \arrow[r] \arrow[d, equal] & L \otimes_R (\prod Y_t) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \varprojlim (L \otimes_R Y_t) \arrow[r] & \prod (L \otimes_R Y_t) \arrow[r] & \prod (L \otimes_R Y_t)) \arrow[r] & 0.
\end{tikzcd}$$

Clearly the vertical morphisms on the right are isomorphisms because the L_i's are free finitely generated. Whence the vertical morphism on the left is an isomorphism as follows by degree-wise inspection. □

Proposition 4.6.6 *Let R be a Noetherian ring and let X be a homologically right-bounded complex with finitely generated homology modules. Let $\mathfrak{Y} = \{\rho_{t,t+1} : Y_{t+1} \to Y_t \mid t \in \mathbb{N}\}$ denote an inverse system of R-complexes that satisfies degree-wise the Mittag-Leffler condition. Then*

$$\operatorname{Tor}_i^R(X, \varprojlim Y_t) \cong \varprojlim \operatorname{Tor}_i^R(X, Y_t)$$

for all $i \in \mathbb{Z}$.

Proof The complex X has a DG-projective resolution $L \xrightarrow{\sim} X$ where L is a right-bounded complex of finitely generated free R-modules (see 1.1.12). Hence the statement is a direct consequence of 4.6.5. □

We have already noted in 1.3.5 that tensor products commute with right-filtered direct limits. Hence we also have the following.

Proposition 4.6.7 *Let R denote a commutative ring. Let $(I, \prec)$ be a right-filtered ordered set and let $\mathfrak{X} = \{\sigma_{ij} : X_i \to X_j \mid 0 \le i \le j \in I\}$ be a direct system of R-complexes over I. Let Y be another R-complex. Then*

$$\operatorname{Tor}_i^R(\varinjlim X_i, Y) \cong \varinjlim \operatorname{Tor}_i^R(X_i, Y)$$

for all $i \in \mathbb{Z}$.

Chapter 5
Koszul Complexes, Depth and Codepth

The notions of Ext-depth and Tor-codepth with respect to an ideal $\mathfrak{a}$ of a commutative ring R were introduced and investigated by Strooker for R-modules. He also proved (see [82, 6.1.6, 6.1.7]) that these can be computed by the use of a Koszul complex when the ideal $\mathfrak{a}$ is finitely generated.

These notions extend naturally to complexes. In this generality they have been studied by Foxby and Iyengar, who extended Strooker's result to unbounded complexes (see [33, Theorems 2.1 and 4.1]). In their paper Foxby and Iyengar worked over a Noetherian ring and also proved additional results. We shall revisit these in a later chapter. However, most of their results hold in greater generality. In particular, the Ext-depth Tor-codepth sensitivity of the Koszul complex holds in general. To present it we follow – more or less – Foxby and Iyengar's presentation. Then we add some remarks on complexes with finite Ext-depth and finite Tor-codepth. First we recall some basic facts on Koszul complexes and also provide some conditions under which its homology is related to completion. In the last section we turn back to modules and recall some facts on their Koszul homology.

5.1 Ext-Depth and Tor-Codepth

Let R be a commutative ring and $\mathfrak{a}$ an ideal of R. In view of 4.4.13 recall that $\operatorname{Ext}^i_R(R/\mathfrak{a}, X)$ and $\operatorname{Tor}^R_i(R/\mathfrak{a}, X)$ are well defined for any R-complex X and all $i \in \mathbb{Z}$. We shall use the following notations.

Notations 5.1.1 For any R complexes X, Y we write

$$\begin{aligned} \operatorname{ext}^-_R(X, Y) &:= \inf\{i \in \mathbb{Z} \mid \operatorname{Ext}^i_R(X, Y) \neq 0\}, \\ \operatorname{tor}^R_-(X, Y) &:= \inf\{i \in \mathbb{Z} \mid \operatorname{Tor}^R_i(X, Y) \neq 0\}. \end{aligned}$$

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Later we shall also write

$$\operatorname{ext}_R^+(X, Y) := \sup\{i \in \mathbb{Z} \mid \operatorname{Ext}_R^i(X, Y) \neq 0\},$$
$$\operatorname{tor}_+^R(X, Y) := \sup\{i \in \mathbb{Z} \mid \operatorname{Tor}_i^R(X, Y) \neq 0\}.$$

As usual inf and sup are taken in the ordered set $\mathbb{Z} \cup \{\pm\infty\}$.

Definition 5.1.2 The Ext-depth and the Tor-codepth of an R-complex X with respect to the ideal $\mathfrak{a}$ are defined by

$$\text{E-dp}(\mathfrak{a}, X) = \inf\{i \in \mathbb{Z} \mid \operatorname{Ext}_R^i(R/\mathfrak{a}, X) \neq 0\} := \operatorname{ext}_R^-(R/\mathfrak{a}, X) \text{ and}$$
$$\text{T-codp}(\mathfrak{a}, X) = \inf\{i \in \mathbb{Z} \mid \operatorname{Tor}_i^R(R/\mathfrak{a}, X) \neq 0\} := \operatorname{tor}_-^R(R/\mathfrak{a}, X).$$

Therefore

$$-\infty \leq \text{E-dp}(\mathfrak{a}, X), \text{T-codp}(\mathfrak{a}, X) \leq \infty.$$

The Tor-codepth is sometimes called *width* (see also [34] or [86] for the case of bounded complexes).

Remark 5.1.3 (a) Let $X \xrightarrow{\sim} J$ be a K-injective resolution of X. It follows by definition that

$$\text{E-dp}(\mathfrak{a}, X) = \inf\{i \in \mathbb{Z} \mid H^i(\operatorname{Hom}_R(R/\mathfrak{a}, J)) \neq 0\}.$$

Dually, let $F \xrightarrow{\sim} X$ be a K-flat resolution of X, then

$$\text{T-codp}(\mathfrak{a}, X) = \inf\{i \in \mathbb{Z} \mid H_i(R/\mathfrak{a} \otimes_R F)) \neq 0\}.$$

(b) If X is exact, then $\text{T-codp}(\mathfrak{a}, X) = \infty = \text{E-dp}(\mathfrak{a}, X)$.
(c) If X and X' are quasi-isomorphic, then

$$\text{E-dp}(\mathfrak{a}, X) = \text{E-dp}(\mathfrak{a}, X') \text{ and } \text{T-codp}(\mathfrak{a}, X) = \text{T-codp}(\mathfrak{a}, X').$$

All this follows from the properties of Ext and Tor recalled in 4.4.13.
(d) Note that $\text{T-codp}(\mathfrak{a}, X) = \text{E-dp}(\mathfrak{a}, X^\vee)$, as follows from the Ext-Tor duality recalled in 4.4.13.

Definitions and observations 5.1.4 We also introduce the class $\mathcal{T}_\mathfrak{a}$ of complexes X with $\text{T-codp}(\mathfrak{a}, X) = \infty$. This class extends the class of modules introduced in 3.2.2 and will play an important rôle in the sequel. We observe the following.
(a) For a complex X the condition $X \in \mathcal{T}_\mathfrak{a}$ is in fact a condition on its K-flat or K-projective resolutions. If two complexes X and X' are quasi-isomorphic and one of them is in $\mathcal{T}_\mathfrak{a}$, then so is the other.
(b) The module of fractions R_x is in $\mathcal{T}_\mathfrak{a}$ for all $x \in \mathfrak{a}$. More generally, any relatively-$\mathfrak{a}$-flat module F as defined in 2.6.1 and such that $F = \mathfrak{a}F$ is in $\mathcal{T}_\mathfrak{a}$. Exact complexes are in $\mathcal{T}_\mathfrak{a}$.

(c) Let $0 \to X \to Y \to Z \to 0$ be a short exact sequence of R-complexes. If two of the three complexes X, Y and Z are in $\mathcal{T}_{\mathfrak{a}}$, then the third one is also in $\mathcal{T}_{\mathfrak{a}}$.

(Let $L \xrightarrow{\sim} R/\mathfrak{a}$ be a free resolution of $R/\mathfrak{a}$. Then we have the short exact sequence $0 \to L \otimes_R X \to L \otimes_R Y \to L \otimes_R Z \to 0$, and, if two of the three complexes $L \otimes_R X$, $L \otimes_R Y$ and $L \otimes_R Z$ are exact, so is the third.)

(d) Let $X \xrightarrow{f} Y$ be a morphism of R-complexes of which we take the cone. If two of the three complexes X, Y and $C(f)$ are in $\mathcal{T}_{\mathfrak{a}}$, then the third one is also in $\mathcal{T}_{\mathfrak{a}}$. (Remember the short exact sequence $0 \to Y \to C(f) \to X_{[-1]} \to 0$.)

(e) The property of being in $\mathcal{T}_{\mathfrak{a}}$ localizes: if $X \in \mathcal{T}_{\mathfrak{a}}$, then $X_{\mathfrak{p}} \in \mathcal{T}_{\mathfrak{a}}$ and $X_{\mathfrak{p}} \in \mathcal{T}_{\mathfrak{a}_{\mathfrak{p}}}$ for any prime ideal $\mathfrak{p}$ of R (in view of 4.4.21).

When $X = M$ is an R-module it is clear that $\text{E-dp}(\mathfrak{a}, M) \geq 0$ and $\text{T-codp}(\mathfrak{a}, M) \geq 0$. We have similar inequalities in the case of complexes.

Lemma 5.1.5 (see [33]) *Let $\mathfrak{a}$ be any ideal of a commutative ring R and X an R-complex.*

(a) We have the inequality

$$\text{E-dp}(\mathfrak{a}, X) \geq \inf\{i \in \mathbb{Z} \mid H^i(X) \neq 0\}.$$

If $\inf\{i \in \mathbb{Z} \mid H^i(X) \neq 0\} = r$ *is finite and* $\operatorname{Hom}_R(R/\mathfrak{a}, H^r(X)) \neq 0$, *then equality holds.*

(b) We have the inequality

$$\text{T-codp}(\mathfrak{a}, X) \geq \inf\{i \mid H_i(X) \neq 0\}.$$

If $\inf\{i \in \mathbb{Z} \mid H_i(X) \neq 0\} = r$ *is finite and* $R/\mathfrak{a} \otimes_R H_r(X) \neq 0$, *then equality holds.*

(c) If X is homologically bounded, then

$$\text{E-dp}(\mathfrak{a}, X) > -\infty \textit{ and } \text{T-codp}(\mathfrak{a}, X) > -\infty.$$

Proof Note that $\inf\{i \in \mathbb{Z} \mid H^i(X) \neq 0\} = \infty$ if and only if $0 \simeq X$ is exact. In that case $\text{E-dp}(\mathfrak{a}, X) = \infty = \text{T-codp}(\mathfrak{a}, X)$ and the assertions are obvious. Assume now that X is not exact.

We prove (a) at first. If $\inf\{i \in \mathbb{Z} \mid H^i(X) \neq 0\} = -\infty$ the inequality in (a) is obvious.

If $\inf\{i \mid H^i(X) \neq 0\} = r$ is finite, then X has an injective resolution $X \xrightarrow{\sim} I$ where $I^i = 0$ for all $i < r$ (see 1.1.12). The inequality in (a) follows immediately. If, moreover, $\operatorname{Hom}(R/\mathfrak{a}, H^r(I)) \cong \operatorname{Hom}_R(R/\mathfrak{a}, H^r(X)) \neq 0$, then the complex

$$\operatorname{Hom}_R(R/\mathfrak{a}, I) : 0 \to \operatorname{Hom}_R(R/\mathfrak{a}, I^r) \to \operatorname{Hom}_R(R/\mathfrak{a}, I^{r+1}) \to \dots$$

has cohomology in degree r. This proves the last assertion in (a).

The assertions in (b) can be deduced from those in (a) applied to the complex $X^\vee$ via the general Matlis duality and the assertions in (c) follow. □

When two ideals $\mathfrak{a}$ and $\mathfrak{a}'$ are topologically equivalent, that is, when $\mathfrak{a}$ contains some power of $\mathfrak{a}'$ and $\mathfrak{a}'$ contains some power of $\mathfrak{a}$, the corresponding Ext-depth and Tor-codepth coincides. More generally, we have comparison formulas of the following type.

Proposition 5.1.6 (see [33]) *Let $\mathfrak{a}$ be an ideal of a commutative ring R. Let X denote an R-complex and M an R-module annihilated by some power $\mathfrak{a}^t$ of $\mathfrak{a}$. Then*

$$\text{E-dp}(\mathfrak{a}, X) \le \operatorname{ext}_R^-(M, X) \textit{ and } \text{T-codp}(\mathfrak{a}, X) \le \operatorname{tor}_-^R(M, X).$$

Proof We first prove the Ext-depth part of the statement. Let $X \xrightarrow{\sim} I$ be a DG-injective resolution of X. We proceed by induction on t. First let $t = 1$, i.e. $\mathfrak{a}M = 0$. Assume there is an integer r such that

$$H^i(\operatorname{Hom}_R(R/\mathfrak{a}, I)) = 0 \text{ for all } i < r.$$

It will be enough to prove that $H^i(\operatorname{Hom}_R(M, I)) = 0$ for all $i < r$. To this end we take a left soft truncation at spot r of the complex $\operatorname{Hom}_R(R/\mathfrak{a}, I)$, the differential of which we denote by $\bar{d}$. Namely $\tau_{[r}(\operatorname{Hom}_R(R/\mathfrak{a}, I))$ is given by

$$0 \to \operatorname{Hom}_R(R/\mathfrak{a}, I^r)/\operatorname{Im}(\bar{d}^{r-1}) \to \operatorname{Hom}_R(R/\mathfrak{a}, I^{r+1}) \to \dots .$$

The complex $\tau_{[r}(\operatorname{Hom}_R(R/\mathfrak{a}, I))$ is a left-bounded $R/\mathfrak{a}$-complex and has an $R/\mathfrak{a}$-injective resolution J such that $J^i = 0$ for all $i < r$ (see 1.1.12). There are quasi-isomorphisms of $R/\mathfrak{a}$-complexes

$$\operatorname{Hom}_R(R/\mathfrak{a}, I) \xrightarrow{\sim} \tau_{[s}(\operatorname{Hom}_R(R/\mathfrak{a}, I)) \xrightarrow{\sim} J.$$

Note that the $R/\mathfrak{a}$-complex $\operatorname{Hom}_R(R/\mathfrak{a}, I)$ is $R/\mathfrak{a}$-DG-injective as is easily seen. The complex J is also an $R/\mathfrak{a}$-DG-injective complex because it is a left-bounded complex of $R/\mathfrak{a}$-injective modules.

As $\mathfrak{a}M = 0$ and in view of 4.4.11 we have an isomorphism and a quasi-isomorphism

$$\operatorname{Hom}_R(M, I) \cong \operatorname{Hom}_{R/\mathfrak{a}}(M, \operatorname{Hom}_R(R/\mathfrak{a}, I)) \xrightarrow{\sim} \operatorname{Hom}_{R/\mathfrak{a}}(M, J).$$

Whence $H^i(\operatorname{Hom}_R(M, I) = 0$ for all $i < r$. This proves the case $t = 1$.

We assume now the statement is proved for all modules annihilated by $\mathfrak{a}^t$ and consider a module M with $\mathfrak{a}^{t+1}M = 0$. As I is a DG-injective complex there is a short exact sequence

$$0 \to \operatorname{Hom}_R(M/\mathfrak{a}^t M, I) \to \operatorname{Hom}_R(M, I) \to \operatorname{Hom}_R(\mathfrak{a}^t M, I) \to 0$$

involving the modules $\mathfrak{a}^t M$ and $M/\mathfrak{a}^t M$, both annihilated by $\mathfrak{a}^t$. It follows that $H^i(\mathrm{Hom}_R(M, I)) = 0$ for all $i < r$ by the induction hypothesis and the associated long exact sequence in cohomology.

The Tor-codepth part of the statement is obtained from the Ext-depth part applied to the complex $X^\vee$ via the general Matlis Ext-Tor duality. $\square$

Corollary 5.1.7 *Let $\mathfrak{a}$ be an ideal of a commutative ring R and $\mathfrak{b}$ an $\mathfrak{a}$-open ideal, that is, an ideal containing some power $\mathfrak{a}^t$ of $\mathfrak{a}$. Let also $\mathfrak{a}'$ be an ideal topologically equivalent to $\mathfrak{a}$. Then*

$$\begin{aligned} \text{E-dp}(\mathfrak{a}', X) &= \text{E-dp}(\mathfrak{a}, X) \leq \text{E-dp}(\mathfrak{b}, X) \quad \textit{and} \\ \text{T-codp}(\mathfrak{a}', X) &= \text{T-codp}(\mathfrak{a}, X) \leq \text{T-codp}(\mathfrak{b}, X) \end{aligned}$$

for all R-complexes X. Hence $\mathcal{T}_{\mathfrak{a}'} = \mathcal{T}_{\mathfrak{a}} \subseteq \mathcal{T}_{\mathfrak{b}}$.

Here is an extension of 5.1.6. Note that the inequality was already proved in [33, 2.2].

Proposition 5.1.8 *Let $\mathfrak{a}$ be an ideal of a commutative ring R and Z be a non-exact homologically bounded R-complex with homology modules annihilated by some power $\mathfrak{a}^t$ of $\mathfrak{a}$. There is an inequality*

$$\text{E-dp}(\mathfrak{a}, X) \leq \mathrm{ext}_R^-(Z, X) - \inf\{i \mid H_i(Z) \neq 0\}$$

for any R-complex X.

Let $b = \inf\{i \mid H_i(Z) \neq 0\}$. Assume that E-dp$(\mathfrak{a}, X) > -\infty$ *and that $H_b(Z) \cong R/\mathfrak{a}'$ for some ideal $\mathfrak{a}'$ topologically equivalent to $\mathfrak{a}$, then equality holds.*

Proof We prove the result by induction on the amplitude of Z. We write $M = H_b(Z)$ and view it as a complex concentrated in degree zero.

If amp(Z)$= 0$, then Z is quasi-isomorphic to $M_{[-b]}$ and

$$\mathrm{Ext}_R^i(Z, X) \cong \mathrm{Ext}_R^i(M_{[-b]}, X) \cong \mathrm{Ext}_R^{i-b}(M, X).$$

Hence $\mathrm{ext}_R^-(Z, X) = \mathrm{ext}_R^-(M, X) + b$ and we conclude by 5.1.6 and 5.1.7.

Suppose amp(Z)> 0. Then we put Z' for the right soft truncation of Z at spot $b + 1$. This gives a short exact sequence $0 \to Z' \to Z \to Z'' \to 0$, where Z'' is quasi-isomorphic to $M_{[-b]}$. Now apply $\mathrm{Hom}_R(\cdot, I)$, where $X \xrightarrow{\sim} I$ is a DG-injective resolution of X. We obtain an exact sequence in cohomology

$$\mathrm{Ext}_R^i(M_{[-b]}, X) \to \mathrm{Ext}_R^i(Z, X) \to \mathrm{Ext}_R^i(Z', X).$$

Now we compute the ext$^-$'s. By the induction hypothesis it follows that

$$\operatorname{ext}^-_R(Z', X) \geq \text{E-dp}(\mathfrak{a}, X) + \inf\{i \mid H_i(Z') \neq 0\} > \text{E-dp}(\mathfrak{a}, X) + b \quad \text{and}$$
$$\operatorname{ext}^-_R(M_{[-b]}, X) \geq \text{E-dp}(\mathfrak{a}, X) + b$$

with an equality in the second case when $M \cong R/\mathfrak{a}'$. We then obtain the result for $\operatorname{ext}^-_R(Z, X)$ by looking at the above exact sequence. □

We have a Tor-codepth counterpart of 5.1.8 with slightly weaker hypotheses.

Proposition 5.1.9 *Let $\mathfrak{a}$ be an ideal of a commutative ring R and Z be a non-exact homologically bounded R-complex with $\mathfrak{a}$-torsion homology modules. Then we have an inequality*

$$\text{T-codp}(\mathfrak{a}, X) \leq \operatorname{tor}^R_-(Z, X) - \inf\{i \mid H_i(Z) \neq 0\}$$

for any R-complex X.

Let $b = \inf\{i \mid H_i(Z) \neq 0\}$. Assume that T-codp$(\mathfrak{a}, X) > -\infty$ *and that $H_b(Z) \cong R/\mathfrak{a}'$ for some ideal $\mathfrak{a}'$ topologically equivalent to $\mathfrak{a}$, then equality holds.*

Proof We first look at the case where $Z = M$ is a complex concentrated in degree 0. We recall that $M = \varinjlim\{M_\lambda\}$, where $\{M_\lambda\}$ denotes the set of finitely generated submodules of M ordered by inclusion. Here the direct limit is taken over a right-filtered ordered set and in this case the direct limit functor commutes with tensor products and is exact. So, if $P \xrightarrow{\sim} X$ is a K-projective resolution of X, then $M \otimes_R P \cong \varinjlim(M_\lambda \otimes_R P)$ and

$$\operatorname{Tor}^R_i(M, X) = H_i(M \otimes_R P) \cong \varinjlim H_i(M_\lambda \otimes_R P).$$

As M is of $\mathfrak{a}$-torsion so are the M_λ's. But the M_λ's are finitely generated, hence each of them is annihilated by some power $\mathfrak{a}^t$ of $\mathfrak{a}$. We thus have that $\operatorname{Tor}^R_i(M, X) = 0$ provided $i <$ T-codp$(\mathfrak{a}, X)$ by 5.1.6. If, moreover, $M \cong R/\mathfrak{a}'$ the equality has already been proved in 5.1.7.

For the general case use induction as in 5.1.8 on the amplitude of Z. We write $M = H_b(Z)$ and view it as a complex concentrated in degree zero. If amp(Z)= 0 the complex Z is quasi-isomorphic to $M_{[-b]}$ and $\operatorname{Tor}^R_i(Z, X) \cong \operatorname{Tor}^R_i(M_{[-b]}, X) \cong \operatorname{Tor}^R_{i-b}(M, X)$. Hence $\operatorname{tor}^R_-(Z, X) = \operatorname{tor}^R_-(M, X) + b$ and the result follows by the first part of the proof.

Let amp(Z)> 0. Then take as in 5.1.8 a right soft truncation Z' of Z at spot $b + 1$. There is a short exact sequence $0 \to Z' \to Z \to Z'' \to 0$, where Z'' is quasi-isomorphic to $M_{[-b]}$. Then we apply the functor $\cdot \otimes_R P$, where $P \xrightarrow{\sim} X$ is a DG-projective resolution of X. Thus we obtain an exact sequence in homology

$$\ldots \to \operatorname{Tor}^R_i(Z', X) \to \operatorname{Tor}^R_i(Z, X) \to \operatorname{Tor}^R_i(M_{[-b]}, X) \to \ldots.$$

Now we compute the tor$_-$'s. By the induction hypothesis we have

$$\operatorname{tor}_-^R(Z', X) \geq \text{T-codp}(\mathfrak{a}, X) + \inf\{i \mid H_i(Z') \neq 0\} > \text{T-codp}(\mathfrak{a}, X) + b \quad \text{and}$$
$$\operatorname{tor}_-^R(M_{[-b]}, X) \geq \text{T-codp}(\mathfrak{a}, X) + b$$

with an equality in the second case when $M \cong R/\mathfrak{a}^t$. We then obtain the result for $\operatorname{tor}_-^R(Z, X)$ by looking at the above exact sequence. $\square$

5.2 Basics About Koszul Complexes

Definitions and observations 5.2.1 Let X denote a complex of R-modules and x an element of R. We have a natural morphism $\varphi(x; X) : X \xrightarrow{x} X$ induced by the multiplication map $X^i \xrightarrow{x} X^i$ for all $i \in \mathbb{Z}$.

We define the ascending and descending Koszul complexes by

$$K^\bullet(x; X) = F(\varphi(x; X)) \text{ and } K_\bullet(x; X) = C(\varphi(x; X)).$$

For a sequence $\underline{x} = x_1, \ldots, x_k$ of elements of R and an element $y \in R$ we denote by $\underline{x}, y$ the sequence $x_1, \ldots, x_k, y$. Then we define inductively

$$K^\bullet(\underline{x}, y; X) = F(\varphi(y; K^\bullet(\underline{x}; X))) \text{ and } K_\bullet(\underline{x}, y; X) = C(\varphi(y; K_\bullet(\underline{x}; X))).$$

In particular, we write $K_\bullet(\underline{x})$ respectively $K^\bullet(\underline{x})$ in the case when $X = R$ considered as a complex in degree zero. By construction we also have

$$K_\bullet(\underline{x}, y; X) \cong K_\bullet(\underline{x}; X) \otimes_R K_\bullet(y) \text{ and } K^\bullet(\underline{x}, y; X) \cong K^\bullet(\underline{x}; X) \otimes_R K^\bullet(y).$$

In particular, the Koszul complexes $K_\bullet(\underline{x}; X)$ and $K^\bullet(\underline{x}; X)$ do not depend on the order of the elements in the sequence $\underline{x}$ and

$$K_\bullet(\underline{x}) \cong K_\bullet(x_1) \otimes_R \cdots \otimes_R K_\bullet(x_k) \text{ and } K^\bullet(\underline{x}) \cong K^\bullet(x_1) \otimes_R \cdots \otimes_R K^\bullet(x_k).$$

For the empty sequence $\underline{x} = \emptyset$ we define $K^\bullet(\underline{x}; X) = X$ and $K_\bullet(\underline{x}; X) = X$.

CAUTION: *Here, as the only exception, the descending Koszul complex is not obtained from the ascending one by lowering the degrees. The precise relation between $K^\bullet(\underline{x}; X)$ and $K_\bullet(\underline{x}; X)$ is given in 5.2.2. But note already that we have $K^\bullet(x) \cong \operatorname{Hom}_R(K_\bullet(x), R)$ by construction and in general $K^\bullet(\underline{x}; X) \cong \operatorname{Hom}_R(K_\bullet(\underline{x}), X)$ by induction and in view of 1.1.9.*

In the following we also write $H_i(\underline{x})$ and $H_i(\underline{x}; X)$ for the homology of $K_\bullet(\underline{x})$ and $K_\bullet(\underline{x}; X)$, $H^i(\underline{x})$ and $H^i(\underline{x}; X)$ for the cohomology of $K^\bullet(\underline{x})$ and $K^\bullet(\underline{x}; X)$. The Koszul complexes $K_\bullet(\underline{x})$ and $K^\bullet(\underline{x})$ are bounded complexes of free modules, hence K-projective and K-flat. Therefore

$$H_i(\underline{x}; X) \cong \operatorname{Tor}_i^R(K_\bullet(\underline{x}), X) \text{ and } H^i(\underline{x}; X) \cong \operatorname{Ext}_R^i(K_\bullet(\underline{x}), X)$$

for any R-complex X.

5.2.2 *The self-duality of the Koszul complex.* Let $\underline{x} = x_1, \dots, x_k$ be a sequence of elements in the commutative ring R and X an R-complex. *When we view the ascending complex $K^\bullet(\underline{x}; X)$ as a descending one by lowering the degrees there is an isomorphism*

$$K^\bullet(\underline{x}; X)_{[k]} \cong K_\bullet(\underline{x}; X).$$

For a length one sequence and $X = R$ this is quite obvious. For the general case we then have

$$\begin{aligned} K^\bullet(\underline{x}; X)_{[k]} &\cong K^\bullet(x_1)_{[1]} \otimes_R \cdots \otimes_R K^\bullet(x_k)_{[1]} \otimes_R X \cong \\ &K_\bullet(x_1) \otimes_R \cdots \otimes_R K_\bullet(x_k) \otimes_R X \cong K_\bullet(\underline{x}; X) \end{aligned}$$

This yields isomorphisms

$$H^i(\underline{x}; X) \cong H_{k-i}(\underline{x}; X)$$

for all $i \in \mathbb{Z}$. In particular, $K_\bullet(\underline{x}; X)$ is exact if and only if $K^\bullet(\underline{x}; X)$ is exact.

In the following we summarize a few more properties of Koszul complexes.

Lemma 5.2.3 *Let $\underline{x} = x_1, \dots, x_k$ denote a sequence of elements of a commutative ring R and $y \in R$. Let X denote a complex of R-modules.*

(a) $K_\bullet(\underline{x}) \otimes_R X \cong K_\bullet(\underline{x}; X)$ *and* $K^\bullet(\underline{x}) \otimes_R X \cong K^\bullet(\underline{x}; X)$.
(b) $\operatorname{Hom}_R(K_\bullet(\underline{x}), X) \cong K^\bullet(\underline{x}; X)$ *and* $\operatorname{Hom}_R(K^\bullet(\underline{x}), X) \cong K_\bullet(\underline{x}; X)$.
(c) There are short exact sequences of complexes

$$0 \to K_\bullet(\underline{x}; X) \to K_\bullet(\underline{x}, y; X) \to K_\bullet(\underline{x}; X)_{[-1]} \to 0,$$
$$0 \to K^\bullet(\underline{x}; X)^{[-1]} \to K^\bullet(\underline{x}, y; X) \to K^\bullet(\underline{x}; X) \to 0.$$

(d) There are long exact sequences in homology and cohomology

$$\cdots \to H_i(\underline{x}, X) \xrightarrow{y} H_i(\underline{x}, X) \to H_i(\underline{x}, y; X) \to H_{i-1}(\underline{x}, X) \xrightarrow{y} \cdots$$
$$\cdots \to H^{i-1}(\underline{x}, X) \xrightarrow{y} H^{i-1}(\underline{x}, X) \to H^i(\underline{x}, y; X) \to H^i(\underline{x}, X) \xrightarrow{y} \cdots$$

Proof The statements in (a) and (b) are direct consequences of the observations in 5.2.1. The exact sequences of complexes in (c) are a consequence of the construction and yield the long exact sequences in (d). □

As an immediate consequence of the long exact sequences in the previous Lemma we can compute the Koszul (co-)homologies of an R-complex X.

Corollary 5.2.4 *With the notation of 5.2.3 we have the following two short exact sequences*

(a) $0 \to H_0(y; H_i(\underline{x}; X)) \to H_i(\underline{x}, y; X) \to H_1(y; H_{i-1}(\underline{x}; X)) \to 0$ *and*
(b) $0 \to H^1(y; H_{i-1}(\underline{x}; X)) \to H^i(\underline{x}, y; X) \to H^0(y; H^i(\underline{x}; X)) \to 0$

for all $i \in \mathbb{Z}$.

Lemma 5.2.5 *Let* $\underline{x} = x_1, \ldots, x_k$ *denote a sequence of elements of a commutative ring* R, *let* $\mathfrak{a} = \underline{x}R$ *and* X *be any* R*-complex. The multiplication by* x_i, $1 \leq i \leq k$, *on the Koszul complex* $K_\bullet(\underline{x}; X)$ *is homotopic to zero. Therefore* $\mathfrak{a}H_i(\underline{x}; X) = 0 = \mathfrak{a}H^i(\underline{x}; X)$ *for any complex* X *and all* i.

Proof For a length one sequence x and $X = R$ the statement is obvious. The general case is now obtained by the remarks in 1.1.4. □

All the above is rather well-known, for more details and information we refer to [4], [58], [77] or [82].

We end this section with a remark on the behaviour of Koszul (co-)homology under completion, in some good cases it is related to completion.

Proposition 5.2.6 *Let* $\mathfrak{a}$ *be an ideal of a commutative ring* R *generated by the sequence* $\underline{x} = x_1, \ldots x_k$ *and assume that the pair* $(R, \mathfrak{a})$ *has the property* C *introduced in 2.6.10, which means that each* $\mathfrak{a}$*-complete module belongs to the class* $\mathcal{C}_\mathfrak{a}$ *(the pair* $(R, \mathfrak{a})$ *has this property when* R *is Noetherian (see 2.5.15)).*

Let X *be a complex with modules* $X_i \in \mathcal{C}_\mathfrak{a}$ *for all* $i \in \mathbb{Z}$. *Write* $\underline{x}^t$ *for the sequence* $x_1^t, \ldots, x_k^t$ *and* $t \geq 1$. *Then the natural homomorphisms* $\tau^\mathfrak{a}_{X_i}; X_i \to \hat{X}_i^\mathfrak{a}$ *induce natural quasi-isomorphisms*

$$K_\bullet(\underline{x}^t; X) \xrightarrow{\sim} K_\bullet(\underline{x}^t; \hat{X}^\mathfrak{a}) \text{ and } K^\bullet(\underline{x}^t; X) \xrightarrow{\sim} K^\bullet(\underline{x}^t; \hat{X}^\mathfrak{a})$$

for all $t \geq 1$. *In particular, we have quasi-isomorphisms*

$$K_\bullet(\underline{x}^t; R) \xrightarrow{\sim} K_\bullet(\underline{x}^t; \hat{R}^\mathfrak{a}) \text{ and } K^\bullet(\underline{x}^t; R) \xrightarrow{\sim} K^\bullet(\underline{x}^t; \hat{R}^\mathfrak{a}),$$

and $K_\bullet(\underline{x}^t; F) \xrightarrow{\sim} K_\bullet(\underline{x}^t; \hat{F}^\mathfrak{a})$ *for any flat complex* F.

Proof By assumption the complexes $K_\bullet(\underline{x}^t; X)$ and $K^\bullet(\underline{x}^t; X)$ are complexes of modules in $\mathcal{C}_\mathfrak{a}$ (finite direct sums of modules in $\mathcal{C}_\mathfrak{a}$ are in $\mathcal{C}_\mathfrak{a}$). As the modules $H_i(\underline{x}^t; X)$ and $H^i(\underline{x}^t; X)$ are annihilated by a power of $\mathfrak{a}$ they are $\mathfrak{a}$-complete, hence they are also in $\mathcal{C}_\mathfrak{a}$. When the complex X is right-bounded so are the complexes $K_\bullet(\underline{x}^t; X)$ and $K^\bullet(\underline{x}^t; X)$ and the result follows by 2.5.13. In general, we note that the modules $H_i(\underline{x}^t; X)$ and $H_i(\underline{x}^t; \hat{X}^\mathfrak{a})$, $H^i(\underline{x}^t; X)$ and $H^i(\underline{x}^t; \hat{X}^\mathfrak{a})$, depend only on a hard truncation $\ldots \to X_i \to \ldots \to X_s \to 0$ for $s \gg 0$. □

5.3 The Ext-Depth Tor-Codepth Sensitivity of the Koszul Complex

In this section we shall investigate the vanishing of the (co-)homology of Koszul complexes in more detail.

Definition 5.3.1 Let $\underline{x} = x_1, \ldots, x_k$ be a sequence in a commutative ring R. Let X denote an R-complex. We define the Koszul-depth and the Koszul-codepth of X by

$$h^-(\underline{x}; X) := \inf\{i \in \mathbb{Z} \mid H^i(\underline{x}; X) \neq 0\} \quad \text{and}$$
$$h_-(\underline{x}; X) := \inf\{i \in \mathbb{Z} \mid H_i(\underline{x}; X) \neq 0\}.$$

Therefore we have that $h_-(\underline{x}; X), h^-(\underline{x}; X) \in \mathbb{Z} \cup \{\infty, -\infty\}$.

Remark 5.3.2 (a) $h_-(\underline{x}, X) = \infty$ if and only if the Koszul complex $K_\bullet(\underline{x}; X)$ is exact and $h^-(\underline{x}, X) = \infty$ if and only if the Koszul complex $K^\bullet(\underline{x}; X)$ is exact. In view of the self-duality of the Koszul complex we have

$$h_-(\underline{x}, X) = \infty \quad \text{if and only if} \quad h^-(\underline{x}, X) = \infty.$$

(b) If the R-complex X is exact, then the Koszul complexes $K_\bullet(\underline{x}; X) \cong K_\bullet(\underline{x}) \otimes_R X$ and $K^\bullet(\underline{x}; X) \cong \operatorname{Hom}_R(K_\bullet(\underline{x}), X)$ are also exact, hence

$$h_-(\underline{x}, X) = \infty = h^-(\underline{x}, X).$$

This follows because the Koszul complex $K_\bullet(\underline{x})$ is K-projective and K-flat.

Note, however, that the converse does not hold. For instance, let $\underline{x}$ be a sequence of elements such that $\underline{x}R$ is the unit ideal. Then, by Lemma 5.2.5 we have the exactness of $K_\bullet(\underline{x}; X)$ and $K^\bullet(\underline{x}; X)$ for any R-complex X.

(c) If the R-complexes X and Y are quasi-isomorphic, then $h_-(\underline{x}, X) = h_-(\underline{x}, Y)$ and $h^-(\underline{x}, X) = h^-(\underline{x}, Y)$

(d) The Koszul depth and codepth are exchanged by the general Matlis duality: with the help of 5.2.3 we have

$$K_\bullet(\underline{x}; X)^\vee \cong K^\bullet(\underline{x}; X^\vee) \text{ and } K_\bullet(\underline{x}; X^\vee) \cong K^\bullet(\underline{x}; X)^\vee$$

and consequently

$$h_-(\underline{x}; X) = h^-(\underline{x}; X^\vee) \text{ and } h^-(\underline{x}; X) = h_-(\underline{x}; X^\vee)$$

for any R-complex X.

Here is the so-called Ext-depth Tor-codepth sensitivity of the Koszul complex.

Theorem 5.3.3 (see [33]) *Let R be a commutative ring, $\mathfrak{a}$ a finitely generated ideal of R and $\underline{x} = x_1, \ldots, x_k$ be a sequence such* $\operatorname{Rad}(\mathfrak{a}) = \operatorname{Rad}(\underline{x}R)$. *Then*

$$\begin{aligned} \text{E-dp}(\mathfrak{a}, X) &= \inf\{i \in \mathbb{N} \mid H^i(\underline{x}; X) \neq 0\} =: h^-(\underline{x}; X) \text{ and} \\ \text{T-codp}(\mathfrak{a}, X) &= \inf\{i \mid H_i(\underline{x}; X) \neq 0\} =: h_-(\underline{x}; X) \end{aligned}$$

for any R-complex X.

Proof Because of the isomorphisms

$$H^i(\underline{x}; X) \cong \operatorname{Ext}^i_R(K_\bullet(\underline{x}), X) \text{ and } H_i(\underline{x}; X) \cong \operatorname{Tor}^R_i(K_\bullet(\underline{x}), X)$$

this is a direct consequence of 5.1.8 and 5.1.9, together with 5.3.11, where we take for Z the complex $K_\bullet(\underline{x})$. By 5.2.3 note that the homology modules of $K_\bullet(\underline{x})$ are annihilated by the ideal $\underline{x}R$ which by assumption contains a power of $\mathfrak{a}$. Note also that $H_0(\underline{x}) = R/\underline{x}R$ and that $\underline{x}R$ is topologically equivalent to $\mathfrak{a}$ by assumption. □

It follows that the Koszul depth and codepth have the same properties as the Ext-depth and Tor-codepth respectively. In particular, we have the following.

Corollary 5.3.4 *Let $\underline{x} = x_1, \ldots, x_k$ and $\underline{y} = y_1, \ldots, y_\ell$ be two sequences of elements in a commutative ring R such that* $\operatorname{Rad}(\underline{x}R) = \operatorname{Rad}(\underline{y}R)$. *Then*

$$\begin{aligned} h^-(\underline{y}; X) &= h^-(\underline{x}; X) \geq \inf\{i \in \mathbb{Z} \mid H^i(X) \neq 0\} \text{ and} \\ h_-(\underline{y}; X) &= h_-(\underline{x}; X) \geq \inf\{i \in \mathbb{Z} \mid H_i(X) \neq 0\} \end{aligned}$$

for any R-complex X.

The following is clear by the previous Theorem 5.3.3, Corollary 5.1.7 and the self-duality of Koszul complexes 5.2.2. Recall also the class $\mathcal{T}_\mathfrak{a}$ of complexes introduced in 5.1.4.

Corollary 5.3.5 *Let R be a commutative ring, $\mathfrak{a}$ a finitely generated ideal of R and $\underline{x} = x_1, \ldots, x_k$ be a sequence such* $\operatorname{Rad}(\mathfrak{a}) = \operatorname{Rad}(\underline{x}R)$. *Then for any R-complex X the following conditions are equivalent:*

(i) T-codp$(\mathfrak{a}, X) = \infty$*, that is, $X \in \mathcal{T}_\mathfrak{a}$.*
(ii) $R/\mathfrak{b} \otimes^{\mathrm{L}}_R X \simeq 0$ *for all $\mathfrak{a}$-open ideals $\mathfrak{b}$ of R.*
(iii) $K_\bullet(\underline{x}^t; X) \simeq 0$ *for some (resp. for every) $t \geq 1$.*
(iv) $K^\bullet(\underline{x}^t; X) \simeq 0$ *for some (resp. for every) $t \geq 1$.*
(v) $\mathrm{R}\operatorname{Hom}_{\mathrm{R}}(\mathrm{R}/\mathfrak{b}, \mathrm{X}) \simeq 0$ *for all $\mathfrak{a}$-open ideals $\mathfrak{b}$ of R.*
(vi) E-dp$(\mathfrak{a}, X) = \infty$.

Corollary 5.3.6 *When the ideal $\mathfrak{a}$ of the commutative ring R is finitely generated the Ext-depth and the Tor-codepth with respect to $\mathfrak{a}$ are exchanged by the general Matlis duality: we have*

$$\text{T-codp}(\mathfrak{a}, X) = \text{E-dp}(\mathfrak{a}, X^\vee) \textit{ and } \text{E-dp}(\mathfrak{a}, X) = \text{T-codp}(\mathfrak{a}, X^\vee)$$

for all R-complex X.

Proof Note that the first equality was already in 5.1.3 without any finiteness conditions. Putting 5.3.3 and the remark in (5.3.2 (d)) together we obtain the second. □

For other relations between depth and codepth some suprema come into play.

Notations and observations 5.3.7 Let $\underline{x} = x_1, \ldots, x_k$ be a sequence in a commutative ring R. For any R-complex X we write

$$h^+(\underline{x}; X) = \sup\{i \mid H^i(\underline{x}; X) \neq 0\},$$
$$h_+(\underline{x}; X) = \sup\{i \mid H_i(\underline{x}; X) \neq 0\}.$$

Therefore $-\infty \leq h_+(\underline{x}; X), h^+(\underline{x}; X) \leq \infty$. By definition and by the self-duality of the Koszul complex we have that $h_+(\underline{x}; X) = -\infty$ if and only if $K_\bullet(\underline{x}; X)$ is exact if and only if $h^+(\underline{x}; X) = -\infty$.

Lemma 5.3.8 *Let $\underline{x} = x_1, \ldots, x_k$ and $\underline{y} = y_1, \ldots, y_\ell$ be two sequences of elements in a commutative ring R such that* $\operatorname{Rad}(\underline{x}R) = \operatorname{Rad}(\underline{y}R)$. *Let X be an R-complex.*

(a) If one of the quantities $h_-(\underline{x}; X)$, $h^+(\underline{x}; X)$ is finite, so is the other. In that case, $h_-(\underline{x}; X) + h^+(\underline{x}; X) = k$ and $h^+(\underline{x}; X) = h^+(\underline{y}; X)$.
(b) If one of the quantities $h_+(\underline{x}; X)$, $h^-(\underline{x}; X)$ is finite, so is the other. In that case, $h_+(\underline{x}; X) + h^-(\underline{x}; X) = k$ and $h_+(\underline{x}; X) = h_+(\underline{y}; X)$.
(c) If both quantities $h_-(\underline{x}; X)$ and $h^-(\underline{x}; X)$ are finite, then so are $h_+(\underline{x}; X)$ and $h^+(\underline{x}; X)$. In that case, the complexes $K_\bullet(\underline{x}; X)$ and $K^\bullet(\underline{x}; X)$ are homologically bounded and not exact.

Proof By the self-duality of the Koszul complex, more precisely by the isomorphisms $H^i(\underline{x}; X) \cong H_{k-i}(\underline{x}; X)$ (see 5.2.2), we have in general $h_-(\underline{x}; X) = k - h^+(\underline{x}; X)$ and $h^-(\underline{x}; X) = k - h_+(\underline{x}; X)$ (we use the sign rule $-(-\infty) = \infty$). The assertions in (a) and (b) follows, recall also that $h_-(\underline{x}; X) = h_-(\underline{y}; X)$ and $h^-(\underline{x}; X) = h^-(\underline{y}; X)$ by 5.3.4.

The assertions in (c) are a direct consequence of the assertions in (a) and (b). □

We are ready to present the complex version of [82, 6.1.8], which was originally concerned with modules.

Proposition 5.3.9 *Let $\mathfrak{a}$ be an ideal of a commutative ring R generated by the sequence $\underline{x} = x_1, \ldots, x_k$. Let X denote a homologically bounded R-complex.*

(a) $\text{E-dp}(\mathfrak{a}, X)$ *is finite if and only if* $\text{T-codp}(\mathfrak{a}, X)$ *is finite.*
(b) When the conditions in (a) are satisfied, then

$$\text{E-dp}(\mathfrak{a}, X) + \text{T-codp}(\mathfrak{a}, X) \leq k.$$

Proof If $\text{E-dp}(\mathfrak{a}, X)$ is finite then $\text{T-codp}(\mathfrak{a}, X) < \infty$ (see 5.3.5). As X is homologically bounded we also have

$$-\infty < \inf\{i \mid H_i(X) \neq 0\} \leq h_-(\underline{x}; X) = \text{T-codp}(\mathfrak{a}, X)$$

(see 5.3.4 and 5.3.3). It follows that $\text{T-codp}(\mathfrak{a}, X)$ is also finite.

If $\text{T-codp}(\mathfrak{a}, X)$ is finite we have in the same way

$$-\infty < \inf\{i \mid H^i(X) \neq 0\} \leq h^-(\underline{x}; X) = \text{E-dp}(\mathfrak{a}, X) < \infty.$$

This proves the first assertion.

Assume now that the conditions in (a) are satisfied. Then the complex $K_\bullet(\underline{x}; X)$ is not exact, so that $h_-(\underline{x}; X) \leq h_+(\underline{x}; X)$. By 5.3.8 (c) we have that the four quantities $h_-(\underline{x}; X)$, $h^-(\underline{x}; X)$, $h_+(\underline{x}; X)$ and $h^+(\underline{x}; X)$ are finite. Therefore by 5.3.8 (b) we also have $h^-(\underline{x}; X) + h_+(\underline{x}; X) = k$. Whence $\text{E-dp}(\mathfrak{a}, X) + \text{T-codp}(\mathfrak{a}, X) \leq k$. □

For later use note the following change of rings result.

Proposition 5.3.10 *Let $\varphi : R \to B$ be a homomorphism of commutative rings. Let $\mathfrak{a}$ be a finitely generated ideal of R and write $\mathfrak{a}^e$ for the ideal of B generated by $\varphi(\mathfrak{a})$. Let also $\underline{y} = y_1, \ldots, y_\ell$ be a sequence in B such that* $\text{Rad}(\underline{y}B) = \text{Rad}(\mathfrak{a}^e)$. *Let X be a B-complex which may be viewed as an R complex via φ. Then we have*

$$\text{E-dp}(\mathfrak{a}, X) = \text{E-dp}(\mathfrak{a}^e, X) = h^-(\underline{y}; X) \quad \textit{and}$$
$$\text{T-codp}(\mathfrak{a}, X) = \text{T-codp}(\mathfrak{a}^e, X) = h_-(\underline{y}; X).$$

Proof Let $\underline{x} = x_1, \ldots, x_k$ be a sequence of elements in R generating the ideal $\mathfrak{a}$ and write $z_i = \varphi(x_i)$, $\underline{z} = z_1, \ldots, z_k$. We have $K_\bullet(\underline{x}) \otimes_R B = K_\bullet(\underline{z})$ as is easily seen. Therefore

$$K_\bullet(\underline{x}) \otimes_R X \cong K_\bullet(\underline{z}) \otimes_B X \text{ and } \operatorname{Hom}_R(K_\bullet(\underline{x}), X) \cong \operatorname{Hom}_B(K_\bullet(\underline{z}), X).$$

Combining this with 5.3.3 and 5.3.4 we obtain

$$\text{E-dp}(\mathfrak{a}, X) = h^-(\underline{x}; X) = h^-(\underline{z}; X) = \text{E-dp}(\mathfrak{a}^e, X) = h^-(\underline{y}; X) \text{ and}$$
$$\text{T-codp}(\mathfrak{a}, X) = h_-(\underline{x}; X) = h_-(\underline{z}; X) = \text{T-codp}(\mathfrak{a}^e, X) = h_-(\underline{y}; X),$$

which completes the proof. □

Lemma 5.3.11 *Let $\underline{x} = x_1, \ldots, x_k$ be a sequence in a commutative ring R generating the ideal $\mathfrak{a}$. Then*

$$\text{E-dp}(\mathfrak{a}, X) = \text{E-dp}(\mathfrak{a}, K^\bullet(\underline{x}; X)) \quad \textit{and}$$
$$\text{T-codp}(\mathfrak{a}, X) = \text{T-codp}(\mathfrak{a}, K_\bullet(\underline{x}; X))$$

for any R-complex X.

Proof Let $X \xrightarrow{\sim} I$ be a K-injective resolution of X. Then

$$K^\bullet(\underline{x}; X) \cong \operatorname{Hom}_R(K_\bullet(\underline{x}), X) \xrightarrow{\sim} \operatorname{Hom}_R(K_\bullet(\underline{x}), I)$$

is a K-injective resolution, as is easily seen. Moreover, we have the isomorphism

$$\operatorname{Hom}_R(R/\mathfrak{a}, \operatorname{Hom}_R(K_\bullet(\underline{x}), I)) \cong \operatorname{Hom}_R(R/\mathfrak{a} \otimes_R K_\bullet(\underline{x}), I).$$

By putting $\bar{K} := R/\mathfrak{a} \otimes_R K_\bullet(\underline{x})$ it follows that

$$\text{E-dp}(\mathfrak{a}, \operatorname{Hom}_R(K_\bullet(\underline{x}), X)) = \inf\{i \in \mathbb{Z} \mid H^i(\operatorname{Hom}_R(\bar{K}, I)) \neq 0\}.$$

The complex $\bar{K}$ is a complex with zero differential and modules $\bar{K}_i = (R/\mathfrak{a})^{n_i}$ with $n_i = \binom{n}{i}$. We have $\bar{K} \cong \oplus_{i=1}^k (R/\mathfrak{a})^{n_i}_{[-i]}$, where $R/\mathfrak{a}$ is viewed as a complex concentrated in degree 0. It is now easy to see that

$$\inf\{i \in \mathbb{Z} \mid H^i(\operatorname{Hom}_R(\bar{K}, I))\} = \text{E-dp}(\mathfrak{a}, I) = \text{E-dp}(\mathfrak{a}, X).$$

The Tor-codepth part of the statement can be obtained from the Ext-depth part via the general Matlis duality or by arguments similar to those above. □

5.4 Koszul Homology of Modules

We turn to properties of a sequence $\underline{x} = x_1, \ldots, x_k$ of elements in a commutative ring R with respect to an R-module M.

5.4.1 *Completely secant sequences and regular sequences*. Let $\underline{x} = x_1, \ldots, x_k$ be a sequence of elements in a commutative ring R and write $\mathfrak{a}$ for the ideal generated by this sequence. Let M denote an R-module.

Following Bourbaki (see [14, Sect. 9, n°6 Definition 2]) we say that the sequence $\underline{x}$ is *completely secant* on N if $H_i(\underline{x}; M) = 0$ for all $i \geq 1$.

Recall also that the sequence $\underline{x}$ is called regular with respect to M or M-regular if $M \neq \mathfrak{a}M$ and

$$(x_1, \ldots, x_{i-1})M :_M x_i = (x_1, \ldots, x_{i-1})M$$

for all $i = 1, \ldots, k$.

We say that the sequence $\underline{x}$ is completely secant (resp. regular) if it is completely secant on the ring R (resp. R-regular).

Remark 5.4.2 Let $\underline{x} = x_1, \ldots, x_k$ be a sequence in a commutative ring R generating the ideal $\mathfrak{a}$ and let M be an R-module.

(a) In view of the self-duality of the Koszul complex (see 5.2.2) and the Ext-depth sensitivity of the Koszul complex (see 5.3.3) note that the sequence $\underline{x}$ is completely secant on M if and only if $H^i(\underline{x}; M) = 0$ for all $i < k$ if and only if $\operatorname{E-dp}(\mathfrak{a}, M) \geq k$.

(b) If $\underline{y} = y_1, \ldots, y_k$ denotes another sequence in R such that $\operatorname{Rad}(\underline{y}R) = \operatorname{Rad}(\mathfrak{a})$, then the sequence $\underline{x}$ is completely secant on the R-module M if and only so is the sequence $\underline{y}$. This follows from the above and 5.3.3. Note also $h^-(\underline{x}; M) = \operatorname{E-dp}(\mathfrak{a}, M) = h^-(\underline{y}; M)$ (see 5.3.4).

In particular, any permutation of a sequence completely secant on M is completely secant on M.

(c) Assume that $\operatorname{E-dp}(\mathfrak{a}, M)$ is finite. Then $\operatorname{E-dp}(\mathfrak{a}, M) = h^-(\underline{x}; M) \leq k$. It follows that the sequence $\underline{x}$ is completely secant on M if and only if $\operatorname{E-dp}(\mathfrak{a}, M) = k$. In that case $M \neq \mathfrak{a}M$.

Regular sequences do not have all the above properties, however there are some relations between these notions.

Theorem 5.4.3 (see [14, Sect. 9, Theorem 1]) *Let $\underline{x} = x_1, \ldots, x_k$ be a sequence of elements in a commutative ring R. Put $\mathfrak{a} = \underline{x}R$ and let M be an R-module. Look at the following conditions.*

(a) The sequence $\underline{x}$ is M-regular.
(b) $H_i(\underline{x}; M) = 0$ for all $i \geq 1$.
(c) $H_1(\underline{x}; M) = 0$.
(d) The natural map $(R/\mathfrak{a})[X_1, \ldots, X_k] \otimes_R M \to \oplus_{n \geq 0} \mathfrak{a}^n M/\mathfrak{a}^{n+1} M$ is bijective.

Then (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d). Suppose that $M/(x_1, \ldots, x_{i-1})M$ is $\mathfrak{a}$-separated for $i = 1, \ldots, k$ and that $M \neq \mathfrak{a}M$. Then (d) $\Rightarrow$ (a).

Let M be a finitely generated module. Assume that the ring R is Noetherian and that the ideal $\mathfrak{a}$ generated by the sequence $\underline{x} = x_1, \ldots, x_k$ is contained in the Jacobson radical of R. Then it is well-known that the Koszul complex $K_\bullet(\underline{x}; M)$ is rigid: $H_i(\underline{x}; M) = 0 \Rightarrow H_{i+1}(\underline{x}; M) = 0$. For an $\mathfrak{a}$-complete module M we have the same result without any further assumptions, and moreover the sequence $\underline{x}$ is M-regular if and only if it is completely secant on M.

Proposition 5.4.4 *(see [78, 8.2]) Let $\mathfrak{a}$ be an ideal of a commutative ring R generated by the sequence $\underline{x} = x_1, \ldots, x_k$. Let $M \neq 0$ denote an $\mathfrak{a}$-complete R-module.*

(a) If $H_i(\underline{x}; M) = 0$ for some i, then $H_{i+1}(\underline{x}; M) = 0$.
(b) If $H_1(\underline{x}; M) = 0$, then the sequence $\underline{x}$ is regular on M.

Proof We proceed by an induction on the length k of the sequence. For $k = 1$ there is nothing to prove. Assume now the statement is proved for sequences of length k. Let $y \in R$, form the sequence $\underline{x}, y$ and write $\mathfrak{a}_1$ for the ideal generated by $\underline{x}, y$. Let also M be an $\mathfrak{a}_1$-complete module.

(a): If $H_i(\underline{x}, y; M) = 0$, then multiplication by y is surjective on $H_i(\underline{x}; M)$ (see the long exact sequences in 5.2.3) and we have $H_i(\underline{x}; M) = \mathfrak{a}_1 H_i(\underline{x}; M)$. But $H_i(\underline{x}; M)$ is a homology module of a complex of $\mathfrak{a}_1$-complete modules. So we obtain $H_i(\underline{x}; M) = 0$ (see 2.2.11). But M is also $\mathfrak{a}$-complete (see 2.2.9). By the induction hypothesis we thus have $H_j(\underline{x}; M) = 0$ for all $j \geq i$. From the long exact homology sequence in 5.2.3 we then obtain $H_j(\underline{x}, y; M) = 0$ for all $j \geq i$.

(b): Assume now that $H_1(\underline{x}, y; M) = 0$. Then multiplication by y is injective on $H_0(\underline{x}; M) = M/(x_1, \ldots, x_k)M$. By the first part of the proof we also have that $H_1(\underline{x}; M) = 0$, so that the sequence $\underline{x}$ is M-regular by the induction hypothesis. All this together gives that the sequence $\underline{x}, y$ is M-regular. □

We now investigate the behaviour of completely secant sequences under completion. The following is a very particular case of a more general result of Strooker (see [82, Theorem 5.2.3]).

Proposition 5.4.5 *Let $\mathfrak{a}$ be an ideal of a commutative ring R generated by the sequence $\underline{x} = x_1, \ldots, x_k$. Let M denote an R-module such that $M \neq \mathfrak{a}M$. Assume that the sequence $\underline{x}$ is completely secant on M. Then it is regular on $\hat{M}^{\mathfrak{a}}$.*

Proof We consider the ring homomorphism

$$\varphi : \mathbb{Z}[X_1, \ldots, X_k] \to R \text{ defined by } \varphi(X_i) = x_i, i = 1, \ldots, k,$$

where the X_i's are indeterminates. We write $\mathfrak{A}$ for the ideal of $\mathbb{Z}[X_1, \ldots, X_k]$ generated by these X_i and $\mathfrak{A}_t$, $t \geq 1$, for the ideal generated by the X_i^t. We also write $\underline{X}^t = X_1^t, \ldots, X_k^t$ and $\underline{x}^t = x_1^t, \ldots, x_k^t$. Then we view M as an $\mathbb{Z}[X_1, \ldots, X_k]$-module via φ. We have that $\hat{M}^{\mathfrak{A}} = \hat{M}^{\mathfrak{a}}$ and $K_\bullet(\underline{X}^t; M) \cong K_\bullet(\underline{x}^t; M)$ for all $t \geq 1$, as is easily seen. We also have

$$\operatorname{Tor}_i^{\mathbb{Z}[X_1,\ldots,X_k]}(\mathbb{Z}[X_1, \ldots, X_k]/\mathfrak{A}_t, M) = H_i(\underline{X}^t; M) \cong H_i(\underline{x}^t; M) = 0$$

for all $i \geq 1$ by assumption. Note that the Koszul complexes $K_\bullet(\underline{X}^t)$ provide free resolutions of $\mathbb{Z}[X_1, \ldots, X_k]/\mathfrak{A}_t$. In view of the exact sequences in 2.5.5 we obtain $\Lambda_i^{\mathfrak{A}}(M) = 0$ for all $i \geq 0$ and $\Lambda_0^{\mathfrak{A}}(M) \cong \hat{M}^{\mathfrak{A}}$. So that M, viewed as a $\mathbb{Z}[X_1, \ldots, X_k]$-module, belongs to the class $\mathcal{C}_{\mathfrak{A}}$ introduced in 2.5.12. By Proposition 5.2.6 we have that the sequences $\underline{X}$ and $\underline{x}$ are completely secant on $\hat{M}^{\mathfrak{a}}$ and the result follows by 5.4.4. □

Let again $\mathfrak{a}$ be an ideal of a commutative ring R generated by the sequence $\underline{x} = x_1, \ldots, x_k$. By 5.3.9 we know that $\text{E-dp}(\mathfrak{a}, M) + \text{T-codp}(\mathfrak{a}, M) \leq k$ for all R-modules M such that $\text{T-codp}(\mathfrak{a}, M)$ is finite. The existence of a module for which equality holds has some consequences.

Lemma 5.4.6 *Let $\mathfrak{a}$ be an ideal of a commutative ring R generated by the sequence $\underline{x} = x_1, \ldots, x_k$ and assume that R is $\mathfrak{a}$-separated or that the ideal $\mathfrak{a}$ is contained in the Jacobson of R. Assume also there is an R-module M such that*

$$\text{E-dp}(\mathfrak{a}, M) + \text{T-codp}(\mathfrak{a}, M) = k.$$

Then there is an R-module N such that $\text{E-dp}(\mathfrak{a}, N) = k$ *and the sequence $\underline{x}$ is regular on $\hat{N}^{\mathfrak{a}}$.*

Proof If $\text{T-codp}(\mathfrak{a}, M) = 0$ we take $M = N$. If $\text{T-codp}(\mathfrak{a}, M) \geq 1$ we have an exact sequence $0 \to M_1 \to F \to M \to 0$ where $\text{T-codp}(\mathfrak{a}, F) = \infty$ (see 2.6.9) and therefore $\text{T-codp}(\mathfrak{a}, M_1) = \text{T-codp}(\mathfrak{a}, M) - 1$. Note also that $\text{E-dp}(\mathfrak{a}, F) = \infty$ (see 5.3.5) and therefore $\text{E-dp}(\mathfrak{a}, M_1) = \text{E-dp}(\mathfrak{a}, M) + 1$. So we obtain the module N by iteration. As $\text{E-dp}(\mathfrak{a}, N) = k$ means that the sequence $\underline{x}$ is completely secant on N (see 5.4.2) the result follows by 5.4.5. □

We apply the above to the parameter systems of a Noetherian local ring.

Corollary 5.4.7 *Let $(R, \mathfrak{m}, \Bbbk)$ be a Noetherian local ring of dimension d with its maximal ideal $\mathfrak{m}$ and residue field $\Bbbk = R/\mathfrak{m}$. If there is an R-module M such that*

$$\text{E-dp}(\mathfrak{m}, M) + \text{T-codp}(\mathfrak{m}, M) = d,$$

then there is a balanced big Cohen–Macaulay R-module, that is an R-module C such that each parameter system of R is regular on C.

Chapter 6
Čech Complexes, Čech Homology and Cohomology

Čech complexes are important tools in various fields of Mathematics, in particular in Algebraic Geometry and Commutative Algebra. In Commutative Algebra the Čech complex is known for its relation to local cohomology in the case when the underlying ring is Noetherian. Here we start with a general investigation of the construction of the Čech complex with respect to a sequence of elements $\underline{x} = x_1, \ldots, x_k$ of a commutative ring R. We investigate Čech homology and cohomology and prove the Ext-depth Tor-codepth sensitivity of the Čech complex as well as some inequalities. One of the new features here is the general assumption of a finite set of elements in a commutative ring R and unbounded R-complexes X.

6.1 The Čech Complex

6.1.1 Let $x \in R$ be an element of a commutative ring R and let $m \geq n$ be two positive integers. Then the two commutative diagrams

$$\begin{tikzcd} 0 \arrow[r] & R \arrow[r, "x^m"] \arrow[d, "x^{m-n}"'] & R \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & R \arrow[r, "x^n"] & R \arrow[r] & 0 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} 0 \arrow[r] & R \arrow[r, "x^n"] \arrow[d, equal] & R \arrow[r] \arrow[d, "x^{m-n}"] & 0 \\ 0 \arrow[r] & R \arrow[r, "x^m"] & R \arrow[r] & 0 \end{tikzcd}$$

induce a map $K_\bullet(x^m) \to K_\bullet(x^n)$ and a map $K^\bullet(x^n) \to K^\bullet(x^m)$. When $\underline{x} = x_1, \ldots, x_k$ is a sequence in the commutative ring R we write $\underline{x}^t = x_1^t, \ldots, x_k^t$ for any integer $t \geq 1$. Following the above diagrams and the constructions of the Koszul complexes we obtain morphisms of complexes

$$K_\bullet(\underline{x}^m) \to K_\bullet(\underline{x}^n) \text{ and } K^\bullet(\underline{x}^n) \to K^\bullet(\underline{x}^m)$$

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for all integers $m \geq n$. It is well-known that these morphisms form an inverse system of complexes $\{K_\bullet(\underline{x}^t)\}$ respectively a direct system of complexes $\{K^\bullet(\underline{x}^t)\}$. For an R-complex X it induces an inverse system of complexes $\{K_\bullet(\underline{x}^t; X)\}_{t\geq 1}$ and a direct system of complexes $\{K^\bullet(\underline{x}^t; X)\}_{t\geq 1}$.

Definition 6.1.2 Let $\underline{x} = x_1, \dots, x_k$ be a sequence of elements in a commutative ring R. The Čech complex on this sequence $\underline{x}$ is defined by $\check{C}_{\underline{x}} := \varinjlim K^\bullet(\underline{x}^t)$.

Observations 6.1.3 (a) For a single element x the Čech complex $\check{C}_x$ is just the complex $0 \to R \to R_x \to 0$, with R in degree 0 and the ring of fractions $R_x = \{\frac{r}{x^t} | r \in R, t \geq 0\}$ in degree 1.

(b) For a sequence $\underline{x} = x_1, \dots, x_k$ we also have

$$\check{C}_{\underline{x}} = \check{C}_{x_1} \otimes_R \cdots \otimes_R \check{C}_{x_k} \text{ and } \check{C}_{\underline{x}} \otimes_R X \cong \varinjlim\{K^\bullet(\underline{x}^t; X)\}$$

for every R-complex X because the direct limit commutes with tensor products. (Remember that $K^\bullet(\underline{x}) \cong K^\bullet(x_1) \otimes_R \cdots \otimes_R K^\bullet(x_k)$.) Moreover, the Čech complex $\check{C}_{\underline{x}}$ has the following form

$$0 \to R \to \oplus_i R_{x_i} \to \oplus_{i<j} R_{x_i x_j} \to \dots \to R_{x_1 \cdots x_k} \to 0$$

with the induced boundary maps sitting in homological degree 0 to k.

(c) We note that the Čech complex is a bounded complex of flat modules, whence DG-flat, such that $\check{C}_{\underline{x}} \otimes_R R/\underline{x}R \cong R/\underline{x}R$.

(d) $\check{C}_{\underline{x}}$ is exact if and only if $\underline{x}R = R$. (If $\check{C}_{\underline{x}} \simeq 0$, then $\check{C}_{\underline{x}} \otimes_R R/\underline{x}R \simeq 0$ in view of 4.4.11, hence $R/\underline{x}R \cong \check{C}_{\underline{x}} \otimes_R R/\underline{x}R \simeq 0$. If $\underline{x}R = R$ then $\check{C}_{\underline{x}} = \varinjlim K^\bullet(\underline{x}^t)$ is exact as a direct limit of exact complexes.)

(e) Let $\underline{x} = x_1, \dots, x_k$ be a sequence of elements in R. Let σ denote a permutation of $\{1, \dots, k\}$ and $\underline{y} = x_{\sigma(1)}, \dots, x_{\sigma(k)}$. Then it is clear that

$$\check{C}_{\underline{y}} \otimes_R X \cong \check{C}_{\underline{x}} \otimes_R X$$

for any R-complex X since the tensor product is commutative.

(f) We also have

$$\operatorname{Hom}_R(\check{C}_{\underline{x}}, X) = \operatorname{Hom}_R(\varinjlim K^\bullet(\underline{x}^t), X) \cong \varprojlim(K_\bullet(\underline{x}^t; X)$$

because $\operatorname{Hom}_R(K^\bullet(\underline{x}^t), X) \cong K_\bullet(\underline{x}^t; X)$ for any R-complex X.

The following result provides a first statement on the structure of the cohomology of the Čech complexes.

Theorem 6.1.4 *Let $\underline{x} = x_1, \dots, x_k$ be a sequence of elements in a commutative ring R, write $\mathfrak{a} = \underline{x}R$ and let X be an R-complex. Then we have*

(a) $H^i(\check{C}_{\underline{x}} \otimes_R X) \cong \varinjlim H^i(\underline{x}^t; X)$ *for all* $i \in \mathbb{Z}$,

(b) $\operatorname{Supp}_R H^i(\check{C}_{\underline{x}} \otimes_R X) \subseteq V(\mathfrak{a})$ *for all* $i \in \mathbb{Z}$ *so that the Čech cohomology modules* $H^i(\check{C}_{\underline{x}} \otimes_R X)$ *are* $\mathfrak{a}$*-torsion and*

$$\operatorname{Hom}_R(R/\mathfrak{a}, H^i(\check{C}_{\underline{x}} \otimes_R X)) \neq 0 \text{ when } H^i(\check{C}_{\underline{x}} \otimes_R X) \neq 0,$$

(c) $\operatorname{Supp}_R \check{C}_{\underline{x}} := \cup_i \operatorname{Supp}_R H^i(\check{C}_{\underline{x}}) = V(\mathfrak{a})$.

Proof As $\check{C}_{\underline{x}} \otimes_R X \cong \varinjlim\{K^\bullet(\underline{x}^t; X)\}$ we have the first isomorphism because $\varinjlim$ is exact. Let $\mathfrak{p} \notin V(\mathfrak{a})$. Then we have

$$H^i(\check{C}_{\underline{x}} \otimes_R X) \otimes_R R_\mathfrak{p} \cong H^i(\check{C}_{\underline{x}} \otimes_R X \otimes_R R_\mathfrak{p})$$

because $R_\mathfrak{p}$ is R-flat. Now we have the isomorphism of complexes

$$\check{C}_{\underline{x}} \otimes_R X \otimes_R R_\mathfrak{p} \cong \varinjlim(K^\bullet(\underline{x}^t; X) \otimes_R R_\mathfrak{p})$$

since the direct limit commutes with tensor products. But the complexes $K^\bullet(\underline{x}^t; X) \otimes_R R_\mathfrak{p}$ are exact (see 5.2.5). Hence the complex $\check{C}_{\underline{x}} \otimes_R X \otimes_R R_\mathfrak{p}$ is exact as a direct limit of exact complexes. That is, $\operatorname{Supp}_R H^i(\check{C}_{\underline{x}} \otimes_R X) \subseteq V(\mathfrak{a})$ and $H^i(\check{C}_{\underline{x}} \otimes_R X)$ is $\mathfrak{a}$-torsion for all $i \in \mathbb{Z}$ (see the remarks in 2.1.13). The last claim in (b) follows.

For all $\mathfrak{p} \in V(\mathfrak{a})$ we have $\mathfrak{a}R_\mathfrak{p} \neq R_\mathfrak{p}$, hence we also have that the complex $\check{C}_{\underline{x}} \otimes_R R_\mathfrak{p}$ is not exact (see 6.1.3 (d)). This together with (b) proves (c). □

We have seen in 2.2.13 that the general Matlis dual of an $\mathfrak{a}$-torsion module is $\mathfrak{a}$-complete, and we know that the general Matlis dual of a flat module is injective (see 1.4.1). Because of this and adjointness we obtain the following.

Corollary 6.1.5 *The general Matlis dual modules* $(H^i(\check{C}_{\underline{x}} \otimes_R X))^\vee$ *are* $\mathfrak{a}$*-complete for all* $i \in \mathbb{Z}$, *where* $\mathfrak{a} = \underline{x}R$.

The complex $\check{C}_{\underline{x}}^\vee$ *is a bounded complex of injective modules with* $\mathfrak{a}$*-complete cohomology modules and* $\operatorname{Hom}_R(R/\mathfrak{a}, \check{C}_{\underline{x}}^\vee) \cong (R/\mathfrak{a})^\vee$.

Notation 6.1.6 The natural morphisms $p_t : K^\bullet(\underline{x}^t) \to R$, defined by $p_t^0 = \mathrm{id}_R$ where R is viewed as a complex concentrated in degree 0, induces a natural morphism $p_{\underline{x}} : \check{C}_{\underline{x}} \to R$. We denote its kernel by $\check{D}_{\underline{x}}$. Therefore we have a short exact sequence of complexes

$$0 \to \check{D}_{\underline{x}} \to \check{C}_{\underline{x}} \xrightarrow{p_{\underline{x}}} R \to 0.$$

Note that this short exact sequence is *semi-split*, which means that it is split-exact in each degree. Note also that the modules $\check{D}_{\underline{x}}^i$ consist of finite direct sums of localizations of the form R_x, $x \in \underline{x}R$.

Remark 6.1.7 The shifted complex $\check{D}_{\underline{x}}^{[1]}$ is the global Čech complex as used in algebraic geometry.

When M is an R-module $H^0(\check{D}_{\underline{x}}^{[1]} \otimes_R M)$ is called the global transform of M. In particular, $H^0(\check{D}_{\underline{x}}^{[1]})$ is the ring of rational functions on $\operatorname{Spec} R \setminus V(\underline{x}R)$.

Theorem 6.1.8 *Let $\underline{x} = x_1, \ldots, x_k$ denote a sequence of elements of R and $\underline{x}R = \mathfrak{a}$. Let X be an R-complex. Then there is a natural morphism*

$$\check{C}_{\underline{x}} \otimes_R X \to X.$$

The following conditions are equivalent:

(i) $\operatorname{Supp}_R H^i(X) \subseteq V(\mathfrak{a})$ *for all $i \in \mathbb{Z}$.*
(ii) The natural morphism $\check{C}_{\underline{x}} \otimes_R X \to X$ is a quasi-isomorphism.
(iii) The complex $\check{D}_{\underline{x}} \otimes_R X$ is homologically trivial.

Proof First recall the short exact sequence of complexes of flat R-modules as in 6.1.6. By tensoring with X it induces the short exact sequence of complexes

$$0 \to \check{D}_{\underline{x}} \otimes_R X \to \check{C}_{\underline{x}} \otimes_R X \to X \to 0.$$

This provides the natural morphism of the statement. Moreover, it is a quasi-isomorphism if and only if $\check{D}_{\underline{x}} \otimes_R X$ is homologically trivial, which proves (ii) $\Longleftrightarrow$ (iii).

Now suppose that $\operatorname{Supp}_R H^i(X) \subseteq V(\mathfrak{a})$ for all $i \in \mathbb{Z}$. Then $R_x \otimes_R X$ is exact for any element $x \in \mathfrak{a}$. Since for all i $\check{D}_{\underline{x}}^i$ is a finite direct sum of some R_x for several elements $x \in \mathfrak{a}$ the complex $\check{D}_{\underline{x}}^i \otimes_R X$ is exact for all i. Therefore $\check{D}_{\underline{x}} \otimes_R X$ is exact too (see 4.1.3), i.e. (i) $\Longrightarrow$ (iii).

The implication (ii) $\Longrightarrow$ (i) is a direct consequence of Theorem 6.1.4. $\square$

The cohomology of the Čech complex built on a sequence $\underline{x} = x_1, \ldots, x_k$ depends only on the radical of the ideal $\mathfrak{a}$ generated by the x_i's. More generally, we have the following.

Corollary 6.1.9 *Let $\underline{x} = x_1, \ldots, x_k$ and $\underline{y} = y_1, \ldots, y_l$ be two sequences in a commutative ring R such that* $\operatorname{Rad}(\underline{x}R) \subseteq \operatorname{Rad}(\underline{y}R)$. *Then there are quasi-isomorphisms*

$$\check{C}_{\underline{x},\underline{y}} \otimes_R X \xrightarrow{\sim} \check{C}_{\underline{y}} \otimes_R X,$$

for all R-complexes X. If, moreover, $\operatorname{Rad}(\underline{y}R) = \operatorname{Rad}(\underline{x}R)$ *then the complexes $\check{C}_{\underline{y}} \otimes_R X$ and $\check{C}_{\underline{x}} \otimes_R X$ are quasi-isomorphic.*

Proof By Theorem 6.1.8 applied to the complex $\check{C}_{\underline{y}} \otimes_R X$ there is a quasi-isomorphism

$$\check{C}_{\underline{x}} \otimes_R (\check{C}_{\underline{y}} \otimes_R X) \xrightarrow{\sim} \check{C}_{\underline{y}} \otimes_R X$$

since $\operatorname{Supp}_R H^i(\check{C}_{\underline{y}} \otimes_R X) \subseteq V(\underline{y}R) \subseteq V(\underline{x}R)$ (see Theorem 6.1.4). Recall now that $\check{C}_{\underline{x},\underline{y}} \cong \check{C}_{\underline{x}} \otimes_R \check{C}_{\underline{y}}$ as follows by the construction.

If $\operatorname{Rad}(\underline{y}R) = \operatorname{Rad}(\underline{x}R)$ we may interchange the rôle of $\underline{x}$ and $\underline{y}$ and get a quasi-isomorphism $\check{C}_{\underline{y},\underline{x}} \otimes_R X \xrightarrow{\sim} \check{C}_{\underline{x}} \otimes_R X$. Because of the isomorphism $\check{C}_{\underline{x},\underline{y}} \cong \check{C}_{\underline{y},\underline{x}}$ this proves the final statement. □

Here is another way to build the Čech complexes.

Remark 6.1.10 Let R_x denote the localization with respect to an element $x \in R$. Let $\iota_{(x;R)} : R \to R_x$ denote the natural homomorphism $r \mapsto r/1$. We observe that

$$\check{C}_x = F(\iota_{(x;R)}).$$

The natural homomorphism $\iota_{(x;R)}$ extends to a morphism of complexes

$$\iota_{(x;X)} : X \to X_x \cong X \otimes_R R_x$$

for any R-complex X. Now let $\underline{x} = x_1, \ldots, x_k$ be a sequence of elements in R, let $y \in R$ and form the sequence $\underline{x}, y$. Let X be an R-complex. By 1.5.4 we obtain

$$\check{C}_{\underline{x},y} \otimes_R X \cong \check{C}_y \otimes_R (\check{C}_{\underline{x}} \otimes_R X) = F(\iota_{(y;\check{C}_{\underline{x}} \otimes_R X)}).$$

We sometimes simplify the notation and write $\check{C}_{\underline{x}}(X) := \check{C}_{\underline{x}} \otimes_R X$.

For inductive arguments the following result could be helpful.

Proposition 6.1.11 *Let $\underline{x} = x_1, \ldots, x_k$ be a sequence of elements in R and $y \in R$. Let X denote an R-complex and write $\check{C}_{\underline{x}}(X) := \check{C}_{\underline{x}} \otimes_R X$. There is a short exact sequence*

$$0 \to H^1(\check{C}_y(H^{i-1}(\check{C}_{\underline{x}}(X)))) \to H^i(\check{C}_{\underline{x},y}(X)) \to H^0(\check{C}_y(H^i(\check{C}_{\underline{x}}(X)))) \to 0$$

for all $i \in \mathbb{Z}$.

Proof By 6.1.10 there is a short exact sequence of complexes

$$0 \to (\check{C}_{\underline{x}} \otimes_R X)_y^{[-1]} \to \check{C}_{\underline{x},y} \otimes_R X \to (\check{C}_{\underline{x}} \otimes_R X) \to 0.$$

It induces a long exact cohomology sequence

$$\ldots \to H^{i-1}((\check{C}_{\underline{x}} \otimes_R X)) \to H^{i-1}((\check{C}_{\underline{x}} \otimes_R X)_y) \to H^i(\check{C}_{\underline{x},y} \otimes_R X)$$
$$\to H^i((\check{C}_{\underline{x}} \otimes_R X)) \to H^i((\check{C}_{\underline{x}} \otimes_R X)_y) \to \ldots$$

Since localization is exact there is an isomorphism $H^i((\check{C}_{\underline{x}} \otimes_R X)_y) \cong H^i(\check{C}_{\underline{x}} \otimes_R X)_y$ and the homomorphism $H^i(\check{C}_{\underline{x}} \otimes_R X) \to H^i((\check{C}_{\underline{x}} \otimes_R X)_y)$ is $\iota_{(y;H^i(\check{C}_{\underline{x}} \otimes_R X))}$.

Then the result follows by considering cokernels and kernels in the long exact cohomology sequence. □

6.2 A Free Resolution of the Čech Complex

In this section we shall construct a free resolution of the Čech complex. First we recall the construction of the Čech complex. By definition we have an exact sequence

$$0 \to \oplus_t K^\bullet(\underline{x}^t) \xrightarrow{\phi} \oplus_t K^\bullet(\underline{x}^t) \to \check{C}_{\underline{x}} \to 0$$

which provides a quasi-isomorphism and a bounded free resolution

$$C(\phi) \xrightarrow{\sim} \check{C}_{\underline{x}}$$

(see 1.5.6). Later we shall put $C(\phi) := T(\underline{x})$). In this section we shall construct another bounded free resolution of $\check{C}_{\underline{x}}$. It will be more helpful for several applications. To this end we also inspect again the localization of a commutative ring R by a single element.

6.2.1 We start with the Čech complex of a single element. Let $x \in R$. We recall that the short exact sequence $0 \to R[T] \xrightarrow{1-xT} R[T] \to R_x \to 0$ provides a free resolution of R_x. Let

$$P_x : 0 \to R[T] \xrightarrow{1-xT} R[T] \to 0$$

denote this free resolution of R_x. Note that another interpretation of this fact is: $R_x \cong \varinjlim\{R, x\}$, where the transition maps are $R \xrightarrow{x} R$, the multiplication by x (see also 4.3.3). There is a natural morphism $g_{\underline{x}} : R \to P_x$ inserted in the commutative diagram

$$\begin{tikzcd} R \arrow[r, equal] \arrow[d, "g_{\underline{x}}"'] & R \arrow[d, "\iota_{(x;R)}"] \\ P_x \arrow[r, "\sim"] & R_x \end{tikzcd}$$

Let $\check{L}_x$ denote the fibre of the natural morphism $g_{\underline{x}} : R \to P_x$. That is,

$$\ldots \to 0 \to R \oplus R[T] \xrightarrow{\rho} R[T] \to 0 \to \ldots, \quad \rho : (r, f(T)) \mapsto r - (1 - xT)f(T).$$

In view of 6.1.10 and 1.5.5 there is a natural quasi-isomorphism

$$F(g_{\underline{x}}) := \check{L}_x \xrightarrow{\sim} \check{C}_x = F(\iota_{(x;R)}).$$

Definition 6.2.2 Let $\underline{x} = x_1, \ldots, x_k$ denote a sequence of elements in R. Then we define $\check{L}_{\underline{x}} = \otimes_{i=1}^k \check{L}_{x_i}$. Note that $\check{L}_{\underline{x}}$ is a bounded complex of free R-modules with $L_{\underline{x}}^i = 0$ for $i < 0$ or $i > k$.

Lemma 6.2.3 (see also [75]) *Let $\underline{x} = x_1, \ldots, x_k$ denote a sequence of elements in the commutative ring R. Then there is a natural quasi-isomorphism $\check{L}_{\underline{x}} \to \check{C}_{\underline{x}}$ which provides a bounded free resolution of $\check{C}_{\underline{x}}$. Moreover, let X denote a complex of R-modules. Then the induced morphism $\check{L}_{\underline{x}} \otimes_R X \to \check{C}_{\underline{x}} \otimes_R X$ is a quasi-isomorphism.*

Proof For the proof of the first statement we proceed by induction on k, the number of elements of the sequence $\underline{x}$. The case $k = 1$ was treated above. Now let $y = x_{k+1}$. Then by the inductive hypothesis for k and the case $k = 1$ there are quasi-isomorphisms

$$\check{L}_{\underline{x},y} \cong \check{L}_{\underline{x}} \otimes \check{L}_y \xrightarrow{\sim} \check{C}_{\underline{x}} \otimes \check{L}_y \xrightarrow{\sim} \check{C}_{\underline{x}} \otimes \check{C}_y \cong \check{C}_{\underline{x},y}.$$

To this end note that $\check{L}_y$ and $\check{C}_{\underline{x}}$ are bounded complexes of flat R-modules, tensoring with them preserves quasi-isomorphisms.

The last claim follows by tensoring the quasi-isomorphism $\check{L}_{\underline{x}} \xrightarrow{\sim} \check{C}_{\underline{x}}$ with X, this yields a quasi-isomorphism again by 4.4.11 since both complexes $\check{L}_{\underline{x}}$ and $\check{C}_{\underline{x}}$ are bounded complexes of flat R-modules. □

Remark 6.2.4 Like the cohomology of $\check{C}_{\underline{x}}$, the cohomology of $\check{L}_{\underline{x}}$ only depends on $\operatorname{Rad}(\underline{x}R)$. This is a consequence of 6.1.9. More generally, let $\underline{x} = x_1, \ldots, x_k$ and $\underline{y} = y_1, \ldots, y_l$ be two sequences of elements in a commutative ring R such that $\operatorname{Rad}(\underline{x}R) \subset \operatorname{Rad}(\underline{y}R)$. We tensor the natural morphism $\check{L}_{\underline{x}} \to R$ by $\check{L}_{\underline{y}}$ and obtain a quasi-isomorphism

$$\check{L}_{\underline{x},\underline{y}} = \check{L}_{\underline{x}} \otimes_R \check{L}_{\underline{y}} \xrightarrow{\sim} \check{L}_{\underline{y}}.$$

This follows from the commutative square

$$\begin{tikzcd}
\check{L}_{\underline{x}} \otimes_R \check{L}_{\underline{y}} \arrow[r] \arrow[d] & \check{L}_{\underline{y}} \arrow[d] \\
\check{C}_{\underline{x}} \otimes \check{C}_{\underline{y}} \arrow[r] & \check{C}_{\underline{y}}
\end{tikzcd}$$

where the two vertical morphisms are the natural quasi-isomorphisms, and where the bottom morphism is a quasi-isomorphism by Corollary 6.1.9.

By commutativity, adjointness and with the help of 4.4.11 we obtain quasi-isomorphisms

$$\check{L}_{\underline{y}} \otimes_R (\check{L}_{\underline{x}} \otimes_R X) \xrightarrow{\sim} \check{L}_{\underline{y}} \otimes_R X,$$
$$\operatorname{Hom}_R(\check{L}_{\underline{y}}, X) \xrightarrow{\sim} \operatorname{Hom}_R(\check{L}_{\underline{y}}, \operatorname{Hom}_R(\check{L}_{\underline{x}}, X))$$

for any R-complex X. We note that these quasi-isomorphisms have an inverse not only in the derived category but also in the category of R-complexes. Indeed, there is a quasi-isomorphism $\check{L}_{\underline{y}} \xrightarrow{\sim} \check{L}_{\underline{x},\underline{y}} \cong \check{L}_{\underline{y}} \otimes_R \check{L}_{\underline{x}}$ by 4.4.8.

The following result could be helpful for induction arguments.

Proposition 6.2.5 *Let R denote a commutative ring. Let $\underline{x} = x_1, \ldots, x_k$ be a sequence of elements of R and $y \in R$ an element. Let X denote an R-complex. Then there is a short exact sequence*

$$0 \to H_0(\operatorname{Hom}_R(\check{L}_y, H_i(\operatorname{Hom}_R(\check{L}_{\underline{x}}, X)))) \to H_i(\operatorname{Hom}_R(\check{L}_{\underline{x},y}, X))$$
$$\to H_1(\operatorname{Hom}_R(\check{L}_y, H_{i-1}(\operatorname{Hom}_R(\check{L}_{\underline{x}}, X)))) \to 0$$

for all $i \in \mathbb{Z}$.

Proof We use the spectral sequence

$$E^2_{i,j} = H_i(\operatorname{Hom}_R(\check{L}_y, H_j(\operatorname{Hom}_R(\check{L}_{\underline{x}}, X)))) \Rightarrow H_{i+j}(\operatorname{Hom}_R(\check{L}_{\underline{x},y}, X))$$

for the homology of $\operatorname{Hom}_R(\check{L}_{\underline{x},y}, X) \cong \operatorname{Hom}_R(\check{L}_y, \operatorname{Hom}_R(\check{L}_{\underline{x}}, X))$. Because $\check{L}_y$ is bounded in degrees $0, 1$ it follows that $E^2_{i,j} = 0$ for all $i \neq 0, 1$ and the spectral sequence degenerates to the short exact sequence of the statement (see e.g. [85, Exercise 5.2.1] or [71, Corollary 10.29]). □

Definition 6.2.6 For a sequence of elements $\underline{x} = x_1, \ldots, x_k$ we compose the natural morphism $p_{\underline{x}} : \check{C}_{\underline{x}} \to R$ with the quasi-isomorphism $\check{L}_{\underline{x}} \xrightarrow{\sim} \check{C}_{\underline{x}}$ and we get a natural morphism $q_{\underline{x}} : \check{L}_{\underline{x}} \to R$, the kernel of which we denote by $\check{L^\flat}_{\underline{x}}$. Moreover, the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \check{L^\flat}_{\underline{x}} \arrow[r] \arrow[d] & \check{L}_{\underline{x}} \arrow[r, "q_{\underline{x}}"] \arrow[d] & R \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \check{D}_{\underline{x}} \arrow[r] & \check{C}_{\underline{x}} \arrow[r, "p_{\underline{x}}"] & R \arrow[r] & 0
\end{tikzcd}$$

shows that there is a quasi isomorphism $\check{L^\flat}_{\underline{x}} \xrightarrow{\sim} \check{D}_{\underline{x}}$. That is, we get a free resolution of the global Čech complex $\check{D}^{[1]}_{\underline{x}}$.

Remark 6.2.7 Let X be an R-complex with $P \xrightarrow{\sim} X$ a K-projective and $X \xrightarrow{\sim} I$ a K-injective resolution. Then we have the quasi-isomorphisms

$$\mathrm{Hom}_R(\check{L}_{\underline{x}}, P) \xrightarrow{\sim} \mathrm{Hom}_R(\check{L}_{\underline{x}}, X) \xrightarrow{\sim} \mathrm{Hom}_R(\check{L}_{\underline{x}}, I) \xleftarrow{\sim} \mathrm{Hom}_R(\check{C}_{\underline{x}}, I).$$

Each of these four complexes may be used as a representative of $\mathrm{RHom}_R(\check{C}_{\underline{x}}, X)$.

Let $F \xrightarrow{\sim} X$ be a K-flat resolution. There are quasi-isomorphisms

$$\check{L}_{\underline{x}} \otimes_R F \xrightarrow{\sim} \check{C}_{\underline{x}} \otimes_R F \xrightarrow{\sim} \check{C}_{\underline{x}} \otimes_R X \xleftarrow{\sim} \check{L}_{\underline{x}} \otimes_R X.$$

Each of these four complexes may be used as a representative of $\check{C}_{\underline{x}} \otimes_R^{\mathrm{L}} X$.

6.3 Čech Homology and Cohomology

As the Čech complex $\check{C}_{\underline{x}}$ is DG-flat the functor $(\check{C}_{\underline{x}} \otimes_R \cdot)$ on the category of complexes can be thought of as a Čech cohomology functor. Similarly the functor $\mathrm{RHom}_R(\check{C}_{\underline{x}}, \cdot)$ defined on the derived category (represented by $\mathrm{Hom}_R(\check{L}_{\underline{x}}, \cdot)$) can be thought of as a Čech homology functor. Under some "weak" conditions (always satisfied when the ring is Noetherian) it is known that for all R-complexes X we have natural isomorphisms $\check{C}_{\underline{x}} \otimes_R^{\mathrm{L}} X \simeq \mathrm{R}\Gamma_{\mathfrak{a}}(X)$ and $\mathrm{RHom}_R(\check{C}_{\underline{x}}, X) \simeq \mathrm{L}\Lambda^{\mathfrak{a}}(X)$ in the derived category, where $\mathfrak{a}$ is the ideal generated by the x_i's. We shall revisit this later. Here we investigate the functors $(\check{C}_{\underline{x}} \otimes_R^{\mathrm{L}} \cdot)$ and $\mathrm{RHom}_R(\check{C}_{\underline{x}}, \cdot)$ for their own sake. In particular, we view the modules $H^i(\check{C}_{\underline{x}} \otimes_R^{\mathrm{L}} X)$ as Čech cohomology modules and the modules $H_i(\mathrm{RHom}_R(\check{C}_{\underline{x}}, X))$ as Čech homology modules. Note that we already have some information on the functor $(\check{C}_{\underline{x}} \otimes_R^{\mathrm{L}} \cdot)$ and its cohomology modules in Sect. 6.1. We also provide the first vanishing criteria involving the class of complexes of infinite Tor-codepth (with respect to an ideal $\mathfrak{a}$ generated by the sequence $\underline{x} = x_1, \ldots, x_k$). While it is quite clear that $\check{C}_{\underline{x}}$ cohomology commutes with right-filtered direct limit, we also investigate the behaviour of $\check{C}_{\underline{x}}$ homology with inverse limits.

6.3.1 Čech homology and cohomology are related: we have an adjointness formula, namely in the derived category we have a natural isomorphism

$$\mathrm{RHom}_R(\check{C}_{\underline{x}} \otimes_R^{\mathrm{L}} X, Y) \simeq \mathrm{RHom}_R(X, \mathrm{RHom}_R(\check{C}_{\underline{x}}, Y))$$

for all R-complexes X, Y, (see 4.4.13). The unit and counit are represented by the natural morphisms

$$\mathrm{Hom}_R(\check{L}_{\underline{x}}, X) \otimes_R \check{L}_{\underline{x}} \to X \text{ and } X \to \mathrm{Hom}_R(\check{L}_{\underline{x}}, X \otimes_R \check{L}_{\underline{x}}).$$

Putting in the above $Y = E$, an injective cogenerator as described in 1.4.8, we obtain the isomorphism

$$(\check{C}_{\underline{x}} \otimes^{\mathrm{L}}_R X)^\vee \simeq \mathrm{RHom}_R(\check{C}_{\underline{x}}, X^\vee)$$

which induces isomorphisms of modules

$$(H^i(\check{C}_{\underline{x}} \otimes^{\mathrm{L}}_R X))^\vee \cong H_i(\mathrm{RHom}_R(\check{C}_{\underline{x}}, X^\vee)) \text{ for all } i \in \mathbb{Z}.$$

It follows that any information on the functor $\mathrm{RHom}_R(\check{C}_{\underline{x}}, \cdot)$ gives information on the functor $(\check{C}_{\underline{x}} \otimes^{\mathrm{L}}_R \cdot)$ via the general Matlis duality. However we cannot go the other way, since complexes are not always Matlis duals.

We obtained information on the Čech cohomology modules $H^i(\check{C}_{\underline{x}} \otimes^{\mathrm{L}}_R X) \cong H^i(\check{C}_{\underline{x}} \otimes_R X)$ in 6.1.4 and also in 6.1.11. We now turn to the Čech homology modules $H_i(\mathrm{RHom}_R(\check{C}_{\underline{x}}, X))$.

Theorem 6.3.2 *Let $\underline{x} = x_1, \ldots, x_k$ be a sequence of elements in a commutative ring R, write $\mathfrak{a} = \underline{x}R$. Let X be an R-complex with one of its K-injective resolutions $X \xrightarrow{\sim} I$. Then*

(a) $H_i(\mathrm{RHom}_R(\check{C}_{\underline{x}}, X)) \cong H_i(\mathrm{Hom}_R(\check{C}_{\underline{x}}, I)) \cong H_i(\mathrm{Hom}_R(\check{L}_{\underline{x}}, X))$,
(b) For all $i \in \mathbb{Z}$ there are functorial short exact sequences

$$0 \to \varprojlim{}^1 H_{i+1}(\underline{x}^t; X)) \to H_i((\mathrm{RHom}_R(\check{C}_{\underline{x}}, X)) \to \varprojlim H_i(\underline{x}^t; X) \to 0,$$

(c) $R/\mathfrak{a} \otimes_R H_i((\mathrm{RHom}_R(\check{C}_{\underline{x}}, X)) \neq 0$ *if* $H_i((\mathrm{RHom}_R(\check{C}_{\underline{x}}, X)) \neq 0$,
(d) $\mathrm{Ext}^j_R(\oplus_{l=1}^k R_{x_l}, H_i(\mathrm{RHom}_R(\check{C}_{\underline{x}}, X))) = 0$ *for all* $j \geq 0$ *and* $i \in \mathbb{Z}$.

Proof For (a) recall that $\mathrm{RHom}_R(\check{C}_{\underline{x}}, X)$ is represented by each of the quasi-isomorphic complexes

$$\mathrm{Hom}_R(\check{C}_{\underline{x}}, I) \xrightarrow{\sim} \mathrm{Hom}_R(\check{L}_{\underline{x}}, I) \xleftarrow{\sim} \mathrm{Hom}_R(\check{L}_{\underline{x}}, X).$$

By the definition of the Čech complex we have an exact sequence

$$0 \to \oplus_t K^\bullet(\underline{x}^t) \xrightarrow{\phi} \oplus_t K^\bullet(\underline{x}^t) \to \check{C}_{\underline{x}} \to 0,$$

where ϕ is the map associated to the direct system $\{K^\bullet(\underline{x}^t)\}$. We may and do take for $X \xrightarrow{\sim} I$ a DG-injective resolution of X, then we obtain a short exact sequence

$$0 \to \mathrm{Hom}_R(\check{C}_{\underline{x}}, I) \to \prod_t \mathrm{Hom}_R(K^\bullet(\underline{x}^t), I) \xrightarrow{\psi} \prod_t \mathrm{Hom}_R(K^\bullet(\underline{x}^t), I) \to 0,$$

where ψ is the map associated to the inverse system $\{\mathrm{Hom}_R(K^\bullet(\underline{x}^t), I)\}$ (see 1.3.3). We consider kernels and cokernels in the associated long exact homology sequence and obtain the short exact sequences in (b). Recall that

$$\operatorname{Hom}_R(K^\bullet(\underline{x}^t), I) \cong K_\bullet(\underline{x}^t; I) \xleftarrow{\sim} K_\bullet(\underline{x}^t; X).$$

Assume now that $H_i((\operatorname{RHom}_R(\check{C}_{\underline{x}}, X)) \neq 0$. Then at least one of the two modules $\varprojlim{}^1 H_{i+1}(\underline{x}^t; X)$ and $\varprojlim H_i(\underline{x}^t; X)$ is not zero. But for all $j \in \mathbb{Z}$ there is an exact sequence

$$0 \to \varprojlim H_j(\underline{x}^t; X) \to \prod_t H_j(\underline{x}^t; X) \to \prod_t H_j(\underline{x}^t; X) \to \varprojlim{}^1 H_j(\underline{x}^t; X) \to 0.$$

In this sequence the module $\prod_t H_j(\underline{x}^t; X)$ is $\mathfrak{a}$-complete as a product of $\mathfrak{a}$-complete modules (the modules $H_j(\underline{x}^t; X)$, annihilated by a power of $\mathfrak{a}$, are $\mathfrak{a}$-complete). It follows that the modules $\varprojlim{}^1 H_{i+1}(\underline{x}^t; X)$ and $\varprojlim H_i(\underline{x}^t; X)$ are not $\mathfrak{a}$-coarse when non-null:

$R/\mathfrak{a} \otimes_R \varprojlim{}^1 H_i(\underline{x}^t; X) \neq 0$ when $\varprojlim{}^1 H_i(\underline{x}^t; X) \neq 0$ and
$R/\mathfrak{a} \otimes_R \varprojlim H_i(\underline{x}^t; X) \neq 0$ when $\varprojlim H_i(\underline{x}^t; X) \neq 0$,

(see 2.2.11). The claim in (c) follows.

Moreover, both modules $\varprojlim{}^1 H_i(\underline{x}^t; X)$ and $\varprojlim H_i(\underline{x}^t; X)$ are also $\mathfrak{a}$-pseudo-complete (see (2.5.7 (b)). We know that $\operatorname{Ext}^j_R(\oplus_{l=1}^k R_{x_l}, M) = 0$ for all $j \geq 0$ and any $\mathfrak{a}$-pseudo-complete module M (see 3.1.8). The claim in (d) follows now with the long exact sequence of the $\operatorname{Ext}^j_R(\oplus_{l=1}^k R_{x_l}, \cdot)$ associated to the short exact sequences in (b). □

Remark 6.3.3 (a) When the ring is Noetherian we also have that the modules $H_i(\operatorname{RHom}_R(\check{C}_{\underline{x}}, X))$ are $\mathfrak{a}$-pseudo-complete. This follows from 2.5.17 applied to the short exact sequences in 6.3.2. (Note, however, that this also hold in a larger generality, see the forthcoming 7.5.13.)

(b) In general, observe the similarity between the short exact sequences in 6.3.2 and those in 2.5.5. When the sequence $\underline{x}$ is completely secant on R note also that $H_i(\underline{x}^t; X) \cong \operatorname{Tor}_i^R(R/\mathfrak{a}_t, X)$, where $\mathfrak{a}_t$ denotes the ideal generated by the sequence $\underline{x}^t = x_1^t, \ldots, x_k^t$.

Next we investigate when $\operatorname{RHom}_R(\check{C}_{\underline{x}}, \cdot)$ and $\check{C}_{\underline{x}} \otimes^{\mathrm{L}}_R \cdot$ vanish. Our criteria provide another characterization of complexes in the class $\mathcal{T}_\mathfrak{a}$ introduced in 5.1.4.

Proposition 6.3.4 *Let $\underline{x} = x_1, \ldots, x_k$ be a sequence of elements in the commutative ring R and let $\mathfrak{a}$ be the ideal generated by the x_i's. For any complex X the following conditions are equivalent:*

(i) $\check{C}_{\underline{x}} \otimes_R X$ *is exact.*
(ii) $X \in \mathcal{T}_\mathfrak{a}$.
(iii) $\operatorname{RHom}_R(\check{C}_{\underline{x}}, X) \simeq 0$.
(iv) $\operatorname{Hom}_R(\check{L}_{\underline{x}}, X)$ *is exact.*

Proof First assume that $X \in \mathcal{T}_{\mathfrak{a}}$. Then for all $i \in \mathbb{Z}$ and all $t \geq 1$ we have

$$H^i(\underline{x}^t; X) = 0 = H_i(\underline{x}^t; X)$$

(see 5.3.5). This yields that $H^i(\check{C}_{\underline{x}} \otimes_R X) \cong \varinjlim H^i(\underline{x}^t; X) = 0$ for all $i \in \mathbb{Z}$. It also follows that $H_i(\operatorname{RHom}_R(\check{C}_{\underline{x}}, X)) = 0$ for all $i \in \mathbb{Z}$ with the short exact sequences in 6.3.2.

Assume now that $\operatorname{RHom}_R(\check{C}_{\underline{x}}, X) \simeq 0$ and let $X \xrightarrow{\sim} I$ be a K-injective resolution of X. By assumption we have $\operatorname{Hom}_R(\check{C}_{\underline{x}}, I) \xrightarrow{\sim} 0$. But the complex $\operatorname{Hom}_R(\check{C}_{\underline{x}}, I)$ is K-injective (see 4.4.3 (a)). Hence we have a quasi-isomorphism

$$\operatorname{Hom}_R(R/\mathfrak{a}, \operatorname{Hom}_R(\check{C}_{\underline{x}}, I)) \xrightarrow{\sim} 0$$

(see 4.4.11). By adjointness and with the isomorphism $R/\mathfrak{a} \otimes \check{C}_{\underline{x}} \cong R/\mathfrak{a}$ we have

$$\operatorname{Hom}_R(R/\mathfrak{a}, \operatorname{Hom}_R(\check{C}_{\underline{x}}, I)) \cong \operatorname{Hom}_R(R/\mathfrak{a} \otimes \check{C}_{\underline{x}}, I) \cong \operatorname{Hom}_R(R/\mathfrak{a}, I).$$

Therefore $\operatorname{Hom}_R(R/\mathfrak{a}, I) \xrightarrow{\sim} 0$, $\operatorname{E-dp}(\mathfrak{a}, X) = \infty$ and $X \in \mathcal{T}_{\mathfrak{a}}$ by 5.3.5.

Next assume that the complex $\check{C}_{\underline{x}} \otimes_R X$ is exact. Let $F \xrightarrow{\sim} X$ be a K-flat resolution of X. Then $\check{C}_{\underline{x}} \otimes_R F \xrightarrow{\sim} 0$. But the complex $\check{C}_{\underline{x}} \otimes_R F$ is K-flat (see 4.4.3 (c)). So we have the quasi-isomorphism

$$R/\mathfrak{a} \otimes_R \check{C}_{\underline{x}} \otimes_R F \xrightarrow{\sim} 0$$

(see 4.4.11). As $R/\mathfrak{a} \otimes_R \check{C}_{\underline{x}} \cong R/\mathfrak{a}$ this means that $R/\mathfrak{a} \otimes_R F \xrightarrow{\sim} 0$ and that $X \in \mathcal{T}_{\mathfrak{a}}$.

The equivalence of the last two conditions is clear since $\check{L}_{\underline{x}} \xrightarrow{\sim} \check{C}_{\underline{x}}$ is a DG-projective resolution of $\check{C}_{\underline{x}}$, so that $\operatorname{RHom}_R(\check{C}_{\underline{x}}, X)$ is also represented by $\operatorname{Hom}_R(\check{L}_{\underline{x}}, X)$ in the derived category. □

We note that the Čech homology and cohomology is impervious with respect to a change of rings. More precisely and more generally we have the following.

Proposition 6.3.5 *Let $\varphi : R \to B$ be a homomorphism of commutative rings. Let $\underline{x} = x_1, \ldots, x_k$ be a sequence in R and $\underline{y} = y_1, \ldots, y_l$ a sequence in B and assume that* $\operatorname{Rad}(\underline{y}B) = \operatorname{Rad}(\varphi(\underline{x})B)$. *Let X be a B-complex which may also be viewed as an R complex via φ. Then*

$$H^i(\check{C}_{\underline{x}} \otimes_R X) \cong H^i(\check{C}_{\underline{y}} \otimes_B X) \text{ and}$$
$$H_i(\operatorname{RHom}_R(\check{C}_{\underline{x}}, X)) \cong H_i(\operatorname{RHom}_B(\check{C}_{\underline{y}}, X))$$

for all $i \in \mathbb{Z}$.

Proof We have $\check{C}_{\underline{x}} \otimes_R B \cong \check{C}_{\varphi(\underline{x})}$ by construction. By the assumption there are quasi-isomorphisms

$$\check{C}_{\underline{y}} \xleftarrow{\sim} \check{C}_{\underline{y}} \otimes_B \check{C}_{\varphi(\underline{x})} \xrightarrow{\sim} \check{C}_{\varphi(\underline{x})},$$

(see 6.1.9). It follows by 4.4.11 that the complexes

$$\check{C}_{\underline{x}} \otimes_R X \cong \check{C}_{\varphi(\underline{x})} \otimes_B X \text{ and } \check{C}_{\underline{y}} \otimes_B X$$

are quasi-isomorphic. This proves the first isomorphisms.

By construction we also have an isomorphism $\check{L}_{\underline{x}} \otimes_R B \cong \check{L}_{\varphi(\underline{x})}$. In view of the quasi-isomorphisms $\check{L}_{\varphi(\underline{x})} \xrightarrow{\sim} \check{C}_{\varphi(\underline{x})}$ and $\check{L}_{\underline{y}} \xrightarrow{\sim} \check{C}_{\underline{y}}$ it follows that the bounded complexes of free B-modules $\check{L}_{\varphi(\underline{x})}$ and $\check{L}_{\underline{y}}$ are quasi-isomorphic. By 4.4.8 we even have a quasi-isomorphism $\check{L}_{\varphi(\underline{x})} \xrightarrow{\sim} \check{L}_{\underline{y}}$. Therefore there are a quasi-isomorphism (see 4.4.11) and an isomorphism

$$\operatorname{Hom}_B(\check{L}_{\underline{y}}, X) \xrightarrow{\sim} \operatorname{Hom}_B(\check{L}_{\varphi(\underline{x})}, X) \cong \operatorname{Hom}_R(\check{L}_{\underline{x}}, X).$$

This provides the second isomorphisms.

When the morphism $\varphi : R \to B$ is flat we can go the other way, at least for the Čech cohomology. The following is rather obvious.

Proposition 6.3.6 *In the situation of 6.3.5 assume that B, viewed as an R-module via φ, is flat and let Y be an R-complex. Then*

(a) $(\check{C}_{\underline{x}} \otimes_R Y) \otimes_R B \cong \check{C}_{\varphi(\underline{x})} \otimes_B (B \otimes_R Y)$ *and*
(b) $H^i(\check{C}_{\underline{x}} \otimes_R Y) \otimes_R B \cong H^i(\check{C}_{\varphi(\underline{x})} \otimes_B (B \otimes_R Y))$ *for all* $i \in \mathbb{Z}$.

Remark 6.3.7 (a) Čech cohomology commutes with right-filtered direct limits. More precisely, let $\underline{x} = x_1, \ldots, x_k$ be a sequence in a commutative ring R, let $(I, \prec)$ be a right-filtered ordered set and let $\mathfrak{X} = \{\sigma_{ij} : X_i \to X_j \mid 0 \le i \le j \in I\}$ be a direct system of R-complexes. In view of 1.3.5 we have

$$\check{C}_{\underline{x}} \otimes_R \varinjlim X_i \cong \varinjlim(\check{C}_{\underline{x}} \otimes_R X_i) \text{ and } H^i(\check{C}_{\underline{x}} \otimes_R \varinjlim X_i) \cong \varinjlim H^i(\check{C}_{\underline{x}} \otimes_R X_i).$$

(b) Čech homology commutes with products. This is because the functor $\operatorname{Hom}_R(\check{L}_{\underline{x}}, \cdot)$ commutes with products.

(c) The behaviour of Čech homology with inverse limits is more subtle. Let $\mathfrak{Y} = \{\rho_{t,t+1} : Y_{t+1} \to Y_t \mid t \in \mathbb{N}\}$ be an inverse system of R-complexes over $\mathbb{N}$ and assume that it satisfies degree-wise the Mittag-Leffler condition. Then we have the short exact sequence

$$0 \to \varprojlim Y_t \to \prod_{t \in \mathbb{N}} Y_t \xrightarrow{\psi_{\mathfrak{Y}}} \prod_{t \in \mathbb{N}} Y_t \to 0$$

(see 1.2.3). It induces the short exact sequence

$$0 \to \operatorname{Hom}_R(\check{L}_{\underline{x}}, \varprojlim Y_t) \to \prod_{t\in\mathbb{N}} \operatorname{Hom}_R(\check{L}_{\underline{x}}, Y_t) \xrightarrow{\psi} \prod_{t\in\mathbb{N}} \operatorname{Hom}_R(\check{L}_{\underline{x}}, Y_t) \to 0$$

because $\check{L}_{\underline{x}}$ is a bounded complex of free modules. Note that

$$\psi = \operatorname{Hom}_R(\check{L}_{\underline{x}}, \psi_{\mathfrak{Y}}) = \psi_{\operatorname{Hom}_R(\check{L}_{\underline{x}}, \mathfrak{Y})}$$

is the morphism associated to the inverse system $\operatorname{Hom}_R(\check{L}_{\underline{x}}, \mathfrak{Y})$ (see 1.2.7). We split off the associated long exact sequence in homology and obtain short exact sequences

$$\begin{aligned} 0 \to \varprojlim{}^1 H_{i+1}(\operatorname{Hom}_R(\check{L}_{\underline{x}}, Y_t) &\to H_i(\operatorname{Hom}_R(\check{L}_{\underline{x}}, \varprojlim Y_t)) \\ &\to \varprojlim H_i(\operatorname{Hom}_R(\check{L}_{\underline{x}}, Y_t)) \to 0 \end{aligned}$$

for all $i \in \mathbb{Z}$.

6.4 Some Classes Related to the Čech Complex

We observed that there are natural morphisms $X \to \operatorname{RHom}_R(\check{C}_{\underline{x}}, X)$ and $\check{C}_{\underline{x}} \otimes^{\mathrm{L}}_R X \to X$ in the derived category and will investigate when they are isomorphisms. Our criteria will involve the class $\mathcal{T}_{\mathfrak{a}}$ introduced in 5.1.4.

Notations and observations 6.4.1 As mentioned before, there is a short exact sequence of DG-flat complexes $0 \to \check{D}_{\underline{x}} \to \check{C}_{\underline{x}} \xrightarrow{p_{\underline{x}}} R \to 0$, degree-wise split-exact. For an R-complex X and a K-injective complex I it induces short exact sequences

$$\begin{aligned} 0 \to \check{D}_{\underline{x}} \otimes_R X \to \check{C}_{\underline{x}} \otimes_R X \to R \otimes_R X \to 0 \text{ and} \\ 0 \to I \to \operatorname{Hom}_R(\check{C}_{\underline{x}}, I) \to \operatorname{Hom}_R(\check{D}_{\underline{x}}, I) \to 0. \end{aligned}$$

In particular, it induces natural morphisms

$$\iota(\check{C}_{\underline{x}}; X) : \check{C}_{\underline{x}} \otimes^{\mathrm{L}}_R X \to X \quad \text{and} \quad \alpha(\check{C}_{\underline{x}}; X) : X \to \operatorname{RHom}_R(\check{C}_{\underline{x}}, X)$$

in the derived category.

The first one is represented by the morphism $p_{\underline{x}} \otimes_R X : \check{C}_{\underline{x}} \otimes_R X \to X$ and also by the composite $(\check{L}_{\underline{x}} \otimes_R X \xrightarrow{\sim} \check{C}_{\underline{x}} \otimes_R X \to X)$. Note that these are quasi-isomorphisms (represent isomorphisms in the derived category) if and only if $\check{D}_{\underline{x}} \otimes_R X$ is exact.

The second one is represented by the composite $X \xrightarrow{\sim} I \to \operatorname{Hom}_R(\check{C}_{\underline{x}}, I)$, where $X \xrightarrow{\sim} I$ is any K-injective resolution of X. Note that it is a quasi-isomorphism (represents an isomorphism in the derived category) if and only if $\operatorname{Hom}_R(\check{D}_{\underline{x}}, I)$ is exact. The map $\alpha(\check{C}_{\underline{x}}; X)$ is also represented by the natural morphism $X \to \operatorname{Hom}_R(\check{L}_{\underline{x}}, X)$ stemming from the natural morphism $q_{\underline{x}} : \check{L}_{\underline{x}} \to R$. To this end, note the commutative diagram

$$\begin{tikzcd}
X \arrow[r, "\sim"] \arrow[d] & I \arrow[r, equal] \arrow[d] & I \arrow[d] \\
\operatorname{Hom}_R(\check{L}_{\underline{x}}, X) \arrow[r, "\sim"] & \operatorname{Hom}_R(\check{L}_{\underline{x}}, I) & \operatorname{Hom}_R(\check{C}_{\underline{x}}, I). \arrow[l, "\sim"']
\end{tikzcd}$$

Lemma 6.4.2 *Let $\mathfrak{a}$ be an ideal of a commutative ring R generated by the sequence $\underline{x} = x_1, \ldots, x_k$. For any K-injective complex I the following conditions are equivalent.*

(i) *The natural morphism $I \to \operatorname{Hom}_R(\check{C}_{\underline{x}}, I)$ is a quasi-isomorphism.*
(ii) $\operatorname{Hom}_R(T, I)$ *is exact for all $T \in \mathcal{T}_{\mathfrak{a}}$.*
(iii) $\operatorname{Hom}_R(R_z, I)$ *is exact for all $z \in \operatorname{Rad}(\mathfrak{a})$.*
(iv) $\operatorname{Hom}_R(\oplus_{i=1}^k R_{x_i}, I)$ *is exact.*

Proof (i) $\Rightarrow$ (ii): Let $T \in \mathcal{T}_{\mathfrak{a}}$. Since the complex $\operatorname{Hom}_R(\check{C}_{\underline{x}}, I)$ is K-injective (see 4.4.3) there is a quasi-isomorphism and an isomorphism

$$\operatorname{Hom}_R(T, I) \xrightarrow{\sim} \operatorname{Hom}_R(T, \operatorname{Hom}_R(\check{C}_{\underline{x}}, I)) \cong \operatorname{Hom}_R(T \otimes_R \check{C}_{\underline{x}}, I).$$

Here the quasi-isomorphism follows from 4.4.11 and the isomorphism follows by adjointness. But the complex $T \otimes_R \check{C}_{\underline{x}}$ is exact by 6.3.4. As I is K-injective it follows that $\operatorname{Hom}_R(T, I) \xrightarrow{\sim} \operatorname{Hom}_R(T \otimes_R \check{C}_{\underline{x}}, I)$ is exact too.

(ii) $\Rightarrow$ (iii): This is true because $R_z \in \mathcal{T}_{\mathfrak{a}}$ when $z \in \operatorname{Rad}(\mathfrak{a})$.

(iii) $\Rightarrow$ (iv): This is obvious.

(iv) $\Rightarrow$ (i): First note that the complex $\operatorname{Hom}_R(R_{x_i}, I)$ is K-injective because the module R_{x_i} is flat. So the quasi-isomorphisms $\operatorname{Hom}_R(R_{x_i}, I) \xrightarrow{\sim} 0$ induces quasi-isomorphisms

$$\operatorname{Hom}_R(R_{xx_i}, I) \cong \operatorname{Hom}_R(R_x, \operatorname{Hom}_R(R_{x_i}, I)) \xrightarrow{\sim} 0 \text{ for any } x \in R.$$

By virtue of 6.4.1 it is enough to prove that the complex $\operatorname{Hom}_R(\check{D}_{\underline{x}}, I)$ is exact.To this end we investigate the double complex $K^{ij} := \operatorname{Hom}_R(\check{D}_{\underline{x}}^{-i}, I^j)$. The modules in $\check{D}_{\underline{x}}$ are direct sums of modules of the form $R_{x_i x}$. Therefore all the columns $\operatorname{Hom}_R(\check{D}_{\underline{x}}^{-i}, I)$ of the double complex are exact by the assumption. Hence the single complex associated to the double complex K^{ij} is exact too (see 4.1.3 and interchange rows and columns). But this single complex is the complex $\operatorname{Hom}_R(\check{D}_{\underline{x}}, I)$ because $\check{D}_{\underline{x}}$ is bounded. Whence the condition (i) is satisfied. □

We reformulate 6.4.2 in the derived category language and obtain.

Theorem and Definition 6.4.3 *Let $\mathfrak{a}$ be an ideal of a commutative ring R generated by the sequence $\underline{x} = x_1, \dots, x_k$. For any R-complex X the following conditions are equivalent.*

(i) *The natural map $\alpha(\check{C}_{\underline{x}}; X) : X \to \operatorname{RHom}_R(\check{C}_{\underline{x}}, X)$ is an isomorphism in the derived category.*
(ii) *The natural morphism $X \to \operatorname{Hom}_R(\check{L}_{\underline{x}}, X)$ is a quasi-isomorphism.*
(iii) $\operatorname{RHom}_R(T, X) \simeq 0$ *for all* $T \in \mathcal{T}_{\mathfrak{a}}$.
(iv) $\operatorname{RHom}_R(R_z, X) \simeq 0$ *for all* $z \in \operatorname{Rad}(\mathfrak{a})$.
(v) $\operatorname{RHom}_R(\oplus_{i=1}^k R_{x_i}, X) \simeq 0$.

We denote by $\mathcal{G}_{\underline{x}}^{L\check{C}}$ the class of R-complexes Z satisfying the above conditions.

Proof Let $X \xrightarrow{\sim} I$ be a K-injective resolution of X. The equivalence of (i), (iii), (iv) and (v) follows by virtue of 6.4.2 applied to the complex I. The equivalence of (i) and (ii) follows by the remarks in 6.4.1. □

(We note the similarity between the conditions in the above theorem and those in the completeness and pseudo-completeness in 3.1.6 and 3.1.11. Recall that $\operatorname{Hom}_R(R_{x_i}, M) = 0$ when M is an $\mathfrak{a}$-separated module and when $x_i \in \mathfrak{a}$ (see 2.1.2).)

Example 6.4.4 Let $\underline{x} = x_1, \dots, x_k$ denote a sequence of elements in a commutative ring R and write $\mathfrak{a} = \underline{x}R$. (a) $\operatorname{Hom}_R(\check{C}_{\underline{x}}, I) \in \mathcal{G}_{\underline{x}}^{L\check{C}}$ for any K-injective complex I. This follows by 6.3.4. We have that $\operatorname{Hom}_R(T, \operatorname{Hom}_R(\check{C}_{\underline{x}}, I)) \cong \operatorname{Hom}_R(T \otimes_R \check{C}_{\underline{x}}, I)$ is exact when $T \in \mathcal{T}_{\mathfrak{a}}$. This means that $\operatorname{RHom}_R(T, \operatorname{Hom}_R(\check{C}_{\underline{x}}, I)) \simeq 0$ for all $T \in \mathcal{T}_{\mathfrak{a}}$ (note that the complex $\operatorname{Hom}_R(\check{C}_{\underline{x}}, I)$ is K-injective).

(b) In particular, the general Matlis dual $\check{C}_{\underline{x}}^{\vee}$ of the Čech complex is a DG-injective complex belonging to $\mathcal{G}_{\underline{x}}^{L\check{C}}$.

(c) We note that $\mathfrak{a}$-pseudo-complete modules belong to $\mathcal{G}_{\underline{x}}^{L\check{C}}$. This follows by 3.1.8: R_{x_i} is a relatively-$\mathfrak{a}$-flat module such that $R_{x_i} = \mathfrak{a}R_{x_i}$. We recall that $\mathfrak{a}$-complete modules are $\mathfrak{a}$-pseudo-complete (see 2.5.7) hence are also in $\mathcal{G}_{\underline{x}}^{L\check{C}}$. When the ring is Noetherian a module belongs to $\mathcal{G}_{\underline{x}}^{L\check{C}}$ if and only if it is $\mathfrak{a}$-pseudo-complete. This follows by Theorem 3.1.11.

(d) In particular, the modules $R/\mathfrak{a}$ and $(R/\mathfrak{a})^{\vee}$, which are both $\mathfrak{a}$-complete, belong to $\mathcal{G}_{\underline{x}}^{L\check{C}}$.

(e) If a module in $\mathcal{G}_{\underline{x}}^{L\check{C}}$ is $\mathfrak{a}$-separated, then it is $\mathfrak{a}$-complete. This follows by 3.1.6.

As a dual result to 6.4.2 we have the following on the Čech cohomology side.

Lemma 6.4.5 *Let $\mathfrak{a}$ be an ideal of a commutative ring R generated by the sequence $\underline{x} = x_1, \ldots, x_k$. For any K-flat complex F the following conditions are equivalent.*

(i) The natural morphism $\check{C}_{\underline{x}} \otimes_R F \to F$ is a quasi-isomorphism.
(ii) $T \otimes_R F$ is exact for all $T \in \mathcal{T}_{\mathfrak{a}}$.
(iii) $R_z \otimes_R F$ is exact for all $z \in \operatorname{Rad}(\mathfrak{a})$.
(iv) $(\oplus_{i=1}^k R_{x_i}) \otimes_R F$ is exact.

Proof The proof of this is similar to the proof of 6.4.2. Here we deduce it from 6.4.2 via the general Matlis duality functor and adjointness. Indeed, the general Matlis dual complex $F^\vee$ is K-injective. Moreover,

$$(\check{C}_{\underline{x}} \otimes_R F)^\vee \cong \operatorname{Hom}_R(\check{C}_{\underline{x}}, F^\vee), \ (T \otimes_R F)^\vee \cong \operatorname{Hom}_R(T, F^\vee) \text{ and}$$
$$(R_z \otimes_R F)^\vee \cong \operatorname{Hom}_R(R_z, F^\vee).$$

This proves the claim by the faithful exactness of the general Matlis duality functor. □

For an arbitrary complex Z we obtain an analogue of Theorem 6.4.3. In fact, this is an extension of Theorem 6.1.8.

Theorem and Definition 6.4.6 *Let $\mathfrak{a}$ be an ideal of a commutative ring R generated by the sequence $\underline{x} = x_1, \ldots, x_k$. For any complex X the following conditions are equivalent.*

(i) The natural map $\iota(\check{C}_{\underline{x}}; X) : \check{C}_{\underline{x}} \otimes^{\mathrm{L}}_R X \to X$ is an isomorphism in the derived category.
(ii) $T \otimes^{\mathrm{L}}_R X \simeq 0$ for all $T \in \mathcal{T}_{\mathfrak{a}}$.
(iii) $R_z \otimes_R X$ is exact for all $z \in \operatorname{Rad}(\mathfrak{a})$.
(iv) $(\oplus_{i=1}^k R_{x_i}) \otimes_R X$ is exact.
(v) $\operatorname{Supp}_R H^i(X) \subseteq V(\mathfrak{a})$ for all $i \in \mathbb{Z}$.
(vi) The natural isomorphism $\check{L}_x \otimes_R X \to X$ is a quasi-isomorphism.

We denote by $\mathcal{G}^{R\check{C}}_{\underline{x}}$ the class of R-complexes X satisfying the above conditions.

Proof Let $F \xrightarrow{\sim} X$ be a K-flat resolution of X. Because of the quasi-isomorphism $\check{C}_{\underline{x}} \otimes_R F \xrightarrow{\sim} \check{C}_{\underline{x}} \otimes_R X$ it follows that the natural map $\iota(\check{C}_{\underline{x}}; X) : \check{C}_{\underline{x}} \otimes^{\mathrm{L}}_R X \to X$ is an isomorphism in the derived category if and only if the natural morphism $\check{C}_{\underline{x}} \otimes_R F \to F$ is a quasi-isomorphism. Hence the equivalence of (i), (ii), (iii) and (iv) follows by 6.4.5 applied to F. The equivalence (i) ⇔ (v) is shown in 6.1.8. The equivalence of (i) and (vi) follows by the remarks in 6.4.1. □

Example 6.4.7 Let $\underline{x} = x_1, \ldots, x_k$ denote a sequence of elements in a commutative ring R and write $\mathfrak{a} = \underline{x}R$.

(a) $\check{C}_{\underline{x}} \otimes_R X \in \mathcal{G}^{R\check{C}}_{\underline{x}}$ for all R-complex X. This follows by 6.1.4.

(b) In particular, the DG-flat complex $\check{C}_{\underline{x}}$ is in $\mathcal{G}^{R\check{C}}_{\underline{x}}$.

(c) We note that an R-module M is in $\mathcal{G}_{\underline{x}}^{R\check{C}}$ if and only if $\operatorname{Supp}_R M \subseteq V(\mathfrak{a})$ if and only if M is $\mathfrak{a}$-torsion (see also 2.1.13).

(d) In particular, the modules $R/\mathfrak{a}$ and $(R/\mathfrak{a})^\vee$, which are both annihilated by $\mathfrak{a}$, hence $\mathfrak{a}$-torsion, belong to $\mathcal{G}_{\underline{x}}^{R\check{C}}$.

The classes $\mathcal{G}_{\underline{x}}^{L\check{C}}$ and $\mathcal{G}_{\underline{x}}^{R\check{C}}$ have nice properties.

Corollary 6.4.8 *Let $\underline{x} = x_1, \ldots, x_k$ be a sequence of elements in a commutative ring R.*

(a) Let $0 \to X' \to X \to X'' \to 0$ be a short exact sequence of complexes. If two of the three complexes X, X' and X'' are in $\mathcal{G}_{\underline{x}}^{L\check{C}}$ (resp. in $\mathcal{G}_{\underline{x}}^{R\check{C}}$), then the third one is also in $\mathcal{G}_{\underline{x}}^{L\check{C}}$ (resp. in $\mathcal{G}_{\underline{x}}^{R\check{C}}$).

(b) If $X \in G_{\underline{x}}^{L\check{C}}$ and if Y is quasi-isomorphic to X, then $Y \in G_{\underline{x}}^{L\check{C}}$. If $X \in \mathcal{G}_{\underline{x}}^{R\check{C}}$ and if Y is quasi-isomorphic to X, then $Y \in \mathcal{G}_{\underline{x}}^{R\check{C}}$. The classes of complexes $\mathcal{G}_{\underline{x}}^{L\check{C}}$ and $\mathcal{G}_{\underline{x}}^{R\check{C}}$ are well-defined in the derived category.

(c) If $X \in \mathcal{G}_{\underline{x}}^{L\check{C}}$ and P is K-projective, then $\operatorname{Hom}_R(P, X) \in \mathcal{G}_{\underline{x}}^{L\check{C}}$.
If $Y \in \mathcal{G}_{\underline{x}}^{R\check{C}}$ and F is K-flat, then $Y \otimes_R F \in \mathcal{G}_{\underline{x}}^{R\check{C}}$.

(d) If $\underline{y} = y_1, \ldots, y_l$ is another sequence such that $\operatorname{Rad}(\underline{x}R) = \operatorname{Rad}(\underline{y}R)$, then $\mathcal{G}_{\underline{x}}^{L\check{C}} = \mathcal{G}_{\underline{y}}^{L\check{C}}$ and $\mathcal{G}_{\underline{x}}^{R\check{C}} = \mathcal{G}_{\underline{y}}^{R\check{C}}$.

Proof (a) Let T be any complex in $\mathcal{T}_\mathfrak{a}$ and $P \xrightarrow{\sim} T$ be a DG-projective resolution. We have the short exact sequence

$$0 \to \operatorname{Hom}_R(P, X') \to \operatorname{Hom}_R(P, X) \to \operatorname{Hom}_R(P, X'') \to 0.$$

If two of the three complexes occurring in this sequence are exact, then so is the third. As $\operatorname{Hom}_R(P, \cdot)$ represents $\operatorname{RHom}_R(T, \cdot)$ in the derived category this together with Theorem 6.4.3 (condition (iii)) proves the $\mathcal{G}_{\underline{x}}^{L\check{C}}$ side of the statement. The proof of the $\mathcal{G}_{\underline{x}}^{R\check{C}}$ side is similar by using the functor $(P \otimes_R \cdot)$ in place of $\operatorname{Hom}_R(P, \cdot)$ and Theorem 6.4.6 in place of Theorem 6.4.3.

(b) These statements are rather obvious in view of the properties of RHom and $\otimes^{\mathrm{L}}$, 6.4.3 condition (iii) and 6.4.6 condition (ii).

(c) Let $X \xrightarrow{\sim} I$ be a K-injective resolution. For all $T \in \mathcal{T}_\mathfrak{a}$ we have the isomorphism

$$\operatorname{Hom}_R(T, \operatorname{Hom}_R(P, I)) \cong \operatorname{Hom}_R(P, \operatorname{Hom}_R(T, I))$$

by adjointness and commutativity. If $X \in \mathcal{G}_{\underline{x}}^{L\check{C}}$, then $\operatorname{Hom}_R(T, I)$ is exact by 6.4.3 condition (iii). Hence the complex $\operatorname{Hom}_R(T, \operatorname{Hom}_R(P, I)) \cong \operatorname{Hom}_R(P, \operatorname{Hom}_R(T, I))$ is exact too and we obtain the first assertion in (c) with 6.4.3 again and the statement in (b). Note the quasi-isomorphism $\operatorname{Hom}_R(P, X) \xrightarrow{\sim} \operatorname{Hom}_R(P, I)$. The second assertion is a direct consequence of the definition of the class $\mathcal{G}_{\underline{x}}^{R\check{C}}$ (see 6.4.6 (ii)).

(d) This follows by 6.4.3, condition (iv), and 6.4.6, condition (iii). □

Starting with a complex in $\mathcal{G}_{\underline{x}}^{R\check{C}}$ we may obtain a complex in $\mathcal{G}_{\underline{x}}^{L\check{C}}$.

Proposition 6.4.9 *Let $\underline{x} = x_1, \ldots, x_k$ be a sequence of elements in a commutative ring R and let X be an R-complex.*

(a) Suppose that $X \in \mathcal{G}_{\underline{x}}^{R\check{C}}$. Then $\operatorname{Hom}_R(X, I) \in \mathcal{G}_{\underline{x}}^{L\check{C}}$ for any K-injective complex I.
(b) $X \in \mathcal{G}_{\underline{x}}^{R\check{C}}$ if and only if $X^{\vee} \in \mathcal{G}_{\underline{x}}^{L\check{C}}$.

Proof Let $X \in \mathcal{G}_{\underline{x}}^{R\check{C}}$ and let T be any complex in $\mathcal{T}_{\mathfrak{a}}$ with one of its K-projective resolutions $P \xrightarrow{\sim} T$. The complex $P \otimes_R X$ is exact (see 6.4.6). Hence the complex $\operatorname{Hom}_R(P \otimes_R X, I) \cong \operatorname{Hom}_R(P, \operatorname{Hom}_R(X, I))$ is exact too. This means that $\operatorname{Hom}_R(X, I) \in \mathcal{G}_{\underline{x}}^{L\check{C}}$ in view of 6.4.3.

The statement in (b) follows by taking for I an injective cogenerator E and by the faithful exactness of the general Matlis duality functor $(\cdot)^{\vee} = \operatorname{Hom}_R(\cdot, E)$. □

For another relation between the classes $\mathcal{G}_{\underline{x}}^{R\check{C}}$ and $\mathcal{G}_{\underline{x}}^{L\check{C}}$, see the forthcoming 6.5.6. Complexes in $\mathcal{G}_{\underline{x}}^{L\check{C}}$ or in $\mathcal{G}_{\underline{x}}^{R\check{C}}$ are also characterized by their (co-)homology.

Theorem 6.4.10 *Let $\underline{x} = x_1, \ldots, x_k$ be a sequence in a commutative ring R. Let X denote an R-complex.*

(a) $X \in \mathcal{G}_{\underline{x}}^{L\check{C}}$ if and only if $H_i(X) \in \mathcal{G}_{\underline{x}}^{L\check{C}}$ for all $i \in \mathbb{Z}$.
(b) $X \in \mathcal{G}_{\underline{x}}^{R\check{C}}$ if and only if $H^i(X) \in \mathcal{G}_{\underline{x}}^{R\check{C}}$ for all $i \in \mathbb{Z}$.

Proof For the proof of (a) let $X \in \mathcal{G}_{\underline{x}}^{L\check{C}}$ and therefore $H_i(X) \cong H_i(\operatorname{RHom}_R(\check{C}_{\underline{x}}, X))$ for all $i \in \mathbb{Z}$. By Theorem 6.3.2 we know that

$$\operatorname{Ext}_R^j(\oplus_{l=1}^k R_{x_l}, H_i(\operatorname{RHom}_R(\check{C}_{\underline{x}}, X))) = 0 \text{ for all } j \geq 0.$$

Whence $H_i(X) \in \mathcal{G}_{\underline{x}}^{L\check{C}}$ by Theorem 6.4.3.

Conversely, assume that $H_i(X) \in \mathcal{G}_{\underline{x}}^{L\check{C}}$ for all $i \in \mathbb{Z}$. Let $X \xrightarrow{\sim} I$ be a DG-injective resolution of X, so that $H_i(X) \cong H_i(I)$. We split the complex I into short exact sequences of modules

$$0 \to B_i \to Z_i \to H_i(X) \to 0 \text{ and } 0 \to Z_i \to I_i \to B_{i-1} \to 0,$$

where $B_i = \operatorname{Im}(d_{i+1}^I)$ and $Z_i = \operatorname{Ker}(d_i^I)$. Now consider the associated long exact sequences of the $\operatorname{Ext}_R^i(R_x, \cdot)$ for any $x \in \operatorname{Rad}(\underline{x}R)$. Note first that

$$\operatorname{Hom}_R(R_x, H_i(X)) = 0 = \operatorname{Ext}^1_R(R_x, H_i(X))$$

by assumption and Theorem 6.4.3. From the short exact sequences on the right we first obtain $\operatorname{Ext}^1_R(R_x, B_{i-1}) = 0$ for all $i \in \mathbb{Z}$. To this end note that I_i is an injective R-module and R_x has projective dimension 1. From the exact sequences on the left we then obtain

$$\operatorname{Hom}_R(R_x, B_i) \cong \operatorname{Hom}_R(R_x, Z_i) \text{ and } \operatorname{Ext}^1_R(R_x, B_i) \cong \operatorname{Ext}^1_R(R_x, Z_i) = 0.$$

Hence our data induce short exact sequences

$$0 \to \operatorname{Hom}_R(R_x, B_i) \to \operatorname{Hom}_R(R_x, Z_i) \to 0 \text{ and}$$
$$0 \to \operatorname{Hom}_R(R_x, Z_i) \to \operatorname{Hom}_R(R_x, I_i) \to \operatorname{Hom}_R(R_x, B_{i-1}) \to 0.$$

We patch these together and obtain that the complex $\operatorname{Hom}_R(R_x, I)$ is exact, that is, $\operatorname{RHom}_R(R_x, X) \simeq 0$. As this holds for all $x \in \operatorname{Rad}(\underline{x}R)$ we have that $X \in \mathcal{G}^{L\check{C}}_{\underline{x}}$ by Theorem 6.4.3.

The statement in (b) is a direct consequence of Theorem 6.4.6, condition (v). □

In the following we investigate some other relations between the classes

$$\mathcal{G}^{L\check{C}}_{\underline{x}}, \; \mathcal{G}^{R\check{C}}_{\underline{x}} \; \text{ and } \; \mathcal{T}_{\mathfrak{a}},$$

where $\mathfrak{a} = \underline{x}R$. It provides a certain kind of converse to 6.4.3 and to 6.4.6 and provides yet another characterization of complexes in $\mathcal{T}_{\mathfrak{a}}$.

Proposition 6.4.11 *Let $\mathfrak{a}$ be an ideal of a commutative ring R generated by the sequence $\underline{x} = x_1, \ldots, x_k$ and let Y be an R-complex. The following conditions are equivalent:*

(i) $\operatorname{RHom}_R(Y, X) \simeq 0$ *for every R-complex $X \in \mathcal{G}^{L\check{C}}_{\underline{x}}$,*
(ii) $Y \in \mathcal{T}_{\mathfrak{a}}$*, and*
(iii) $Y \otimes^{\mathrm{L}}_R Z \simeq 0$ *for every complex $Z \in \mathcal{G}^{R\check{C}}_{\underline{x}}$.*

Proof When condition (i) is satisfied we have $\operatorname{RHom}_R(Y, (R/\mathfrak{a})^{\vee}) \simeq 0$ because $(R/\mathfrak{a})^{\vee} \in \mathcal{G}^{L\check{C}}_{\underline{x}}$ (see 6.4.4). Let $P \xrightarrow{\sim} Y$ be a K projective resolution of Y. This means that $\operatorname{Hom}_R(P, (R/\mathfrak{a})^{\vee})$ is exact. By adjointness we have that $(P \otimes_R R/\mathfrak{a})^{\vee}$ is exact and so is $P \otimes_R R/\mathfrak{a}$. Hence $Y \in \mathcal{T}_{\mathfrak{a}}$ by definition.

When condition (iii) is satisfied we have $Y \otimes^{\mathrm{L}}_R R/\mathfrak{a} \simeq 0$ because $R/\mathfrak{a} \in \mathcal{G}^{R\check{C}}_{\underline{x}}$ (see 6.4.7). By the definition it yields that $Y \in \mathcal{T}_{\mathfrak{a}}$.

The implications (ii) ⇒ (i) and (ii) ⇒ (iii) have already been proved in 6.4.3 and 6.4.6. □

We now consider two sequences $\underline{x}$ and $\underline{y}$ and investigate the relations between the corresponding classes.

Proposition 6.4.12 *Let* $\underline{x} = x_1, \ldots, x_k$ *and* $\underline{y} = y_1, \ldots, y_l$ *be two sequences of elements in a commutative ring* R *such that* $\operatorname{Rad}(\underline{x}R) \subseteq \operatorname{Rad}(\underline{y}R)$. *Then*

$$\mathcal{G}_{\underline{y}}^{L\check{C}} \subseteq \mathcal{G}_{\underline{x}}^{L\check{C}} \quad \textit{and} \quad \mathcal{G}_{\underline{y}}^{R\check{C}} \subseteq \mathcal{G}_{\underline{x}}^{R\check{C}}.$$

Proof For any $X \in \mathcal{G}_{\underline{y}}^{L\check{C}}$ we have

$$\operatorname{RHom}_R(R_z, X) \simeq 0 \text{ for all } z \in \operatorname{Rad}(\underline{y}R)$$

(see 6.4.3 applied to sequence $\underline{y}$). Because $\operatorname{Rad}(\underline{x}R) \subseteq \operatorname{Rad}(\underline{y}R)$ this also holds for all $z \in \operatorname{Rad}(\underline{x}R)$. This provides the first inclusion by 6.4.3. The second one is proved in the same way, using 6.4.6 in place of 6.4.3. □

Remark 6.4.13 In the situation of 6.4.12 recall that the modules in $\mathcal{G}_{\underline{y}}^{R\check{C}}$ are exactly the $\underline{y}R$-torsion modules (see 6.4.7). The inclusion on the right may be viewed as a large generalization of the following rather obvious fact: if an R-module M is $\underline{y}R$-torsion it is also $\underline{x}R$-torsion when $\operatorname{Rad}(\underline{x}R) \subseteq \operatorname{Rad}(\underline{y}R)$.

Recall also the modules in $\mathcal{G}_{\underline{y}}^{L\check{C}}$ are exactly the $\underline{y}R$-pseudo-complete modules when the ring is Noetherian (see 6.4.4 (c)). In that case, the inclusion on the left provides a generalization of 2.5.9.

For two arbitrary sequences we note the following.

Proposition 6.4.14 *Let* $\underline{x} = x_1, \ldots, x_k$ *and* $\underline{y} = y_1, \ldots, y_l$ *be two sequences of elements in a commutative ring* R *and let us form the sequence* $\underline{x}, \underline{y}$. *We have the equalities*

$$\mathcal{G}_{\underline{x},\underline{y}}^{L\check{C}} = \mathcal{G}_{\underline{x}}^{L\check{C}} \cap \mathcal{G}_{\underline{y}}^{L\check{C}} \textit{ and } \mathcal{G}_{\underline{x},\underline{y}}^{R\check{C}} = \mathcal{G}_{\underline{x}}^{R\check{C}} \cap \mathcal{G}_{\underline{y}}^{R\check{C}}.$$

Proof This is a direct consequence of condition (v) in 6.4.3 and of condition (iv) in 6.4.6. □

6.5 Composites

In this section we shall investigate the composite of the Čech homology and cohomology functors for various sequences of elements. On the way we shall prove an extension of the known fact that the functors $\check{C}_{\underline{x}} \otimes_R^{\mathrm{L}} \cdot$ and $\operatorname{RHom}_R(\check{C}_{\underline{x}}, \cdot)$ are idempotent in the derived category.

Observations 6.5.1 Let $\underline{x} = x_1, \ldots, x_k$ and $\underline{y} = y_1, \ldots, y_l$ be two sequences of elements in a commutative ring R. We form the sequences $\underline{x}, \underline{y} = x_1, \ldots, x_k, y_1, \ldots, y_l$ and $\underline{y}, \underline{x} = y_1, \ldots, y_l, x_1, \ldots, x_k$. Then recall that $\check{C}_{\underline{y}} \otimes_R \check{C}_{\underline{x}} \cong \check{C}_{\underline{y},\underline{x}} \cong \check{C}_{\underline{x},\underline{y}}$ by construction.

(a) For any R-complex X we have natural isomorphisms

$$\check{C}_{\underline{y}} \otimes^{\mathrm{L}}_R (\check{C}_{\underline{x}} \otimes^{\mathrm{L}}_R X) \xrightarrow{\sim} \check{C}_{\underline{y},\underline{x}} \otimes^{\mathrm{L}}_R X \text{ and}$$
$$\mathrm{RHom}_R(\check{C}_{\underline{y}}, \mathrm{RHom}_R(\check{C}_{\underline{x}}, X)) \xrightarrow{\sim} \mathrm{RHom}_R(\check{C}_{\underline{y},\underline{x}}, X)$$

in the derived category. This follows by associativity, respectively by adjointness.

(b) If $\mathrm{Rad}(\underline{x}R) \subseteq \mathrm{Rad}(\underline{y}R)$, then we have the quasi-isomorphisms

$$\check{C}_{\underline{y}} \otimes^{\mathrm{L}}_R (\check{C}_{\underline{x}} \otimes^{\mathrm{L}}_R X) \xrightarrow{\sim} \check{C}_{\underline{y}} \otimes^{\mathrm{L}}_R X \text{ and}$$
$$\mathrm{RHom}_R(\check{C}_{\underline{y}}, \mathrm{RHom}_R(\check{C}_{\underline{x}}, X)) \xrightarrow{\sim} \mathrm{RHom}_R(\check{C}_{\underline{y}}, X)$$

in the derived category. This follows from the quasi-isomorphism of complexes $\check{C}_{\underline{y},\underline{x}} \xrightarrow{\sim} \check{C}_{\underline{y}}$ obtained in 6.1.9.

(c) Putting $\underline{x} = \underline{y}$ in (b) we see that the functors

$$(\check{C}_{\underline{x}} \otimes^{\mathrm{L}} \cdot) \text{ and } \mathrm{RHom}_R(\check{C}_{\underline{x}}, \cdot)$$

are idempotent in the derived category.

Observations 6.5.2 (a) We note that the natural isomorphisms from 6.5.1 are naturally compatible with the natural map $\iota(\check{C}_{\underline{x}}; X)$ and $\alpha(\check{C}_{\underline{x}}; X)$ defined in 6.4.1. Recall that the natural map $\iota(\check{C}_{\underline{x}}; X)$ is represented by the natural morphism $\check{L}_{\underline{x}} \otimes_R X \to X$ and that the natural map $\alpha(\check{C}_{\underline{x}}; X)$ is represented by the natural morphism $X \to \mathrm{Hom}_R(\check{L}_{\underline{x}}, X)$. At the complex level we also have commutative triangles

$$\begin{tikzcd}
\check{L}_{\underline{y},\underline{x}} \otimes_R X \cong \check{L}_{\underline{y}} \otimes_R (\check{L}_{\underline{x}} \otimes_R X) \arrow[rr] \arrow[dr] & & \check{L}_{\underline{x}} \otimes_R X \arrow[dl] \\
& X &
\end{tikzcd}$$

and

$$\begin{tikzcd}
& X \arrow[dl] \arrow[dr] & \\
\mathrm{Hom}_R(\check{L}_{\underline{x}}, X) \arrow[rr] & & \mathrm{Hom}_R(\check{L}_{\underline{y}}, \mathrm{Hom}_R(\check{L}_{\underline{x}}, X)) \cong \mathrm{Hom}_R(\check{L}_{\underline{y},\underline{x}}, X)
\end{tikzcd}$$

where all the morphisms are the natural ones.

(b) Assume now that $\mathrm{Rad}(\underline{x}R) \subseteq \mathrm{Rad}(\underline{y}R)$. Then we have quasi-isomorphisms $\check{L}_{\underline{y}} \xrightarrow{\sim} \check{L}_{\underline{y},\underline{x}} \xrightarrow{\sim} \check{L}_{\underline{y}}$ (see 6.2.4), hence also quasi-isomorphisms

$$\check{L}_{\underline{y}} \otimes_R X \xrightarrow{\sim} \check{L}_{\underline{y},\underline{x}} \otimes_R X \quad \text{and} \quad \mathrm{Hom}_R(\check{L}_{\underline{y},\underline{x}}, X) \xrightarrow{\sim} \mathrm{Hom}_R(\check{L}_{\underline{y}}, X).$$

The reader is invited to insert these quasi-isomorphisms in the above commutative triangles.

We are now prepared for results relating Čech cohomology and Čech homology, that is, results involving both functors $(\check{C}_{\underline{x}} \otimes_R \cdot)$ and $\mathrm{RHom}_R(\check{C}_{\underline{x}}, \cdot)$. To this end recall the free resolution $\check{L}^{\flat}{}_{\underline{x}} \xrightarrow{\sim} \check{D}_{\underline{x}}$ of the shifted global Čech complex as introduced in 6.2.6.

Lemma 6.5.3 *Let $\underline{x} = x_1, \ldots, x_k$ be a sequence of elements in a commutative ring R and write $\mathfrak{a} = \underline{x}R$. Let X be any R-complex and I any K-injective R-complex. Then*

(a) $\check{L}^{\flat}{}_{\underline{x}} \otimes_R X \xrightarrow{\sim} \check{D}_{\underline{x}} \otimes_R X \in \mathcal{T}_{\mathfrak{a}}$,
(b) $\mathrm{Hom}_R(\check{D}_{\underline{x}}, I) \xrightarrow{\sim} \mathrm{Hom}_R(\check{L}^{\flat}{}_{\underline{x}}, I) \in \mathcal{T}_{\mathfrak{a}}$.

Proof (a) It will be enough to prove the statement for $\check{D}_{\underline{x}} \otimes_R X$. Let $F \xrightarrow{\sim} X$ be a K-flat resolution of X. Then $\check{D}_{\underline{x}} \otimes_R F \xrightarrow{\sim} \check{D}_{\underline{x}} \otimes_R X$ is a K-flat resolution of $\check{D}_{\underline{x}} \otimes_R X$. To prove the assertion we have to show that $R/\mathfrak{a} \otimes_R \check{D}_{\underline{x}} \otimes_R F \simeq 0$. But this is obvious since $R/\mathfrak{a} \otimes_R \check{D}_{\underline{x}} \cong 0$.

(b) Again it suffices to show the claim for $\mathrm{Hom}_R(\check{D}_{\underline{x}}, I)$. This complex is K-injective and $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{Hom}_R(\check{D}_{\underline{x}}, I)) \cong \mathrm{Hom}_R(R/\mathfrak{a} \otimes_R \check{D}_{\underline{x}}, I)$ is exact because $R/\mathfrak{a} \otimes_R \check{D}_{\underline{x}} \cong 0$. Hence $\text{E-dp}(\mathfrak{a}, \mathrm{Hom}_R(\check{D}_{\underline{x}}, I)) = \infty$ and $\mathrm{Hom}_R(\check{D}_{\underline{x}}, I) \in \mathcal{T}_{\mathfrak{a}}$ by 5.3.5. □

Lemma 6.5.4 *Let $\underline{x} = x_1, \ldots, x_k$ and $\underline{y} = y_1, \ldots, y_l$ denote two sequences of elements in a commutative ring R, write $\mathfrak{a} = \underline{x}R$ and $\mathfrak{b} = \underline{y}R$. Suppose* $\mathrm{Rad}(\mathfrak{a}) \subseteq \mathrm{Rad}(\mathfrak{b})$ *and let X denote an R-complex. Then the following natural morphisms are quasi-isomorphisms.*

(a) $\mathrm{Hom}_R(\check{L}_{\underline{y}}, \check{C}_{\underline{x}} \otimes_R X) \to \mathrm{Hom}_R(\check{L}_{\underline{y}}, X)$.
(b) $\mathrm{Hom}_R(\check{L}_{\underline{y}}, \check{L}_{\underline{x}} \otimes_R X) \to \mathrm{Hom}_R(\check{L}_{\underline{y}}, X)$.
(c) $\check{C}_{\underline{y}} \otimes_R X \to \check{C}_{\underline{y}} \otimes_R \mathrm{Hom}_R(\check{L}_{\underline{x}}, X)$.
(d) $\check{L}_{\underline{y}} \otimes_R X \to \check{L}_{\underline{y}} \otimes_R \mathrm{Hom}_R(\check{L}_{\underline{x}}, X)$.

Proof (a) We apply the functor $\mathrm{Hom}_R(\check{L}_{\underline{y}}, \cdot)$ to the short exact sequence of complexes $0 \to \check{D}_{\underline{x}} \otimes_R X \to \check{C}_{\underline{x}} \otimes_R X \to X \to 0$ and obtain the short exact sequence

$$0 \to \mathrm{Hom}_R(\check{L}_{\underline{y}}, \check{D}_{\underline{x}} \otimes_R X) \to \mathrm{Hom}_R(\check{L}_{\underline{y}}, \check{C}_{\underline{x}} \otimes_R X) \to \mathrm{Hom}_R(\check{L}_{\underline{y}}, X) \to 0.$$

Therefore it will be sufficient to show that the first complex in the previous sequence is exact. We have seen in 6.5.3 that $\check{D}_{\underline{x}} \otimes_R X \in \mathcal{T}_{\mathfrak{a}}$. But $\mathcal{T}_{\mathfrak{a}} \subseteq \mathcal{T}_{\mathfrak{b}}$, as follows by 5.1.7.

Note that the hypothesis $\operatorname{Rad}(\mathfrak{a}) \subseteq \operatorname{Rad}(\mathfrak{b})$ implies that $\mathfrak{b}$ is $\mathfrak{a}$-open because the ideal $\mathfrak{a}$ is finitely generated. Now we obtain by 6.3.4 that the complex $\operatorname{Hom}_R(\check{L}_{\underline{y}}, \check{D}_{\underline{x}} \otimes_R X)$ is exact.

(b) The quasi-isomorphism $\check{L}_{\underline{x}} \to \check{C}_{\underline{x}}$ induces a quasi-isomorphism

$$\operatorname{Hom}_R(\check{L}_{\underline{y}}, \check{L}_{\underline{x}} \otimes_R X) \to \operatorname{Hom}_R(\check{L}_{\underline{y}}, \check{C}_{\underline{x}} \otimes_R X)$$

because $\check{L}_{\underline{x}}$ is DG-projective and DG-flat. Whence (b) follows from (a).

(c) We use the short exact sequence $0 \to \check{L^\flat}_{\underline{x}} \to \check{L}_{\underline{x}} \to R \to 0$ as described in 6.2.6. This is a sequence of bounded free complexes and gives rise to a short exact sequence $0 \to X \to \operatorname{Hom}_R(\check{L}_{\underline{x}}, X) \to \operatorname{Hom}_R(\check{L^\flat}_{\underline{x}}, X) \to 0$. By tensoring with $\check{C}_{\underline{y}}$ we obtain the short exact sequence of complexes

$$0 \to \check{C}_{\underline{y}} \otimes_R X \to \check{C}_{\underline{y}} \otimes_R \operatorname{Hom}_R(\check{L}_{\underline{x}}, X) \to \check{C}_{\underline{y}} \otimes_R \operatorname{Hom}_R(\check{L^\flat}_{\underline{x}}, X) \to 0.$$

We need prove that the last complex in the sequence is exact. To this end, let $X \xrightarrow{\sim} I$ denote a K-injective resolution. Then

$$\operatorname{Hom}_R(\check{L^\flat}_{\underline{x}}, X) \xrightarrow{\sim} \operatorname{Hom}_R(\check{L^\flat}_{\underline{x}}, I) \in \mathcal{T}_{\mathfrak{a}},$$

(see 6.5.3). As $\mathcal{T}_{\mathfrak{a}} \subseteq \mathcal{T}_{\mathfrak{b}}$ we obtain with 6.3.4 that the complex $\check{C}_{\underline{y}} \otimes_R \operatorname{Hom}_R(\check{L^\flat}_{\underline{x}}, X)$ is exact.

(d) Because of the quasi-isomorphism $\check{L}_{\underline{y}} \to \check{C}_{\underline{y}}$, and with the help of 4.4.11, the claim in (d) is a consequence of (c). $\square$

Now we get the following theorem, which provides some more composites.

Theorem 6.5.5 *Let $\underline{x} = x_1, \ldots, x_k$ and $\underline{y} = y_1, \ldots, y_l$ be two sequences of elements in a commutative ring R. Suppose $\operatorname{Rad}(\underline{x}R) \subseteq \operatorname{Rad}(\underline{y}R)$ and let X denote an R-complex. Then we have the following isomorphisms*

$$\operatorname{RHom}_R(\check{C}_{\underline{y}}, \check{C}_{\underline{x}} \otimes^{\mathrm{L}}_R X) \xrightarrow{\sim} \operatorname{RHom}(\check{C}_{\underline{y}}, X) \text{ and}$$
$$\check{C}_{\underline{y}} \otimes^{\mathrm{L}}_R X \xrightarrow{\sim} \check{C}_{\underline{y}} \otimes^{\mathrm{L}}_R \operatorname{RHom}_R(\check{C}_{\underline{x}}, X)$$

in the derived category.

Proof The statements are a consequence of the results of 6.5.4. $\square$

Putting $\underline{x} = \underline{y}$ in the above theorem, and recalling that both functors $\operatorname{RHom}_R(\check{C}_{\underline{x}}, \cdot)$ and $(\check{C}_{\underline{x}} \otimes^{\mathrm{L}} \cdot)$ are idempotent in the derived category, we obtain one more relation between the classes $\mathcal{G}_{\underline{x}}^{L\check{C}}$ and $\mathcal{G}_{\underline{x}}^{R\check{C}}$ introduced in 6.4.3 and 6.4.6. Recall also that these classes are well defined in the derived category.

Corollary 6.5.6 *The functor* $\operatorname{RHom}_R(\check{C}_{\underline{x}}, \cdot)$ *induces a bijection*

$$\mathcal{G}_{\underline{x}}^{R\check{C}} \to \mathcal{G}_{\underline{x}}^{L\check{C}}$$

in the derived category. The inverse bijection is given by the functor $(\check{C}_{\underline{x}} \otimes^{\mathrm{L}} \cdot)$.

Remark 6.5.7 Note also the following particular case of 6.5.4 (b). There is a quasi-isomorphism

$$\operatorname{Hom}_R(\check{L}_{\underline{x}}, \check{L}_{\underline{x}}) \xrightarrow{\sim} \operatorname{Hom}_R(\check{L}_{\underline{x}}, R).$$

We shall investigate the endomorphism complex $\operatorname{Hom}_R(\check{L}_{\underline{x}}, \check{L}_{\underline{x}})$ in more detail later.

6.6 Depth, Codepth and Čech Complexes

In this section we investigate when the Čech homology and cohomology modules vanish. We provide some inequalities.We first show how the Ext-depth and the Tor-codepth of a complex can be computed with Čech cohomology and Čech homology respectively. That is, we prove the Ext-depth Tor-codepth sensitivity of the Čech complex. To this end, the following lemma will play a substantial rôle.

Lemma 6.6.1 *For any sequence* $\underline{x} = x_1, \ldots, x_k$ *in a commutative ring* R *and all* $t \geq 1$ *we have a quasi-isomorphism*

$$K^\bullet(\underline{x}^t) \otimes_R \check{C}_{\underline{x}} \xrightarrow{\sim} K^\bullet(\underline{x}^t).$$

Proof This a direct consequence of Theorem 6.1.8 applied to the Koszul complexes $K^\bullet(\underline{x}^t)$, the cohomology of which is annihilated by a power of $\mathfrak{a}$. □

Theorem 6.6.2 *Let* $\underline{x} = x_1, \ldots, x_k$ *be a sequence in a commutative ring* R *generating the ideal* $\mathfrak{a}$. *For every complex* X *we have*

$$\text{E-dp}(\mathfrak{a}, X) = \text{E-dp}(\mathfrak{a}, \check{C}_{\underline{x}} \otimes_R X) = \inf\{i \mid H^i(\check{C}_{\underline{x}} \otimes_R X) \neq 0\}.$$

Proof In the whole proof we keep in mind that

$$\text{E-dp}(\mathfrak{a}, X) = h^-(\underline{x}; X) = h^-(\underline{x}^t; X)$$

for all $t \geq 1$ and every complex X (see 5.3.3). The quasi-isomorphism of Lemma 6.6.1 is a quasi-isomorphism between bounded complexes of flat R-modules. Therefore it provides a quasi-isomorphism

$$K^\bullet(\underline{x}) \otimes_R \check{C}_{\underline{x}} \otimes_R X \xrightarrow{\sim} K^\bullet(\underline{x}) \otimes_R X$$

(see 4.4.11) and therefore

$$h^-(\underline{x}; X) = h^-(\underline{x}; \check{C}_{\underline{x}} \otimes_R X) \geq \inf\{i \mid H^i(\check{C}_{\underline{x}} \otimes_R X) \neq 0\}$$

by 5.3.4. We already have that $\operatorname{E-dp}(\mathfrak{a}, X) = \operatorname{E-dp}(\mathfrak{a}, \check{C}_{\underline{x}} \otimes_R X)$. When

$$\inf\{i \mid H^i(\check{C}_{\underline{x}} \otimes_R X) \neq 0\} = \infty$$

we then have the equality $h^-(\underline{x}; X) = \inf\{i \mid H^i(\check{C}_{\underline{x}} \otimes_R X) \neq 0\}$ and the statement is proved. Assume now that $\inf\{i \mid H^i(\check{C}_{\underline{x}} \otimes_R X) \neq 0\} = -\infty$. Then for all $j \in \mathbb{Z}$ there is an $i \leq j$ such that $H^i(\check{C}_{\underline{x}} \otimes_R X) \neq 0$, hence also such that $H^i(\underline{x}^t; X) \neq 0$ for some $t \geq 1$. Recall that $H^i(\check{C}_{\underline{x}} \otimes_R X) = \varinjlim H^i(\underline{x}^t; X)$. But $h^-(\underline{x}^t; X) = h^-(\underline{x}; X)$. It follows that $\operatorname{E-dp}(\mathfrak{a}, X) = h^-(\underline{x}; X) = -\infty$, which finishes the proof of the statement in this case.

Finally, assume that $\inf\{i \mid H^i(\check{C}_{\underline{x}} \otimes_R Y) \neq 0\} = r$ is finite. We obtain the equality $\operatorname{E-dp}(\mathfrak{a}, \check{C}_{\underline{x}} \otimes_R X) = \inf\{i \mid H^i(\check{C}_{\underline{x}} \otimes_R X) \neq 0\}$ by 5.15 since $\operatorname{Hom}_R(R/\mathfrak{a}, H^i(\check{C}_{\underline{x}} \otimes_R X)) \neq 0$ when $H^i(\check{C}_{\underline{x}} \otimes_R X) \neq 0$ (see 6.1.4). □

Theorem 6.6.3 *Let $\underline{x} = x_1, \ldots, x_k$ be a sequence in a commutative ring R generating the ideal $\mathfrak{a}$. Let X denote an R-complex with one of its K-injective resolutions $X \xrightarrow{\sim} I$. Then*

$$\operatorname{T-codp}(\mathfrak{a}, X) = \operatorname{T-codp}(\mathfrak{a}, \operatorname{Hom}_R(\check{C}_{\underline{x}}, I)) = \inf\{i \mid H_i(\operatorname{RHom}_R(\check{C}_{\underline{x}}, X)) \neq 0\}.$$

Proof Here we keep in mind that

$$\operatorname{T-codp}(\mathfrak{a}, X) = h_-(\underline{x}; X) = h_-(\underline{x}^t; X)$$

for all $t \geq 1$ and any R-complex X (see 5.3.3). Moreover, $\operatorname{RHom}_R(\check{C}_{\underline{x}}, X)$ is represented by $\operatorname{Hom}_R(\check{C}_{\underline{x}}, I)$ so that $H_i(\operatorname{RHom}_R(\check{C}_{\underline{x}}, X)) \cong H_i(\operatorname{Hom}_R(\check{C}_{\underline{x}}, I))$. From the quasi-isomorphism of Lemma 6.6.1 we obtain a quasi-isomorphism

$$\operatorname{Hom}_R(K^\bullet(\underline{x}), I) \xrightarrow{\sim} \operatorname{Hom}_R(K^\bullet(\underline{x}) \otimes_R \check{C}_{\underline{x}}, I).$$

By 5.2.3 we have $\operatorname{Hom}_R(K^\bullet(\underline{x}), I) \cong K_\bullet(\underline{x}) \otimes_R I$. By adjointness and 5.2.3 we also have $\operatorname{Hom}_R(K^\bullet(\underline{x}) \otimes_R \check{C}_{\underline{x}}, I) \cong K_\bullet(\underline{x}) \otimes_R \operatorname{Hom}_R(\check{C}_{\underline{x}}, I)$. Hence we have a quasi-isomorphism

$$K_\bullet(\underline{x}) \otimes_R I \xrightarrow{\sim} K_\bullet(\underline{x}) \otimes_R \operatorname{Hom}_R(\check{C}_{\underline{x}}, I).$$

As $h_-(\underline{x}; I) = h_-(\underline{x}; X)$ it follows that

$$h_-(\underline{x}; X) = h_-(\underline{x}; \operatorname{Hom}_R(\check{C}_{\underline{x}}, I)) \geq \inf\{i \mid H_i(\operatorname{Hom}_R(\check{C}_{\underline{x}}, I)) \neq 0\}$$

(see 5.3.4). We already have that $\operatorname{T-codp}(\mathfrak{a}, X) = \operatorname{T-codp}(\mathfrak{a}, \operatorname{Hom}_R(\check{C}_{\underline{x}}, I))$.

When $\inf\{i \mid H_i(\mathrm{RHom}_R(\check{C}_{\underline{x}}, X)) \neq 0\} = \infty$ we then have the equality $h_-(\underline{x}; X) = \inf\{i \mid H_i(\mathrm{RHom}_R(\check{C}_{\underline{x}}, X)) \neq 0\}$ and the statement is proved.

Assume that $\inf\{i \mid H_i(\mathrm{RHom}_R(\check{C}_{\underline{x}}, X)) \neq 0\} = -\infty$. Then for all $j \in \mathbb{Z}$ there is an $i \leq j$ such that $H_i(\mathrm{RHom}_R(\check{C}_{\underline{x}}, X)) \neq 0$. Hence we also get that $H_i(\underline{x}^t; X) \neq 0$ or $H_{i+1}(\underline{x}^t; X) \neq 0$ for some $t \geq 1$ in view of the short exact sequences in 6.3.2. But $h_-(\underline{x}^t; X) = h_-(\underline{x}; X)$. It follows that $\text{T-codp}(\mathfrak{a}, X) = h_-(\underline{x}; X) = -\infty$, which finishes the proof of the statement in this case too.

Finally, assume that $\inf\{i \mid H_i(\mathrm{Hom}_R(\check{C}_{\underline{x}}, I)) \neq 0\} = r$ is finite. We obtain the equality

$$\text{T-codp}(\mathfrak{a}, \mathrm{Hom}_R(\check{C}_{\underline{x}}, I)) = \inf\{i \mid H_i(\mathrm{Hom}_R(\check{C}_{\underline{x}}, I)) \neq 0\}$$

by 5.1.5 since $R/\mathfrak{a} \otimes_R H_i(\mathrm{Hom}_R(\check{C}_{\underline{x}}, I)) \neq 0$ when $H_i(\mathrm{Hom}_R(\check{C}_{\underline{x}}, I) \neq 0$ (see 6.3.2). □

Remark 6.6.4 Note that Theorem 6.6.2 may also be deduced from Theorem 6.6.3 applied to the general Matlis dual complex $X^\vee$. Indeed, $\text{E-dp}(\mathfrak{a}, X) = \text{T-codp}(\mathfrak{a}, X^\vee)$ when $\mathfrak{a}$ is finitely generated and $\mathrm{RHom}_R(\check{C}_{\underline{x}}, X^\vee) \simeq (\check{C}_{\underline{x}} \otimes^{\mathrm{L}}_R X)^\vee$ by adjointness.

The vanishing of the Čech homology modules and the vanishing of the Čech cohomology modules are interrelated by the following

Theorem 6.6.5 *Let $\varphi : R \to B$ be a homomorphism of commutative rings. Let $\underline{x} = x_1, \ldots, x_k$ be a sequence of elements in R and let $\underline{y} = y_1, \ldots, y_l$ be a sequence in B such that* $\mathrm{Rad}(\underline{y}B) = \mathrm{Rad}(\varphi(\underline{x})B)$, *$l \leq k$. Let X be a B-complex, which may be viewed as an R complex via φ. Then*

(a) $\sup\{i \mid H^i(\check{C}_{\underline{x}} \otimes_R X) \neq 0\} \leq l - \inf\{i \mid H_i(\mathrm{RHom}_R(\check{C}_{\underline{x}}, X)) \neq 0\}$ *and*
(b) $\sup\{i \mid H_i(\mathrm{RHom}_R(\check{C}_{\underline{x}}, X)) \neq 0\} \leq l - \inf\{i \mid H^i(\check{C}_{\underline{x}} \otimes_R X) \neq 0\}$.

Proof (a): We have seen that $\text{T-codp}(\underline{x}R, X) = \inf\{i \mid H_i(\mathrm{RHom}_R(\check{C}_{\underline{x}}, X) \neq 0))\}$ (see 6.6.3). Because $K_\bullet(\underline{x}^t) \otimes_R X \cong K_\bullet(\varphi(\underline{x}^t)) \otimes_B X$ and because of the Tor-codepth sensitivity of the Koszul complex (see 5.3.3) it yields that $\text{T-codp}(\underline{x}R, X) = h_-(\underline{x}; X) = h_-(\varphi(\underline{x}); X) = h_-(\underline{y}; X)$.

Assume first that $\text{T-codp}(\underline{x}R, X) = h_-(\underline{y}; X)$ is finite. Then $h^+(\underline{y}; X)$ is also finite,

$$h^+(\underline{y}; X) = l - h_-(\underline{y}; X)$$

and $h^+(\underline{y}; X) = h^+(\underline{y}^t; X)$ for all $t > 0$ (see 5.3.8 (a)). Then note that $\check{C}_{\underline{x}} \otimes_R X \cong \check{C}_{\varphi(\underline{x})} \otimes_B X \simeq \check{C}_{\underline{y}} \otimes_B X$ (see 6.1.9). Recall now that $H^i(\check{C}_{\underline{y}} \otimes_B X) = \varinjlim H^i(\underline{y}^t; X)$. As the $h^+(\underline{y}^t; X)$ are all equal for all $t \geq 1$ we observe that

$$\sup\{i \mid H^i(\check{C}_{\underline{x}} \otimes_R X) \neq 0\} = \sup\{i \mid H^i(\check{C}_{\underline{y}} \otimes_B X) \neq 0\} \leq h^+(\underline{y}; X).$$

The inequality in (a) follows from the above.

If T-codp$(\underline{x}R, X) = h_-(\underline{y}; X) = \infty$ the complexes $K_\bullet(\underline{x}^t; X)$ are exact and so are the complexes $K^\bullet(\underline{x}^t; X)$ by the self-duality of the Koszul complex. It follows that the complex $\check{C}_{\underline{x}} \otimes_R X$ is exact so that $\sup\{i \mid H^i(\check{C}_{\underline{x}} \otimes_R X) \neq 0\} = -\infty$ and the claim is obvious.

If T-codp$(\underline{x}R, X) - \infty$ then also $\inf\{i \mid H_i(\mathrm{RHom}_R(\check{C}_{\underline{x}}, X)) \neq 0\} = -\infty$ (see 6.6.3) and in this case again the claim is obvious.

This finishes the proof of (a).

(b): We have seen that E-dp$(\underline{x}R, X) = \inf\{i \mid H_i(\check{C}_{\underline{x}} \otimes_R X) \neq 0))\}$ (see 6.6.2). Because $K^\bullet(\underline{x}^t) \otimes_R X \cong K^\bullet(\varphi(\underline{x}^t)) \otimes_B X$ and because of the Ext-depth sensitivity of the Koszul complex it yields E-dp$(\underline{x}R, X) = h^-(\underline{x}; X) = h^-(\varphi(\underline{x}); X) = h^-(\underline{y}; X)$.

Assume first that E-dp$(\underline{x}R, X) = h^-(\underline{y}; X)$ is finite. Then $h_+(\underline{y}; X)$ is also finite,

$$h_+(\underline{y}; X) = l - h^-(\underline{y}; X)$$

and $h_+(\underline{y}; X) = h_+(\underline{y}^t; X)$ for all $t > 0$ (see 5.3.8 (b)). We recall the isomorphisms $H_i(\mathrm{RHom}_R(\check{C}_{\underline{x}}, X)) \cong H_i(\mathrm{RHom}_B(\check{C}_{\underline{y}}, X))$ (see 6.3.5). We also recall the short exact sequences in 6.3.2 (for the B-complex X and the sequence $\underline{y}$). As the $h_+(\underline{y}^t; X)$ are all equal we observe that

$$\sup\{i \mid H_i(\mathrm{RHom}_R(\check{C}_{\underline{x}}, X)) \neq 0\} = \sup\{i \mid H_i(\mathrm{RHom}_B(\check{C}_{\underline{y}}, X)) \neq 0\} \leq h_+(\underline{y}; X)$$

and the inequality in (b) follows.

If E-dp$(\underline{x}R, X) = \infty$ then also T-codp$(\mathfrak{a}, X) = \infty$ (see 5.3.5). In view of Theorem 6.6.3 it follows that $\inf\{i \mid H_i(\mathrm{RHom}_R(\check{C}_{\underline{x}}, X)) \neq 0\} = \infty$ and that $\mathrm{RHom}_R(\check{C}_{\underline{x}}, X) \simeq 0$, so that $\sup\{i \mid H_i(\mathrm{RHom}_R(\check{C}_{\underline{x}}, X)) \neq 0\} = -\infty$. In this case the claim is obvious.

If E-dp$(\underline{x}R, X) = -\infty$ then also $\inf\{i \mid H^i(\check{C}_{\underline{x}} \otimes_R X) \neq 0\} = -\infty$ (see 6.6.2). In this case again the claim is obvious.

This finishes the proof of (b). □

Remark 6.6.6 In the above theorem note as in 6.6.4 that statement (a) may be deduced from statement (b) via the general Matlis duality.

Let $\underline{x} = x_1, \ldots, x_k$ denote a sequence of elements in a commutative ring R and let M be an R-module. Then $H^i(\check{C}_{\underline{x}} \otimes_R M) = 0$ and $H_i(\mathrm{RHom}_R(\check{C}_{\underline{x}}, M)) \cong H_i(\mathrm{Hom}_R(\check{L}_{\underline{x}}, M)) = 0$ for all $i > k$ since $\check{C}_{\underline{x}}$ and $\check{L}_{\underline{x}}$ are bounded by k. In fact, a little more is true.

Proposition 6.6.7 *Let $\underline{x} = x_1, \ldots, x_k$ be a sequence in a commutative ring R and let M be an R-module. Assume there is a sequence $\underline{y} = y_1, \ldots, y_l$, $l \leq k$, such that* $\mathrm{Rad}(\underline{x}R + \mathrm{Ann}_R M) = \mathrm{Rad}(\underline{y}R + \mathrm{Ann}_R M)$. *Then*

(a) $H^i(\check{C}_{\underline{x}} \otimes_R M) = 0$ *for all* $i > l -$ T-codp$(\underline{x}R, M)$ *and*
(b) $H_i(\mathrm{RHom}_R(\check{C}_{\underline{x}}, M)) = 0$ *for all* $i > l -$ E-dp$(\underline{x}R, M)$.

Moreover, if T-codp$(\underline{x}R, M)$ *is finite, (equivalently if* E-dp$(\underline{x}R, M)$ *is finite) then*

(c) $0 \leq$ T-codp$(\underline{x}R, M) \leq l -$ E-dp$(\underline{x}R, M)$ *and*
(d) $0 \leq$ E-dp$(\underline{x}R, M) \leq l -$ T-codp$(\underline{x}R, M)$.

Proof The first assertions are a direct consequence of the above Theorems.

The last ones follows by 5.3.9. Note that T-codp$(\underline{x}R, M) =$ T-codp$(\underline{y}R, M)$, that E-dp$(\underline{x}R, M) =$ E-dp$(\underline{y}R, M)$, and that these quantities are finite simultaneously. In that case, T-codp$(\underline{x}R, M) +$ E-dp$(\underline{x}R, M) =$ T-codp$(\underline{y}R, M) +$ E-dp$(\underline{y}R, M) \leq l$. For an R-module M note also that T-codp$(\underline{x}R, M) \geq 0$ and E-dp$(\underline{x}R, M) \geq 0$. □

We end the section with some more equalities involving the Ext-depth and the Tor-codepth of a complex.

Proposition 6.6.8 *Let $\underline{x} = x_1, \ldots, x_k$ be a sequence in a commutative ring R and let X be an R-complex with one of its K-injective resolution $X \xrightarrow{\sim} I$. Then*

$$\text{E-dp}(\mathfrak{a}, X) = \text{E-dp}(\mathfrak{a}, \operatorname{Hom}_R(\check{C}_{\underline{x}}, I)) \quad \textit{and} \quad \text{T-codp}(\mathfrak{a}, X) = \text{T-codp}(\mathfrak{a}, \check{C}_{\underline{x}} \otimes_R X).$$

Proof We have

$$\operatorname{Hom}_R(R/\mathfrak{a}, \operatorname{Hom}_R(\check{C}_{\underline{x}}, I)) \cong \operatorname{Hom}_R(R/\mathfrak{a} \otimes \check{C}_{\underline{x}}, I) \cong \operatorname{Hom}_R(R/\mathfrak{a}, I).$$

As the complex $\operatorname{Hom}_R(\check{C}_{\underline{x}}, I)$ is K-injective, the first equality follows.

For the second we take a K-flat resolution $F \xrightarrow{\sim} X$. Then we have the quasi-isomorphism $\check{C}_{\underline{x}} \otimes_R F \xrightarrow{\sim} \check{C}_{\underline{x}} \otimes_R X$, which provides a K-flat resolution of $\check{C}_{\underline{x}} \otimes_R X$. As $R/\mathfrak{a} \otimes_R \check{C}_{\underline{x}} \otimes_R F \cong R/\mathfrak{a} \otimes_R F$, the second equality follows. □

Chapter 7
Local Cohomology and Local Homology

Let R denote a commutative ring and $\mathfrak{a}$ an ideal of R. The $\mathfrak{a}$-torsion functor and the $\mathfrak{a}$-completion functor, defined respectively by $\Gamma_{\mathfrak{a}}(M) := \{m \in M \mid \mathfrak{a}^t m = 0 \text{ for some } t \in \mathbb{N}\}$ and $\Lambda^{\mathfrak{a}}(M) := \varprojlim(R/\mathfrak{a}^t \otimes_R M)$ for any R-module M, extends naturally to complexes. In this chapter we first recall that $\mathrm{L}\Lambda^{\mathfrak{a}}$ and $\mathrm{R}\Gamma_{\mathfrak{a}}$ are well defined in the derived category and fix some notations. Then we investigate when $\mathrm{L}\Lambda^{\mathfrak{a}}(X)$ and $\mathrm{R}\Gamma_{\mathfrak{a}}(X)$ vanish. For complexes homologically-bounded on the *good size* we obtain more precise results, previously known when the ring is Noetherian, possibly new in the present generality. We also provide a description of $\mathrm{L}\Lambda^{\mathfrak{a}}(X)$ and $\mathrm{R}\Gamma_{\mathfrak{a}}(X)$ in terms of microscope and telescope. These descriptions do not refer to the resolutions of X, which could be an advantage.

In the case when the ideal $\mathfrak{a}$ of R is generated by a particular finite sequence $\underline{x}$ of elements of R, that is, by a so-called weakly pro-regular sequence, the local homology and cohomology of any R-complex with respect to the ideal $\mathfrak{a}$ coincide respectively with its Čech homology and cohomology with respect to the sequence $\underline{x}$. For the local and Čech cohomology this a complex version of a result of Grothendieck. In this generality it was already observed in [2] and in [75, Theorem 3.2]. For the local and Čech homology it appears in [75, Theorem 4.5] in the case when the complex is bounded, in [1] under some additional hypotheses and in general in [64]. In Sects. 7.4 and 7.5 we revisit these results with somewhat different methods and add some remarks. We follow an idea of Greenlees and May and first show that the local cohomology and homology with respect to an ideal generated by a weakly pro-regular sequence can be described in terms of telescope and microscope of certain Koszul complexes. This will also allow us to extend results previously known for modules over a Noetherian ring to unbounded complexes in the more general setting of an ideal generated by a weakly pro-regular sequence.

In particular, in the last section we show how the Ext-depth and the Tor-codepth of a complex with respect to an ideal generated by a weakly pro-regular sequence may be computed with the local cohomology and homology with respect to that ideal. For complexes over a Noetherian ring this was originally proved by Foxby and Iyengar in [33, Theorem 2.1, Theorem 4.1]. Note, however, that for the local cohomology

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the module case was already in [82, 5.3.15] without any assumption on the ideal, while for the local homology the module case requires only that the ideal is finitely generated, see 3.4.2. Then we improve some bounds for the vanishing of the local homology and cohomology of a module. Finally, we explore the behaviour of the Ext-depth and Tor-codepth under $\mathrm{R}\Gamma$ and $\mathrm{L}\Lambda$.

7.1 The General Case

Let $\mathfrak{a}$ be an ideal of a commutative ring R. We first check that $\mathrm{L}\Lambda^{\mathfrak{a}}$ is well defined in the derived category. We shall prove a little more, we are going on to relate the microscope construction to the $\mathfrak{a}$-adic-completion functor $\Lambda^{\mathfrak{a}}$, also denoted by $\hat{(\cdot)}^{\mathfrak{a}}$, and to its left-derived. To this end we recall the results of 2.1.3 and 2.5.1: for an R-module M we have natural maps $\tau_M^{\mathfrak{a}}$, $\eta_M^{\mathfrak{a}}$ and $\gamma_M^{\mathfrak{a}}$ making the following triangle

$$\begin{tikzcd} & \Lambda_0^{\mathfrak{a}}(M) \arrow[dr, "\gamma_M^{\mathfrak{a}}"] & \\ M \arrow[ur, "\eta_M^{\mathfrak{a}}"] \arrow[rr, "\tau_M^{\mathfrak{a}}"] & & \hat{M}^{\mathfrak{a}} \end{tikzcd}$$

commutative and $\gamma_M^{\mathfrak{a}}$ is always surjective. Recall also 2.5.4: $\gamma_M^{\mathfrak{a}} = \tau_{\Lambda_0^{\mathfrak{a}}(M)}^{\mathfrak{a}}$ when the ideal $\mathfrak{a}$ is finitely generated.

7.1.1 *The inverse system* $\mathfrak{L}^{\mathfrak{a}}$. Let $\mathfrak{a}$ be an arbitrary ideal of a commutative ring R and let $\{\mathfrak{a}_t\}_{t\geq 0}$ be a descending sequence of ideals which form a base of open neighbourhoods of the origin for the $\mathfrak{a}$-adic topology of R. Let L_t be a free resolution of $R/\mathfrak{a}_t$ with $L_{t,0} = R$. The surjective homomorphism $R/\mathfrak{a}_{t+1} \to R/\mathfrak{a}_t$ can be lifted to a morphism $L_{t+1} \to L_t$. We choose such liftings and obtain an inverse system of complexes, denoted by $\mathfrak{L}^{\mathfrak{a}}$. (Of course, this $\mathfrak{L}^{\mathfrak{a}}$ depends on the choice of the $\mathfrak{a}_t$, the L_t and the liftings $L_{t+1} \to L_t$.)

Observations 7.1.2 We use the inverse system $\mathfrak{L}^{\mathfrak{a}}$ introduced in 7.1.1. Note first that the morphisms $h_t : R \to L_t$ defined by $h_{t,0} = \mathrm{id}_R$, where R is viewed as a complex concentrated in degree 0, induce a morphism of inverse systems $X \to \mathfrak{L}^{\mathfrak{a}} \otimes_R X$, where X is viewed as a constant inverse system. Hence we have a composite morphism $X \xrightarrow{\sim} \mathrm{Mic}(X) \to \mathrm{Mic}(\mathfrak{L}^{\mathfrak{a}} \otimes_R X)$ for any complex X (see also 4.2.4). Note that this morphism is functorial. That is, a morphism of R-complexes $X \to Y$ induces a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ \mathrm{Mic}(\mathfrak{L}^{\mathfrak{a}} \otimes_R X) \arrow[r] & \mathrm{Mic}(\mathfrak{L}^{\mathfrak{a}} \otimes_R Y). \end{tikzcd}$$

If the upper horizontal morphism is a quasi-isomorphism then the lower one is a quasi-isomorphism too (see 4.2.8).

Example 7.1.3 The quasi-isomorphisms $L_t \xrightarrow{\sim} R/\mathfrak{a}_t$ induce quasi-isomorphisms $\mathrm{Mic}(\mathfrak{L}^{\mathfrak{a}}) \xrightarrow{\sim} \mathrm{Mic}(\{R/\mathfrak{a}_t\}) \xleftarrow{\sim} \hat{R}^{\mathfrak{a}}$ (see 4.2.5 and 4.2.3). Hence $\hat{R}^{\mathfrak{a}} = H_0(\mathrm{Mic}(\mathfrak{L}^{\mathfrak{a}}))$ while $H_i(\mathrm{Mic}(\mathfrak{L}^{\mathfrak{a}})) = 0$ for all $i \neq 0$. This is a very particular case of a more general phenomenon as investigated in the following.

Proposition 7.1.4 *Let $\mathfrak{a}$ be an ideal of a commutative ring R. Let X be an arbitrary R-complex and $F \xrightarrow{\sim} X$ a K-flat resolution. Then there are quasi-isomorphisms making the following diagram commutative*

$$\begin{tikzcd}
F \arrow[r, equal] \arrow[d, "\tau_F^{\mathfrak{a}}"'] & F \arrow[r, equal] \arrow[d] & F \arrow[r, "\sim"] \arrow[d] & X \arrow[d] \\
\Lambda^{\mathfrak{a}}(F) \arrow[r, "\sim"] & \mathrm{Mic}(\{R/\mathfrak{a}^t \otimes_R F\}) & \mathrm{Mic}(\mathfrak{L}^{\mathfrak{a}} \otimes_R F) \arrow[l, "\sim"'] \arrow[r, "\sim"] & \mathrm{Mic}(\mathfrak{L}^{\mathfrak{a}} \otimes_R X)
\end{tikzcd}$$

where the vertical maps are the natural ones.

Proof The quasi-isomorphism

$$\Lambda^{\mathfrak{a}}(F) = \varprojlim F/\mathfrak{a}^t F \xrightarrow{\sim} \mathrm{Mic}(\{R/\mathfrak{a}^t \otimes_R F\})$$

follows by Lemma 4.2.3 (a). Note that the inverse system $\{F/\mathfrak{a}^t F\}$ is degree-wise surjective.

The quasi-isomorphism $\mathrm{Mic}(\mathfrak{L}^{\mathfrak{a}} \otimes_R F) \xrightarrow{\sim} \mathrm{Mic}(\mathfrak{L}^{\mathfrak{a}} \otimes_R X)$ follows by Proposition 4.2.8.

As F is K-flat we have an inverse system of quasi-isomorphisms $L_t \otimes_R F \xrightarrow{\sim} R/\mathfrak{a}^t \otimes_R F$. Hence the quasi-isomorphism in the middle follows by Lemma 4.2.5 (b). The commutativity of the diagram is rather obvious. □

The conclusion of the above is twofold: on one hand the homology of the complex $\mathrm{Mic}(\mathfrak{L}^{\mathfrak{a}} \otimes_R X)$ does not depend upon the particular choices made in the construction of $\mathfrak{L}^{\mathfrak{a}}$. On the other hand if $F \xrightarrow{\sim} X$ and $F' \xrightarrow{\sim} X$ are two K-flat resolutions of X, then $\Lambda^{\mathfrak{a}}(F)$ and $\Lambda^{\mathfrak{a}}(F')$ are quasi-isomorphic. More precisely, we have the following consequences.

Corollary 7.1.5 *Let $h : F \xrightarrow{\sim} F'$ be a quasi-isomorphism of K-flat complexes. Then $\Lambda^{\mathfrak{a}}(h)$ is also a quasi-isomorphism.*

Proof This is an immediate consequence of Proposition 7.1.4. Namely, put $X = F'$. □

By virtue of 4.4.12 we obtain the following, an extension to complexes of [38, Proposition 1.1] which was stated there in the case of modules.

Theorem 7.1.6 *Let $\mathfrak{a}$ be an ideal of a commutative ring R. Then the left-derived functor $\mathrm{L}\Lambda^{\mathfrak{a}}$ is well defined in the derived category: if $F \xrightarrow{\sim} X$ is any K-flat resolution of the R-complex X, then $\mathrm{L}\Lambda^{\mathfrak{a}}(X)$ is represented by the complex $\Lambda^{\mathfrak{a}}(F)$ and also by the complex $\mathrm{Mic}(\mathfrak{L}^{\mathfrak{a}} \otimes_R X)$, where $\mathfrak{L}^{\mathfrak{a}}$ is the inverse system described in 7.1.1.*

Moreover, there is a natural map $\alpha_X^{\mathfrak{a}} : X \to \mathrm{L}\Lambda^{\mathfrak{a}}(X)$ in the derived category, represented by the composite $X \xleftarrow{\sim} F \xrightarrow{\tau_F^{\mathfrak{a}}} \Lambda^{\mathfrak{a}}(F)$ and also by the natural morphism $X \to \mathrm{Mic}(\mathfrak{L}^{\mathfrak{a}} \otimes_R X)$.

The i^{th}-local homology module of the complex X is now given by

$$\Lambda_i^{\mathfrak{a}}(X) := H_i(\mathrm{L}\Lambda^{\mathfrak{a}}(X) \cong H_i(\Lambda^{\mathfrak{a}} F) \cong H_i(\mathrm{Mic}(\mathfrak{L}^{\mathfrak{a}} \otimes_R X))$$

for all $i \in \mathbb{Z}$.

Proof The existence of $\mathrm{L}\Lambda^{\mathfrak{a}}(X)$ is a direct consequence of Corollary 7.1.5 (see the remarks in 4.4.12). Strictly speaking $\mathrm{L}\Lambda^{\mathfrak{a}}(X)$ is represented by $\Lambda^{\mathfrak{a}}(P')$, where $P' \xrightarrow{\sim} X$ is any K-projective resolution of X. We take a K-projective resolution $P \xrightarrow{\sim} F$ of F and consider the composite $P \xrightarrow{\sim} F \xrightarrow{\sim} X$, noting that $\Lambda^{\mathfrak{a}}(P) \xrightarrow{\sim} \Lambda^{\mathfrak{a}}(F)$ (see 7.1.5). This together with 7.1.4 proves the first assertions. The last ones follow by the commutativity of the diagram in 7.1.4. □

Remark 7.1.7 Assume that the ideal $\mathfrak{a}$ of the commutative ring R is finitely generated. Then the homology modules $\Lambda_i^{\mathfrak{a}}(X)$ are $\mathfrak{a}$-pseudo-complete for all R-complexes X.

(Since $\Lambda_i^{\mathfrak{a}}(X) \cong H_i(\hat{F}^{\mathfrak{a}})$, where $F \xrightarrow{\sim} X$ is a K-flat resolution, as $\hat{F}^{\mathfrak{a}}$ is a complex of $\mathfrak{a}$-complete modules, this follows by 2.5.7.)

Remark 7.1.8 Let $F \xrightarrow{\sim} X$ be any K-flat resolution of the R-complex X. We use the notations from 7.1.1. As the inverse system $\{R/\mathfrak{a}_t\}$ is surjective the local homology modules $\Lambda_i^{\mathfrak{a}}(X) \cong H_i(\varprojlim F/\mathfrak{a}_t F)$ are inserted in the short exact sequences

$$0 \to \varprojlim{}^1 \operatorname{Tor}_{i+1}^R(R/\mathfrak{a}_t, X) \to \Lambda_i^{\mathfrak{a}}(X) \to \varprojlim \operatorname{Tor}_i^R(R/\mathfrak{a}_t, X) \to 0$$

(see 1.2.8). Note that the module case was first observed in [38] and recalled in 2.5.5.

Remark 7.1.9 (a) For an R-module M the natural map $\alpha_M^{\mathfrak{a}} : M \to \mathrm{L}\Lambda^{\mathfrak{a}}(M)$ and the natural morphism $M \to \mathrm{Mic}(\mathfrak{L}^{\mathfrak{a}} \otimes_R M)$ induces in homological degree 0 the natural homomorphism $\eta_M^{\mathfrak{a}} : M \to \Lambda_0^{\mathfrak{a}}(M)$ defined in 2.5.1. In other words, we have

$$H_0(\alpha_M^{\mathfrak{a}}(M)) = \eta_M^{\mathfrak{a}}.$$

This also follows from the commutative diagram in 7.1.4. Indeed, let $F \xrightarrow{\sim} M$ be a free resolution of M. The right-bounded complex F is K-flat and the natural map $\tau_F^{\mathfrak{a}} : F \to \Lambda^{\mathfrak{a}}(F)$ induces by definition the map $\eta_M^{\mathfrak{a}}$ in homological degree zero.

(b) Assume that the ring is Noetherian. If M is $\mathfrak{a}$-complete or more generally $\mathfrak{a}$-pseudo-complete, then the natural morphism

$$M \to \mathrm{Mic}(\mathfrak{L}^{\mathfrak{a}} \otimes_R M)$$

is a quasi-isomorphism. In other words, the natural map $\alpha_M^{\mathfrak{a}}(M))$ is an isomorphism in the derived category. (In this case, we know by 2.5.15 that $\Lambda_i^{\mathfrak{a}}(M) = 0$ for all $i \neq 0$. But we shall see later that this holds in greater generality (see the forthcoming 7.5.13).)

We now turn to the local cohomology side. For the existence of $\mathrm{R}\Gamma_{\mathfrak{a}}$ we take a more direct approach.

Lemma 7.1.10 *Let $\mathfrak{a}$ be an ideal of a commutative ring R and let $h : I \xrightarrow{\sim} I'$ be a quasi-isomorphism between K-injective R-complexes. The morphism h induces a quasi-isomorphism*

$$\Gamma_{\mathfrak{a}}(h) : \Gamma_{\mathfrak{a}}(I) \xrightarrow{\sim} \Gamma_{\mathfrak{a}}(I').$$

Proof The quasi-isomorphism $h : I \xrightarrow{\sim} I'$ yields a direct system of quasi-isomorphisms

$$\mathrm{Hom}_R(R/\mathfrak{a}^t, h) : \mathrm{Hom}_R(R/\mathfrak{a}^t, I) \xrightarrow{\sim} \mathrm{Hom}_R(R/\mathfrak{a}^t, I')$$

(see 4.4.11). By passing to the direct limit the conclusion follows since $\varinjlim$ is exact and commutes with cohomology. □

The above statement and the remarks in 4.4.12 provide the following result.

Theorem 7.1.11 *For an ideal $\mathfrak{a}$ of a commutative ring R the right-derived functor $\mathrm{R}\Gamma_{\mathfrak{a}}$ is well defined in the derived category: if $X \xrightarrow{\sim} I$ is a K-injective resolution of the R-complex X, then $\mathrm{R}\Gamma_{\mathfrak{a}}(X)$ is represented by the complex $\Gamma_{\mathfrak{a}}(I)$.*

Moreover, there is a natural map $\iota_{\mathfrak{a}}^X : \mathrm{R}\Gamma_{\mathfrak{a}}(X) \to X$ in the derived category, represented by the composite $\Gamma_{\mathfrak{a}}(I) \hookrightarrow I \xleftarrow{\sim} X$.

The ith local cohomology module of the complex X is now given by

$$H_{\mathfrak{a}}^i(X) := H^i(\mathrm{R}\Gamma_{\mathfrak{a}}(X) = H^i(\Gamma_{\mathfrak{a}}(I))$$

for all $i \in \mathbb{Z}$.

Remark 7.1.12 By definition $\Gamma_{\mathfrak{a}}(I) = \varinjlim \mathrm{Hom}_R(R/\mathfrak{a}^t, I)$. As $\varinjlim$ is exact and commutes with cohomology we have

$$H_{\mathfrak{a}}^i(X) = \varinjlim \mathrm{Ext}_R^i(R/\mathfrak{a}^t, X).$$

For an ideal $\mathfrak{a}$ of a commutative ring R and any R-complex X the local cohomology modules $H_{\mathfrak{a}}^i(X)$ are $\mathfrak{a}$-torsion: $\Gamma_{\mathfrak{a}}(H_{\mathfrak{a}}^i(X)) = H_{\mathfrak{a}}^i(X)$, hence supported in $V(\mathfrak{a})$.

Remark 7.1.13 Let M be an R-module. The natural map

$$\iota_{\mathfrak{a}}^{M} : \mathrm{R}\Gamma_{\mathfrak{a}}(M) \to M$$

induces in homological degree 0 the inclusion $\Gamma_{\mathfrak{a}}(M) \hookrightarrow M$.

If the ring R is Noetherian and if the R-module M is $\mathfrak{a}$-torsion, then the natural map $\iota_{\mathfrak{a}}^{M} : \mathrm{R}\Gamma_{\mathfrak{a}}(M) \to M$ is an isomorphism in the derived category. (In that case it is known that $H_{\mathfrak{a}}^{i}(M) = 0$ for all $i \neq 0$. See also the forthcoming 7.4.7 for a slightly more general result.)

We also have a description of $\mathrm{R}\Gamma_{\mathfrak{a}}$ in terms of telescope.

Observations 7.1.14 For any R-complex X we have the direct system

$$\{\mathrm{Hom}_R(L_t, X)\} := \mathrm{Hom}_R(\mathfrak{L}^{\mathfrak{a}}, X)$$

where $\mathfrak{L}^{\mathfrak{a}}$ is the inverse system described in 7.1.1, from which we take the notations. Note that the morphisms $h_t : R \to L_t$ induce a morphism of direct systems $\mathrm{Hom}_R(\mathfrak{L}^{\mathfrak{a}}, X) \to X$, where X is viewed as a constant direct system. Hence we have a composite morphism $\mathrm{Tel}(\mathrm{Hom}_R(\mathfrak{L}^{\mathfrak{a}}, X)) \to \mathrm{Tel}(X) \xrightarrow{\sim} X$ for any complex X (see also 4.3.2). Note that this composite morphism is functorial.

Putting 4.3.4 and 4.3.2 together with 7.1.11 we obtain the following result. Recall also that $\mathrm{Tel}(\mathrm{Hom}_R(R/\mathfrak{a}_t, I)) \xrightarrow{\sim} \varinjlim \mathrm{Hom}_R(R/\mathfrak{a}_t, I) \cong \Gamma_{\mathfrak{a}}(I)$.

Proposition 7.1.15 *Let $\mathfrak{a}$ denote an arbitrary ideal of a commutative ring R. Let $X \xrightarrow{\sim} I$ be any K-injective resolution of an R-complex X. With the notations in 7.1.1 we have direct systems of quasi-isomorphisms*

$$\mathrm{Hom}_R(L_t, X) \xrightarrow{\sim} \mathrm{Hom}_R(L_t, I) \xleftarrow{\sim} \mathrm{Hom}_R(R/\mathfrak{a}_t, I).$$

Then there are quasi-isomorphisms making the following diagram commutative

$$\begin{tikzcd}
\mathrm{Tel}(\mathrm{Hom}_R(\mathfrak{L}^{\mathfrak{a}}, X)) \arrow[r, "\sim"] \arrow[d] & \mathrm{Tel}(\mathrm{Hom}_R(\mathfrak{L}^{\mathfrak{a}}, I)) \arrow[d] & \mathrm{Tel}\{\mathrm{Hom}_R(R/\mathfrak{a}^t, I)\} \arrow[l, "\sim"'] \arrow[d, "\wr"] \\
X \arrow[r, "\sim"] & I & \Gamma_{\mathfrak{a}}(I) \arrow[l]
\end{tikzcd}$$

where the vertical maps are the natural ones. Up to a quasi-isomorphism

$$\mathrm{Tel}(\mathrm{Hom}_R(\mathfrak{L}^{\mathfrak{a}}, X))$$

depends only on X and not on the choice of $\mathfrak{L}^{\mathfrak{a}}$, and $\mathrm{Tel}(\mathrm{Hom}_R(\mathfrak{L}^{\mathfrak{a}}, X))$ *represents* $\mathrm{R}\Gamma_{\mathfrak{a}}(X)$ *in the derived category.*

Moreover, the natural map $\iota_{\mathfrak{a}}^{X} : \mathrm{R}\Gamma_{\mathfrak{a}}(X) \to X$ is also represented by the natural morphism $\mathrm{Tel}(\mathrm{Hom}_R(\mathfrak{L}^{\mathfrak{a}}, X)) \to X$.

We end this section with a final remark.

7.1.16 Let $\mathfrak{a}$ be a finitely generated ideal of a commutative ring R and $\underline{x} = x_1, \ldots, x_k$ be a sequence in R such that $\operatorname{Rad}(\mathfrak{a}) = \operatorname{Rad}(\underline{x}R)$. The ideals $\mathfrak{a}$ and $\underline{x}R$ are topologically equivalent, hence $\Lambda^{\mathfrak{a}} = \Lambda^{\underline{x}R}$ and $\Gamma_{\mathfrak{a}} = \Gamma_{\underline{x}R}$. As far as we are concerned with the local homology and cohomology with respect to the ideal $\mathfrak{a}$ we may as well assume that $\mathfrak{a} = \underline{x}R$.

7.2 First Vanishing Results with Applications to the Class $\mathcal{T}_{\mathfrak{a}}$

Of particular interest for local homology as well as local cohomology is its vanishing under certain circumstances. We start with a first approach to these questions in the general case. This provides further characterizations and some more properties of complexes in the class $\mathcal{T}_{\mathfrak{a}}$ introduced in 5.1.4.

Lemma 7.2.1 *Let $\mathfrak{a}$ be any ideal of a commutative ring R. Let X denote an R-complex with* $\operatorname{T-codp}(\mathfrak{a}, X) = \infty$. *Then* $\mathrm{L}\Lambda^{\mathfrak{a}}(X) \simeq 0$, *that is,* $\Lambda_i^{\mathfrak{a}}(X) = 0$ *for all* $i \in \mathbb{Z}$.

Proof We have $\operatorname{Tor}_i^R(R/\mathfrak{a}^t, X) = 0$ for all $i \in \mathbb{Z}$, as follows by 5.1.7. By virtue of 7.1.8 this implies that $\Lambda_i^{\mathfrak{a}}(X) = 0$ for all $i \in \mathbb{Z}$. $\square$

To go on, that is, to have the converse of 7.2.1, we shall assume that the ideal $\mathfrak{a}$ is finitely generated. Then the following lemma will be available.

Lemma 7.2.2 *Let $\mathfrak{a}$ be a finitely generated ideal of a commutative ring R and let Y be an R-complex. Then the natural morphism $\tau_Y^{\mathfrak{a}} : Y \to \Lambda^{\mathfrak{a}}(Y)$ induces an isomorphism*

$$\operatorname{Hom}_R(\Lambda^{\mathfrak{a}}(Y), \Lambda^{\mathfrak{a}}(X)) \cong \operatorname{Hom}_R(Y, \Lambda^{\mathfrak{a}}(X))$$

for any R-complex X.

Proof As the ideal $\mathfrak{a}$ is finitely generated the complex $\Lambda^{\mathfrak{a}}(X)$ is a complex of $\mathfrak{a}$-complete modules. Note that the statement is known when Y and X are modules (see 2.2.3). Its extension to complexes is immediate by the definition of $\operatorname{Hom}_R(\cdot, \cdot)$ for complexes. $\square$

The following contains a complex version of Corollary 3.4.2 (b).

Proposition 7.2.3 *Let $\mathfrak{a}$ be a finitely generated ideal of a commutative ring R. Let X denote an R-complex with one of its K-projective resolutions $P \xrightarrow{\sim} X$. The following conditions are equivalent.*

(i) $\mathrm{L}\Lambda^{\mathfrak{a}}(X) \simeq 0$, *that is,* $\Lambda^{\mathfrak{a}}(P)$ *is exact.*
(ii) $\Lambda^{\mathfrak{a}}(P)$ *is split-exact.*

(iii) *For every $\mathfrak{a}$-open ideal $\mathfrak{b}$ of R the complex $R/\mathfrak{b} \otimes_R P$ is exact.*

(iv) $\operatorname{T-codp}(\mathfrak{a}, X) = \infty$, *in other words X belongs to the class $\mathcal{T}_{\mathfrak{a}}$ introduced in 5.1.4.*

Proof Assume that $\Lambda^{\mathfrak{a}}(P)$ is exact. Then $\Lambda^{\mathfrak{a}}(P) \xrightarrow{\sim} 0$. This induces a quasi-isomorphism $\operatorname{Hom}_R(P, \Lambda^{\mathfrak{a}}(P)) \xrightarrow{\sim} \operatorname{Hom}_R(P, 0) \cong 0$. But

$$\operatorname{Hom}_R(P, \Lambda^{\mathfrak{a}}(P)) \cong \operatorname{Hom}_R(\Lambda^{\mathfrak{a}}(P), \Lambda^{\mathfrak{a}}(P))$$

(see 7.2.2). Hence $\operatorname{Hom}_R(\Lambda^{\mathfrak{a}}(P), \Lambda^{\mathfrak{a}}(P)) \simeq 0$ is exact. In view of the definition of Hom_R this implies that the endomorphisms of $\Lambda^{\mathfrak{a}}(P)$ are homotopic to zero. In particular we have that $\mathrm{id}_{\Lambda^{\mathfrak{a}}(P)}$ is homotopic to zero, so that $\Lambda^{\mathfrak{a}}(P)$ is split-exact.

For any $\mathfrak{a}$-open ideal $\mathfrak{b}$ we have $R/\mathfrak{b} \otimes_R P \cong R/\mathfrak{b} \otimes_R \Lambda^{\mathfrak{a}}(P)$ (see 2.2.2). Hence condition (ii) implies condition (iii), remember the properties of split-exact complexes recalled in 1.1.8. Note that conditions (iii) and (iv) are equivalent in view of 5.1.7. The implication (iv) $\Rightarrow$ (i) is shown in 7.2.1. $\square$

As an application we obtain another property of complexes in the class $\mathcal{T}_{\mathfrak{a}}$, a complex version and a generalization of [76, Theorem 3.1]. (This theorem stated that $\operatorname{Ext}^i_R(F, M) = 0$ for all $i \in \mathbb{Z}$ when M is an $\mathfrak{a}$-complete module over a Noetherian ring R and when F is a flat R-module such that $F/\mathfrak{a}F = 0$. Note that such an M satisfies the hypothesis of the following theorem (see the remarks in 7.1.9).)

Theorem 7.2.4 *Let $\mathfrak{a}$ be a finitely generated ideal of a commutative ring R. Let T, X denote two R-complexes and $T \in \mathcal{T}_{\mathfrak{a}}$. Suppose that the natural map $\alpha^{\mathfrak{a}}_X : X \to \mathrm{L}\Lambda^{\mathfrak{a}}(X)$ is an isomorphism in the derived category. Then* $\operatorname{RHom}_R(T, X) \simeq 0$ *in the derived category.*

Proof Let $P \xrightarrow{\sim} T$ and $L \xrightarrow{\sim} X$ be K-projective resolutions of T and X respectively. By the assumption there is a quasi-isomorphism $L \xrightarrow{\sim} \Lambda^{\mathfrak{a}}(L)$, hence a quasi-isomorphism $\operatorname{Hom}_R(P, L) \xrightarrow{\sim} \operatorname{Hom}_R(P, \Lambda^{\mathfrak{a}}(L))$. But $\operatorname{Hom}_R(P, \Lambda^{\mathfrak{a}}(L)) \cong \operatorname{Hom}_R(\Lambda^{\mathfrak{a}}(P), \Lambda^{\mathfrak{a}}(L))$ (see 7.2.2). Hence $\operatorname{RHom}_R(T, X)$ is represented by any of the quasi-isomorphic complexes

$$\operatorname{Hom}_R(P, X) \xleftarrow{\sim} \operatorname{Hom}_R(P, L) \xrightarrow{\sim} \operatorname{Hom}_R(P, \Lambda^{\mathfrak{a}}(L)) \cong \operatorname{Hom}_R(\Lambda^{\mathfrak{a}}(P), \Lambda^{\mathfrak{a}}(L)).$$

By 7.2.3 and the assumption on T the complex $\Lambda^{\mathfrak{a}}(P)$ is split-exact. It follows that the complex $\operatorname{Hom}_R(\Lambda^{\mathfrak{a}}(P), \Lambda^{\mathfrak{a}}(L))$ is exact. To this end recall the properties of split-exact complexes (see 1.1.8). Whence the complex $\operatorname{Hom}_R(P, X)$ is exact too. This proves the claim. $\square$

The local cohomology side is easier and does not require any finiteness condition on the ideal $\mathfrak{a}$ of R.

Lemma 7.2.5 *Let $\mathfrak{a}$ be any ideal of a commutative ring R and let X be an R-complex. Then* $\operatorname{E-dp}(\mathfrak{a}, X) \le \inf\{i \in \mathbb{Z} \mid H^i_{\mathfrak{a}}(X) \neq 0\}$.

Proof Put $r = \operatorname{E-dp}(\mathfrak{a}, X) \le \infty$. Then for $i < r$ and every $t \in \mathbb{N}$ we have the vanishing $\operatorname{Ext}^i_R(R/\mathfrak{a}^t, X) = 0$ as follows by 5.1.7. By passing to the direct limit and in view of 7.1.12 we obtain that $H^i_{\mathfrak{a}}(X) = 0$ for all $i < r$. □

The following contains a complex version of Proposition (3.2.7 (a)).

Proposition 7.2.6 *Let $\mathfrak{a}$ be an ideal of a commutative ring R and let X be an R-complex with one of its K-injective resolutions $X \xrightarrow{\sim} I$. The following conditions are equivalent:*

(i) $\mathrm{R}\Gamma_{\mathfrak{a}}(X) \simeq 0$*, that is $\Gamma_{\mathfrak{a}}(I)$ is exact.*
(ii) *$\Gamma_{\mathfrak{a}}(I)$ is split-exact.*
(iii) *For any $\mathfrak{a}$-open ideal $\mathfrak{b}$ of R the complex* $\operatorname{Hom}_R(R/\mathfrak{b}, I)$ *is exact.*
(iv) $\operatorname{E-dp}(\mathfrak{a}, X) = \infty$.

Proof Assume that $\Gamma_{\mathfrak{a}}(I)$ is exact. Then there is a quasi-isomorphism $0 \xrightarrow{\sim} \Gamma_{\mathfrak{a}}(I)$. It induces a quasi-isomorphism $\operatorname{Hom}_R(\Gamma_{\mathfrak{a}}(I), I) \xrightarrow{\sim} \operatorname{Hom}_R(0, I) \cong 0$. But

$$\operatorname{Hom}_R(\Gamma_{\mathfrak{a}}(I), I) = \operatorname{Hom}_R(\Gamma_{\mathfrak{a}}(I), \Gamma_{\mathfrak{a}}(I))$$

as is easily seen because the statement is true for an R-module. Hence the complex $\operatorname{Hom}_R(\Gamma_{\mathfrak{a}}(I), \Gamma_{\mathfrak{a}}(I))$ is exact. In view of the definition of Hom_R this implies that $\mathrm{id}_{\Gamma_{\mathfrak{a}}(I)}$ is homotopic to zero. Thus $\Gamma_{\mathfrak{a}}(I)$ is split-exact.

For an $\mathfrak{a}$-open ideal $\mathfrak{b}$ we have $\operatorname{Hom}_R(R/\mathfrak{b}, I) = \operatorname{Hom}_R(R/\mathfrak{b}, \Gamma_{\mathfrak{a}}(I))$ as is easily seen. Hence condition (ii) implies condition (iii), remember the properties of split-exact complexes recalled in 1.1.8. Finally, note that conditions (iii) and (iv) are equivalent in view of 5.1.7. They imply condition (i) by 7.2.5. □

Here is a counterpart to Theorem 7.2.4.

Theorem 7.2.7 *Let $\mathfrak{a}$ be any ideal of a commutative ring R. Let T, X denote two R-complexes. Suppose that* $\operatorname{E-dp}(\mathfrak{a}, T) = \infty$ *and that the natural map* $\iota^X_{\mathfrak{a}} : \mathrm{R}\Gamma_{\mathfrak{a}}(X) \to X$ *is an isomorphism in the derived category. Then* $\operatorname{RHom}_R(X, T) \simeq 0$ *in the derived category.*

Proof Let $T \xrightarrow{\sim} I$ and $X \xrightarrow{\sim} J$ be K-injective resolutions of T and X, respectively. By the assumption there is a quasi-isomorphism $\Gamma_{\mathfrak{a}}(J) \xrightarrow{\sim} J$, hence a quasi-isomorphism $\operatorname{Hom}_R(J, I) \xrightarrow{\sim} \operatorname{Hom}_R(\Gamma_{\mathfrak{a}}(J), I)$. But we have that $\operatorname{Hom}_R(\Gamma_{\mathfrak{a}}(J), I) \cong \operatorname{Hom}_R(\Gamma_{\mathfrak{a}}(J), \Gamma_{\mathfrak{a}}(I))$ as is easily seen. Hence $\operatorname{RHom}_R(X, T)$ is represented by any of the quasi-isomorphic complexes

$$\operatorname{Hom}_R(X, I) \xleftarrow{\sim} \operatorname{Hom}_R(J, I) \xrightarrow{\sim} \operatorname{Hom}_R(\Gamma_{\mathfrak{a}}(J), I) \cong \operatorname{Hom}_R(\Gamma_{\mathfrak{a}}(J), \Gamma_{\mathfrak{a}}(I)).$$

By 7.2.6 and the assumption on T the complex $\varGamma_{\mathfrak{a}}(I)$ is split-exact. It follows that the complex $\operatorname{Hom}_R(\varGamma_{\mathfrak{a}}(J), \varGamma_{\mathfrak{a}}(I))$ is exact (see 1.1.8). Whence the complex $\operatorname{Hom}_R(X, I)$ is exact too. This proves the claim. □

Here is another counterpart to Theorem 7.2.4.

Theorem 7.2.8 *Let $\mathfrak{a}$ be any ideal of a commutative ring R. Let T, X denote two R-complexes. Suppose that* T-codp$(\mathfrak{a}, T) = \infty$ *and that the natural map $\iota_{\mathfrak{a}}^X$: $\mathrm{R}\varGamma_{\mathfrak{a}}(X) \to X$ is an isomorphism in the derived category. Then $T \otimes_R^{\mathrm{L}} X \simeq 0$ in the derived category.*

Proof Let $P \xrightarrow{\sim} T$ be a K-projective resolution of T and $X \xrightarrow{\sim} J$ be a K-injective resolution of X. By the assumption there is a quasi-isomorphism $\varGamma_{\mathfrak{a}}(J) \xrightarrow{\sim} J$, hence a quasi-isomorphism $P \otimes_R \varGamma_{\mathfrak{a}}(J) \xrightarrow{\sim} P \otimes_R J$. Then $T \otimes_R^{\mathrm{L}} X$ is represented by any of the quasi-isomorphic complexes

$$\begin{aligned} P \otimes_R J \xleftarrow{\sim} P \otimes_R \varGamma_{\mathfrak{a}}(J) = P \otimes_R \varinjlim \operatorname{Hom}_R(R/\mathfrak{a}^t, I) &\cong \varinjlim (P \otimes_R \operatorname{Hom}_R(R/\mathfrak{a}^t, I)) \\ &\cong \varinjlim (P/\mathfrak{a}^t P \otimes_{R/\mathfrak{a}^t} \operatorname{Hom}_R(R/\mathfrak{a}^t, I)), \end{aligned}$$

recall that $\varinjlim$ commutes with tensor products (see 1.3.3). By the assumption on T the complexes $P/\mathfrak{a}^t P$ are exact. Moreover, these complexes $P/\mathfrak{a}^t P$, viewed as $R/\mathfrak{a}^t$-complexes, are K-projective, as is easily seen. By 4.4.4 it follows that they are split-exact. Whence the complexes $P/\mathfrak{a}^t P \otimes_{R/\mathfrak{a}^t} \operatorname{Hom}_R(R/\mathfrak{a}^t, I)$ are exact (see 1.1.8) and so is their direct limit. □

When the ideal $\mathfrak{a}$ is finitely generated we put 7.2.3 and 7.2.6 together with 5.3.5 to obtain another characterization of complexes in the class $\mathcal{T}_{\mathfrak{a}}$ introduced in 5.3.5.

Proposition 7.2.9 *Let $\mathfrak{a}$ be a finitely generated ideal of a commutative ring R and let X be any R-complex. The following conditions are equivalent:*

(i) $\mathrm{L}\varLambda^{\mathfrak{a}}(X) \simeq 0$,
(ii) $X \in \mathcal{T}_{\mathfrak{a}}$,
(iii) $\mathrm{R}\varGamma_{\mathfrak{a}}(X) \simeq 0$.

For complexes homologically-bounded on the good side, Lemmas 7.2.1 and 7.2.5 can be refined.

Proposition 7.2.10 *For any ideal $\mathfrak{a}$ of a commutative ring R and any homologically-left-bounded complex X it follows that*

$$\inf\{i \in \mathbb{Z} \mid H_{\mathfrak{a}}^i(X) \neq 0\} = \text{E-dp}(\mathfrak{a}, X).$$

Proof The homologically-left-bounded complex X has a left-bounded injective resolution $X \xrightarrow{\sim} J$. In view of 5.1.7 the statement is a reformulation of 3.4.4. □

Proposition 7.2.11 *For any finitely generated ideal* $\mathfrak{a}$ *of a commutative ring* R *and any homologically-right-bounded complex* X *it follows that*

$$\inf\{i \in \mathbb{Z} \mid \Lambda_i^{\mathfrak{a}}(X) \neq 0\} = \text{T-codp}(\mathfrak{a}, X).$$

Proof The homologically-right-bounded complex X has a right-bounded free resolution $P \xrightarrow{\sim} X$. In view of 5.1.7 the statement is a reformulation of 3.4.1. □

To get rid of the boundedness assumptions in 7.2.10 and 7.2.11 we need some more assumptions on the ideal $\mathfrak{a}$ of R. When the ideal $\mathfrak{a}$ is generated by a weakly pro-regular sequence we shall see in 7.6.2 that both Propositions 7.2.10 and 7.2.11 also hold without any boundedness conditions.

When the ideal $\mathfrak{a}$ is generated by a weakly pro-regular sequence we shall also have a kind of converse to Theorem 7.2.8 (see 9.6.11), a kind of converse to Theorem 7.2.7 (see 9.6.15), and a kind of converse to Theorem 7.2.4 (see Theorem 9.6.8).

7.3 Weakly Pro-regular Sequences

In this section we are prepared for a bridge between Čech (co-)homology and local (co)-homology in a commutative ring R. Note that the Čech complex is constructed for a finite number of elements while local (co-)homology is done for an arbitrary ideal. So the connection between both can hold only for certain finitely generated ideals. It will be given via a certain property of a finite sequence of elements.

Definition 7.3.1 (see [2], [75, Definition 2.3]) A sequence of elements $\underline{x} = x_1, \ldots, x_k$ of a commutative ring R is called *weakly pro-regular* if for each $i \neq 0$ the inverse system of Koszul homology modules $\{H_i(\underline{x}^t)\}$ is pro-zero. That is, for each $t \in \mathbb{N}$ there is an $s \geq t$ such that the natural morphism $H_i(\underline{x}^s) \to H_i(\underline{x}^t)$ is the zero map.

(Here the inverse system $\{H_i(\underline{x}^t)\}$ is the one induced by the inverse system $\{K_\bullet(\underline{x}^t)\}$ described in 6.1.1.)

More generally, we say that the sequence $\underline{x}$ is weakly pro-regular with respect to a module M (M-weakly pro-regular for short) if for each $i \neq 0$ the inverse system of Koszul homology modules $\{H_i(\underline{x}^t; M)\}$ is pro-zero.

Example 7.3.2 (a) Clearly any regular sequence $\underline{x} = x_1, \ldots, x_k$ is weakly pro-regular.

(b) Completely secant sequences in Bourbaki's terminology are also weakly pro-regular. This follows by the remarks in 5.4.2.

(c) For a length one sequence x and an R-module M the following conditions are equivalent:

(i) the increasing sequence of submodules $0 :_M x^t, t > 0$ stabilizes,
(ii) x is M-weakly pro-regular.

When these conditions are satisfied we also say that M is of xR-bounded torsion.

(d) It is known that any finite sequence in a Noetherian ring R is weakly pro-regular (see e.g. [82, Lemma 4.3.3] or A.2.3 for a slightly more general result).

The importance of the notion of weakly pro-regular sequences comes from the following result. It originates from the work of Grothendieck in [40].

Lemma 7.3.3 (see [75, Lemma 2.4]) *Let $\underline{x} = x_1, \dots, x_k$ denote a sequence of elements of a commutative ring R. Then the following conditions are equivalent:*

(i) $\underline{x}$ *is a weakly pro-regular sequence,*
(ii) $\{H_i(\underline{x}^t; F)\}$ *is pro-zero for all* $i \neq 0$ *and any flat* R*-module* F*,*
(iii) $\varinjlim H^i(\underline{x}^t; I) = 0$ *for all* $i \neq 0$ *and any injective* R*-module* I*,*
(iv) $H^i(\check{C}_{\underline{x}} \otimes_R I) = 0$ *for all* $i \neq 0$ *and any injective* R*-module* I*.*

Proof The equivalence (i) $\Leftrightarrow$ (ii) is clear since $H_i(\underline{x}^t) \otimes_R F \cong H_i(\underline{x}^t; F)$ for all i when F is a flat R-module. The equivalence (iii) $\Leftrightarrow$ (iv) follows since $\varinjlim H^i(\underline{x}^t; I) = H^i(\check{C}_{\underline{x}} \otimes_R I)$ (see 6.1.4).

Now let us prove (i) $\Rightarrow$ (iii). Since I is an injective R-module

$$H^i(\mathrm{Hom}_R(K_\bullet(\underline{x}^t), I)) \cong \mathrm{Hom}_R(H_i(\underline{x}^t), I)$$

for all i. Therefore, and because of $H^i(\mathrm{Hom}_R(K_\bullet(\underline{x}^t), I)) \cong H^i(\underline{x}^t, I)$,

$$\varinjlim H^i(\underline{x}^t; I) \cong \varinjlim \mathrm{Hom}_R(H_i(\underline{x}^t), I).$$

By the assumption the inverse system $\{H_i(\underline{x}^t)\}$ is pro-zero for $i \neq 0$. Whence the direct limit $\varinjlim H^i(\underline{x}^t; I)$ vanishes, as required.

In order to complete the proof we have to show that (iii) $\Rightarrow$ (i). Let $f : H_i(\underline{x}^t) \hookrightarrow I$ denote an injection into an injective R-module I. Then

$$f \in \mathrm{Hom}_R(H_i(\underline{x}^t), I) \cong H^i(\underline{x}^t; I)$$

since I is an injective R-module. Because of the assumption we have the vanishing $\varinjlim H^i(\underline{x}^t; I) = 0$. So there must be an integer $s \geq t$ such that the image of f in $H^i(\underline{x}^s; I)$ has to be zero. In other words, the composite of the maps

$$H_i(\underline{x}^s) \to H_i(\underline{x}^t) \overset{f}{\hookrightarrow} I$$

is zero. Since f is an injection it follows that the first map has to be zero. $\square$

The following lemma provides some more information on the notion of weakly pro-regularity.

Lemma 7.3.4 (see [75]) *Let $\underline{x} = x_1, \dots, x_k$ and $\underline{y} = y_1, \dots, y_l$ be two sequences in a commutative ring R such that* $\mathrm{Rad}(\underline{x}R) = \mathrm{Rad}(\underline{y}R)$. *If one of the three sequences $\underline{x}$, $\underline{y}$ and $\underline{x}, \underline{y}$ is weakly pro-regular, then so are the other two.*

Proof In view of 6.1.9 this is direct consequence of the equivalence 7.3.3 (i) $\Leftrightarrow$ (iv). □

As we plan to investigate the local homology and cohomology of complexes we also need to investigate the behaviour of weakly pro-regular sequences with respect to flat and injective complexes. Though the following lemmas are quite important for the sequel, we did not find any trace of them in the literature. But first let us introduce some more notation.

Definitions and observations 7.3.5 Let $\underline{x} = x_1, \ldots, x_k$ be a sequence of elements of the commutative ring R. Let $t \in \mathbb{N}$ be a natural number. We denote by $C_\bullet(\underline{x}^t)$ the kernel of the natural morphism $K_\bullet(\underline{x}^t) \to H_0(\underline{x}^t) = R/\underline{x}^t R$, where $R/\underline{x}^t R$ is considered as a complex concentrated in degree zero. Thus we have short exact sequences of complexes

$$0 \to C_\bullet(\underline{x}^t) \to K_\bullet(\underline{x}^t) \to R/\underline{x}^t R \to 0.$$

Note that $H_i(C_\bullet(\underline{x}^t)) = H_i(K_\bullet(\underline{x}^t))$ for all $i \geq 1$ and $H_0(C_\bullet(\underline{x}^t)) = 0$.

For a complex F of flat R-modules it induces a short exact sequence of inverse systems of complexes

$$0 \to C_\bullet(\underline{x}^t) \otimes_R F \to K_\bullet(\underline{x}^t; F) \to F/\underline{x}^t F \to 0.$$

For a complex I of injective R-modules it induces a short exact sequence of direct system of complexes

$$0 \to \operatorname{Hom}_R(R/\underline{x}^t R, I) \to K^\bullet(\underline{x}^t; I) \to \operatorname{Hom}_R(C_\bullet(\underline{x}^t), I) \to 0.$$

The following result is an extension of 7.3.3 to the case of a complex of flat R-modules.

Lemma 7.3.6 *Let $\underline{x} = x_1, \ldots, x_k$ be a weakly pro-regular sequence of a commutative ring R. Let F denote a complex of flat R-modules. Then the inverse system of homology modules $\{H_i(C_\bullet(\underline{x}^t) \otimes_R F)\}$ is pro-zero for all $i \in \mathbb{Z}$.*

Proof First note that the complex $C_\bullet(\underline{x}^t) \otimes_R F$, simply denoted by $K(t)$, is the single complex associated to the double complex $K_{p,q}(t) := C_p(\underline{x}^t) \otimes_R F_q$. Therefore, for each $t \geq 1$, there is a convergent spectral sequence

$$E(t)^1_{p,q} = H_q(C_\bullet(\underline{x}^t) \otimes_R F_p) \Longrightarrow H_{p+q}(C_\bullet(\underline{x}^t) \otimes_R F)$$

associated to the filtration defined by $K^q(t) = \oplus_{j \leq q} K_{i,j}(t)$. The natural morphism $C_\bullet(\underline{x}^s) \to C_\bullet(\underline{x}^t)$ for $s \geq t$ induces a morphism of the corresponding spectral sequences (see e.g. [39] or [85]). In particular, for the subsequent stages of the spectral sequences there is a natural homomorphism $E(s)^r_{p,q} \to E(t)^r_{p,q}$ for each $r \geq 1$ so that there are inverse systems $\{E(t)^r_{p,q} \mid t > 0\}$. We claim that these inverse systems are pro-zero for each $r \geq 1$ and all p, q. We proceed by induction on r. For $r = 1$ the claim is true by assumption since $\underline{x}$ is weakly pro-regular (see 7.3.3 and the definition of $C_\bullet(\underline{x}^t) \otimes_R F_p$ as given in 7.3.5). For the subsequent stages we proceed by induction. Note that $E(t)^{r+1}_{p,q}$ is a subquotient of $E(t)^r_{p,q}$, there are homomorphisms $d(t)^r_{p,q} : E(t)^r_{p,q} \to E(t)^r_{p-r,q+r-1}$ for $t \geq 1$ and $E(t)^{r+1}_{p,q} = \operatorname{Ker}\ d(t)^r_{p,q} / \operatorname{Im}\ d(t)^r_{p+r,q-r+1}$. By virtue of the observation in 1.2.4, and assuming the claim is true for r, it follows that it is true for $r+1$, and that the inverse system $\{E(t)^{r+1}_{p,q}\}$ is pro-zero. Because $E(t)^\infty_{p,q} = E(t)^r_{p,q}$ for $r \gg 1$, the inverse system $\{E(t)^\infty_{p,q} \mid t \geq 1\}$ is pro-zero too, and this holds for all p, q.

We now investigate the module $H(t)_i := H_i(C_\bullet(\underline{x}^t) \otimes_R F)$ for a given $i \in \mathbb{Z}$. The filtration on the double complex $K_{i,j}(t)$ induces a filtration on $H(t)_i$ defined by $\Phi^q(H(t)_i) = \operatorname{Im}(H_i(K^q(t) \to H_i(K(t)))$. Note that $\Phi^q(H(t)_i) = 0$ for $i > q + k$, in particular $\Phi^{i-(k+1)}(H(t)_i) = 0$. Therefore the filtration on $H(t)_i$ is bounded and there is a decreasing sequence of submodules

$$0 \subseteq \Phi^{i-k} H(t)_i \subseteq \ldots \subseteq \Phi^{i-1} H(t)_i \subseteq \Phi^i H(t)_i \subseteq \Phi^{i+1} H(t)_i = H(t)_i.$$

By construction we have

$$E^\infty_{pq}(t) \cong \Phi^q(H(t)_{p+q}) / \Phi^{q-1}(H(t)_{p+q}).$$

For the subquotients we thus have

$$\Phi^{i-p} H(t)_i / \Phi^{i-(p+1)} H(t)_i \cong E^\infty_{p,i-p}(t) \text{ for } -1 \leq p \leq k.$$

We have seen that all of the inverse systems $\{E^\infty_{p,i-p}(t)\}$ are pro-zero. Therefore by induction on the length of the filtration it follows (with the observation in 1.2.4) that the inverse system $\{H(t)_i\}_{t \geq 1}$ is pro-zero too. This completes the proof. □

For the readers who do not like spectral sequences we provide a second proof of 7.3.6 based on a diagram chase.

7.3.7 *Second proof of* 7.3.6. As the sequence $\underline{x}$ is weakly proregular we have for any given $t \in \mathbb{N}$ an increasing sequence of natural numbers $t = t_1 \leq t_2 \leq \cdots \leq t_k$ such that for all i, $0 \leq i \leq n$, all the homomorphisms in the following sequence

$$H_i(C_\bullet(\underline{x}^{t_k})) \to \cdots \to H_i(C_\bullet(\underline{x}^{t_2})) \to H_i(C_\bullet(\underline{x}^{t_1}))$$

are zero.

We investigate the inverse system of double complexes $K_{ij}(t) = C_i(\underline{x}^t) \otimes_R F_j$ and their rows $C_\bullet(\underline{x}^t) \otimes_R F_j$ for $j \in \mathbb{Z}$. The inverse system of homology modules $\{H_i(C_\bullet(\underline{x}^t) \otimes_R F_j)\}$ of these rows are pro-zero for all $i \in \mathbb{Z}$. This follows because F_j is flat and

$$H_i(C_\bullet(\underline{x}^t) \otimes_R F_j) = H_i(C_\bullet(\underline{x}^t)) \otimes_R F_j = H_i(\underline{x}^t) \otimes_R F_j$$

for $i \neq 0$ and $H_0(C_\bullet(\underline{x}^t) \otimes_R F_j) = 0$.

In order to handle the inverse system of double complexes $K_{ij}(t)$ we write d_h^t for the differential of their rows and d_v^t for the differential of their columns. We also write d^t for the differential of the associated single complexes: for $w_{ij} \in C_i(\underline{x}^t) \otimes_R F_j$ we have $d^t(w_{ij}) = d_h^t(w_{ij}) + (-1)^i d_v^t(w_{ij})$. A chain of degree n in the single complex associated to $K_{ij}(t)$ will be denoted by $w_n^t = (w_{0,n}^t, w_{1,n-1}^t, \dots, w_{k,n-k}^t)$, where $w_{i,n-i}^t$ denotes the component of w_n^t in $C_i(\underline{x}^t) \otimes_R F_{n-i}$. If $s \leq t$ we freely denote by w_n^s the image of w_n^t in $K_{ij}(s)$.

Now let us fix an integer $n \in \mathbb{Z}$ and take $z_n^{t_k} = (z_{0,n}^{t_k}, z_{1,n-1}^{t_k}, \dots, z_{k,n-k}^{t_k})$, a cycle of degree n in the single complex associated to $K_{ij}(t_k)$. As $z_n^{t_k}$ is a cycle we have

$$(-1)^i d_v(z_{i,n-i}^{t_k}) + d_h(z_{i+1,n-i-1}^{t_k}) = 0.$$

With the help of the preceding observations and some diagram chasing we shall build – step by step – a chain $y_{n+1}^{t_1}$ of degree $n+1$ in the single complex associated to $K_{ij}(t_1)$ such that $d^{t_1}(y_{n+1}^{t_1}) = z_n^{t_1}$. Let us see how.

We put $y_{0,n+1}^{t_k} = 0$ and, as $H_0(C_\bullet(\underline{x}^{t_k} \otimes_R F_n)) = 0$, we may pick an element

$$y_{1,n}^{t_k} \in K_{1n}(t_k) = C_1(\underline{x}^{t_k}) \otimes_R F_n \text{ such that } d_h(y_{1,n}^{t_k}) = z_{0,n}^{t_k}.$$

Now we have

$$d_h(d_v(y_{1,n}^{t_k})) = d_v(d_h(y_{1,n}^{t_k})) = d_v(z_{0,n}^{t_k}) = -d_h(z_{1,n-1}^{t_k}).$$

Therefore $d_h(d_v(y_{1,n}^{t_k}) + z_{1,n-1}^{t_k}) = 0$ and $d_v(y_{1,n}^{t_k}) + z_{1,n-1}^{t_k}$ is a cycle of degree 1 in the row complex $C_\bullet(\underline{x}^{t_k}) \otimes_R F_{n-1}$. Then we go to the level t_{k-1}. Because the homomorphism $H_1(C_\bullet(\underline{x}^{t_k}) \otimes_R F_{k-1}) \to H_1(C_\bullet(\underline{x}^{t_{k-1}}) \otimes_R F_{k-1})$ is zero, at this level the element $d_v(y_{1,n}^{t_{k-1}}) + z_{1,n-1}^{t_{k-1}}$ is a boundary of degree 1 in the row complex $C_\bullet(\underline{x}^{t_{k-1}}) \otimes_R F_{n-1}$. Thus we may choose an element

$$y_{2,n-1}^{t_{k-1}} \in C_2(\underline{x}^{t_{k-1}}) \otimes_R F_{n-1} \text{ such that } d_h(y_{2,n-1}^{t_{k-1}}) = z_{1,n-1}^{t_{k-1}} + d_v(y_{1,n}^{t_{k-1}}).$$

At this homological level we now have

$$d_h(y_{1,n}^{t_{k-1}}) + (-1)^0 d_v(y_{0,n+1}^{t_{k-1}}) = z_{0,n}^{t_{k-1}} \text{ and}$$
$$d_h(y_{2,n-1}^{t_{k-1}}) + (-1)^1 d_v(y_{1,n}^{t_{k-1}}) = z_{1,n-1}^{t_{k-1}}.$$

Successively we proceed in the same way by going to the levels $t_{k-2}, \dots, t_1 = t$. Finally, we obtain at level t_1 the wanted chain $y_{n+1}^{t_1}$. Therefore the map

$$H_k(C_\bullet(\underline{x}^{t_k}) \otimes_R F) \to H_k(C_\bullet(\underline{x}^{t_1}) \otimes_R F)$$

is zero. This holds for all $n \in \mathbb{Z}$ and all $t = t_1 \in \mathbb{N}$ and the statement is proved. $\square$

Similarly to the result in 7.3.6 there is an extension of 7.3.3 to the situation of a complex of injective modules.

Lemma 7.3.8 *Let $\underline{x} = x_1, \dots, x_k$ denote a weakly pro-regular sequence of the commutative ring R. Let I be a complex of injective R-modules.*

Then the complex $\varinjlim \operatorname{Hom}_R(C_\bullet(\underline{x}^t), I)$ is exact.

Proof The complex $\operatorname{Hom}_R(C_\bullet(\underline{x}^t), I)$ is the single complex associated to the double complex $K^{i,j}(t) = \operatorname{Hom}_R(C_i(\underline{x}^t), I^j)$ because $C_\bullet(\underline{x}^t)$ is bounded. Let us investigate its rows $\operatorname{Hom}_R(C_\bullet(\underline{x}^t), I^j)$. As the R-module I^j is injective we have

$$H^0(\operatorname{Hom}_R(C_\bullet(\underline{x}^t), I^j)) = 0 \text{ and}$$
$$H^n(\operatorname{Hom}_R(C_\bullet(\underline{x}^t), I^j)) \cong \operatorname{Hom}_R(H_n(\underline{x}^t), I^j) \cong H^n(\underline{x}^t, I^j)$$

for all $n > 0$. As the direct limit functor commutes with taking cohomology we have that $\varinjlim \operatorname{Hom}_R(C_\bullet(\underline{x}^t), I^j))$ is exact for all $j \in \mathbb{Z}$ (see 7.3.3). Hence the double complex $\varinjlim_t K^{i,j}(t)$ has exact rows. When the complex I is right-bounded the complex $\varinjlim \operatorname{Hom}_R(C_\bullet(\underline{x}^t), I)$, which is associated to the double complex $\varinjlim_t K^{i,j}(t)$, is exact by Lemma 4.1.3. In the general case of an unbounded complex, note that $H^n(\varinjlim \operatorname{Hom}_R(C_\bullet(\underline{x}^t), I))$ depends only on a hard truncation $\ldots \to I^{m-1} \to I^m \to 0$ for $m \gg n$. $\square$

These last results on weakly pro-regularity will give rise to a concrete relation between local and Čech (co)homology.

7.4 Local and Čech Cohomology with Telescope

When $\mathfrak{a}$ is an ideal of a Noetherian ring R a result of Grothendieck asserts that the local cohomology modules $H^i_{\mathfrak{a}}(M)$ of an R-module M can be computed via Čech complexes, that is, $H^i_{\mathfrak{a}}(M) \cong H^i(\check{C}_{\underline{x}} \otimes_R M)$ where $\underline{x} = x_1, \dots, x_k$ is a sequence generating the ideal $\mathfrak{a}$ (see e.g. [40]).

This holds in greater generality and also admits a complex version (see [1]). In this section we prove that $\mathrm{R}\Gamma_{\mathfrak{a}}(X)$ is represented by $\check{C}_{\underline{x}} \otimes_R X$ for all unbounded R-complexes X when $\underline{x} = x_1, \dots, x_k$ is a weakly pro-regular sequence generating the ideal $\mathfrak{a}$. We also clarify the relation between local and Čech cohomology as done in [75]. Then we present a description of $\check{C}_{\underline{x}} \otimes_R X$ in terms of telescope, suggested by the work of Greenlees and May in [38] and more helpful for further investigations.

Though the local cohomology of an R-complex X is related to its K-injective resolution (see 7.1.11), we first look at injective complexes and injective resolutions. It will turn out that $\mathrm{R}\Gamma_{\mathfrak{a}}(X)$ may also be computed with an ordinary injective resolution of X.

Proposition 7.4.1 *Let $\underline{x} = x_1, \ldots, x_k$ be a sequence of elements of a commutative ring R and $\mathfrak{a} = \underline{x}R$ the ideal generated by it.*

(a) Let X denote a complex of R-modules. There is a natural degree-wise injective morphism $\Gamma_{\mathfrak{a}}(X) \to \check{C}_{\underline{x}} \otimes_R X$. The composition with the natural morphism $\check{C}_{\underline{x}} \otimes_R X \to X$ is the inclusion $\Gamma_{\mathfrak{a}}(X) \hookrightarrow X$.

(b) The natural morphism $\Gamma_{\mathfrak{a}}(I) \to \check{C}_{\underline{x}} \otimes_R I$ is a quasi-isomorphism for each complex of injective R-modules provided the sequence $\underline{x}$ is weakly pro-regular.

Proof In order to prove the first part of the claim we consider the commutative triangles

$$\begin{tikzcd} & R \arrow[dl] \arrow[dr] & \\ K_\bullet(\underline{x}^t) \arrow[rr] & & R/\mathfrak{a}_t \end{tikzcd}$$

where $\mathfrak{a}_t$ denotes the ideal generated by the sequence $\underline{x}^t = x_1^t, \ldots, x_k^t$. We apply the functor $\operatorname{Hom}_R(\cdot, X)$ and obtain commutative triangles for $t \geq 1$

$$\begin{tikzcd} & X & \\ \operatorname{Hom}_R(R/\mathfrak{a}^t, X) \arrow[ur] \arrow[rr] & & \operatorname{Hom}_R(K_\bullet(\underline{x}^t), X) \arrow[ul] \end{tikzcd}$$

Then we take direct limits and obtain the claim since $\varinjlim \operatorname{Hom}_R(K_\bullet(\underline{x}^t), X) \cong \varinjlim K^\bullet(\underline{x}^t; X) \cong \check{C}_{\underline{x}} \otimes_R X$.

Now suppose that the sequence $\underline{x}$ is weakly pro-regular and let $X = I$ be a complex of injective R-modules. With the notation of 7.3.5 we have a short exact sequence of direct systems

$$0 \to \operatorname{Hom}_R(R/\underline{x}^t R, I) \to K^\bullet(\underline{x}^t; I) \to \operatorname{Hom}_R(C_\bullet(\underline{x}^t), I) \to 0.$$

We take direct limits. As the direct limit functor is exact we obtain the quasi-isomorphism $\Gamma_{\mathfrak{a}}(I) \xrightarrow{\sim} \check{C}_{\underline{x}} \otimes_R I$ since the complex $\varinjlim \operatorname{Hom}_R(C_\bullet(\underline{x}^t), I)$ is exact by 7.3.8. $\square$

We already have a variant of 7.1.10 with the same conclusion under different hypotheses.

Corollary 7.4.2 *Assume that the ideal $\mathfrak{a}$ of a commutative ring R is generated by a weakly pro-regular sequence $\underline{x} = x_1, \ldots, x_k$. If I and I' are complexes of injective R-modules and if $f : I \xrightarrow{\sim} I'$ is a quasi-isomorphism, then $\Gamma_{\mathfrak{a}}(h)$ is also a quasi-isomorphism.*

Proof By 7.4.1 (b) there is commutative square

$$\begin{tikzcd} \Gamma_{\mathfrak{a}}(I) \arrow[r] \arrow[d] & \Gamma_{\mathfrak{a}}(I') \arrow[d] \\ \check{C}_{\underline{x}} \otimes_R I \arrow[r, "\sim"] & \check{C}_{\underline{x}} \otimes_R I' \end{tikzcd}$$

where the vertical morphisms are quasi-isomorphisms (see 7.4.1 (b)). Note that we have the quasi-isomorphism $\check{C}_{\underline{x}} \otimes_R I \xrightarrow{\sim} \check{C}_{\underline{x}} \otimes_R I'$ because the complex $\check{C}_{\underline{x}}$ is DG-flat. The conclusion follows. □

More important is the following.

Corollary 7.4.3 *Let $\mathfrak{a}$ be an ideal of a commutative ring R generated by a weakly pro-regular sequence $\underline{x} = x_1, \ldots, x_k$. Let X be an arbitrary R-complex and $X \xrightarrow{\sim} I$ one of its injective resolutions. Then we have quasi-isomorphisms such that the following diagram is commutative*

$$\begin{tikzcd} \check{C}_{\underline{x}} \otimes_R X \arrow[r, "\sim"] \arrow[d] & \check{C}_{\underline{x}} \otimes_R I \arrow[d] & \Gamma_{\mathfrak{a}}(I) \arrow[l, "\sim"'] \arrow[d, equal] \\ X \arrow[r, "\sim"] & I & \Gamma_{\mathfrak{a}}(I) \arrow[l, hook'] \end{tikzcd}$$

where the vertical morphisms are the natural ones.

Proof The commutativity of the right square follows from (7.4.1 (a)) and that of the left square by functoriality. We have the quasi-isomorphism $\check{C}_{\underline{x}} \otimes_R X \xrightarrow{\sim} \check{C}_{\underline{x}} \otimes_R I$ because the complex $\check{C}_{\underline{x}}$ is DG-flat and the quasi-isomorphism $\Gamma_{\mathfrak{a}}(I) \xrightarrow{\sim} \check{C}_{\underline{x}} \otimes_R I$ is in 7.4.1 (b). □

Here is the bridge between local and Čech cohomology, and even a little more.

Theorem 7.4.4 (see also [75, Theorem 3.2]) *Let $\mathfrak{a}$ be a finitely generated ideal of a commutative ring R and let $\underline{x} = x_1, \ldots, x_k$ be a sequence in R such that* $\mathrm{Rad}(\mathfrak{a}) = \mathrm{Rad}(\underline{x}R)$. *Then:*

(a) The following conditions are equivalent:

(i) The sequence $\underline{x} = x_1, \ldots, x_k$ is weakly pro-regular,
(ii) $H^i_{\mathfrak{a}}(M) \cong H^i(\check{C}_{\underline{x}} \otimes_R M)$ for any R-module M.

(b) When the conditions in (a) are satisfied $\mathrm{R}\Gamma_{\mathfrak{a}}(X)$ *is represented by $\check{C}_{\underline{x}} \otimes_R X$ for any R-complex X and the natural map $\iota^X_{\mathfrak{a}} : \mathrm{R}\Gamma_{\mathfrak{a}}(X) \to X$ in the derived category is represented by the natural morphism $\check{C}_{\underline{x}} \otimes_R X \to X$.*

If $X \xrightarrow{\sim} I$ is an injective resolution then $\mathrm{R}\Gamma_{\mathfrak{a}}(X)$ *is also represented by* $\Gamma_{\mathfrak{a}}(I)$ *and the natural map* $\iota_{\mathfrak{a}}^{X} : \mathrm{R}\Gamma_{\mathfrak{a}}(X) \to X$ *is also represented by the composite* $\Gamma_{\mathfrak{a}}(I) \to I \leftarrow X$.

Proof We may and do assume $\mathfrak{a} = \underline{x}R$ (see 7.1.16). Let $I \xrightarrow{\sim} I'$ be a DG-injective resolution of I, so that the composite $X \xrightarrow{\sim} I \xrightarrow{\sim} I'$ provides a DG-injective resolution of X. Then $\mathrm{R}\Gamma_{\mathfrak{a}}(X)$ is represented by $\Gamma_{\mathfrak{a}}(I')$ and the natural map $\iota_{\mathfrak{a}}^{X} : \mathrm{R}\Gamma_{\mathfrak{a}}(X) \to X$ is represented by the composite $\Gamma_{\mathfrak{a}}(I') \to I' \leftarrow X$ (see 7.1.11).

Assume first that the sequence $\underline{x} = x_1, \ldots, x_k$ is weakly pro-regular. Corollary 7.4.3 asserts that $\Gamma_{\mathfrak{a}}(I)$ and $\Gamma_{\mathfrak{a}}(I')$ are both quasi-isomorphic to $\check{C}_{\underline{x}} \otimes_R X$. Hence $\Gamma_{\mathfrak{a}}(I)$ and $\check{C}_{\underline{x}} \otimes_R X$ both represent $\mathrm{R}\Gamma_{\mathfrak{a}}(X)$, in particular condition (ii) is satisfied. We obtain the remaining claims in (b) by the commutative diagram in 7.4.3.

Suppose now that condition (ii) is satisfied. For an injective R-module J this means $H^i(\check{C}_{\underline{x}} \otimes J) \cong H_{\mathfrak{a}}^i(J) = 0$ for all $i \neq 0$. So the sequence $\underline{x}$ is weakly pro-regular by Lemma 7.3.3. □

We note that part (a) of the above theorem and the first assertion in part (b) were also proved in [1].

Corollary 7.4.5 *Let* $\mathfrak{a}$ *be a finitely generated ideal of a commutative ring R, generated up to radicals by a weakly pro-regular sequence* $\underline{x} = x_1, \ldots, x_k$. *Then*

$$H_{\mathfrak{a}}^i(X) \cong H^i(\check{C}_{\underline{x}} \otimes_R X) \cong \varinjlim H^i(\underline{x}^t; X)$$

for all R-complexes X and all $i \in \mathbb{Z}$.

Proof This is a direct consequence of the above and 6.1.4. □

Corollary 7.4.6 *Let* $\mathfrak{a}$ *be a finitely generated ideal of a commutative ring R, generated up to radicals by a weakly pro-regular sequence* $\underline{x} = x_1, \ldots, x_k$*, and let M be an R-module. Then* $H_{\mathfrak{a}}^i(M) = 0$ *for all $i > k$.*

For an improved bound of the vanishing of $H_{\mathfrak{a}}^i(M)$ we refer to the forthcoming 7.6.5.

Corollary 7.4.7 *Let* $\mathfrak{a}$ *be a finitely generated ideal of a commutative ring R, generated up to radicals by a weakly pro-regular sequence* $\underline{x} = x_1, \ldots, x_k$. *Let M be an R-module. Then M is* $\mathfrak{a}$*-torsion if and only if the natural map* $\iota_{\mathfrak{a}}^{M} : \mathrm{R}\Gamma_{\mathfrak{a}}(M) \to M$ *is an isomorphism in the derived category. In that case,* $H_{\mathfrak{a}}^i(M) = 0$ *for all $i \neq 0$.*

It follows that the pair $(R, \mathfrak{a})$ *has the property B introduced in 2.7.8.*

Proof If M is $\mathfrak{a}$-torsion then $\operatorname{Supp}_R M \subseteq V(\mathfrak{a})$. Hence the natural morphism $\check{C}_{\underline{x}} \otimes_R M \to M$ is a quasi-isomorphism by Theorem 6.1.8. It follows that $H_{\mathfrak{a}}^i(M) = 0$ for all $i \neq 0$. If the natural map $\iota_{\mathfrak{a}}^{M} : \mathrm{R}\Gamma_{\mathfrak{a}}(M) \to M$ is an isomorphism in the derived category, then $M \cong H^0(\check{C}_{\underline{x}} \otimes_R M) = \Gamma_{\mathfrak{a}}(M)$. Whence M is $\mathfrak{a}$-torsion. □

We note that the local cohomology commutes with right-filtered direct limits.

Corollary 7.4.8 *Let $\mathfrak{a}$ be a finitely generated ideal of a commutative ring R, generated up to radicals by a weakly pro-regular sequence $\underline{x} = x_1, \ldots, x_k$. Then the local homology modules $H^i_{\mathfrak{a}}(\cdot)$ commute with right-filtered direct limits. Namely, if $(I, \prec)$ is a right-filtered ordered set and if $\mathfrak{X} = \{\sigma_{ij} : X_i \to X_j \mid 0 \leq i \leq j \in I\}$ is a direct system of R-complexes, then*

$$H^i_{\mathfrak{a}}(\varinjlim X_i) \cong \varinjlim H^i_{\mathfrak{a}}(X_i).$$

Proof This is a direct consequence of 6.3.7. $\square$

For later use we need another representative of $\mathrm{R}\Gamma_{\mathfrak{a}}(X)$. To this end we are mainly concerned with the direct systems of complexes $\{K^\bullet(\underline{x}^t; X)\}$ and their telescope, following the work of Greenlees and May in [38].

Definition 7.4.9 Let $\underline{x} = x_1, \ldots, x_k$ be a sequence of elements in the commutative ring R and let X be an R-complex. Then we have the direct system $\{K^\bullet(\underline{x}^t; X) \mid t \geq 1\}$. We define the complex

$$T(\underline{x}; X) = \mathrm{Tel}(\{K^\bullet(\underline{x}^t; X)\}).$$

For simplicity we also write $T(\underline{x}) = T(\underline{x}; R) = \mathrm{Tel}(\{K^\bullet(\underline{x}^t)\})$.

Observations 7.4.10 Let $\underline{x} = x_1, \ldots, x_k$ be a sequence in a commutative ring R.

(a) Note that $T(\underline{x})$ is a bounded complex of free R-modules, hence DG-projective and DG-flat, and $T(\underline{x})^i = 0$ for $i \notin [-1, k]$.

(b) $T(\underline{x}; X)$ is a functor in X (because $K^\bullet(\underline{x}^t; X) \cong K^\bullet(\underline{x}^t) \otimes_R X$). Moreover, there are functorial isomorphisms $T(\underline{x}; X) \cong T(\underline{x}) \otimes_R X$ (see 4.3.5).

(c) It follows that the functor $T(\underline{x}; \cdot)$ preserves quasi-isomorphisms and is exact. In particular, the complex $T(\underline{x}; X)$ is exact as soon as X is exact.

(d) There is a natural quasi-isomorphism

$$T(\underline{x}; X) \xrightarrow{\sim} \varinjlim K^\bullet(\underline{x}^t; X) = \check{C}_{\underline{x}} \otimes_R X$$

(see 4.3.2). We compose it with the natural morphism $\check{C}_{\underline{x}} \otimes_R X \to X$ and obtain a natural morphism

$$\iota^X_{\underline{x}} : T(\underline{x}; X) \to X.$$

(e) For the case $X = R$ there is a quasi-isomorphism $T(\underline{x}) \xrightarrow{\sim} \check{C}_{\underline{x}}$. This yields a bounded free resolution of $\check{C}_{\underline{x}}$. Another free resolution $\check{L}_{\underline{x}} \xrightarrow{\sim} \check{C}_{\underline{x}}$ was given in 6.2.3. Thus the complexes $\check{L}_{\underline{x}}$ and $T(\underline{x})$ are quasi-isomorphic. Moreover, by 4.4.8 we have quasi-isomorphisms

$$\check{L}_{\underline{x}} \xrightarrow{\sim} T(\underline{x}) \xrightarrow{\sim} \check{L}_{\underline{x}}.$$

Note also that $\check{L}^i_{\underline{x}} = 0$ for $i \notin [0, k]$, $\check{L}_{\underline{x}}$ is a complex of length k while $T(\underline{x})$ is a complex of length $k + 1$.

We are ready for a description of $\mathrm{R}\Gamma_{\mathfrak{a}}$ in terms of the telescope. Putting 7.4.10 (d) and 7.4.4 together we obtain the following.

Theorem 7.4.11 *Let $\mathfrak{a}$ be an ideal of a commutative ring R generated by a weakly pro-regular sequence $\underline{x} = x_1, \ldots, x_k$. Then in the derived category $\mathrm{R}\Gamma_{\mathfrak{a}}(X)$ is represented by $T(\underline{x}; X)$ and therefore $H^i_{\mathfrak{a}}(X) \cong H^i(T(\underline{x}; X))$ for all $i \in \mathbb{Z}$.*

Moreover, the natural morphism $\iota^X_{\mathfrak{a}}; \mathrm{R}\Gamma_{\mathfrak{a}}(X) \to X$ in the derived category may be represented by the natural morphism $\iota^X_{\underline{x}} : T(\underline{x}; X) \to X$.

We note that the module case of Theorem 7.4.11 was already in [38] under slightly stronger hypotheses.

7.5 Local and Čech Homology with Microscope

Here we show that the local homology with respect to an ideal $\mathfrak{a}$ generated by a weakly pro-regular sequence $\underline{x} = x_1, \ldots, x_k$ coincides with the Čech homology, a result independently obtained in [64] by somewhat different methods. Here we first present a description of the local homology in terms of microscope, a description suggested by the work of Greenlees and May in [38]. It will turn out that $\Lambda^{\mathfrak{a}}(X)$ may be computed by using an ordinary flat resolution of X when the ideal $\mathfrak{a}$ is generated by a weakly pro-sequence sequence. Then we obtain the result by the duality between telescope and microscope and provide some first consequences. As we shall be mainly concerned with microscopes of inverse systems of Koszul complexes we simplify notations and introduce definitions.

Definitions and observations 7.5.1 Let $\underline{x} = x_1, \ldots, x_k$ denote a sequence of elements of a commutative ring R. Let X be any complex of R-modules. We construct the microscope with respect to the inverse system $\{K_\bullet(\underline{x}^t; X)\}$ and put

$$M(\underline{x}; X) = \mathrm{Mic}(\{K_\bullet(\underline{x}^t; X)\}).$$

This gives a functor $M(\underline{x}; \cdot)$ which is exact and preserves quasi-isomorphisms (see 4.2.8). In particular, $M(\underline{x}, X)$ is exact as soon as the complex X is exact.

We also put $M(\underline{x}) = M(\underline{x}; R) = \mathrm{Mic}(\{K_\bullet(\underline{x}^t)\})$. For an arbitrary R-module N note that $M(\underline{x}, N)_i = 0$ for $i \notin [-1, k]$.

Local homology is related to K-flat resolutions (see 7.1.6). Nevertheless we first look at flat complexes and flat resolutions.

Proposition 7.5.2 *Let $\underline{x} = x_1, \ldots, x_k$ be a sequence of elements in the commutative ring R and write $\underline{x}^t$ for the sequence $\underline{x} = x^t_1, \ldots, x^t_k$. Let X denote an R-complex.*

(a) There are natural morphisms $\alpha_X^{\underline{x}}$ *and* $\delta_X^{\underline{x}}$ *inserted in a commutative triangle*

$$\begin{tikzcd} & X \arrow[dl, "\alpha_X^{\underline{x}}"'] \arrow[dr] & \\ M(\underline{x}; X) \arrow[rr, "\delta_X^{\underline{x}}"'] & & \mathrm{Mic}(\{R/\underline{x}^t R \otimes_R X\}). \end{tikzcd}$$

(b) If $X = F$ *is a complex of flat* R*-modules and if the sequence* $\underline{x}$ *is weakly pro-regular, then the natural morphism* $\delta_F^{\underline{x}}$ *is a quasi-isomorphism.*

Proof For the proof of (a) we consider the inverse system of commutative triangles

$$\begin{tikzcd} & X \arrow[dl] \arrow[dr] & \\ K_\bullet(\underline{x}^t; X) \arrow[rr] & & R/\underline{x}^t R \otimes_R X. \end{tikzcd}$$

We take microscopes and compose with the natural quasi-isomorphism $X \xrightarrow{\sim} \mathrm{Mic}(X)$, where X is viewed as a constant inverse system.

For the proof of (b) we consider the inverse system of short exact sequences

$$0 \to C_\bullet(\underline{x}^t) \otimes_R F \to K_\bullet(\underline{x}^t) \otimes_R F \to R/\underline{x}^t R \otimes_R F \to 0$$

described in 7.3.5. Whence there is a short exact sequence of complexes

$$0 \to \mathrm{Mic}(\{C_\bullet(\underline{x}^t) \otimes_R F\}) \to M(\underline{x}; F) \to \mathrm{Mic}(\{R/\underline{x}^t R \otimes_R F\}) \to 0$$

(see 4.2.5). By 7.3.6 we know that for all $i \in \mathbb{Z}$ the inverse system of homology modules $\{H_i(C_\bullet(\underline{x}^t) \otimes_R F))\}$ is pro-zero. By 4.2.3 we now obtain that the complex $\mathrm{Mic}(\{C_\bullet(\underline{x}^t) \otimes_R F\})$ is exact. Whence the morphism $M(\underline{x}; F) \to \mathrm{Mic}(\{R/\underline{x}^t R \otimes_R F\})$ is a quasi-isomorphism. □

Corollary 7.5.3 *Let* $\mathfrak{a}$ *be an ideal of a commutative ring* R *generated by a weakly pro-regular sequence* $\underline{x} = x_1, \ldots, x_k$*. Let* X *be an arbitrary* R*-complex and* $F \xrightarrow{\sim} X$ *one of its flat resolutions. Then there are quasi-isomorphisms such that the following diagram is commutative*

$$\begin{tikzcd} F \arrow[r, equal] \arrow[d] & F \arrow[r, equal] \arrow[d] & F \arrow[r, "\sim"] \arrow[d] & X \arrow[d] \\ \Lambda^{\mathfrak{a}}(F) \arrow[r, "\sim"] & \mathrm{Mic}(\{R/\underline{x}^t R \otimes_R F\}) & M(\underline{x}; F) \arrow[l, "\sim"'] \arrow[r, "\sim"] & M(\underline{x}; X) \end{tikzcd}$$

where the vertical maps are the natural ones.

Proof We have the commutative square on the left and its bottom quasi-isomorphism by Lemma (4.2.3 (a)). We have the middle commutative square and its bottom quasi-isomorphism by Proposition 7.5.2. We have the commutative square on the right by naturality and its bottom quasi-isomorphism because the functor $M(\underline{x};\cdot)$ preserves quasi-isomorphisms (see 7.5.1). □

Here is another consequence of the preceding considerations. Note the slight but important difference in the hypotheses between the following and the previous 7.1.5.

Corollary 7.5.4 *Let $\mathfrak{a}$ denote an ideal of a commutative ring R generated by a weakly pro-regular sequence $\underline{x} = x_1, \ldots, x_k$. Let $h : F \xrightarrow{\sim} F'$ be a quasi-isomorphism between flat complexes. Then $\Lambda^{\mathfrak{a}}(h) : \Lambda^{\mathfrak{a}}(F) \xrightarrow{\sim} \Lambda^{\mathfrak{a}}(F')$ is also a quasi-isomorphism.*

Proof The claim follows from the commutative diagram

$$\begin{array}{ccccc}
\Lambda^{\mathfrak{a}}(F) & \xrightarrow{\sim} & \mathrm{Mic}(\{R/\underline{x}^t R \otimes_R F\}) & \xleftarrow{\sim} & M(\underline{x}; F) \\
\downarrow & & \downarrow & & \downarrow \\
\Lambda^{\mathfrak{a}}(F') & \xrightarrow{\sim} & \mathrm{Mic}(\{R/\underline{x}^t R \otimes_R F'\}) & \xleftarrow{\sim} & M(\underline{x}; F')
\end{array}$$

noting that the right vertical morphism is a quasi-isomorphism (see 7.5.1). □

Now we are ready for the descriptions of $\mathrm{L}\Lambda^{\mathfrak{a}}$ we had in mind.

Theorem 7.5.5 *Let $\mathfrak{a}$ be a finitely generated ideal of a commutative ring R and assume there is a weakly pro-regular sequence $\underline{x} = x_1, \ldots, x_k$ in R such that* $\mathrm{Rad}(\mathfrak{a}) = \mathrm{Rad}(\underline{x}R)$.

Then, for each R-complex X and each flat resolution $F \xrightarrow{\sim} X$ of X, we have that the complexes $M(\underline{x}; X)$ and $\Lambda^{\mathfrak{a}}(F)$ are quasi-isomorphic and both represent $\mathrm{L}\Lambda^{\mathfrak{a}}(X)$ *in the derived category. Hence*

$$\Lambda_i^{\mathfrak{a}}(X) \cong H_i(M(\underline{x}; X)) \cong H_i(\Lambda^{\mathfrak{a}}(F)) \text{ for all } i \in \mathbb{Z}.$$

Moreover, the natural map $\alpha_X^{\mathfrak{a}} : X \to \mathrm{L}\Lambda^{\mathfrak{a}}(X)$ in the derived category is represented by the composite $X \xleftarrow{\sim} F \longrightarrow \Lambda^{\mathfrak{a}}(F)$ and also by the natural morphism $\alpha_X^{\underline{x}} : X \to M(\underline{x}; X)$.

Proof We may and do assume $\mathfrak{a} = \underline{x}R$ (see 7.1.16). Let $P \xrightarrow{\sim} F$ be a DG-projective resolution of F, so that the composite $P \xrightarrow{\sim} F \xrightarrow{\sim} X$ provides a DG-projective resolution of X. Then $\mathrm{L}\Lambda^{\mathfrak{a}}(X)$ is represented by $\Lambda^{\mathfrak{a}}(P)$ and the natural morphism $\alpha_X^{\mathfrak{a}} : X \to \mathrm{L}\Lambda^{\mathfrak{a}}(X)$ is represented by the composite $X \xleftarrow{\sim} F \xleftarrow{\sim} P \longrightarrow \Lambda^{\mathfrak{a}}(P)$ (see 7.1.6). But there is a quasi-isomorphism $\Lambda^{\mathfrak{a}}(P) \xrightarrow{\sim} \Lambda^{\mathfrak{a}}(F)$ (see 7.5.4). Hence our statements are a direct consequence of 7.5.3. □

We note that the module case of the above theorem was already in [38] under slightly stronger hypotheses.

Corollary 7.5.6 *In the situation of 7.5.5 there are natural short exact sequences*

$$0 \to \varprojlim{}^1 H_{i+1}(\underline{x}^t; X)) \to \Lambda_i^{\mathfrak{a}}(X) \to \varprojlim H_i(\underline{x}^t; X) \to 0$$

for any R-complex X and all $i \in \mathbb{Z}$

Proof This is a direct consequence of the above theorem and of Lemma 4.2.3. $\square$

Corollary 7.5.7 *Let M be an R-module. With the assumptions of 7.5.5 it follows that $\Lambda_i^{\mathfrak{a}}(M) = 0$ for all $i > k$.*

Proof The proof follows since $M(\underline{x}; M)_i = 0$ for $i \notin [-1, k]$. $\square$

For an improved bound of the vanishing of $\Lambda_i^{\mathfrak{a}}(M)$, see the forthcoming 7.6.5.

Remark 7.5.8 In the situation of 7.5.5 let M denote an R-module. As the natural morphism $\alpha_M^{\underline{x}} : M \to M(\underline{x}; M)$ represents $\alpha_M^{\mathfrak{a}}$, we note that $\alpha_M^{\underline{x}}$ induces in homological degree 0 the natural homomorphism $\eta_M^{\mathfrak{a}} : M \to \Lambda_0^{\mathfrak{a}}(M)$.

In the case of $M = R$ it follows that the complex $M(\underline{x}) = M(\underline{x}; R)$ has its homology concentrated in degree 0, where it is $\hat{R}^{\mathfrak{a}}$. Hence $M(\underline{x})$ and $\hat{R}^{\mathfrak{a}}$ are quasi-isomorphic (for a more precise result, see 7.5.17).

If R is $\mathfrak{a}$-complete then the natural morphism $\alpha_R^{\underline{x}}$ is a quasi-isomorphism.

Now we prepare for a bridge between local and Čech homology.

Lemma 7.5.9 *Let $\underline{x} = x_1, \ldots, x_k$ be a sequence in a commutative ring R and let X denote an R-complex. There is a natural isomorphism of complexes* $\operatorname{Hom}_R(T(\underline{x}), X) \cong M(\underline{x}; X)$ *and a commutative triangle*

$$\begin{tikzcd} & X \arrow[dl, "\alpha_X^{\underline{x}}"'] \arrow[dr, "\operatorname{Hom}_R(\iota_{\underline{x}}^R, X)"] & \\ M(\underline{x}; X) \arrow[rr, equal, "\sim"] & & \operatorname{Hom}_R(T(\underline{x}), X). \end{tikzcd}$$

Proof First we recall 4.3.6: for all direct systems $\mathfrak{D}$ over $\mathbb{N}$ and all complexes X there is a natural isomorphism $\operatorname{Hom}_R(\operatorname{Tel}(\mathfrak{D}), X)) \cong \operatorname{Mic}(\operatorname{Hom}_R(\mathfrak{D}, X))$. Next we put $\mathfrak{D} = \{K^\bullet(\underline{x}^t)\}$ and obtain the isomorphism by taking into account that $\operatorname{Hom}_R(K^\bullet(\underline{x}^t), X) \cong K_\bullet(\underline{x}^t; X)$. We now check the commutativity of the triangle. To this end we consider the morphism of direct systems

$$\{K^\bullet(\underline{x}^t)\} \to R$$

where R is viewed as a constant direct system. Note first that the composite $T(\underline{x}) \to \operatorname{Tel}(R) \xrightarrow{\sim} \varinjlim R = R$ is the natural morphism $\iota_{\underline{x}}^R$ as described in 7.4.10. By naturality we have a commutative diagram

$$\begin{tikzcd}
X = \mathrm{Hom}_R(\varinjlim R, X) \arrow[r] \arrow[d, equal] & \mathrm{Hom}_R(\mathrm{Tel}(R), X) \arrow[r] \arrow[d] & \mathrm{Hom}_R(T(\underline{x}), X) \arrow[d] \\
X = \varprojlim \mathrm{Hom}_R(R, X) \arrow[r] & \mathrm{Mic}(X) \arrow[r] & M(\underline{x}; X)
\end{tikzcd}$$

where the two vertical morphisms on the right are the isomorphisms given by 4.3.6. In this diagram the composite of the top row is the morphism $\mathrm{Hom}_R(\iota^R_{\underline{x}}, X)$. By a little reflection we observe that the morphism

$$\mathrm{Mic}(X) \to M(\underline{x}, X)$$

in the bottom row stems from the morphism of inverse systems $X \to \{K_\bullet(\underline{x}^t; X)\}$. Hence the composite of the bottom row is the morphism $\alpha^{\underline{x}}_X$ described in 7.5.2. $\square$

We first use microscopes and telescopes to obtain another description of the Čech homology.

Theorem 7.5.10 *Let $\underline{x} = x_1, \ldots, x_k$ be a sequence in a commutative ring R and let X be an R-complex with one of its K-injective resolution $X \xrightarrow{\sim} I$. Then the complexes*

$$M(\underline{x}; X),\ \mathrm{Hom}_R(T(\underline{x}), X) \text{ and } \mathrm{Hom}_R(\check{C}_{\underline{x}}, I)$$

are quasi-isomorphic. Each of them represents $\mathrm{RHom}_R(\check{C}_{\underline{x}}, X)$ in the derived category. Moreover, the natural map $X \to \mathrm{RHom}_R(\check{C}_{\underline{x}}, X))$ in the derived category is represented by both natural morphisms

$$\alpha^{\underline{x}}_X : X \to M(\underline{x}; X) \text{ and } \mathrm{Hom}_R(\iota^R_{\underline{x}}, X) : X \to \mathrm{Hom}_R(T(\underline{x}), X).$$

Proof There are quasi-isomorphisms that make the following diagram commutative

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d] & X \arrow[r, "\sim"] \arrow[d] & I \arrow[r, equal] \arrow[d] & I \arrow[d] \\
M(\underline{x}; X) \arrow[r, equal, "\sim"] & \mathrm{Hom}_R(T(\underline{x}), X) \arrow[r, "\sim"] & \mathrm{Hom}_R(T(\underline{x}), I) & \mathrm{Hom}_R(\check{C}_{\underline{x}}, I) \arrow[l, "\sim"']
\end{tikzcd}$$

where the vertical morphisms are the natural ones. Note that the commutative square on the left is given by Lemma 7.5.9, while the others are given by naturality, taking into account that $T(\underline{x})$ is DG-projective and I is K-injective by assumption. As the natural map $X \to \mathrm{RHom}_R(\check{C}_{\underline{x}}, X)$ is represented by the composite $X \xrightarrow{\sim} I \to \mathrm{Hom}_R(\check{C}_{\underline{x}}, I)$ (see 6.4.1) the conclusion follows. $\square$

Remark 7.5.11 Let X be an R-complex with one of its K-injective resolutions $X \xrightarrow{\sim} I$ and recall the free resolution $\check{L}_{\underline{x}} \xrightarrow{\sim} \check{C}_{\underline{x}}$ obtained in 6.2.3. Then we have the quasi-isomorphisms

$$\mathrm{Hom}_R(\check{L}_{\underline{x}}, X) \xrightarrow{\sim} \mathrm{Hom}_R(\check{L}_{\underline{x}}, I) \xleftarrow{\sim} \mathrm{Hom}_R(\check{C}_{\underline{x}}, I).$$

Hence $\mathrm{RHom}_R(\check{C}_{\underline{x}}, X)$ is also represented by $\mathrm{Hom}_R(\check{L}_{\underline{x}}, X)$ and the natural map $X \to \mathrm{RHom}_R(\check{C}_{\underline{x}}, X)$ is also represented by the composite $X \xrightarrow{\sim} I \to \mathrm{Hom}_R(\check{L}_{\underline{x}}, I)$ or by the natural morphism $X \to \mathrm{Hom}_R(\check{L}_{\underline{x}}, X)$.

We now have the relation between the local homology with respect to an ideal generated by a weakly pro-regular sequence $\underline{x}$ and the Čech homology with respect to that sequence. Putting together Theorems 7.5.5, 7.5.10 and Remark 7.5.11 we obtain the following.

Theorem 7.5.12 *Let $\mathfrak{a}$ be a finitely generated ideal of a commutative ring R and $\underline{x} = x_1, \ldots, x_k$ be a sequence in R such that $\mathrm{Rad}(\mathfrak{a}) = \mathrm{Rad}(\underline{x}R)$. Assume that the sequence $\underline{x}$ is weakly pro-regular.*

Then $\mathrm{L}\Lambda^{\mathfrak{a}}(X)$ and $\mathrm{RHom}_R(\check{C}_{\underline{x}}, I)$ are isomorphic in the derived category for any R-complex X. They are represented by each of the quasi-isomorphic complexes

$$M(\underline{x}; X),\ \mathrm{Hom}_R(T(\underline{x}, X),\ \mathrm{Hom}_R(\check{L}_{\underline{x}}, X) \ \textit{and}\ \mathrm{Hom}_R(\check{C}_{\underline{x}}, I),$$

where $X \xrightarrow{\sim} I$ is a K-injective resolution. In particular,

$$\begin{aligned}\Lambda_i^{\mathfrak{a}}(X) &\cong H_i(\mathrm{Hom}_R(T(\underline{x}), X)) \cong H_i(\mathrm{Hom}_R(\check{L}_{\underline{x}}, X)) \\ &\cong H_i(\mathrm{Hom}_R(\check{C}_{\underline{x}}, I)) \cong \mathrm{Ext}_R^{-i}(\check{C}_{\underline{x}}, X).\end{aligned}$$

Moreover, the natural map $\alpha_X^{\mathfrak{a}} : X \to \mathrm{L}\Lambda^{\mathfrak{a}}(X)$ in the derived category, which is represented by $\alpha_X^{\underline{x}} : X \to M(\underline{x}; X)$, is also represented by the composite $X \xrightarrow{\sim} I \to \mathrm{Hom}_R(\check{C}_{\underline{x}}, I)$ or by the composite $X \xrightarrow{\sim} I \to \mathrm{Hom}_R(\check{L}_{\underline{x}}, I)$, and by the natural morphism $\mathrm{Hom}_R(\iota_{\underline{x}}^R, X) : X \to \mathrm{Hom}_R(T(\underline{x}), X)$ or by the natural morphism $X \to \mathrm{Hom}_R(\check{L}_{\underline{x}}, X)$.

In the above theorem the fact that $\mathrm{L}\Lambda^{\mathfrak{a}}(X)$ is represented by $\mathrm{Hom}_R(\check{L}_{\underline{x}}, X)$ is also proved in [64] by quite different methods, a reduction to the case when X is a flat R-module. The fact that $\mathrm{L}\Lambda^{\mathfrak{a}}(X)$ is represented by $\mathrm{Hom}_R(\check{C}_{\underline{x}}, I)$ was already proved in [1] under slightly stronger hypotheses, and in [75] when X is bounded.

Here is the local homology counterpart of Corollary 7.4.7. Note also that the following generalizes results previously known when the ring is Noetherian and already observed in 2.5.15, in 2.5.17 and in 2.5.18.

Corollary 7.5.13 *In the situation of 7.5.12 let M be an R-module and X an R-complex.*

(a) M is $\mathfrak{a}$-pseudo-complete if and only if the natural map $\alpha_M^{\mathfrak{a}} : M \to \mathrm{L}\Lambda^{\mathfrak{a}}(M)$ is an isomorphism in the derived category. In that case, $\Lambda_i^{\mathfrak{a}}(M) = 0$ for all $i \neq 0$.

(b) The pair $(R, \mathfrak{a})$ has the property C introduced in 2.6.10, that is, every $\mathfrak{a}$-complete module M belongs to the class $\mathcal{C}_{\mathfrak{a}}$ introduced in 2.5.12.

(c) *Let* $0 \to M_1 \to M_2 \to M_3 \to 0$ *be a short exact sequence of* R*-modules. If two of the three modules* M_1, M_2, M_3 *are* $\mathfrak{a}$*-pseudo-complete, so is the third.*
(d) *If* X *is a complex of* $\mathfrak{a}$*-pseudo-complete* R*-modules, then its homology modules* $H_i(X)$ *are* $\mathfrak{a}$*-pseudo-complete for all* $i \in \mathbb{Z}$.

Proof Recall that the natural map $\alpha_M^{\mathfrak{a}} : M \to \mathrm{L}\Lambda^{\mathfrak{a}}(M)$ induces the homomorphism $\eta_M^{\mathfrak{a}}$ in homological degree 0 (see 7.1.9). Hence, if $\alpha_M^{\mathfrak{a}}$ is an isomorphism in the derived category then M is $\mathfrak{a}$-pseudo-complete and $\Lambda_i^{\mathfrak{a}}(M) = 0$ for all $i \neq 0$.

Assume now that M is $\mathfrak{a}$-pseudo-complete. Then $\operatorname{Ext}_R^j(\oplus_{i=1}^k R_{x_i}, M) = 0$ for all $j \geq 0$ by 3.1.8. If $M \xrightarrow{\sim} I$ is an injective resolution of the R-module M it follows that the composite $M \xrightarrow{\sim} I \to \operatorname{Hom}_R(\check{C}_{\underline{x}}, I)$ is a quasi-isomorphism (see 6.4.2). Hence $\alpha_M^{\mathfrak{a}}$ is an isomorphism in the derived category.

For the assertion in (b) recall 2.5.7, valid when the ideal $\mathfrak{a}$ is finitely generated: if M is $\mathfrak{a}$-complete then M is also $\mathfrak{a}$-pseudo-complete and the natural homomorphism $\eta_M^{\mathfrak{a}} : M \to \Lambda_0^{\mathfrak{a}}(M)$ is an isomorphism. It follows that the natural homomorphism $\gamma_M^{\mathfrak{a}} : \Lambda_0^{\mathfrak{a}}(M) \to \hat{M}^{\mathfrak{a}}$ is also an isomorphism. This together with (a) gives (b).

The assertion in (c) follows by the assertion in (a) and the long exact homology sequence of the $\Lambda_i^{\mathfrak{a}}$'s.

Now the proof of the assertion in (d) is the same as that in 2.5.18. □

Also important is the following. Note that in general $\hat{R}^{\mathfrak{a}}$ is not R-flat (see Example 2.8.7).

Corollary 7.5.14 *Let* $\mathfrak{a}$ *be an ideal of a commutative ring* R *generated by the sequence* $\underline{x} = (x_1, \ldots, x_k)$*. If this sequence is weakly pro-regular on* R *it is also weakly pro-regular on* $\hat{R}^{\mathfrak{a}}$.

Proof This is a direct consequence of the above and 5.2.6. □

Here is a partial converse of the previous Theorem 7.5.12 and another characterization of weakly pro-regular sequences when the ring is coherent. For the notion of a coherent ring, see 1.4.2. Remember also Lemma 1.4.5: for two injective R-modules I, J over a coherent ring R the R-module $\operatorname{Hom}_R(I, J)$ is R-flat.

Proposition 7.5.15 *Let* $\underline{x} = x_1, \ldots, x_k$ *denote a sequence of elements of a coherent ring* R*. Let* $\mathfrak{a} = \underline{x}R$*. Then the following conditions are equivalent:*

(i) *The sequence* $\underline{x}$ *is weakly pro-regular.*
(ii) $\Lambda_i^{\mathfrak{a}}(M) \cong H_i(\operatorname{RHom}_R(\check{C}_{\underline{x}}, M)) \cong H_i(\operatorname{Hom}_R(T(\underline{x}), M))$ *for all* R*-modules* M.
(iii) $H_i(\operatorname{Hom}_R(T(\underline{x}), F)) = 0$ *for any flat* R*-module* F *and all* $i \neq 0$.

Proof The implication (i) $\Longrightarrow$ (ii) is a direct consequence of Theorem 7.5.12 and the implication (ii) $\Longrightarrow$ (iii) is obvious.

Assume now that condition (iii) is satisfied. Let I denote an injective R-module and consider its general Matlis dual $I^{\vee}$, which is a flat R-module (see 1.4.5). We have an isomorphism and a quasi-isomorphism

$$\operatorname{Hom}_R(T(\underline{x}), I^\vee) \cong (T(\underline{x}) \otimes_R I)^\vee \xleftarrow{\sim} (\check{C}_{\underline{x}} \otimes_R I)^\vee.$$

Because the general Matlis duality functor is faithfully exact it follows that $H^i(\check{C}_{\underline{x}} \otimes_R I) = 0$ for all $i \neq 0$ and any injective R-module I. By 7.3.3 this shows that the sequence $\underline{x}$ is weakly pro-regular. $\square$

We do not know whether the previous result holds without the assumption that R is coherent. This was erroneously claimed in [75].

Next there will be a dual version of 7.4.1. In a certain sense it provides a more direct argument for the presentation of $\mathrm{L}\Lambda^{\mathfrak{a}}(X)$ in the derived category.

Proposition 7.5.16 *Let $\underline{x} = x_1, \ldots, x_k$ denote a weakly pro-regular sequence of a commutative ring R and $\mathfrak{a} = \underline{x}R$. Let F be a complex of flat R-modules. Then there is a natural quasi-isomorphism of complexes*

$$\operatorname{Hom}_R(\check{L}_{\underline{x}}, F) \xrightarrow{\sim} \Lambda^{\mathfrak{a}}(F)$$

that is degree-wise surjective. The composition with the natural morphism $F \to \operatorname{Hom}_R(\check{L}_{\underline{x}}, F)$ is the natural morphism $\tau_F^{\mathfrak{a}} : F \to \Lambda^{\mathfrak{a}}(F)$.

Proof First note that the natural morphism $\check{L}_{\underline{x}} \to R$ (see 6.2.6) induces a morphism $X \to \operatorname{Hom}_R(\check{L}_{\underline{x}}, X)$ for any R-complex X. Now recall that $\check{L}^i_{\underline{x}} = 0$ for all $i < 0$ and all $i > k$. Therefore, for a flat R-module G we obtain a complex

$$\operatorname{Hom}_R(\check{L}_{\underline{x}}, G) : 0 \to \operatorname{Hom}_R(\check{L}^k_{\underline{x}}, G) \to \ldots \to \operatorname{Hom}_R(\check{L}^0_{\underline{x}}, G) \to 0.$$

By virtue of 7.5.12 it follows that $H_i(\operatorname{Hom}_R(\check{L}_{\underline{x}}, G)) = 0$ for all $i \neq 0$ and

$$H_0(\operatorname{Hom}_R(\check{L}_{\underline{x}}, G)) = \Lambda^{\mathfrak{a}}(G).$$

Thus $\operatorname{Hom}_R(\check{L}_{\underline{x}}, G)$ is a left resolution of $\Lambda^{\mathfrak{a}}(G)$ and induces a natural quasi-isomorphism $\operatorname{Hom}_R(\check{L}_{\underline{x}}, G) \xrightarrow{\sim} \Lambda^{\mathfrak{a}}(G)$. Recall also that the natural morphism $G \to \operatorname{Hom}_R(\check{L}_{\underline{x}}, G)$, which represents the natural map $\alpha_G^{\mathfrak{a}}$ (see 7.5.12), induces in homological degree zero the homomorphism $\eta_G^{\mathfrak{a}}$ (see 7.1.9) and that $\eta_G^{\mathfrak{a}} = \tau_G^{\mathfrak{a}}$ because G is flat. Hence the composite $G \to \operatorname{Hom}_R(\check{L}_{\underline{x}}, G) \xrightarrow{\sim} \Lambda^{\mathfrak{a}}(G)$ is the homomorphism $\tau_G^{\mathfrak{a}}$.

Now let F denote a complex of flat R-modules. Let F^i, $i \in \mathbb{Z}$, denote the ith module of the complex F. By the above the complex $\operatorname{Hom}_R(\check{L}_{\underline{x}}, F^i)$ is a left resolution of $\Lambda^{\mathfrak{a}}(F^i)$ for all $i \in \mathbb{Z}$. Clearly it is functorial. Whence there is a commutative diagram

$$\begin{array}{ccc} \operatorname{Hom}_R(\check{L}_{\underline{x}}, F^{i+1}) & \xrightarrow{\sim} & \Lambda^{\mathfrak{a}}(F^{i+1}) \\ \uparrow & & \uparrow \\ \operatorname{Hom}_R(\check{L}_{\underline{x}}, F^{i}) & \xrightarrow{\sim} & \Lambda^{\mathfrak{a}}(F^{i}) \end{array}$$

where the horizontal morphisms are quasi isomorphisms. These data induce a morphism of complexes

$$h : \operatorname{Hom}_R(\check{L}_{\underline{x}}, F) \to \Lambda^{\mathfrak{a}}(F) \text{ with}$$
$$\operatorname{Hom}_R(\check{L}_{\underline{x}}, F)^n := \prod_i \operatorname{Hom}_R(\check{L}^i_{\underline{x}}, F^{i+n}) \to \Lambda^{\mathfrak{a}}(F^n)$$

which is degree-wise surjective. To finish the proof we observe that the cone $C(h)$ is the single complex associated to the double complex defined by $K^{i,j} = \operatorname{Hom}_R(\check{L}^{-i}_{\underline{x}}, F^j)$ for $-k \le i \le 0$ and $K^{1,j} = \Lambda^{\mathfrak{a}}(F^j)$. This double complex has exact rows and only a finite number of non-zero columns, hence its second spectral sequence is zero and convergent. Therefore $C(h)$ is exact, h is a quasi-isomorphism and the last claim follows. □

With the hypotheses of 7.5.16 we have seen in 7.5.3 that the complexes $M(\underline{x}; F)$ and $\Lambda^{\mathfrak{a}}(F)$ are quasi-isomorphic. We now have a more precise statement.

Corollary 7.5.17 *Let $\mathfrak{a}$ be an ideal of a commutative ring R generated by a weakly pro-regular sequence $\underline{x} = x_1, \dots, x_k$ and let F be a complex of flat R-modules. Then there is a quasi-isomorphism of complexes*

$$M(\underline{x}; F) \xrightarrow{\sim} \Lambda^{\mathfrak{a}}(F)$$

such that the composition with the natural morphism $F \to M(\underline{x}; F)$ is homotopic to the natural morphism $\tau^{\mathfrak{a}}_F : F \to \Lambda^{\mathfrak{a}}(F)$.

In particular, there is a quasi-isomorphism $M(\underline{x}) \xrightarrow{\sim} \Lambda^{\mathfrak{a}}(R)$. Its composition with the natural morphism $R \to M(\underline{x}; R)$ is the natural homomorphism $\tau^{\mathfrak{a}}_R$.

Proof We first recall that $M(\underline{x}; F) \cong \operatorname{Hom}_R(T(\underline{x}), F)$ (see 7.5.9). Then we choose a quasi-isomorphism $h : \check{L}_{\underline{x}} \xrightarrow{\sim} T(\underline{x})$ such that the triangle

$$\begin{tikzcd} \check{L}_{\underline{x}} \arrow[rr, "h"] \arrow[dr] & & T(\underline{x}) \arrow[dl] \\ & \check{C}_{\underline{x}}, & \end{tikzcd}$$

where the oblique morphisms are the natural ones, is commutative up to homotopy (such a quasi-isomorphism exists by 4.4.6). It induces a quasi-isomorphism $\operatorname{Hom}_R(T(\underline{x}), F) \xrightarrow{\sim} \operatorname{Hom}_R(\check{L}_{\underline{x}}, F)$ (see 4.4.11) which we compose with the quasi-isomorphism $\operatorname{Hom}_R(\check{L}_{\underline{x}}, F) \xrightarrow{\sim} \Lambda^{\mathfrak{a}}(F)$ obtained in 7.5.16. This gives the required quasi-isomorphism. Then we observe that the triangle

$$\begin{tikzcd} & F \arrow[dl] \arrow[dr] & \\ \operatorname{Hom}_R(T(\underline{x}), F) \arrow[rr, "\sim"] & & \operatorname{Hom}_R(\check{L}_{\underline{x}}, F), \end{tikzcd}$$

where the oblique morphisms are the natural ones, is also commutative up to homotopy by the choice of h. This together with the commutative triangle in 7.5.9 yields the last claims. □

As a first application we compute the local homology of a finitely presented module over a coherent ring.

Proposition 7.5.18 *Let R denote a coherent ring and let $\mathfrak{a} \subset R$ be an ideal generated by a weakly pro-regular sequence $\underline{x} = x_1, \ldots, x_k$. Let N denote a finitely presented R-module. Then*

$$\Lambda_i^{\mathfrak{a}}(N) \cong \operatorname{Tor}_i^R(\hat{R}^{\mathfrak{a}}, N) \text{ for all } i \geq 0.$$

Proof We have $\Lambda_i^{\mathfrak{a}}(N) \cong H_i(M(\underline{x}; N))$ because the sequence $\underline{x}$ is weakly pro-regular and $M(\underline{x}; N) = \operatorname{Mic}(K_\bullet(\underline{x}^t) \otimes_R N) \cong M(\underline{x}) \otimes_R N$ because N is finitely presented (see 4.2.9). We also have a quasi-isomorphism $M(\underline{x}) \xrightarrow{\sim} \hat{R}^{\mathfrak{a}}$, see 7.5.17. On the other hand $M(\underline{x})$ is a bounded complex of flat R-modules because R is coherent (see 1.4.4). It follows that $H_i(M(\underline{x}; N)) \cong H_i(M(\underline{x}) \otimes_R N) \cong \operatorname{Tor}_i^R(\hat{R}^{\mathfrak{a}}, N)$. □

We end the section with a remark on the local homology of products.

Proposition 7.5.19 *Let $\mathfrak{a}$ be an ideal of a commutative ring R generated by a weakly pro-regular sequence $\underline{x} = x_1, \ldots, x_k$. Then $\mathrm{L}\Lambda^{\mathfrak{a}}$ commutes with products: for a family of R-complexes X_α, $\alpha \in I$, we have that $\mathrm{L}\Lambda^{\mathfrak{a}}(\prod_\alpha X_\alpha) \simeq \prod_\alpha \mathrm{L}\Lambda^{\mathfrak{a}}(X_\alpha)$ in the derived category. In particular,*

$$\Lambda_i^{\mathfrak{a}}(\prod_\alpha X_\alpha) \cong \prod_\alpha \Lambda_i^{\mathfrak{a}}(X_\alpha)$$

for all $i \in \mathbb{Z}$.

Proof Since $\operatorname{Hom}_R(T(\underline{x}), \prod_\alpha X_\alpha) \cong \prod_\alpha \operatorname{Hom}_R(T(\underline{x}), X_\alpha)$ this is also a consequence of Theorem 7.5.12. □

For the local homology of an inverse system with respect to an ideal generated by a weakly pro-regular sequence and again in view of Theorem 7.5.12 we only refer to 6.3.7.

7.6 Depth and Codepth with Local (Co-)Homology

In this section we reinterpret the results of Chapter 6, section 6, in terms of local homology and cohomology. That is, we provide lower and upper bounds for the non-vanishing of $\Lambda_i^{\mathfrak{a}}(X)$ and $H_{\mathfrak{a}}^i(X)$ when the ideal $\mathfrak{a}$ is generated by a weakly pro-regular sequence. While the lower bounds are known, at least when the ring is Noetherian (see [33]), our upper bounds seem to be new. We also investigate the behaviour of the Ext-depth. When the ring is Noetherian, we also present another upper bound for the non-vanishing of $\Lambda_i^{\mathfrak{a}}(X)$, due to Frankild (see [34]). Via the Matlis duality this also provides another upper bound for the non-vanishing of $H_{\mathfrak{a}}^i(X)$.

Definition 7.6.1 We define the local depth of an R-complex X with respect to an ideal $\mathfrak{a}$ of R by

$$h_{\mathfrak{a}}^-(X) = \inf\{i \in \mathbb{Z} \mid H_{\mathfrak{a}}^i(X) \neq 0\}.$$

Also we define the local codepth of the R-complex X with respect to the ideal $\mathfrak{a}$ by

$$\lambda_-^{\mathfrak{a}}(X) = \inf\{i \in \mathbb{Z} \mid \Lambda_i^{\mathfrak{a}}(X) \neq 0\}.$$

Here is a generalization of Foxby and Iyengar's result [33, Theorem 2.1, Theorem 4.1], originally stated for Noetherian rings, and the promised extension of 7.2.10 and 7.2.11.

Theorem 7.6.2 *Let $\mathfrak{a}$ be an ideal of a commutative ring R generated by a weakly pro-regular sequence $\underline{x} = x_1, \ldots, x_k$. Then for any R-complex X we have*

$$h_{\mathfrak{a}}^-(X) = \text{E-dp}(\mathfrak{a}, X) \text{ and}$$
$$\lambda_-^{\mathfrak{a}}(X) = \text{T-codp}(\mathfrak{a}, X).$$

Proof By definition $H_{\mathfrak{a}}^i(X) = H^i(\mathrm{R}\Gamma_{\mathfrak{a}}(X))$ and $\Lambda_i^{\mathfrak{a}}(X) = H_i(\mathrm{L}\Lambda^{\mathfrak{a}}(X))$. When the sequence $\underline{x}$ is weakly pro-regular we have seen in 7.4.4 that $\mathrm{R}\Gamma_{\mathfrak{a}}(X)$ is represented by $\check{C}_{\underline{x}} \otimes_R X$, while $\mathrm{L}\Lambda^{\mathfrak{a}}(X) \simeq \mathrm{RHom}_R(\check{C}_{\underline{x}}, X)$ in the derived category (see 7.5.12). Hence the result is a direct consequence of 6.6.2 and 6.6.3. □

As for the Čech homology and cohomology we consider some supremum analogous to the infimum in 7.6.1.

Definition 7.6.3 For any ideal of a commutative ring R and any R-complex X we write

$$h_{\mathfrak{a}}^+(X) := \sup\{i \in \mathbb{Z} \mid H_{\mathfrak{a}}^i(X) \neq 0\} \text{ and } \lambda_+^{\mathfrak{a}}(X) := \sup\{i \in \mathbb{Z} \mid \Lambda_i^{\mathfrak{a}}(X) \neq 0\}.$$

These quantities can be viewed as a kind of relative dimension.

The following is reformulation of Theorem 6.6.5. It is a generalization of [79, Theorem 3.7] originally stated for modules over a Noetherian ring. It will also provide some bounds for the non-vanishing of the local homology and cohomology of an R-module.

Theorem 7.6.4 *Let $\mathfrak{a}$ be an ideal of a commutative ring R generated by the weakly pro-regular sequence $\underline{x} = x_1, \ldots, x_k$. Let $\varphi : R \to B$ be a homomorphism of commutative rings and let $\underline{y} = y_1, \ldots, y_l$ be a sequence in B such that* $\mathrm{Rad}(\underline{y}B) = \mathrm{Rad}(\varphi(\underline{x})B)$*, $l \leq k$. Let X be a B-complex viewed as an R complex via φ. Then*

$$h^{+}_{\mathfrak{a}}(X) \le l - \text{T-codp}(\underline{x}R, X) = l - \lambda^{\mathfrak{a}}_{-}(X) \textit{ and}$$
$$\lambda^{\mathfrak{a}}_{+}(X) \le l - \text{E-dp}(\underline{x}R, X) = l - h^{-}_{\mathfrak{a}}(X).$$

Proof In view of 7.4.4 and 7.5.12 this is a consequence of 6.6.5 and 7.6.2. □

We also obtain the bounds announced in 7.4.6 and in 7.5.7.

Corollary 7.6.5 *Let $\mathfrak{a}$ be an ideal of a commutative ring R generated by a weakly pro-regular sequence $\underline{x} = x_1, \dots, x_k$. Let M denote an R-module. Assume there is a sequence $\underline{y} = y_1, \dots, y_l$, $l \le k$ such that* $\operatorname{Rad}(\underline{x}R + \operatorname{Ann}_R M) = \operatorname{Rad}(\underline{y}R + \operatorname{Ann}_R M)$. *Then*

$$H^{i}_{\mathfrak{a}}(M) = 0 \textit{ for all } i > l - \text{T-codp}(\mathfrak{a}, M) \textit{ and}$$
$$\Lambda^{\mathfrak{a}}_{i}(M) = 0 \textit{ for all } i > l - \text{E-dp}(\mathfrak{a}R, M).$$

Proof This is a reformulation of 6.6.7, which also contains more information. □

We now investigate the behaviour of the Ext-depth and Tor-codepth under local homology and cohomology. Recall that two quasi-isomorphic complexes have the same Ext-depth and the same Tor-codepth, so that we may speak of the Ext-depth and Tor-codepth of $\mathrm{R}\Gamma_{\mathfrak{a}}(X)$ and $\mathrm{L}\Lambda^{\mathfrak{a}}(X)$ without ambiguity.

Proposition 7.6.6 *Let $\mathfrak{a}$ be a finitely generated ideal of a commutative ring R and $\underline{x} = x_1, \dots, x_k$ be a weakly pro-regular sequence such that* $\operatorname{Rad}(\mathfrak{a}) = \operatorname{Rad}(\underline{x}R)$. *Let X be an R-complex. Then*

(a) $\text{E-dp}(\mathfrak{a}, X) = \text{E-dp}(\mathfrak{a}, \mathrm{R}\Gamma_{\mathfrak{a}}(X)) = \text{E-dp}(\mathfrak{a}, \mathrm{L}\Lambda^{\mathfrak{a}}(X))$,
(b) $\text{T-codp}(\mathfrak{a}, X) = \text{T-codp}(\mathfrak{a}, \mathrm{R}\Gamma_{\mathfrak{a}}(X)) = \text{T-codp}(\mathfrak{a}, \mathrm{L}\Lambda^{\mathfrak{a}}(X))$, *and in particular,*
(c) $\text{E-dp}(\mathfrak{a}, R) = \text{E-dp}(\mathfrak{a}, \hat{R}^{\mathfrak{a}})$.

Proof By 7.4.4 and 7.5.12 $\mathrm{R}\Gamma_{\mathfrak{a}}(X)$ is represented by $\check{C}_{\underline{x}} \otimes_R X$ and $\mathrm{L}\Lambda^{\mathfrak{a}}(X)$ is represented by $\operatorname{Hom}_R(\check{C}_{\underline{x}}, I)$, where $X \xrightarrow{\sim} I$ is a K-injective resolution. Hence the claims follow by 6.6.2, 6.6.3 and 6.6.8. Note also that $\mathrm{L}\Lambda^{\mathfrak{a}}(R)$ is represented by $\hat{R}^{\mathfrak{a}}$. □

Note, however, that the Ext-depth and the Tor-codepth are not preserved by the $\mathfrak{a}$-torsion functor and the $\mathfrak{a}$-completion functor. For example, let $(R, \mathfrak{m}, \Bbbk)$ be a Noetherian local ring of positive depth and let E be the injective hull of the residue field. Then $0 < \text{E-dp}(\mathfrak{m}, R) = \text{T-codp}(\mathfrak{m}, E) < \infty$ and $\text{T-codp}(\mathfrak{m}, R) = 0 = \text{E-dp}(\mathfrak{m}, E)$. But $\Gamma_{\mathfrak{m}}(R) = 0 = \Lambda^{\mathfrak{m}}(E)$.

When the ring R is Noetherian, we also have another bound for $\lambda^{\mathfrak{a}}_{+}(X)$, due to Frankild. To present it we first recall Grothendieck's vanishing theorem. It involves the dimension of an R-module M, defined by $\dim_R M = \dim \operatorname{Supp}_R M$. Note that if M is a finitely generated R-module then $\dim_R M = \dim_R R/\operatorname{Ann}_R M$ (see e.g. [58] for some details). Here is Grothendieck's vanishing result.

Theorem 7.6.7 (see [40, 41]) *Let $\mathfrak{a}$ denote an ideal of a Noetherian ring R. Let M be an R-module with $d = \dim_R M$. Then $H^i_{\mathfrak{a}}(M) = 0$ for all $i > d$.*

Proof First note that we may assume $d < \infty$ because otherwise there is nothing to prove. Let $\mathfrak{m}$ be a maximal ideal of R. Then $H^i_{\mathfrak{a}R_{\mathfrak{m}}}(M_{\mathfrak{m}}) \cong H^i_{\mathfrak{a}}(M) \otimes_R R_{\mathfrak{m}}$ for all $i \in \mathbb{N}$. Hence we may assume that R is a Noetherian local ring since $\dim M_{\mathfrak{m}} \leq d$ and $H^i_{\mathfrak{a}}(M) = 0$ if and only if $H^i_{\mathfrak{a}}(M) \otimes_R R_{\mathfrak{m}}$ vanishes for all $\mathfrak{m}$.

Moreover, M is the direct limit $\varinjlim M_\lambda$ of its finitely generated submodules. Since $\operatorname{Supp}_R M_\lambda \subseteq \operatorname{Supp}_R M$ we may assume that M is finitely generated in view of 7.4.8.

Therefore we have to prove $H^i_{\mathfrak{a}}(M) = 0$ for $i > \dim_R M$ when M is a finitely generated R-module over a Noetherian local ring $(R, \mathfrak{m})$. We proceed by induction on $d = \dim_R M$. First let $d = 0$, then M is of $\mathfrak{m}$-torsion and therefore also of $\mathfrak{a}$-torsion. Whence $H^i_{\mathfrak{a}}(M) = 0$ for $i > 0$ (see 7.4.7). Now let $d \geq 1$ and $N = H^0_{\mathfrak{a}}(M)$. The short exact sequence $0 \to N \to M \to M/N \to 0$ provides isomorphisms $H^i_{\mathfrak{a}}(M) \cong H^i_{\mathfrak{a}}(M/N)$ for $i > 0$. Because $\dim_R M \geq \dim_R M/N$ we may assume that $H^0_{\mathfrak{a}}(M) = 0$. That is, there exists an M-regular element $x \in \mathfrak{a}$. Because $\dim_R M/xM = d - 1$ we have $H^i_{\mathfrak{a}}(M/xM) = 0$ for $i > d - 1$ by the induction hypothesis. The short exact sequence $0 \to M \xrightarrow{x} M \to M/xM \to 0$ yields injections $0 \to H^i_{\mathfrak{a}}(M) \xrightarrow{x} H^i_{\mathfrak{a}}(M)$ for $i > d$. Since $H^i_{\mathfrak{a}}(M)$ is of $\mathfrak{a}$-torsion (see 7.1.12) this implies the vanishing of $H^i_{\mathfrak{a}}(M)$ for $i > d$. □

Corollary 7.6.8 *Let $\mathfrak{a}$ be an ideal of a Noetherian ring generated by a sequence $\underline{x} = x_1, \ldots, x_k$. Then $H^i(\check{C}_{\underline{x}}) = 0$ for all $i > \dim R$.*

Proof This follows by 7.6.7 because $H^i(\check{C}_{\underline{x}}) = H^i_{\mathfrak{a}}(R)$ when R is Noetherian (see 7.4.4 and recall that any finite sequence in a Noetherian ring is weakly pro-regular). □

We need the following well-known lemma. It is a variant of [30, Lemma 2.1].

Lemma 7.6.9 *Let R denote a commutative ring and let P be a bounded non-exact complex of projective R-modules. Then*

$$\inf\{i \mid H^i(\operatorname{Hom}_R(P, X) \neq 0\} \geq \inf\{i \mid H^i(X) \neq 0\} - \sup\{i \mid H^i(P) \neq 0\}$$

for any R-complex X.

Proof First note that $\sup\{i \mid H^i(P) \neq 0\}$ is finite by the hypothesis on P.

If $\inf\{i \mid H^i(X) \neq 0\} = -\infty$ there is nothing to prove. If $\inf\{i \mid H^i(X) \neq 0\} = \infty$ then X is exact and so is $\operatorname{Hom}_R(P, X)$. It follows that $\inf\{i \mid H^i(\operatorname{Hom}_R(P, X) \neq 0\} = \infty$.

Assume now that $\inf\{i \mid H^i(X) \neq 0\}$ is finite. We write $r = \inf\{i \mid H^i(X) \neq 0\}$ and $s = \sup\{i \mid H^i(P) \neq 0\}$. By cancellation there is a quasi-isomorphism $P' \xrightarrow{\sim} P$, where P' is a bounded non-exact complex of projective R-modules with $H^i(P') = 0$ for all $i > s$. There is also a quasi-isomorphism $X \xrightarrow{\sim} I$, where I is a left-bounded complex of injective R-modules with $H^j(I) = 0$ for all $j < r$ (see 1.1.12).

Then $H^i(\mathrm{Hom}_R(P, X)) \cong H^i(\mathrm{Hom}_R(P', I))$ for all $i \in \mathbb{Z}$ and $\mathrm{Hom}_R(P', I)$ is left-bounded. Note also that $(\mathrm{Hom}_R(P', I))^n = \prod_{i+j=n} \mathrm{Hom}_R(P'^{-i}, I^j) \neq 0$ only if there exist i, j such that $j \geq r$ and $i \geq -s$. Hence $(\mathrm{Hom}_R(P', I))^n = 0$ for all $n < r - s$ and the claim follows. □

We are now prepared to present Frankild's result.

Theorem 7.6.10 (see [34, 2.12]) *Let R denote a Noetherian ring. Let $\mathfrak{a}$ be a proper ideal of R. Then*

$$\lambda_+^{\mathfrak{a}}(X) \leq \dim R - \text{E-dp}(\mathfrak{a}, X)$$

for all R-complexes X.

Proof Let $\underline{x} = x_1, \ldots, x_k$ be a sequence generating the ideal $\mathfrak{a}$. We recall the bounded free resolution $\check{L}_{\underline{x}}$ of $\check{C}_{\underline{x}}$ obtained in 6.2.3. Then $\mathrm{L}\Lambda^{\mathfrak{a}}(X)$ is represented by $\mathrm{Hom}_R(\check{L}_{\underline{x}}, X) \simeq \mathrm{Hom}_R(\check{L}_{\underline{x}}, \check{C}_{\underline{x}} \otimes_R X)$ (see 7.5.12 and 6.5.4). Hence

$$\begin{aligned}\lambda_+^{\mathfrak{a}}(X) &= \sup\{i \mid H_i(\mathrm{Hom}_R(\check{L}_{\underline{x}}, \check{C}_{\underline{x}} \otimes_R X)) \neq 0\} \\ &= -\inf\{i \mid H^i(\mathrm{Hom}_R(\check{L}_{\underline{x}}, \check{C}_{\underline{x}} \otimes_R X)) \neq 0\}.\end{aligned}$$

The conclusion follows by applying Lemma 7.6.9 to the complexes $\check{L}_{\underline{x}}$ and $\check{C}_{\underline{x}} \otimes_R X$. Note that $\check{L}_{\underline{x}}$ is not exact (since $\mathfrak{a}$ is proper), that $\sup\{i \mid H^i(\check{L}_{\underline{x}}) \neq 0\} \leq \dim R$ (see 7.6.8) and recall that $\inf\{i \mid H^i(\check{C}_{\underline{x}} \otimes_R X) \neq 0\} = \text{E-dp}(\mathfrak{a}, X)$ (see 6.6.2). □

On the local cohomology side we get the following.

Corollary 7.6.11 *Let R denote a Noetherian ring and let $\mathfrak{a}$ be a proper ideal of R. Then*

$$h_{\mathfrak{a}}^+(X) \leq \dim R - \text{T-codp}(\mathfrak{a}, X)$$

for all R-complexes X.

Proof First recall that $\mathrm{R}\Gamma_{\mathfrak{a}}(X)$ is represented by $\check{L}_{\underline{x}} \otimes_R X$ (see 7.4.4). By adjointness note that $(\check{L}_{\underline{x}} \otimes_R X)^\vee \cong \mathrm{Hom}_R(\check{L}_{\underline{x}}, X^\vee)$, where $(\cdot)^\vee$ denotes the general Matlis duality functor. Hence $(H_{\mathfrak{a}}^i(X))^\vee \cong \Lambda_i^{\mathfrak{a}}(X^\vee)$ and $h_{\mathfrak{a}}^+(X) = \lambda_+^{\mathfrak{a}}(X^\vee)$. Now the claim follows by Theorem 7.6.10 applied to the complex $X^\vee$. Recall that $\text{E-dp}(\mathfrak{a}, X^\vee) = \text{T-codp}(\mathfrak{a}, X)$ (see 5.3.6). □

The above results also have some unexpected consequences.

Corollary 7.6.12 *Let R denote a Noetherian ring, $\mathfrak{a}$ a proper ideal of R and M an R-module. Assume that* E-dp$(\mathfrak{a}, M)$ *is finite. Then*

$$0 \leq \text{E-dp}(\mathfrak{a}, M) \leq \dim R/\mathrm{Ann}_R(M) \quad \textit{and} \quad 0 \leq \text{T-codp}(\mathfrak{a}, M) \leq \dim R/\mathrm{Ann}_R(M).$$

Proof We write $\bar{\cdot}$ for the images modulo $\mathrm{Ann}_R(M)$ and view M as an $\bar{R}$-module. Note that $\text{E-dp}(\mathfrak{a}, M) = \text{E-dp}(\bar{\mathfrak{a}}, M)$ (see 5.3.10). Note also that $\text{T-codp}(\bar{\mathfrak{a}}, M) \geq 0$ and $\text{E-dp}(\bar{\mathfrak{a}}, M) \geq 0$ because M is an $\bar{R}$-module. Then recall that $\text{E-dp}(\bar{\mathfrak{a}}, M)$ is finite if and only if $\text{T-codp}(\bar{\mathfrak{a}}, M)$ is finite (see 5.3.9). In that case, and because of the equalities $h^-_{\bar{\mathfrak{a}}}(M) = \text{E-dp}(\bar{\mathfrak{a}}, M)$ and $\lambda^{\bar{\mathfrak{a}}}_-(M) = \text{T-codp}(\bar{\mathfrak{a}}, M)$ (see Theorem 7.6.2) we have $h^+_{\bar{\mathfrak{a}}}(M) \geq 0$ and $\lambda^{\bar{\mathfrak{a}}}_+(M) \geq 0$. The claims now follow by 7.6.10 and 7.6.11 applied to the $\bar{R}$-module M. $\square$

Chapter 8
The Formal Power Series Koszul Complex

In his book Matsumura proved the following Theorem (see [58, Theorem 8.12]).

Let $\mathfrak{a}$ *be an ideal of a Noetherian ring* R *generated by the sequence* $\underline{x} = x_1, \ldots, x_k$. *There is a natural isomorphism* $\hat{R}^{\mathfrak{a}} \cong R[[\underline{U}]]/(\underline{U} - \underline{x})$, *where* $R[[\underline{U}]]$ *denotes the formal power series ring in the variables* $U_1, \ldots, U_k$ *over* R *and* $\underline{U} - \underline{x}$ *denotes the sequence* $U_1 - x_1, \ldots, U_k - x_k$.

When reading this we wondered if it is necessary to suppose that the ring R is Noetherian. More generally we were also interested in an extension of Matsmura's result to modules and complexes. As an outcome of our investigations we will provide a Koszul complex over the polynomial ring $R[\underline{U}] = R[U_1, \ldots, U_k]$ which represents $\operatorname{RHom}_R(\check{C}_{\underline{x}}, X)$ in the derived category, hence also $\mathrm{L}\Lambda^{\mathfrak{a}}(X)$ provided the involved sequence $\underline{x}$ is weakly pro-regular, generating the ideal $\mathfrak{a}$. As a particular case we will recover Matsumura's Theorem and some new descriptions of the local homology modules $\Lambda_i^{\mathfrak{a}}(X)$. Furthermore, we get another property of M-weakly pro-regular sequences for an R-module M.

8.1 Čech Homology and Koszul Complexes

In order to present the result we need some notations for formal power series rings and its extension to modules.

Notations 8.1.1 Let R denote a commutative ring, M an R-module and $\underline{U} = U_1, \ldots, U_k$ a set of variables over R. We need the polynomial ring $R[\underline{U}] := R[U_1, \ldots, U_k]$ and identify the $R[\underline{U}]$-module $\operatorname{Hom}_R(R[\underline{U}], M)$ with the formal power series module $M[[\underline{U}]] = M[[U_1, \ldots, U_k]]$.

That is, we identify a homomorphism $s \in \operatorname{Hom}_R(R[\underline{U}], M)$ with the formal power series $\sum_{i_1,\ldots,i_k \geq 0} s(U_1^{i_1} \cdots U_k^{i_k}) U_1^{i_1} \cdots U_k^{i_k}$.

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We also consider the polynomial module $M[\underline{U}] = R[\underline{U}] \otimes_R M$. We note that $M[[\underline{U}]] \cong \varprojlim M[\underline{U}]/(\underline{U})^t M[\underline{U}]$ and that $R[[\underline{U}]] \cong \varprojlim R[\underline{U}]/(\underline{U})^t R[\underline{U}]$. It follows that $M[[\underline{U}]]$ also has the structure of an $R[[\underline{U}]]$-module (in view of 2.1.3 applied to the ideal $\underline{U} R[\underline{U}]$ of $R[\underline{U}]$).

Now let X denote an R-complex. Similarly we also have the degree-wise identification of the $R[\underline{U}]$-complex $\operatorname{Hom}_R(R[\underline{U}], X)$ with $X[[\underline{U}]] := X[[U_1, \ldots, U_k]]$. Moreover, $X[[\underline{U}]]$ is an $R[\underline{U}]$-complex and an $R[[\underline{U}]]$-complex hence also an R-complex. There is an obvious isomorphism of R-complexes $X[[\underline{U}]] \cong X^{\mathbb{N}^k}$. For a single variable U we will identify $X[[U]]$ with $X^{\mathbb{N}}$.

This construction is functorial in X, a morphism of R-complexes $X \to Y$ induces a morphism $X[[\underline{U}]] \to Y[[\underline{U}]]$ of $R[[\underline{U}]]$-complexes. Moreover, if $X \to Y$ is a quasi-isomorphism, then $X[[\underline{U}]] \to Y[[\underline{U}]]$ is a quasi-isomorphism too. This follows easily since direct products commute with homology.

8.1.2 *The formal power series Koszul complex.* Let x denote an element of a commutative ring R and X an R-complex. We add a variable U, then we define

$$\mathcal{K}_\bullet(x; X) = C(\mu_{U-x}),$$

the cone of the multiplication map by $U - x$ on the $R[[U]]$-complex $X[[U]]$. For a sequence of elements $\underline{x} = x_1, \ldots, x_k$ and an element y in R we form the sequence $\underline{x}, y = x_1, \ldots, x_k, y$ and define inductively

$$\mathcal{K}_\bullet(\underline{x}, y; X) = \mathcal{K}_\bullet(y; \mathcal{K}_\bullet(\underline{x}; X)).$$

By 5.2.1 note that $\mathcal{K}_\bullet(\underline{x}; X)$ is isomorphic to the Koszul complex $K_\bullet(\underline{U} - \underline{x}, X[[\underline{U}]])$ of the $R[[U]]$-complex $X[[U]]$ with respect to the sequence $\underline{U} - \underline{x} = U_1 - x_1, \ldots, U_k - x_k$.

We recall that a morphism of R-complexes $f : X \to Y$ induces morphisms $X[[\underline{U}]] \to Y[[\underline{U}]]$ and $\mathcal{K}_\bullet(\underline{x}; X) \to \mathcal{K}_\bullet(\underline{x}; Y)$ of $R[[\underline{U}]]$-complexes. Moreover, if f is a quasi-isomorphism, then these induced morphisms are also quasi-isomorphisms.

We will prove that this Koszul complex $\mathcal{K}_\bullet(\underline{x}; X)$, viewed as an R-complex, represents $\operatorname{RHom}_R(\check{C}_{\underline{x}}, X)$ in the derived category. We will do this by induction on r, the length of the sequence $\underline{x}$.

Remark 8.1.3 Let M denote an R-module and U a variable. As a first step we describe the multiplication by $U - x$ on the $R[[U]]$-module $M[[U]]$:

$$\mu_{U-x} : M[[U]] \to M[[U]],$$
$$m(U) = \textstyle\sum_{i \geq 0} m_i U^i \in M[[U]] \mapsto (U - x)m(U) = -xm_0 + \sum_{i \geq 1} (m_{i-1} - xm_i)U^i.$$

We also recall that $\operatorname{RHom}_R(\check{C}_{\underline{x}}, X)$ has already been represented by the complex $\operatorname{Hom}_R(\check{L}_{\underline{x}}, X)$ (see 6.2.7). First we analyse the description of the complex $\operatorname{Hom}_R(\check{L}_x, X)$ for a single element $x \in R$.

Remark 8.1.4 On the complex $\operatorname{Hom}_R(\check{L}_x, X)$. Let x be an element of a commutative ring R and let $\check{L}_x$ denote the free resolution of the Čech complex $\check{C}_x$ obtained in 6.2.1. It is the complex

$$\ldots \to 0 \to R \oplus R[U] \xrightarrow{\rho} R[U] \to 0 \to \ldots, \ \rho : (r, f(U)) \mapsto r - (1 - xU) f(U)$$

with $\check{L}_x^0 = R \oplus R[U]$ and $\check{L}_x^1 = R[U]$. Let X denote an R-complex. We apply the functor $\operatorname{Hom}_R(\cdot, X)$ and get a morphism of R-complexes

$$\rho_X^* = \operatorname{Hom}_R(\rho, X) : \operatorname{Hom}_R(R[U], X) \to \operatorname{Hom}_R(R \oplus R[U], X),$$

with $\operatorname{Hom}_R(R[U], X) = \operatorname{Hom}_R(\check{L}_x^1, X)$ and $\operatorname{Hom}_R(R \oplus R[U], X) = \operatorname{Hom}_R(\check{L}_x^0, X)$. Since the ascending complex $\check{L}_x$ is the fiber of ρ, the complex $\operatorname{Hom}_R(\check{L}_x, X)$ is isomorphic to the cone of this morphism ρ_X^* (see 1.5.4).

As indicated in 8.1.1 we identify $\operatorname{Hom}_R(R[U], X)$ with $X[[U]] = \prod_{i \geq 0} XU^i$, and we identify $\operatorname{Hom}_R(R \oplus R[U], X)$ with $XU^{-1} \oplus X[[U]] = \prod_{i \geq -1} XU^i$, where XU^{-1} denotes a copy of X. With these identifications we now describe the morphism ρ_X^*. It is described degree-wise. For $X_i = M$ we have

$$M[[U]] \xrightarrow{\rho_M^*} MU^{-1} \oplus M[[U]], \quad \textstyle\sum_{i \geq 0} m_i U^i \mapsto m_0 U^{-1} - \sum_{i \geq 0} (m_i - x m_{i+1}) U^i.$$

Lemma 8.1.5 *Let R denote a commutative ring, X an R-complex and $x \in R$ an element. There is a quasi-isomorphism of R-complexes, functorial in X*

$$\mathcal{K}_\bullet(x; X) \xrightarrow{\sim} \operatorname{Hom}_R(\check{L}_x, X).$$

More precisely, there is a short exact sequence of R-complexes

$$0 \to \mathcal{K}_\bullet(x; X) \to \operatorname{Hom}_R(\check{L}_x, X) \to \mathcal{X} \to 0,$$

degree-wise split exact, where $\mathcal{X}$ denotes the cone of the identity morphism $\mathrm{id}_X : X \to X$.

Proof For the proof we first note that we have a commutative diagram

$$\begin{tikzcd}
X[[U]] \arrow[r, "\mu_{U-x}"] \arrow[d, "\mu_{-U}"] & X[[U]] \arrow[d, "i"] \\
X[[U]] \arrow[r, "\rho_X^*"] & XU^{-1} \oplus X[[U]]
\end{tikzcd}$$

where μ_{U-x} and μ_{-U} denote the multiplication by $U - x$ and $-U$, respectively, where i is the natural injection and where ρ_X^* is the morphism described in 8.1.4.

We have to check the commutativity and we do it degree-wise. For $X_i = M$ and $m(U) = \sum_{i\geq 0} m_i U^i$ we get

$$\rho_M^*(-U \cdot m(U)) = -xm_0 + \textstyle\sum_{i\geq 1}(m_i - xm_{i+1})U^i = (U - x) \cdot m(U).$$

We note that μ_{-U} and i viewed as morphisms of R-complexes are degree-wise split-injective, with cokernel XU^0 and XU^{-1}, respectively. Hence we have a commutative diagram of R-complex morphisms with exact rows

$$\begin{tikzcd}
0 \arrow[r] & X[[U]] \arrow[r, "\mu_{-U}"] \arrow[d, "\mu_{U-x}"] & X[[U]] \arrow[r, "p_1"] \arrow[d, "\rho_X^*"] & XU^0 \arrow[r] \arrow[d, "\tau"] & 0 \\
0 \arrow[r] & X[[U]] \arrow[r, "i"] & XU^{-1} \oplus X[[U]] \arrow[r, "p_0"] & XU^{-1} \arrow[r] & 0
\end{tikzcd}$$

where p_0 and p_1 denote the natural projections, where τ is an isomorphism. We take cones and obtain a short exact sequence of R-complexes

$$0 \to C(\mu_{U-x}) \to C(\rho_X^*) \to C(\tau) \to 0.$$

Now $\mathcal{K}_\bullet(x; X) = C(\mu_{U-x})$ by definition (see 8.1.2) and $\operatorname{Hom}_R(\check{L}_x, X) \cong C(\rho_X^*)$ (see 8.1.4). Moreover, $C(\tau)$ is homologically trivial since τ is an isomorphism. Hence we obtain the quasi-isomorphism of the statement, it is clearly functorial in X. □

We are ready for our main theorem. That is, we extend Lemma 8.1.5 to the situation of a sequence $\underline{x}$ of elements of R.

Theorem 8.1.6 *Let $\underline{x} = x_1, \ldots, x_k$ denote a sequence of elements in the commutative ring R. There is a quasi-isomorphism of R-complexes*

$$\mathcal{K}_\bullet(\underline{x}; X) \xrightarrow{\sim} \operatorname{Hom}_R(\check{L}_{\underline{x}}, X),$$

functorial in X.

Proof We proceed by induction on k. For $k = 1$ the statement is shown in 8.1.5. Assume the claim is true for $k \geq 1$. Let $y \in R$ and form the sequence $\underline{x}, y = x_1, \ldots, x_k, y$. Then by the inductive hypothesis the quasi-isomorphism $\mathcal{K}_\bullet(\underline{x}; X) \xrightarrow{\sim} \operatorname{Hom}_R(\check{L}_{\underline{x}}, X)$ induces quasi-isomorphisms

$$\mathcal{K}_\bullet(y; \mathcal{K}_\bullet(\underline{x}; X)) \xrightarrow{\sim} \operatorname{Hom}_R(\check{L}_y, \mathcal{K}_\bullet(\underline{x}; X)) \text{ and}$$
$$\operatorname{Hom}_R(\check{L}_y, \mathcal{K}_\bullet(\underline{x}; X)) \xrightarrow{\sim} \operatorname{Hom}_R(\check{L}_y, \operatorname{Hom}_R(\check{L}_{\underline{x}}, X))$$

(see 8.1.5 applied to y and $\mathcal{K}_\bullet(\underline{x}; X)$, then note that $\check{L}_y$ is DG-projective.) By definition we have

$$\mathcal{K}_\bullet(\underline{x}, y; X) = \mathcal{K}_\bullet(y; \mathcal{K}_\bullet(\underline{x}; X))$$

(see 8.1.2). By adjointness and the definition of $\check{L}_{\underline{x},y}$ we have the isomorphisms

$$\operatorname{Hom}_R(\check{L}_y, \operatorname{Hom}_R(\check{L}_{\underline{x}}, X)) \cong \operatorname{Hom}_R(\check{L}_y \otimes_R \check{L}_{\underline{x}}, X) \cong \operatorname{Hom}_R(\check{L}_{\underline{x},y}, X).$$

This completes the inductive step and finishes the proof. □

The previous result has several important consequences summarized in the next section. Here we have an application concerning the homology of $\mathcal{K}_\bullet(\underline{x}; X)$.

Corollary 8.1.7 *Let $\underline{x} = x_1, \ldots, x_k$ denote a sequence of elements of R. For an R-complex X the homology modules of $\mathcal{K}_\bullet(\underline{x}; X)$ are inserted in the short exact sequences*

$$0 \to \varprojlim{}^1 H_{i+1}(\underline{x}^t; X) \to H_i(\mathcal{K}_\bullet(\underline{x}; X)) \to \varprojlim H_i(\underline{x}^t; X) \to 0$$

for all $i \in \mathbb{Z}$.

Proof For all $i \in \mathbb{Z}$ there are functorial short exact sequences

$$0 \to \varprojlim{}^1 H_{i+1}(\underline{x}^t; X) \to H_i((\operatorname{RHom}_R(\check{C}_{\underline{x}}, X)) \to \varprojlim H_i(\underline{x}^t; X) \to 0$$

(see 6.3.2). Then recall that $\operatorname{Hom}_R(\check{L}_{\underline{x}}, X)$ represents $\operatorname{RHom}_R(\check{C}_{\underline{x}}, X)$ in the derived category since $\check{L}_{\underline{x}} \xrightarrow{\sim} \check{C}_{\underline{x}}$ is a bounded free resolution of $\check{C}_{\underline{x}}$. Because of the quasi-isomorphism $\mathcal{K}_\bullet(\underline{x}; X) \xrightarrow{\sim} \operatorname{Hom}_R(\check{L}_{\underline{x}}, X)$ the result follows. □

8.2 Applications to Weakly Pro-regular Sequences

In the following we shall investigate the homology of $\mathcal{K}_\bullet(\underline{x}; M)$ for an R-module M. Recall that $\mathcal{K}_\bullet(\underline{x}; M)$ is by definition isomorphic to the Koszul complex $K_\bullet(\underline{U} - \underline{x}; M[[\underline{U}]])$. By 7.3.1 a sequence $\underline{x}$ is called M-weakly pro-regular if the inverse system of Koszul homology $\{H_i(\underline{x}^t; M)\}$ is pro-zero for all $i > 0$.

Proposition 8.2.1 *Let $\mathfrak{a}$ be an ideal of a commutative ring R generated by the sequence $\underline{x} = x_1, \ldots, x_k$. Let $\underline{U} = U_1, \ldots, U_k$ be a set of variables over R. For an R-module M the following holds:*

(a) There is an isomorphism and an epimorphism

$$H_0(\mathcal{K}_\bullet(\underline{x}; M)) \cong M[[\underline{U}]]/(\underline{U} - \underline{x})M[[\underline{U}]] \twoheadrightarrow \hat{M}^{\mathfrak{a}}.$$

The epimorphism is an isomorphism if and only if $\varinjlim{}^1 H_1(\underline{x}^t; M) = 0$.

(b) *Suppose that the sequence $\underline{x}$ is M-weakly pro-regular. Then*

$$H_0(\mathcal{K}_\bullet(\underline{x}; M)) \cong \hat{M}^{\mathfrak{a}} \text{ and } H_i(\mathcal{K}_\bullet(\underline{x}; M)) = 0 \text{ for all } i > 0.$$

(c) *If the sequence $\underline{x}$ is M-weakly pro-regular, then the sequence $\underline{U} - \underline{x} = U_1 - x_1, \dots, U_k - x_k$ is completely secant on the $R[\underline{U}]$-module $M[[\underline{U}]]$.*

Proof (a) follows by 8.1.7 for $i = 0$. Recall that $\varprojlim H_0(\underline{x}^t; M) \cong \varprojlim M/\underline{x}^t M \cong \hat{M}^{\mathfrak{a}}$.

(b) follows by the short exact sequences in 8.1.7 since the inverse system $\{H_i(\underline{x}^t; M)\}$ is pro-zero for all $i > 0$ and therefore $\varprojlim{}^1 H_i(\underline{x}^t; M) = \varprojlim H_i(\underline{x}^t; M) = 0$ for all $i > 0$ (see 1.2.4).

(c) follows from (b) in view of the definition of a completely secant sequence (see 5.4.1). $\square$

We do not know whether the converse of 8.2.1 (c) is true.

Remark 8.2.2 In the situation of 8.2.1 assume again that the sequence $\underline{x}$ is M-weakly pro-regular. Then it follows that the Koszul complex $\mathcal{K}_\bullet(\underline{x}; M)$ is homologically trivial if and only if $M = \underline{x}M$ as follows by 2.5.1.

Now let us investigate the case where the sequence $\underline{x}$ is weakly pro-regular. It provides another presentation of $\mathrm{L}\Lambda^{\mathfrak{a}}(X)$ in the derived category.

Proposition 8.2.3 *Let $\mathfrak{a}$ be an ideal of a commutative ring R generated by a weakly pro-regular sequence $\underline{x} = x_1, \dots, x_k$. Then the Koszul complex $\mathcal{K}_\bullet(\underline{x}; X)$ represents $\mathrm{L}\Lambda^{\mathfrak{a}}(X)$ in the derived category. In particular, $H_i(\underline{U} - \underline{x}; X[[\underline{U}]]) \cong \Lambda_i^{\mathfrak{a}}(X)$ for all $i \in \mathbb{Z}$.*

Proof This is an immediate consequence of Theorem 8.1.6 in view of 7.5.12. $\square$

We now obtain a slight generalization of Matsumura's result mentioned in the introduction of this chapter. Note that any sequence is weakly pro-regular in a Noetherian ring. We will also obtain some new descriptions of the local homology modules.

Proposition 8.2.4 *Let $\mathfrak{a}$ be an ideal of a commutative ring R generated by a weakly pro-regular sequence $\underline{x} = x_1, \dots, x_k$. Let $\underline{U} = U_1, \dots, U_k$ be a set of variables over R.*

(a) *$H_0(\mathcal{K}_\bullet(\underline{x}; R)) = R[[\underline{U}]]/(\underline{U} - \underline{x})R[[\underline{U}]] \cong \hat{R}^{\mathfrak{a}}$ and $H_i(\mathcal{K}_\bullet(\underline{x}; R)) = 0$ for all $i > 0$.*

(b) *$\hat{R}^{\mathfrak{a}}$ has the structure of an $R[[\underline{U}]]$-module and the evaluation homomorphism*

$$\mathrm{ev}_{\underline{x}} : R[\underline{U}] \to R : f(U_1, \dots, U_k) \mapsto f(x_1, \dots, x_k)$$

extends to the evaluation homomorphism

$$\mathrm{ev}_{\underline{x}}^e : R[[\underline{U}]] \to \hat{R}^{\mathfrak{a}} : s(U_1, \dots, U_k) \mapsto s(x_1, \dots, x_k).$$

(c) *The Koszul complex* $\mathcal{K}_\bullet(\underline{x}; R) \cong K_\bullet(\underline{U} - \underline{x}; R[[\underline{U}]])$ *provides a bounded free resolution of* $\hat{R}^{\mathfrak{a}}$ *viewed as an* $R[[\underline{U}]]$*-module.*

Proof (a) is a consequence of 8.2.1 (b) applied to R. The evaluation homomorphism $\mathrm{ev}_{\underline{x}} : R[\underline{U}] \to R : f(U_1, \ldots, U_k) \mapsto f(x_1, \ldots, x_k)$ is clear and provides R with the structure of an $R[\underline{U}]$-module. This evaluation homomorphism $\mathrm{ev}_{\underline{x}}$ induces homomorphisms $R[\underline{U}]/(\underline{U}R[\underline{U}])^t \to R/\mathfrak{a}^t$ for all $t > 0$. By passing to the inverse limit we get the evaluation homomorphism $\mathrm{ev}_{\underline{x}}^e$ in (b). Statement (c) is a direct consequence of (a). □

For the evaluation homomorphism $\mathrm{ev}_{\underline{x}}$, see also 8.3.1 for more information. By virtue of 8.2.4 (c) there is another interpretation of the left derived functors $\Lambda_i^{\mathfrak{a}}(X)$ for a certain finitely generated ideal $\mathfrak{a}$.

Corollary 8.2.5 *We fix the notation and assumptions of 8.2.4. Let X denote an R-complex. Then there are isomorphisms*

$$\Lambda_i^{\mathfrak{a}}(X) \cong H_i(\mathcal{K}_\bullet(\underline{x}; X)) \cong \mathrm{Tor}_i^{R[[\underline{U}]]}(\hat{R}^{\mathfrak{a}}, X[[\underline{U}]])$$

for all $i \in \mathbb{Z}$, where $\hat{R}^{\mathfrak{a}}$ has the structure of an $R[[\underline{U}]]$-module as obtained in 8.2.4 (b).

Proof We have the first isomorphism because $\mathcal{K}_\bullet(\underline{x}; X)$ represents $\mathrm{L}\Lambda^{\mathfrak{a}}(X)$ in the derived category (see 8.2.3). We get the second isomorphism because of

$$\mathcal{K}_\bullet(\underline{x}; X) \cong K_\bullet(\underline{U} - \underline{x}; X[[\underline{U}]]) \cong K_\bullet(\underline{U} - \underline{x}; R[[\underline{U}]]) \otimes_{R[[\underline{U}]]} X[[\underline{U}]]$$

and because the complex $K_\bullet(\underline{U} - \underline{x}; R[[\underline{U}]])$ provides a bounded free resolution of $\hat{R}^{\mathfrak{a}}$ viewed as an $R[[\underline{U}]]$-module (see 8.2.4). □

8.3 Applications to Koszul Homology

In this section we continue with the study of the homology of $\mathcal{K}_\bullet(\underline{x}; X)$. In the previous section we were mainly interested in the case where $\underline{x}$ satisfies additional conditions. In order to relate the homology to further homological data we also consider general sequences. First a general remark.

Remark 8.3.1 Let R denote a commutative ring and $\underline{x} = x_1, \ldots, x_k$ be a sequence of elements in R. As before, let $\underline{U} = U_1, \ldots, U_k$ be a sequence of variables over R and $R[\underline{U}] = R[U_1, \ldots, U_k]$. As in 8.2.4 (b) R possesses the structure of an $R[\underline{U}]$-module via the evaluation homomorphism

$$\mathrm{ev}_{\underline{x}} : R[\underline{U}] \to R : f(U_1, \ldots, U_k) \mapsto f(x_1, \ldots, x_k).$$

Moreover, there is an isomorphism $R \cong R[\underline{U}]/(\underline{U} - \underline{x})R[\underline{U}]$ of $R[\underline{U}]$-modules. The sequence $\underline{U} - \underline{x} = U_1 - x_1, \ldots, U_k - x_k$ is regular on $R[\underline{U}]$, as is easily seen. Therefore the $R[\underline{U}]$-Koszul complex $K_\bullet(\underline{U} - \underline{x}; R[\underline{U}])$ provides a bounded free resolution of R viewed as an $R[\underline{U}]$-module via the evaluation homomorphism $\mathrm{ev}_{\underline{x}}$.

Proposition 8.3.2 *With the notations and conventions of 8.3.1 let $\mathfrak{a} = \underline{x}R$. Let X be an R-complex and let M be an R-module.*

(a) For all $i \in \mathbb{Z}$

$$H_i(\mathcal{K}_\bullet(\underline{x}; X)) \cong \mathrm{Tor}_i^{R[\underline{U}]}(R, X[[\underline{U}]]) \cong \mathrm{Ext}_{R[\underline{U}]}^{k-i}(R, X[[\underline{U}]]).$$

(b) If the sequence $\underline{x}$ is weakly pro-regular, then

$$\mathrm{Tor}_i^{R[\underline{U}]}(R, X[[\underline{U}]]) \cong \mathrm{Ext}_{R[\underline{U}]}^{k-i}(R, X[[\underline{U}]]) \cong \Lambda_i^{\mathfrak{a}}(X)$$

for all $i \in \mathbb{Z}$.

(c) If the sequence $\underline{x}$ is M-weakly pro-regular, then

$$\mathrm{Tor}_i^{R[\underline{U}]}(R, M[[\underline{U}]]) = \mathrm{Ext}_{R[\underline{U}]}^{k-i}(R, M[[\underline{U}]]) = 0 \text{ for } i > 0 \text{ and}$$
$$R \otimes_{R[\underline{U}]} M[[\underline{U}]] \cong \mathrm{Ext}_{R[\underline{U}]}^{k}(R, M[[\underline{U}]]) \cong \hat{M}^{\mathfrak{a}}.$$

Here R is considered as an $R[\underline{U}]$-module via the evaluation map $\mathrm{ev}_{\underline{x}}$.

Proof By definition the complex $\mathcal{K}_\bullet(\underline{x}; X)$ is isomorphic to the Koszul complex $K_\bullet(\underline{U} - \underline{x}; X[[\underline{U}]])$. Because of the structure of $X[[\underline{U}]]$ as an $R[\underline{U}]$-complex it follows that

$$K_\bullet(\underline{U} - \underline{x}; X[[\underline{U}]]) \cong K_\bullet(\underline{U} - \underline{x}; R[\underline{U}]) \otimes_{R[\underline{U}]} X[[\underline{U}]].$$

By 8.3.1 the sequence $\underline{U} - \underline{x}$ is $R[\underline{U}]$-regular and the complex $K_\bullet(\underline{U} - \underline{x}; R[\underline{U}])$ is an $R[\underline{U}]$-free resolution of $R[\underline{U}]/(\underline{U} - \underline{x})R[\underline{U}] \cong R$. Hence the statements in (a) follow by the definitions and the self-duality of the Koszul complex.

The proof of (b) follows by (a) and the first isomorphism in 8.2.5. For the proof of (c) we use (a) and 8.2.1 (b). □

In the case when the sequence $\underline{x}$ is weakly pro-regular we have already noted that the complex $\mathcal{K}_\bullet(\underline{x}; X)$ represents $\mathrm{L}\Lambda^{\mathfrak{a}}(X)$ in the derived category. In general, there is also a relation between $\mathcal{K}_\bullet(\underline{x}; X)$ and $\mathrm{L}\Lambda^{\mathfrak{a}}(X)$. To present it we need the following.

Lemma 8.3.3 *Let $\underline{x}$ denote a sequence of elements in a commutative ring R. There is a natural quasi-isomorphism*

$$\mathcal{K}_\bullet(\underline{x}; X) \xrightarrow{\sim} M(\underline{x}, X)$$

for any R-complex X.

Proof We recall that we had two bounded free resolutions of $\check{C}_{\underline{x}}$, namely $T(\underline{x}) \xrightarrow{\sim} \check{C}_{\underline{x}}$ and $\check{L}_{\underline{x}} \xrightarrow{\sim} \check{C}_{\underline{x}}$.

Hence there is a quasi-isomorphism $T(x) \xrightarrow{\sim} \check{L}_{\underline{x}}$ (see 4.4.8). Now we have quasi-isomorphisms and an isomorphism

$$\mathcal{K}_\bullet(\underline{x}; X) \xrightarrow{\sim} \operatorname{Hom}_R(\check{L}_{\underline{x}}, X) \xrightarrow{\sim} \operatorname{Hom}_R(T(\underline{x}), X) \cong M(\underline{x}; X).$$

For the first quasi-isomorphism we refer to 8.1.6. The second one is a particular case of 4.4.11. Finally, for the last isomorphism, see 7.5.9. □

Theorem 8.3.4 *Let $\mathfrak{a}$ be an ideal of a commutative ring R generated by the sequence $\underline{x} = x_1, \ldots, x_k$. Let X denote an R-complex with one of its DG-flat resolutions $F \xrightarrow{\sim} X$. There is a natural map*

$$\mathcal{K}_\bullet(\underline{x}; X) \to \mathrm{L}\Lambda^{\mathfrak{a}} X$$

in the derived category, represented by the composite

$$\mathcal{K}_\bullet(\underline{x}; X) \xleftarrow{\sim} \mathcal{K}_\bullet(\underline{x}; F) \xrightarrow{\sim} M(\underline{x}, F) \to \operatorname{Mic}(\{R/\underline{x}^t R \otimes_R F\}) \xleftarrow{\sim} \Lambda^{\mathfrak{a}}(F).$$

At the homology level it induces natural homomorphisms

$$H_i(\mathcal{K}_\bullet(\underline{x}; X)) \to \Lambda_i^{\mathfrak{a}}(X) \ \textit{ for all } \ i \in \mathbb{Z}.$$

Suppose that the sequence $\underline{x}$ is weakly pro-regular. Then the natural map $\mathcal{K}_\bullet(\underline{x}; F) \to \mathrm{L}\Lambda^{\mathfrak{a}}(X)$ is an isomorphism in the derived category and the induced homomorphisms are isomorphisms.

Proof First note that $\Lambda^{\mathfrak{a}}(F)$ represents $\mathrm{L}\Lambda^{\mathfrak{a}}(X)$ in the derived category (see 7.1.6).

The quasi-isomorphism on the left is clear since $\mathcal{K}_\bullet(\underline{x}; \cdot)$ preserves quasi-isomorphisms (see 8.1.2). The quasi-isomorphism on the right follows by Lemma (4.2.3 (a)). The quasi-isomorphism $\mathcal{K}_\bullet(\underline{x}; F) \xrightarrow{\sim} M(\underline{x}, F)$ is shown above in 8.3.3. The natural morphism $M(\underline{x}; F) \to \operatorname{Mic}(\{R/\underline{x}^t R \otimes_R F\})$ is described in 7.5.2, where it is also shown that it is a quasi-isomorphism when the sequence is weakly pro-regular. □

At the homology level we can say a bit more.

Proposition 8.3.5 *Let $\mathfrak{a}$ be an ideal of a commutative ring R generated by the sequence $\underline{x} = x_1, \ldots, x_k$. Let us denote by $\mathfrak{a}_t$ the ideal generated by the sequence $\underline{x}^t = x_1^t, \ldots, x_k^t$. Then there are commutative diagrams with exact rows*

$$
\begin{tikzcd}
0 \arrow[r] & \varprojlim^1 H_{i+1}(\underline{x}^t; X) \arrow[r] \arrow[d] & H_i(\mathcal{K}_\bullet(x; X)) \arrow[r] \arrow[d] & \varprojlim H_i(\underline{x}^t; X) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \varprojlim^1 \operatorname{Tor}^R_{i+1}(R/\mathfrak{a}_t, X) \arrow[r] & \Lambda_i^{\mathfrak{a}}(X) \arrow[r] & \varprojlim \operatorname{Tor}^R_i(R/\mathfrak{a}_t, X) \arrow[r] & 0
\end{tikzcd}
$$

for all $i \in \mathbb{Z}$.

Proof The first short exact sequences were already shown in 8.2.1. By view of 7.1.8 the second ones follow. We have to show that they fit into a commutative diagram. To this end remember where they come from. There is an inverse system of morphisms $K_\bullet(\underline{x}^t) \to R/\mathfrak{a}_t$ which induces the inverse system of morphisms

$$\{K_\bullet(\underline{x}^t; F)\} \to \{F/\mathfrak{a}_t F\},$$

where $F \xrightarrow{\sim} X$ denotes a DG-flat resolution of X. We take microscopes and recall that $\operatorname{Mic}(\{K_\bullet(\underline{x}^t; F)\}) = M(\underline{x}; F)$ by definition. By the definition of the microscope we have a commutative diagram with exact rows

$$
\begin{tikzcd}
0 \arrow[r] & (\prod_t K_\bullet(\underline{x}^t; F))_{[1]} \arrow[r] \arrow[d] & M(\underline{x}; F) \arrow[r] \arrow[d] & \prod_t K_\bullet(\underline{x}^t; F) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (\prod_t F/\mathfrak{a}_t F)_{[1]} \arrow[r] & \operatorname{Mic}(\{F/\mathfrak{a}_t F\}) \arrow[r] & \prod_t F/\mathfrak{a}_t F) \arrow[r] & 0.
\end{tikzcd}
$$

This diagram induces a morphism between the long exact homology sequences associated to its rows. We break these into short exact sequences as in 4.2.3 and obtain the commutative diagrams of the statements. Recall that $\mathrm{L}\Lambda^{\mathfrak{a}}(X)$ is represented as $\varprojlim F/\mathfrak{a}_t F$ and that we have a quasi-isomorphism $\varprojlim F/\mathfrak{a}_t F \xrightarrow{\sim} \operatorname{Mic}(\{F/\mathfrak{a}_t F\})$ (see 4.2.3). Finally, remember the quasi-isomorphisms $\mathcal{K}_\bullet(\underline{x}; X) \xleftarrow{\sim} \mathcal{K}_\bullet(\underline{x}; F) \xrightarrow{\sim} M(\underline{x}, F)$ (see the last remark in 8.1.2 and 8.3.3). □

8.4 The Case of a Single Element

In this section we again investigate the case of a sequence consisting of a single element. It turns out that a bit more can be said in that particular case.

8.4.1 Let x denote an element of a commutative ring R and M an R-module. First recall that $H_1(x^t; M) \cong 0 :_M x^t$. The multiplication by x induces an homomorphism $0 :_M x^{t+1} \to 0 :_M x^t$ and provides us with an inverse system. In the following we denote by $\{0 :_M x^t, x\}$ this inverse system.

Note that x is M-weakly pro-regular if and only if M is of bounded xR-torsion. This is equivalent to saying that the inverse system $\{0 :_M x^t, x\}$ is pro-zero.

We need a technical lemma, which originally appeared in the proof of [38, Proposition 1.5].

Lemma 8.4.2 *Let x denote an element of a commutative ring R. Then*

$$\operatorname{Tor}_1^R(R/xR, M) \cong (0 :_M x)/(0 :_R x)M$$

for any R-module M.

Proof We tensor the short exact sequence $0 \to R/(0 :_R x) \xrightarrow{x} R \to R/xR \to 0$ by M and get the exact sequence

$$0 \to \operatorname{Tor}_1^R(R/xR, M) \to M/(0 :_R x)M \xrightarrow{x} M \to M/xM \to 0,$$

where $M/(0 :_R x)M \xrightarrow{x} M, m + (0 :_R x)M \mapsto xm$. Therefore the kernel of this homomorphism is $(0 :_M x)/(0 :_R x)M$, which proves the claim. $\square$

In the following we shall improve the results of 8.2.1 in the case of a length one sequence and an R-module M.

Theorem 8.4.3 *Let x denote an element of a commutative ring R. For an R-module M there is a commutative diagram with exact rows*

$$\begin{tikzcd}
0 \arrow[r] & \varprojlim{}^1\{0 :_M x^t, x\} \arrow[r] \arrow[d, two heads, "g"] & H_0(\mathcal{K}_\bullet(x; M)) \arrow[r] \arrow[d, two heads, "h"] & \hat{M}^{xR} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \varprojlim{}^1 \operatorname{Tor}_1^R(R/x^tR, M) \arrow[r] & \Lambda_0^{xR}(M) \arrow[r] & \hat{M}^{xR} \arrow[r] & 0.
\end{tikzcd}$$

where both g and h are surjective.

If M is of bounded xR-torsion, then $H_1(\mathcal{K}_\bullet(x; M)) = 0$ and there are isomorphisms $H_0(\mathcal{K}_\bullet(x; M)) \cong \Lambda_0^{xR}(M) \cong \hat{M}^{xR}$.

Proof The commutative diagram is a particular case of those shown in 8.3.5 (for $i = 0$ and $\mathfrak{a} = xR$). We have to prove that the homomorphisms g and h in this diagram are surjective. Clearly it is enough to prove that g is surjective (by the snake lemma). Because of the isomorphism

$$\operatorname{Tor}_1^R(R/x^tR, M) \cong (0 :_M x^t)/(0 :_R x^t)M$$

(see 8.4.2) there is a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & (0 :_R x^{t+1})M \arrow[r] \arrow[d, "x"] & 0 :_M x^{t+1} \arrow[r] \arrow[d, "x"] & \operatorname{Tor}_1^R(R/x^{t+1}R, M) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (0 :_R x^t)M \arrow[r] & 0 :_M x^t \arrow[r] & \operatorname{Tor}_1^R(R/x^tR, M) \arrow[r] & 0.
\end{tikzcd}$$

Whence it yields (see 1.2.2) a six-term exact sequence

$$
\begin{aligned}
0 \to \varprojlim (0 :_R x^t)M \to \varprojlim (0 :_M x^t) \to \varprojlim \operatorname{Tor}_1^R(R/x^t R, M) \\
\to \varprojlim{}^1 (0 :_R x^t)M \to \varprojlim{}^1 (0 :_M x^t) \to \varprojlim{}^1 \operatorname{Tor}_1^R(R/x^t R, M) \to 0.
\end{aligned}
$$

This proves that the homomorphism g is surjective.

We now prove the last statement. If M is of bounded xR-torsion the inverse system $\{0 :_M x^t, x\}$ is pro-zero. Hence $\varprojlim (0 :_M x^t) = \varprojlim{}^1 (0 :_M x^t) = 0$ (see e.g. 1.2.4). As the homomorphism g in the diagram of the statement is surjective we also have $\varprojlim{}^1 \operatorname{Tor}_1^R(R/x^t R, M) = 0$. Whence $H_0(\mathcal{K}_\bullet(x; M)) \cong \Lambda_0^{xR}(M) \cong \hat{M}^{xR}$, as follows by the above commutative diagram. The equality $H_1(\mathcal{K}_\bullet(x; M)) = 0$ was already in 8.2.1. □

Proposition 8.4.4 *Let $x \in R$ and let M be an R-module. Then $H_i(\mathcal{K}_\bullet(x; M)) = 0$ for $i \neq 0, 1$ and there is a five-term exact sequence of R-modules*

$$
0 \to H_1(\mathcal{K}_\bullet(x; M)) \to \operatorname{Hom}_R(R_x, M) \to M \to H_0(\mathcal{K}_\bullet(x; M)) \to \operatorname{Ext}_R^1(R_x, M) \to 0.
$$

Proof The first statement on the vanishing is trivial. Now recall the quasi-isomorphism $\mathcal{K}_\bullet(x; M) \xrightarrow{\sim} \operatorname{Hom}_R(\check{L}_x, M)$ (see 8.1.6). By the definition of $\check{L}_x$ there is a short exact sequence of R-free complexes

$$
0 \to P_x^{[-1]} \to \check{L}_x \to R \to 0.
$$

This is because $\check{L}_x$ is the fibre of the morphism $g : R \to P_x$, where the R-complex $P_x : 0 \to R[U] \xrightarrow{1-xU} R[U] \to 0$ provides a free resolution of R_x and where g is viewed as a morphism of ascending complexes (see 6.2.1). We apply $\operatorname{Hom}_R(\cdot, M)$ to the above short exact sequence of complexes of free R-modules and get an exact sequence of R-complexes

$$
0 \to \operatorname{Hom}_R(R, M) \to \operatorname{Hom}_R(\check{L}_x, M) \to \operatorname{Hom}_R(P_x, M)^{[1]} \to 0.
$$

Since P_x is a free resolution of R_x the long exact homology sequence yields the five-term exact sequence. □

Remark 8.4.5 The previous result shows that $H_1(\mathcal{K}_\bullet(x; M))$ is the kernel of the natural homomorphism $\operatorname{Hom}_R(R_x, M) \to M$, $\phi \mapsto \phi(1)$. By virtue of 8.1.7 (applied to a length one sequence) we also have that $H_1(\mathcal{K}_\bullet(x; M)) \cong \varprojlim H_1(x^t; M) = \varprojlim \{0 :_M x^n, x\}$.

Corollary 8.4.6 *Let x be an element of a commutative ring R and M an R-module. Assume that the length one sequence x is weakly pro-regular. Then there is a five-term exact sequence*

$$
0 \to \Lambda_1^{xR}(M) \to \operatorname{Hom}_R(R_x, M) \to M \to \Lambda_0^{xR}(M) \to \operatorname{Ext}_R^1(R_x, M) \to 0.
$$

Hence the cokernel of the natural map $\eta_M^{xR} : M \to \Lambda_0^{xR}(M)$ *is* $\operatorname{Ext}_R^1(R_x, M)$. *Moreover, if* $\operatorname{Hom}_R(R_x, M) = 0$*, then* $\Lambda_1^{xR}(M) = 0$.

Proof This follows by Proposition 8.4.4 since in our case $\mathcal{K}_\bullet(x; M)$ represents $\mathrm{L}\Lambda^{xR}$ in the derived category. □

Chapter 9
Complements and Applications

Here we will add some extensions and applications to the previous chapters. We present various aspects of local homology and cohomology, some more properties of the completion and torsion functors.

We start with the composites of the derived functors of the completion and the torsion. Their interplay with the Hom-functors and the tensor product is also considered. In the second section there are results about duality and adjointness linking local homology and local cohomology with respect to an ideal generated by a weakly proregular sequence. It also provides a first version of local duality. In the third section we prove results about the endomorphism complex of $\check{C}_{\underline{x}}$ and related objects. Then the Mayer–Vietoris sequences for local homology and local cohomology are proved. We also consider Mayer–Vietoris sequences for Čech homology and local cohomology. There are also applications of the classes $\mathcal{C}_{\mathfrak{a}}$ and $\mathcal{B}_{\mathfrak{a}}$ to unbounded complexes. Homologically complete and cohomologically torsion complexes are studied in Sect. 9.6. This is completed in Sect. 9.7 by studying the cosupport. In the final section there are change of rings theorems.

9.1 Composites

As applications of our previous investigations on the composite of Čech homology and cohomology functors we give a few interpretations in terms of local cohomology and completion.

Some of the results of this section are not new, in particular note that the case $\mathfrak{a} = \mathfrak{b}$ of Theorem 9.1.3 was independently proved in [64] by different methods. One of the features here is the realization of isomorphisms in the derived category by explicit natural isomorphisms or quasi-isomorphisms of a complex.

P. Schenzel and A.-M. Simon, *Completion, Čech and Local Homology and Cohomology*, Springer Monographs in Mathematics,
https://doi.org/10.1007/978-3-319-96517-8_9

Proposition 9.1.1 *Let $\underline{x} = x_1, \ldots, x_k$ and $\underline{y} = y_1, \ldots, y_l$ denote two weakly pro-regular sequences of a commutative ring R. Suppose that the sequence $\underline{x}, \underline{y} = x_1, \ldots, x_k, y_1, \ldots, y_l$ is also weakly pro-regular. Put $\mathfrak{a} = \underline{x}R$ and $\mathfrak{b} = \underline{y}R$. Then there are isomorphisms in the derived category*

(a) $\mathrm{R}\Gamma_{\mathfrak{a}}(\mathrm{R}\Gamma_{\mathfrak{b}}(X)) \simeq \mathrm{R}\Gamma_{\mathfrak{a}+\mathfrak{b}}(X)$ *and*
(b) $\mathrm{L}\Lambda^{\mathfrak{a}}(\mathrm{L}\Lambda^{\mathfrak{b}}(X)) \simeq \mathrm{L}\Lambda^{\mathfrak{a}+\mathfrak{b}}(X)$

for any R-complex X.

Proof With the notation of 6.2.2 it follows that $\operatorname{Hom}_R(\check{L}_{\underline{z}}, X)$ resp. $\check{L}_{\underline{z}} \otimes_R X$ is a representative of $\mathrm{L}\Lambda^{\mathfrak{c}}(X)$ resp. $\mathrm{R}\Gamma_{\mathfrak{c}}(X)$ for a weakly pro-regular sequence $\underline{z} = z_1, \ldots, z_m$ of R and $\mathfrak{c} = \underline{z}R$. This follows by virtue of the quasi-isomorphism $\check{L}_{\underline{z}} \xrightarrow{\sim} T(\underline{z})$ and the results of 7.5.12 and 7.4.11. Recall that $\check{L}_{\underline{x},\underline{y}} = \check{L}_{\underline{x}} \otimes_R \check{L}_{\underline{y}}$ by definition.

Then (a) is a consequence of the isomorphism

$$\check{L}_{\underline{x}} \otimes_R (\check{L}_{\underline{y}} \otimes_R X) \cong \check{L}_{\underline{x},\underline{y}} \otimes_R X.$$

Statement (b) follows from

$$\operatorname{Hom}_R(\check{L}_{\underline{x},\underline{y}}, X) \cong \operatorname{Hom}_R(\check{L}_{\underline{x}}, \operatorname{Hom}_R(\check{L}_{\underline{y}}, X))$$

because of the adjointness formula. □

In the above, statement (a) already appeared in [75], and (b) appeared in [75] under the additional assumption that each x_i, y_i forms a length one weakly pro-regular sequence.

As an application we get that $\mathrm{R}\Gamma_{\mathfrak{a}}$ and $\mathrm{L}\Lambda^{\mathfrak{a}}$ are idempotent when the ideal $\mathfrak{a}$ is generated by a weakly pro-regular sequence, a fact independently proved in [1] and [64]. We even get a little bit more.

Proposition 9.1.2 *Let $\underline{x} = x_1, \ldots, x_k$ and $\underline{y} = y_1, \ldots, y_l$ denote two weakly pro-regular sequences of a commutative ring R such that* $\operatorname{Rad}(\underline{x}R) \subseteq \operatorname{Rad}(\underline{y}R)$. *Put $\mathfrak{a} = \underline{x}R$ and $\mathfrak{b} = \underline{y}R$. Then:*

(a) $\mathrm{R}\Gamma_{\mathfrak{b}}(\mathrm{R}\Gamma_{\mathfrak{a}}(X)) \simeq \mathrm{R}\Gamma_{\mathfrak{a}}(\mathrm{R}\Gamma_{\mathfrak{b}}(X)) \simeq \mathrm{R}\Gamma_{\mathfrak{b}}(X)$ *and*
(b) $\mathrm{L}\Lambda^{\mathfrak{b}}(\mathrm{L}\Lambda^{\mathfrak{a}}(X)) \simeq \mathrm{L}\Lambda^{\mathfrak{a}}(\mathrm{L}\Lambda^{\mathfrak{b}}(X)) \simeq \mathrm{L}\Lambda^{\mathfrak{b}}(X)$

for any R-complex X.

Proof The proof is similar to that of 9.1.1, taking into account the quasi-isomorphism $\check{L}_{\underline{x}} \otimes_R \check{L}_{\underline{y}} \xrightarrow{\sim} \check{L}_{\underline{y}}$ (see 6.2.4). □

Another application is the composite of $\mathrm{L}\Lambda^{\mathfrak{a}}$ and $\mathrm{R}\Gamma_{\mathfrak{a}}$ under certain circumstances of the ideals. It provides that a certain mixture of the composites behaves well.

Theorem 9.1.3 *Let $\underline{x} = x_1, \ldots, x_k$ and $\underline{y} = y_1, \ldots, y_l$ denote two weakly pro-regular sequences of a commutative ring R. Put $\mathfrak{a} = \underline{x}R$ and $\mathfrak{b} = \underline{y}R$ and suppose that* $\operatorname{Rad}(\mathfrak{a}) \subseteq \operatorname{Rad}(\mathfrak{b})$. *Then there are isomorphisms in the derived category*

(a) $\mathrm{R}\Gamma_{\mathfrak{b}}(\mathrm{L}\Lambda^{\mathfrak{a}}(X)) \simeq \mathrm{R}\Gamma_{\mathfrak{b}}(X)$ *and*
(b) $\mathrm{L}\Lambda^{\mathfrak{b}}(\mathrm{R}\Gamma_{\mathfrak{a}}(X)) \simeq \mathrm{L}\Lambda^{\mathfrak{b}}(X)$

for any R-complex X.

Proof As at the beginning of the proof of 9.1.1 we have that $\check{L}_{\underline{z}} \otimes_R X$ resp. $\operatorname{Hom}_R(\check{L}_{\underline{z}}, X)$ is a representative of $\mathrm{R}\Gamma_{\mathfrak{c}}(X)$ resp. $\mathrm{L}\Lambda^{\mathfrak{c}}(X)$ for a weakly pro-regular sequence $\underline{z} = z_1, \ldots, z_m$ of R and $\mathfrak{c} = \underline{z}R$. Then the statements are consequences of the quasi-isomorphisms shown in 6.5.4 (d) and (b). The theorem may also be viewed as a reinterpretation of Theorem 6.5.5. □

Second proof of 9.1.3. Let $X \xrightarrow{\sim} I$ be a DG-injective resolution of X. Then $\mathrm{L}\Lambda^{\mathfrak{a}}(X)$ is represented by $\operatorname{Hom}_R(\check{C}_{\underline{x}}, I)$, which is DG-injective. Hence $\mathrm{R}\Gamma_{\mathfrak{b}}(\mathrm{L}\Lambda^{\mathfrak{a}}(X))$ is represented by $\varinjlim \operatorname{Hom}_R(R/\mathfrak{b}^t, \operatorname{Hom}_R(\check{C}_{\underline{x}}, I))$. As $\operatorname{Hom}_R(R/\mathfrak{b}^t, \operatorname{Hom}_R(\check{C}_{\underline{x}}, I)) \cong \operatorname{Hom}_R(R/\mathfrak{b}^t \otimes_R \check{C}_{\underline{x}}, I) \cong \operatorname{Hom}_R(R/\mathfrak{b}^t, I)$ the statement in (a) follows.

We obtain a second proof of (b) in a similar way, taking a DG-flat resolution $F \xrightarrow{\sim} X$ of X and noting that $\check{C}_{\underline{x}} \otimes_R F$, which represents $\mathrm{R}\Gamma_{\mathfrak{a}}(X)$, is DG-flat. □

In the following we will add some more new results about composites of local homology and cohomology.

Lemma 9.1.4 *Let $\underline{x} = x_1, \ldots, x_k$ and $\underline{y} = y_1, \ldots, y_l$ denote two sequences of elements in a commutative ring R. Write $\mathfrak{a} = \underline{x}R$ and $\mathfrak{b} = \underline{y}R$. Suppose* $\operatorname{Rad}(\mathfrak{a}) \subseteq \operatorname{Rad}(\mathfrak{b})$. *Let X denote an R-complex and I a K-injective complex. Then there is a natural quasi-isomorphism:*

$$\operatorname{Hom}_R(\operatorname{Hom}_R(\check{L}_{\underline{x}}, X), \operatorname{Hom}_R(\check{L}_{\underline{y}}, I)) \xrightarrow{\sim} \operatorname{Hom}_R(\check{L}_{\underline{y}}, \operatorname{Hom}_R(X, I)).$$

Proof By 6.5.4 there is a natural quasi-isomorphism of R-complexes $\check{L}_{\underline{y}} \otimes_R X \xrightarrow{\sim} \check{L}_{\underline{y}} \otimes_R \operatorname{Hom}_R(\check{L}_{\underline{x}}, X)$. By applying the functor $\operatorname{Hom}_R(\cdot, I)$ it induces a natural quasi-isomorphism

$$\operatorname{Hom}_R(\check{L}_{\underline{y}} \otimes_R \operatorname{Hom}_R(\check{L}_{\underline{x}}, X), I) \xrightarrow{\sim} \operatorname{Hom}_R(\check{L}_{\underline{y}} \otimes_R X, I).$$

By adjointness this proves the statement. Namely we have that

$$\operatorname{Hom}_R(\check{L}_{\underline{y}} \otimes_R X, I) \cong \operatorname{Hom}_R(\check{L}_{\underline{y}}, \operatorname{Hom}_R(X, I)) \text{ and}$$
$$\operatorname{Hom}_R(\check{L}_{\underline{y}} \otimes_R \operatorname{Hom}_R(\check{L}_{\underline{x}}, X), I) \cong \operatorname{Hom}_R(\operatorname{Hom}_R(\check{L}_{\underline{x}}, X), \operatorname{Hom}_R(\check{L}_{\underline{y}}, I)),$$

as required. □

The previous lemma allows us to establish another result about composites.

Theorem 9.1.5 *Let $\underline{x} = x_1, \ldots, x_k$ and $\underline{y} = y_1, \ldots, y_l$ denote two weakly pro-regular sequences of a commutative ring R. Put $\mathfrak{a} = \underline{x}R$ and $\mathfrak{b} = \underline{y}R$ and suppose that* $\operatorname{Rad}(\mathfrak{a}) \subseteq \operatorname{Rad}(\mathfrak{b})$. *Then there are isomorphisms in the derived category*

(a) $\mathrm{RHom}_R(\mathrm{L}\Lambda^{\mathfrak{a}}(X), \mathrm{L}\Lambda^{\mathfrak{b}}(Y)) \simeq \mathrm{L}\Lambda^{\mathfrak{b}}(\mathrm{RHom}_R(X, Y))$ *and*
(b) $\mathrm{RHom}_R(\mathrm{L}\Lambda^{\mathfrak{a}}(X), \mathrm{L}\Lambda^{\mathfrak{b}}(Y)) \simeq \mathrm{RHom}_R(X, \mathrm{L}\Lambda^{\mathfrak{b}}(Y))$

for two R-complexes X, Y.

Proof For the proof of (a) first note that $\mathrm{Hom}_R(\check{L}_{\underline{y}}, \mathrm{Hom}_R(X, I))$ is a representative of $\mathrm{L}\Lambda^{\mathfrak{b}}(\mathrm{RHom}_R(X, Y))$ in the derived category, where $Y \xrightarrow{\sim} I$ is a K-injective resolution of Y. Note also that $\mathrm{L}\Lambda^{\mathfrak{b}}(Y)$ is represented by $\mathrm{Hom}_R(\check{L}_{\underline{y}}, I)$, which is a K-injective complex. Therefore $\mathrm{RHom}_R(\mathrm{L}\Lambda^{\mathfrak{a}}(X), \mathrm{L}\Lambda^{\mathfrak{b}}(Y))$ is represented by

$$\mathrm{Hom}_R(\mathrm{Hom}_R(\check{L}_{\underline{x}}, X), \mathrm{Hom}_R(\check{L}_{\underline{y}}, I)).$$

This together with the above lemma proves the claim in (a).

For the proof of (b) we follow similar arguments. □

Theorem 9.1.6 *Let $\underline{x} = x_1, \ldots, x_k$ and $\underline{y} = y_1, \ldots, y_l$ denote two weakly pro-regular sequences of a commutative ring R. Put $\mathfrak{a} = \underline{x}R$ and $\mathfrak{b} = \underline{y}R$ and suppose that* $\mathrm{Rad}(\mathfrak{a}) \subseteq \mathrm{Rad}(\mathfrak{b})$. *Then there are isomorphisms in the derived category*

(a) $\mathrm{R}\Gamma_{\mathfrak{b}}(X \otimes^{\mathrm{L}}_R Y) \simeq \mathrm{R}\Gamma_{\mathfrak{b}}(X) \otimes^{\mathrm{L}}_R \mathrm{R}\Gamma_{\mathfrak{a}}(Y)$ *and*
(b) $\mathrm{R}\Gamma_{\mathfrak{b}}(X \otimes^{\mathrm{L}}_R Y) \simeq \mathrm{R}\Gamma_{\mathfrak{b}}(X) \otimes^{\mathrm{L}}_R Y$

for two R-complexes X, Y.

Proof Let $F \xrightarrow{\sim} Y$ be a K-flat resolution of Y. Then $\mathrm{R}\Gamma_{\mathfrak{b}}(X \otimes^{\mathrm{L}}_R Y)$ is represented by $\check{L}_{\underline{y}} \otimes_R X \otimes_R F$. The statement in (b) follows. For (a) we recall the quasi-isomorphism $\check{L}_{\underline{x}} \otimes_R \check{L}_{\underline{y}} \xrightarrow{\sim} \check{L}_{\underline{y}}$ (see 6.2.4) and the result follows by associativity. □

Our last composite results require stronger hypotheses.

Lemma 9.1.7 *Let R be a Noetherian ring and $\mathfrak{a} \subset R$ an ideal. Let X be a complex of finitely generated R-modules and Y an R-complex. Assume that one of the following conditions is satisfied*

(a) X is right-bounded and Y is left-bounded,
(b) Y is bounded.

Then there is a natural isomorphism

$$\Gamma_{\mathfrak{a}}(\mathrm{Hom}_R(X, Y)) \cong \mathrm{Hom}_R(X, \Gamma_{\mathfrak{a}}(Y)).$$

Proof In the case of a finitely generated R-module X and an arbitrary R-module Y the claim is correct because it is true for $X = R$. For the case of complexes note that $\mathrm{Hom}_R(X, Y)^n = \prod_{i+j=n} \mathrm{Hom}_R(X_i, Y^j)$ is a finite product of the $\mathrm{Hom}_R(X_i, Y^j)$ when one of the two conditions is satisfied. Since $\Gamma_{\mathfrak{a}}$ commutes with finite direct products the result in the case of complexes as assumed in the statement follows. □

Theorem 9.1.8 *Let $\mathfrak{a} \subset R$ denote an ideal of a Noetherian ring R and let X, Y denote two R-complexes. Assume that X is homologically right-bounded with finitely generated homology modules and that Y is homologically left-bounded. Then there is an isomorphism*

$$\mathrm{R}\Gamma_{\mathfrak{a}}(\mathrm{RHom}_R(X, Y)) \simeq \mathrm{RHom}_R(X, \mathrm{R}\Gamma_{\mathfrak{a}}(Y))$$

in the derived category.

Proof The complex X has a right-bounded free resolution $L \xrightarrow{\sim} X$ where the L_i are finitely generated and the complex Y has a left-bounded injective resolution $Y \xrightarrow{\sim} I$ (see 1.1.12). Then $\mathrm{RHom}_R(X, Y)$ is represented by the complex $\mathrm{Hom}_R(L, I)$, which is K-injective and $\mathrm{R}\Gamma_{\mathfrak{a}}(\mathrm{RHom}_R(X, Y))$ is represented by the complex $\Gamma_{\mathfrak{a}}(\mathrm{Hom}_R(L, I))$. Since $\Gamma_{\mathfrak{a}}(\mathrm{Hom}_R(L, I)) \cong \mathrm{Hom}_R(L, \Gamma_{\mathfrak{a}}(I))$ (see 9.1.7) and since this last complex represents $\mathrm{RHom}_R(X, \mathrm{R}\Gamma_{\mathfrak{a}}(Y))$ the result follows. $\square$

9.2 Adjointness and Duality

Here we present the duality between local cohomology and local homology with respect to an ideal generated by a weakly pro-regular sequence. We provide an explicit isomorphism of complexes representing an adjointness formula linking local homology and cohomology. To this end we first provide some mixed isomorphisms involving the functors telescope $T(\underline{x}; \cdot)$ and microscope $M(\underline{x}; \cdot)$. Then we also provide some other duality formulas concerning Čech and local cohomology.

Recalls 9.2.1 *Let $\underline{x} = x_1, \ldots, x_k$ denote a sequence of elements of a commutative ring R. We recall the functors $T(\underline{x}; \cdot)$ and $M(\underline{x}; \cdot)$ introduced in 7.4.9 and 7.5.1, defined respectively by*

$$T(\underline{x}; X) = \mathrm{Tel}(\{K^{\bullet}(\underline{x}^t; X)\}) \text{ and } M(\underline{x}; X) = \mathrm{Mic}(\{K_{\bullet}(\underline{x}^t; X)\})$$

for an R-complex X and the complexes $T(\underline{x}) := T(\underline{x}; R)$ and $M(\underline{x}) := M(\underline{x}; R)$. Recall that $T(\underline{x}; X) \cong T(\underline{x}) \otimes_R X$ and $\mathrm{Hom}_R(T(\underline{x}), X) \cong M(\underline{x}; X)$ (see 7.4.10 (b) and 7.5.9). Recall also that $K_{\bullet}(\underline{x}^t) \otimes X \cong \mathrm{Hom}_R(K^{\bullet}(\underline{x}^t), X)$.

With the results on Čech complexes we first note that the complexes $T(\underline{x}; X)$ and $M(\underline{x}; X)$ only depend on $\mathrm{Rad}(\underline{x}R)$.

Lemma 9.2.2 *Let $\underline{x} = x_1, \ldots, x_k$ and $\underline{y} = y_1, \ldots, y_l$ be two sequences in a commutative ring R such that $\mathrm{Rad}(\underline{x}R) = \mathrm{Rad}(\underline{y}R)$. Then there are quasi-isomorphisms*

(a) $T(\underline{x}; X) \xrightarrow{\sim} T(\underline{y}; X)$ *and* $M(\underline{y}; X) \xrightarrow{\sim} M(\underline{x}; X)$ *and*

(b) $\check{L}_{\underline{x}} \otimes_R X \xrightarrow{\sim} \check{L}_{\underline{y}} \otimes_R X$ *and* $\mathrm{Hom}_R(\check{L}_{\underline{y}}, X) \xrightarrow{\sim} \mathrm{Hom}_R(\check{L}_{\underline{x}}, X)$

for any R-complex X.

Proof We begin with the proof of (a). We have the quasi-isomorphisms $T(\underline{x}) \xrightarrow{\sim} \check{C}_{\underline{x}}$ and $T(\underline{y}) \xrightarrow{\sim} \check{C}_{\underline{y}}$. It follows that the complexes $T(\underline{x})$ and $T(\underline{y})$ are quasi-isomorphic because $\check{C}_{\underline{x}}$ and $\check{C}_{\underline{y}}$ are quasi-isomorphic (see 6.1.9). We even have a quasi-isomorphism $T(\underline{x}) \xrightarrow{\sim} T(\underline{y})$ (see 4.4.6). We tensor it by X and obtain a quasi-isomorphism $T(\underline{x}) \otimes_R X \xrightarrow{\sim} T(\underline{y}) \otimes_R X$ (see 4.4.11). Then the first quasi-isomorphism of the statement follows (see 7.4.10 (b)). We also have a quasi-isomorphism

$$\operatorname{Hom}_R(T(\underline{y}), X) \xrightarrow{\sim} \operatorname{Hom}_R(T(\underline{x}), X)$$

(see 4.4.11 again). But we have isomorphisms $\operatorname{Hom}_R(T(\underline{x}), X) \cong M(\underline{x}; X)$ and $\operatorname{Hom}_R(T(\underline{y}), X) \cong M(\underline{y}; X)$ (see 7.5.9). The second quasi-isomorphism follows.

For the proof of (b) note that there is a quasi-isomorphism $\check{L}_{\underline{x}} \xrightarrow{\sim} \check{L}_{\underline{y}}$ of bounded complexes of free R-modules (this follows by 6.2.4). Hence the statements follow by 4.4.11. □

We shall take advantage of telescope and microscope in order to obtain more information on local homology and cohomology. First we need the following results.

Theorem 9.2.3 *Let $\underline{x} = x_1, \ldots, x_k$ denote a sequence of elements in a commutative ring R. Let X, Y be two complexes of R-modules. We have the following natural isomorphisms of complexes:*

(a) $T(\underline{x}; X) \otimes_R Y \cong T(\underline{x}; X \otimes_R Y)$,
(b) $\operatorname{Hom}_R(T(\underline{x}; X), Y) \cong \operatorname{Hom}_R(X, M(\underline{x}; Y))$,
(c) $\operatorname{Hom}_R(T(\underline{x}; X), Y) \cong M(\underline{x}; \operatorname{Hom}_R(X, Y))$.

Proof As $T(\underline{x}; X) \cong T(\underline{x}) \otimes_R X$ (see 7.4.10) the statement in (a) follows by associativity.

For (b) recall the isomorphisms

$$\operatorname{Hom}_R(T(\underline{x}; X), Y) \cong \operatorname{Hom}_R(T(\underline{x}) \otimes_R X, Y) \cong \operatorname{Hom}_R(X, \operatorname{Hom}_R(T(\underline{x}), Y)$$

by adjointness. As $\operatorname{Hom}_R(T(\underline{x}), Y) \cong M(\underline{x}; Y)$ (see 7.5.9) the statement in (b) follows.

By adjointness and 7.5.9 again it yields the isomorphisms

$$\operatorname{Hom}_R(T(\underline{x}; X), Y) \cong \operatorname{Hom}_R(T(\underline{x}), \operatorname{Hom}_R(X, Y) \cong M(\underline{x}; \operatorname{Hom}_R(X, Y)),$$

whence the statement in (c). □

We obtain the adjointness formula linking local homology and local cohomology and more. This adjointness formula was also obtained in [64] in a different way, and previously in [1] under slightly stronger hypotheses.

Corollary 9.2.4 *Let $\mathfrak{a}$ be an ideal of a commutative ring R generated by a weakly pro-regular sequence $\underline{x} = x_1, \ldots, x_k$ and let X, Y be two R-complexes. There is a natural isomorphism in the derived category*

$$\mathrm{RHom}_R(\mathrm{R}\Gamma_{\mathfrak{a}}(X), Y) \simeq \mathrm{RHom}_R(X, \mathrm{L}\Lambda^{\mathfrak{a}}(Y))$$

represented by the isomorphism of complexes

$$\mathrm{Hom}_R(T(\underline{x}; X), I) \cong \mathrm{Hom}_R(X, M(\underline{x}; I))$$

where $Y \xrightarrow{\sim} I$ is a K-injective resolution. There is also a natural isomorphism

$$\mathrm{RHom}_R(\mathrm{R}\Gamma_{\mathfrak{a}}(X), Y) \simeq \mathrm{L}\Lambda^{\mathfrak{a}}(\mathrm{RHom}_R(X, Y))$$

in the derived category, represented by the isomorphism of complexes

$$\mathrm{Hom}_R(T(\underline{x}; X), I) \cong M(\underline{x}; \mathrm{Hom}_R(X, I)).$$

Proof The isomorphisms of complexes are shown in 9.2.3.

As $\mathrm{R}\Gamma_{\mathfrak{a}}(X)$ is represented by $T(\underline{x}; X)$ and $\mathrm{L}\Lambda^{\mathfrak{a}}(Y)$ is represented by $M(\underline{x}; Y) \xrightarrow{\sim} M(\underline{x}; I)$ (see 7.4.11 and 7.5.5) the first isomorphism in the derived category follows. Note that $M(\underline{x}; I) \cong \mathrm{Hom}_R(T(\underline{x}), I)$ is K-injective.

For the second note that $\mathrm{RHom}_R(X, Y)$ is represented by $\mathrm{Hom}_R(X, I)$. Hence $\mathrm{L}\Lambda^{\mathfrak{a}}(\mathrm{RHom}_R(X, Y))$ is represented by the microscope $M(\underline{x}; \mathrm{Hom}_R(X, I))$. The second isomorphism in the derived category follows. □

The above formula becomes particular suggestive when we put in place of Y an injective cogenerator E. It gives rise to a general Matlis duality functor $(\cdot)^\vee = \mathrm{Hom}_R(\cdot, E)$ (see 1.4.8). The following corollary, interesting for the study of the local cohomology modules $H^i_{\mathfrak{a}}(R)$, could help to achieve information about local cohomology in view of information about completion. Note that this corollary also generalizes the remark in 2.5.20.

Corollary 9.2.5 *In the situation of 9.2.4 let X denote an R-complex and let E be an injective cogenerator. Then in the derived category there are isomorphisms*

$$(\mathrm{R}\Gamma_{\mathfrak{a}}(X))^\vee \simeq \mathrm{RHom}_R(X, \mathrm{L}\Lambda^{\mathfrak{a}}(E)) \simeq \mathrm{L}\Lambda^{\mathfrak{a}}(X^\vee),$$

where $(\cdot)^\vee = \mathrm{Hom}_R(\cdot, E)$. In particular,

$$(H^i_{\mathfrak{a}}(X))^\vee \cong \Lambda^{\mathfrak{a}}_i(X^\vee) \text{ and } (H^i_{\mathfrak{a}}(R))^\vee \cong \Lambda^{\mathfrak{a}}_i(E).$$

Proof By the definitions the proof is an immediate consequence of 9.2.4 for $Y = E$. □

Remark 9.2.6 In the situation of 9.2.4 and 9.2.5 let $X = M$ denote an R-module. Then $(H^i_{\mathfrak{a}}(M))^\vee \cong \Lambda^{\mathfrak{a}}_i(M^\vee)$ for all $i \geq 0$ as seen above. For $i = 0$ note also that

$$\Lambda_0^{\mathfrak{a}}(M^\vee) \cong (H_{\mathfrak{a}}^0(M))^\vee = (\Gamma_{\mathfrak{a}}(M))^\vee \cong \Lambda^{\mathfrak{a}}(M^\vee)$$

is $\mathfrak{a}$-complete (see 2.2.13).

More down to earth, note also the following.

Proposition 9.2.7 *Let $\mathfrak{a}$ be a finitely generated ideal of a commutative ring R. Let X be any R-complex and I a complex of injective R-modules. Then there is an isomorphism*

$$\Lambda^{\mathfrak{a}}(\operatorname{Hom}_R(X, I) \cong \operatorname{Hom}_R(\Gamma_{\mathfrak{a}}(X), I) \cong \operatorname{Hom}_R(\Gamma_{\mathfrak{a}}(X), \Gamma_{\mathfrak{a}}(I)).$$

In particular, $\Lambda^{\mathfrak{a}}(X^\vee) \cong (\Gamma_{\mathfrak{a}}(X))^\vee$, where $(\cdot)^\vee$ denotes, as usual, the general Matlis duality functor.

Proof There is an inverse system of isomorphisms

$$R/\mathfrak{a}^t \otimes_R \operatorname{Hom}_R(X, I) \cong \operatorname{Hom}_R(\operatorname{Hom}_R(R/\mathfrak{a}^t, X), I)$$

(see 1.4.6). The first isomorphism follows by taking inverse limits (see also 1.3.3), the second one is obvious. □

Our next aim is to obtain some first local duality formula, generalizing the classical Grothendieck local duality for Cohen–Macaulay or Gorenstein local rings (see also Part III for more information around this subject).

Observations 9.2.8 (a) If the ideal $\mathfrak{a}$ of a commutative ring R is generated by a weakly pro-regular sequence $\underline{x} = x_1, \ldots, x_k$, recall that $\mathrm{L}\Lambda^{\mathfrak{a}}(E)$ is represented by the following complexes

$$(\check{C}_{\underline{x}})^\vee, \ (\check{L}_{\underline{x}})^\vee \text{ and } (T(\underline{x}))^\vee$$

(see 7.5.12). We note that in general $(\check{C}_{\underline{x}})^\vee$ is a bounded complex of injective modules, whence it is DG-injective.

(b) Assume now that the ideal $\mathfrak{a}$ of a commutative ring R is generated by a completely secant sequence $\underline{x} = x_1, \ldots, x_k$ in Bourbaki's terminology. This means that $H_i(\underline{x}^t) = 0$ for all $i > 0$ and all $t \geq 1$, equivalently that $H^i(\underline{x}^t) = 0$ for all $i < k$ and all $t \geq 1$ (see 5.4.2). At first this implies that the sequence $\underline{x}$ is weakly-pro-regular. This also implies that the complex $\check{C}_{\underline{x}}$ provides up to a shift a bounded flat resolution of $H^k(\check{C}_{\underline{x}})$. We note that $H^k(\check{C}_{\underline{x}}) \cong H^k_{\mathfrak{a}}(R)$ by 7.4.5. Then the complex $\check{C}_{\underline{x}}^\vee$ provides an injective resolution J of $(H^k_{\mathfrak{a}}(R))^\vee$ by putting $J = \check{C}_{\underline{x}}^{\vee[k]}$.

Proposition 9.2.9 *Let $\mathfrak{a}$ be an ideal of a commutative ring R generated by the sequence $\underline{x} = x_1, \ldots, x_k$ and let X be any R-complex.*

(a) If the sequence $\underline{x}$ is weakly pro-regular then

$$(H^i_{\mathfrak{a}}(X))^\vee \cong H^{-i}(\operatorname{Hom}_R(X, \check{C}_{\underline{x}}^\vee)) \cong \operatorname{Ext}_R^{-i}(X, \check{C}_{\underline{x}}^\vee).$$

(b) If the sequence $\underline{x}$ is completely secant in Bourbaki's terminology, then

$$(H^i_{\mathfrak{a}}(X))^{\vee} \cong \operatorname{Ext}^{k-i}_R(X, (H^k_{\mathfrak{a}}(R))^{\vee}).$$

Proof For the proof of (a) take the general Matlis dual of the isomorphism $H^i_{\mathfrak{a}}(X) \cong H^i(X \otimes_R \check{C}_{\underline{x}})$ obtained in 7.4.5 and recall that the complex $\check{C}^{\vee}_{\underline{x}}$ is DG-injective. Then (b) follows by the observations in 9.2.8. □

We end the section with some other quasi-isomorphisms involving the functors $T(\underline{x}; \cdot)$ and $M(\underline{x}; \cdot)$. The following provides yet another proof of Theorem 9.1.3.

Proposition 9.2.10 *Let $\underline{x} = x_1, \ldots, x_k$ and $\underline{y} = y_1, \ldots, y_l$ be two sequences in a commutative ring R such that $\operatorname{Rad}(\underline{x}R) \subseteq \operatorname{Rad}(\underline{y}R)$. Then there are quasi-isomorphisms*

(a) $T(\underline{y}; X) \xrightarrow{\sim} T(\underline{y}; M(\underline{x}; X))$ and
(b) $M(\underline{y}; T(\underline{x}; X)) \xrightarrow{\sim} M(\underline{y}; X)$

for any R-complex X.

Proof For the proof of (a) first recall that there are isomorphisms

$$T(\underline{y}; M(\underline{x}; X)) \cong T(\underline{y}) \otimes_R M(\underline{x}; X) \cong T(\underline{y}) \otimes_R \operatorname{Hom}_R(T(\underline{x}), X)$$

as they follow by 9.2.3. Now the morphism in (a) stems from the natural morphism $X \to \operatorname{Hom}_R(T(\underline{x}), X)$. But there are quasi-isomorphisms $\check{L}_{\underline{x}} \xrightarrow{\sim} T(\underline{x})$ and $T(\underline{y}) \xrightarrow{\sim} \check{L}_{\underline{y}}$ of complexes of free R-modules. Therefore there is a commutative square

$$\begin{tikzcd}
\check{L}_{\underline{y}} \otimes_R X \arrow[r] & \check{L}_{\underline{y}} \otimes_R \operatorname{Hom}_R(\check{L}_{\underline{x}}, X) \\
T(\underline{y}) \otimes_R X \arrow[r] \arrow[u] & T(\underline{y}) \otimes_R \operatorname{Hom}_R(T(\underline{x}), X) \arrow[u]
\end{tikzcd}$$

in which the vertical morphisms are quasi-isomorphisms. As the top horizontal morphism is a quasi-isomorphism (see 6.5.4 (d)) so is the bottom one.

For the proof of (b) we argue similarly. Namely $\operatorname{Hom}_R(\check{L}_{\underline{y}}, \check{L}_{\underline{x}} \otimes_R X)$ is quasi-isomorphic to $M(\underline{y}; T(\underline{x}; X))$ and $\operatorname{Hom}_R(\check{L}_{\underline{y}}, X)$ is quasi-isomorphic to $M(\underline{y}; X)$. Then the statement is a consequence of 6.5.4 (b). □

While the results in 6.5.4 grow from natural morphisms, it is not clear whether the results in 9.2.10 are also consequences of some natural morphisms between the telescope and the microscope.

9.3 Some Endomorphisms

This section is devoted to the endomorphism complexes of some particular complexes. We start with the case of $\check{L}_{\underline{x}}$.

Theorem 9.3.1 *Let $\underline{x} = x_1, \dots, x_k$ denote a weakly pro-regular sequence in a commutative ring R and $\mathfrak{a} = \underline{x}R$. Then there is a natural quasi-isomorphism*

$$\operatorname{Hom}_R(\check{L}_{\underline{x}}, \check{L}_{\underline{x}}) \xrightarrow{\sim} \hat{R}^{\mathfrak{a}}.$$

The natural homomorphism $\tau_R^{\mathfrak{a}} : R \to \hat{R}^{\mathfrak{a}}$ factors through this quasi-isomorphism.

Proof By virtue of 7.5.16 there is a natural quasi-isomorphism

$$\operatorname{Hom}_R(\check{L}_{\underline{x}}, R) \xrightarrow{\sim} \hat{R}^{\mathfrak{a}}.$$

In view of 6.5.4 (b) there is also a quasi-isomorphism

$$\operatorname{Hom}_R(\check{L}_{\underline{x}}, \check{L}_{\underline{x}}) \xrightarrow{\sim} \operatorname{Hom}_R(\check{L}_{\underline{x}}, R).$$

This provides the quasi-isomorphism of the statement. Moreover, there is the homothety morphism $h : R \to \operatorname{Hom}_R(\check{L}_{\underline{x}}, \check{L}_{\underline{x}})$ given by multiplication of elements of R. These morphisms are inserted in the diagram

$$\begin{tikzcd} & R \arrow[dl, "h"'] \arrow[d] \arrow[dr, "\tau_R^{\mathfrak{a}}"] & \\ \operatorname{Hom}_R(\check{L}_{\underline{x}}, \check{L}_{\underline{x}}) \arrow[r] & \operatorname{Hom}_R(\check{L}_{\underline{x}}, R) \arrow[r] & \hat{R}^{\mathfrak{a}}. \end{tikzcd}$$

In this diagram the triangle on the right is commutative (see 7.5.16 again) and so is the one on the left. The last assertion follows. □

This interesting result for the endomorphism complex of $\check{L}_{\underline{x}}$ also has some other consequences, namely for the endomorphism complex of $T(\underline{x})$ when the sequence $\underline{x}$ is weakly pro-regular.

Theorem 9.3.2 *Let $\underline{x} = x_1, \dots, x_k$ denote a sequence of elements of a commutative ring R and $\mathfrak{a} = \underline{x}R$. Suppose that $\underline{x}$ is a weakly pro-regular sequence. Then there is a quasi-isomorphism*

$$\operatorname{Hom}_R(T(\underline{x}), T(\underline{x})) \xrightarrow{\sim} \hat{R}^{\mathfrak{a}}.$$

That is, $H_0(\operatorname{Hom}_R(T(\underline{x}), T(\underline{x}))) \cong \hat{R}^{\mathfrak{a}}$ and $H_i(\operatorname{Hom}_R(T(\underline{x}), T(\underline{x}))) = 0$ for all $i \neq$ 0.

Proof As $T(\underline{x}) \xrightarrow{\sim} \check{C}_{\underline{x}}$ and $\check{L}_{\underline{x}} \xrightarrow{\sim} \check{C}_{\underline{x}}$ are DG-projective resolutions of $\check{C}_{\underline{x}}$ there are quasi -isomorphisms $T(\underline{x}) \xrightarrow{\sim} \check{L}_{\underline{x}}$ and $\check{L}_{\underline{x}} \xrightarrow{\sim} T(\underline{x})$ (see 4.4.8). They provide a quasi-isomorphism

$$\operatorname{Hom}_R(T(\underline{x}), T(\underline{x})) \xrightarrow{\sim} \operatorname{Hom}_R(\check{L}_{\underline{x}}, \check{L}_{\underline{x}}).$$

The result follows by 9.3.1. □

In the following we shall investigate the endomorphism complex of $\check{C}_{\underline{x}}$.

Theorem 9.3.3 *Let $\underline{x} = x_1, \ldots, x_k$ denote a sequence of elements of a commutative ring R. Then the natural morphism*

$$f : \operatorname{Hom}_R(\check{C}_{\underline{x}}, \check{C}_{\underline{x}}) \to \operatorname{Hom}_R(\check{C}_{\underline{x}}, R)$$

stemming from the surjection $\check{C}_{\underline{x}} \to R$ is a quasi-isomorphism.

Proof First note that the length one complexes $\check{C}_{x_i} \otimes_R R_{x_i}$ are contractible, $1 \leq i \leq k$. Whence for all $z \in R$ the complexes $\check{C}_{\underline{x}} \otimes_R R_{x_i z} \cong \check{C}_{x_1} \otimes_R \ldots \otimes_R \check{C}_{x_k} \otimes_R R_{x_i} \otimes_R R_z$ and $\operatorname{Hom}_R(\check{C}_{\underline{x}}, R_{x_i z}) \cong \operatorname{Hom}_{R_{x_i z}}(\check{C}_{\underline{x}} \otimes_R R_{x_i z}, R_{x_i z})$ are contractible too, in particular they are exact. Now it follows that $\operatorname{Hom}_R(\check{C}_{\underline{x}}, \check{C}^i_{\underline{x}})$ is exact for all $1 \leq i \leq k$.

The complex $\operatorname{Hom}_R(\check{C}_{\underline{x}}, \check{C}_{\underline{x}})$ is the single complex associated to the double complex $K^{ij} = \operatorname{Hom}_R(\check{C}^{-i}_{\underline{x}}, \check{C}^j_{\underline{x}})$ because $\check{C}_{\underline{x}}$ is bounded. As the rows $\operatorname{Hom}_R(\check{C}_{\underline{x}}, \check{C}^j_{\underline{x}})$ of this double complex are exact for $j > 0$ as shown above the natural surjection $\operatorname{Hom}_R(\check{C}_{\underline{x}}, \check{C}_{\underline{x}}) \to \operatorname{Hom}_R(\check{C}_{\underline{x}}, \check{C}^0_{\underline{x}}) = \operatorname{Hom}_R(\check{C}_{\underline{x}}, R)$ is a quasi-isomorphism. □

Lemma 9.3.4 *Let $\underline{x} = x_1, \ldots, x_k$ denote a sequence of elements of a commutative ring R and put $\mathfrak{a} = \underline{x}R$. There is a commutative triangle*

$$\begin{tikzcd} & R \arrow[dl, "h"'] \arrow[dr, "a"] & \\ \operatorname{Hom}_R(\check{C}_{\underline{x}}, \check{C}_{\underline{x}}) \arrow[rr, "f"] & & \operatorname{Hom}_R(\check{C}_{\underline{x}}, R) \end{tikzcd}$$

where the morphisms a and f stem from the surjection $\check{C}_{\underline{x}} \to R$ and where h is the homothety morphism given by multiplication by the elements r of R. The morphisms a and h are injective.

Suppose moreover that R is $\mathfrak{a}$-separated. Then the morphism a is an isomorphism, i.e., $R \cong \operatorname{Hom}_R(\check{C}_{\underline{x}}, R)$.

Proof Clearly the triangle is commutative. The morphism a is injective because the morphism $\check{C}_{\underline{x}} \to R$ is degree-wise surjective. Hence the natural morphism h is also injective.

For the proof of the last statement observe that $\operatorname{Hom}_R(\check{C}^i_{\underline{x}}, R) = 0$ for all $i \neq 0$ when R is $\mathfrak{a}$-separated. This follows because $\check{C}^i_{\underline{x}}$, $i \neq 0$, is the direct sum of localizations R_y, $y \in \mathfrak{a}$, and $\operatorname{Hom}_R(R_y, R) = 0$ for $y \in \mathfrak{a}$ when R is $\mathfrak{a}$-separated. □

Putting 9.3.3 and 9.3.4 together we obtain the following.

Corollary 9.3.5 *Let $\underline{x} = x_1, \ldots, x_k$ denote a sequence of elements of a commutative ring R and put $\mathfrak{a} = \underline{x}R$. Assume that R is $\mathfrak{a}$-separated. Then the homothety morphism $h : R \to \operatorname{Hom}_R(\check{C}_{\underline{x}}, \check{C}_{\underline{x}})$ is a quasi-isomorphism.*

We continue the investigation of $\operatorname{Hom}_R(\check{C}_{\underline{x}}, \check{C}_{\underline{x}})$ and $\operatorname{Hom}_R(\check{C}_{\underline{x}}, R)$.

Lemma 9.3.6 *With the notation of 9.3.4 there is a morphism of complexes $a_1 : \operatorname{Hom}_R(\check{C}_{\underline{x}}, R) \to \hat{R}^{\mathfrak{a}}$ such that the following diagram*

$$\begin{tikzcd} R \arrow[rr, "\tau_R^{\mathfrak{a}}"] \arrow[dr, "a"'] & & \hat{R}^{\mathfrak{a}} \\ & \operatorname{Hom}_R(\check{C}_{\underline{x}}, R) \arrow[ur, "a_1"'] & \end{tikzcd}$$

is commutative.

Proof There is an inverse system of morphisms

$$R \to K_\bullet(\underline{x}^t) \to R/\underline{x}^t R.$$

By passing to the inverse limit we obtain the morphism a_1 and the commutativity of the diagram since $\operatorname{Hom}_R(\check{C}_{\underline{x}}, R) = \operatorname{Hom}_R(\varinjlim K^\bullet(\underline{x}^t), R \cong \varprojlim K_\bullet(\underline{x}^t)$. □

Proposition 9.3.7 *Let $\underline{x} = x_1, \ldots, x_k$ denote a sequence of elements of a commutative ring R and $\mathfrak{a} = \underline{x}R$. There is a natural morphism*

$$g : \operatorname{Hom}_R(\check{C}_{\underline{x}}, \check{C}_{\underline{x}}) \to \hat{R}^{\mathfrak{a}}.$$

It is a quasi-isomorphism if and only if R is $\mathfrak{a}$-complete.

Proof The morphism g is defined by $g = a_1 \circ f$, where a_1 is the morphism defined in 9.3.6 and f is the quasi-isomorphism defined in 9.3.3. If R is $\mathfrak{a}$-complete it is $\mathfrak{a}$-separated and the morphism $a : R \to \operatorname{Hom}_R(\check{C}_{\underline{x}}, R)$ is an isomorphism (see 9.3.4). Hence the morphism a_1 is a quasi-isomorphism by 9.3.6 and so is g.

Conversely assume that the natural morphism g is a quasi-isomorphism. Then the natural morphism $a_1 : \operatorname{Hom}_R(\check{C}_{\underline{x}}, R) \to \hat{R}^{\mathfrak{a}}$ is also a quasi-isomorphism. By looking in cohomological degree 0 we observe that the natural morphism $\tau_R^{\mathfrak{a}} = (R \to \operatorname{Hom}_R(\check{C}_{\underline{x}}, R) \to \hat{R}^{\mathfrak{a}})$ is injective because the natural map $R \to \operatorname{Hom}_R(\check{C}_{\underline{x}}, R)$ is injective. This means that R is $\mathfrak{a}$-separated, and the result follows by 9.3.4 again. □

9.3.8 The previous result shows that $\operatorname{RHom}_R(\check{C}_{\underline{x}}, \check{C}_{\underline{x}})$ is not represented by the complex $\operatorname{Hom}_R(\check{C}_{\underline{x}}, \check{C}_{\underline{x}})$. In general, note that $\operatorname{RHom}_R(\check{C}_{\underline{x}}, \check{C}_{\underline{x}})$ is represented by $\operatorname{Hom}_R(\check{L}_{\underline{x}}, \check{L}_{\underline{x}})$ as well as by $\operatorname{Hom}_R(T(\underline{x}), T(\underline{x}))$. This is clear because $T(\underline{x}) \xrightarrow{\sim} \check{C}_{\underline{x}}$ and $\check{L}_{\underline{x}} \xrightarrow{\sim} \check{C}_{\underline{x}}$ are bounded free resolutions of $\check{C}_{\underline{x}}$. When the sequence $\underline{x}$ generating the ideal $\mathfrak{a}$ is weakly-proregular note also by 9.3.1 that $\operatorname{RHom}_R(\check{C}_{\underline{x}}, \check{C}_{\underline{x}})$ is isomorphic to $\hat{R}^{\mathfrak{a}}$ in the derived category.

Let $R \xrightarrow{\sim} J$ be an injective resolution of R and let $\mathfrak{a}$ be an ideal of R generated by a weakly-pro-regular sequence. Then the complexes $\Gamma_{\mathfrak{a}}(J)$ and $\check{C}_{\underline{x}}$ are quasi-isomorphic. So it will be of some interest to look at the endomorphism complex of $\Gamma_{\mathfrak{a}}(J)$.

Proposition 9.3.9 *Let $\mathfrak{a}$ denote an ideal of a commutative ring R generated by a weakly pro-regular sequence $\underline{x} = x_1, \ldots, x_k$ and let $R \xrightarrow{\sim} J$ be an injective resolution. Then the complex $\operatorname{Hom}_R(\Gamma_{\mathfrak{a}}(J), \Gamma_{\mathfrak{a}}(J))$ is quasi-isomorphic to $\hat{R}^{\mathfrak{a}}$. That is,*

$$H_0(\operatorname{Hom}_R(\Gamma_{\mathfrak{a}}(J), \Gamma_{\mathfrak{a}}(J))) = \hat{R}^{\mathfrak{a}} \text{ and}$$
$$H_i(\operatorname{Hom}_R(\Gamma_{\mathfrak{a}}(J), \Gamma_{\mathfrak{a}}(J))) = 0$$

for all $i \neq 0$.

Proof First note that $\operatorname{Hom}_R(\Gamma_{\mathfrak{a}}(J), \Gamma_{\mathfrak{a}}(J)) \cong \operatorname{Hom}_R(\Gamma_{\mathfrak{a}}(J), J)$. Then recall the quasi-isomorphism $\Gamma_{\mathfrak{a}}(J) \xrightarrow{\sim} \check{C}_{\underline{x}} \otimes_R J$ (see 7.4.1). This together with the quasi-isomorphism $\check{C}_{\underline{x}} \otimes_R R \xrightarrow{\sim} \check{C}_{\underline{x}} \otimes_R J$ implies quasi-isomorphisms

$$\operatorname{Hom}_R(\check{C}_{\underline{x}}, J) \xleftarrow{\sim} \operatorname{Hom}_R(\check{C}_{\underline{x}} \otimes_R J, J) \xrightarrow{\sim} \operatorname{Hom}_R(\Gamma_{\mathfrak{a}}(J), J).$$

Because $\check{L}_{\underline{x}} \xrightarrow{\sim} \check{C}_{\underline{x}}$ is a free resolution of the Čech complex there are quasi-isomorphisms

$$\operatorname{Hom}_R(\check{C}_{\underline{x}}, J) \xrightarrow{\sim} \operatorname{Hom}_R(\check{L}_{\underline{x}}, J) \xleftarrow{\sim} \operatorname{Hom}_R(\check{L}_{\underline{x}}, R).$$

We now recall the quasi-isomorphism $\operatorname{Hom}_R(\check{L}_{\underline{x}}, R) \xrightarrow{\sim} \hat{R}^{\mathfrak{a}}$ (see 7.5.16). All this together finally proves the claim. □

When the ring is coherent we have a slightly better result.

Proposition 9.3.10 *Let $\mathfrak{a}$ be an ideal of a coherent ring R generated by a weakly pro-regular sequence $\underline{x} = x_1, \ldots, x_k$. Let $R \xrightarrow{\sim} J$ denote an injective resolution of R. Then there is a quasi-isomorphism*

$$\hat{R}^{\mathfrak{a}} \xrightarrow{\sim} \operatorname{Hom}_R(\Gamma_{\mathfrak{a}}(J), \Gamma_{\mathfrak{a}}(J)).$$

Proof Since J is an injective resolution of R there are quasi-isomorphisms

$$R \xrightarrow{\sim} J = \operatorname{Hom}_R(R, J) \xleftarrow{\sim} \operatorname{Hom}_R(J, J).$$

Therefore the homothety morphism $h : R \to \operatorname{Hom}_R(J, J)$ is a quasi-isomorphism (see 1.1.7). Note that $\operatorname{Hom}_R(J, J)$ is a complex of flat modules (see 1.4.5). It follows that h induces a quasi-isomorphism

$$\hat{R}^{\mathfrak{a}} \xrightarrow{\sim} \Lambda^{\mathfrak{a}}(\operatorname{Hom}_R(J, J)$$

(see 7.5.4). But $\Lambda^{\mathfrak{a}}(\operatorname{Hom}_R(J, J)) \cong \operatorname{Hom}_R(\Gamma_{\mathfrak{a}}(J), \Gamma_{\mathfrak{a}}(J))$ (see 9.2.7). This completes the proof. $\square$

As another feature of our investigations we are interested in the endomorphisms of the complex $\mathrm{L}\Lambda^{\mathfrak{a}}(E)$, where $E = E_R(\Bbbk)$ denotes the injective hull of the residue field $\Bbbk = R/\mathfrak{m}$ of a Noetherian local ring $(R, \mathfrak{m})$.

Proposition 9.3.11 *Let $(R, \mathfrak{m}, \Bbbk)$ denote a Noetherian local ring with $E = E_R(\Bbbk)$ the injective hull of the residue field $\Bbbk$. Let $\underline{x} = x_1, \ldots, x_k$ be a sequence of elements in R generating a proper ideal. Then there is a natural quasi-isomorphism*

$$\operatorname{Hom}_R(\operatorname{Hom}_R(\check{L}_{\underline{x}}, E), \operatorname{Hom}_R(\check{L}_{\underline{x}}, E)) \xrightarrow{\sim} \hat{R}^{\mathfrak{m}}.$$

Moreover, the natural homomorphism $\tau_R^{\mathfrak{m}} : R \to \hat{R}^{\mathfrak{m}}$ factors through this previous quasi-isomorphism.

Proof By 9.1.4 there is a natural quasi-isomorphism

$$\operatorname{Hom}_R(\operatorname{Hom}_R(\check{L}_{\underline{x}}, E), \operatorname{Hom}_R(\check{L}_{\underline{x}}, E)) \xrightarrow{\sim} \operatorname{Hom}_R(\check{L}_{\underline{x}}, \operatorname{Hom}_R(E, E)).$$

Now let us investigate the second complex. Because $\operatorname{Hom}_R(E, E) \cong \hat{R}^{\mathfrak{m}}$ and because $\hat{R}^{\mathfrak{m}}$ is R-flat we have an isomorphism and a quasi-isomorphism

$$\operatorname{Hom}_R(\check{L}_{\underline{x}}, \operatorname{Hom}_R(E, E)) \cong \operatorname{Hom}_R(\check{L}_{\underline{x}}, \hat{R}^{\mathfrak{m}}) \xrightarrow{\sim} \Lambda^{\underline{x}R}(\hat{R}^{\mathfrak{m}})$$

(see 7.5.16). Recall now that $\hat{R}^{\mathfrak{m}}$ is $\underline{x}R$-complete (see 2.2.9), so that $\Lambda^{\underline{x}R}(\hat{R}^{\mathfrak{m}}) = \hat{R}^{\mathfrak{m}}$. All this together provides the quasi-isomorphism.

By 7.5.16 we also have that the composite $R \to \operatorname{Hom}_R(\check{L}_{\underline{x}}, \hat{R}^{\mathfrak{m}}) \to \hat{R}^{\mathfrak{m}}$ is the natural homomorphism $\tau_R^{\mathfrak{m}} : R \to \hat{R}^{\mathfrak{m}}$. It follows that the composite

$$R \to \operatorname{Hom}_R(\operatorname{Hom}_R(\check{L}_{\underline{x}}, E), \operatorname{Hom}_R(\check{L}_{\underline{x}}, E)) \to \hat{R}^{\mathfrak{m}}$$

is this natural homomorphism $\tau_R^{\mathfrak{m}}$. $\square$

Corollary 9.3.12 *Let $(R, \mathfrak{m}, \Bbbk)$ be a Noetherian local ring with E the injective hull of the residue field $\Bbbk$. For any proper ideal $\mathfrak{a}$ of R there is an isomorphism*

$$\operatorname{RHom}_R(\mathrm{L}\Lambda^{\mathfrak{a}}(E), \mathrm{L}\Lambda^{\mathfrak{a}}(E)) \simeq \hat{R}^{\mathfrak{m}}$$

in the derived category.

Proof Let $\underline{x} = x_1, \ldots, x_k$ be a generating set of the ideal $\mathfrak{a}$. Then $\mathrm{L}\Lambda^{\mathfrak{a}}(E)$ is represented by the complex $\operatorname{Hom}_R(\check{L}_{\underline{x}}, E)$. As this complex is a bounded complex of injective modules, the statement follows by 9.3.11. □

9.4 Mayer–Vietoris Sequences for Local and Čech (Co-)Homology

We first present the Mayer–Vietoris long exact sequence concerning the local cohomology of a complex with respect to two ideals $\mathfrak{a}$ and $\mathfrak{b}$ of a Noetherian ring and its counterpart in local homology. Taking advantage of these we then present long exact sequences of Mayer–Vietoris type for the Čech homology and cohomology of a complex with respect to two arbitrary sequences $\underline{x}$ and $\underline{y}$ in an arbitrary commutative ring. This in turn has some unexpected consequences concerning weakly pro-regular sequences. Moreover, it also allows Mayer–Vietoris type sequences for local homology and cohomology for certain pairs of ideals of an arbitrary commutative ring, provided that these ideals and their sum are generated by weakly pro-regular sequences.

We end the section with short exact sequences involving the enlargement of an ideal by a single element.

9.4.1 Let $\mathfrak{a}, \mathfrak{b}$ be two ideals of a commutative ring R. Then there is an inverse system of short exact sequences

$$0 \to R/\mathfrak{a}^t \cap \mathfrak{b}^t \to R/\mathfrak{a}^t \oplus R/\mathfrak{b}^t \to R/(\mathfrak{a}^t + \mathfrak{b}^t) \to 0$$

for $t \geq 1$. When the ring R is Noetherian ring the decreasing sequence of ideals $\mathfrak{a}^t \cap \mathfrak{b}^t$ forms a base of open neighbourhoods of the $(\mathfrak{a} \cap \mathfrak{b})$-adic topology which also coincides with the $(\mathfrak{a} \cdot \mathfrak{b})$-adic topology and the decreasing sequence of ideals $(\mathfrak{a}^t + \mathfrak{b}^t)$ forms a base of open neighbourhoods of the $(\mathfrak{a} + \mathfrak{b})$-adic topology.

Theorem 9.4.2 *Let $\mathfrak{a}, \mathfrak{b}$ two ideals of a Noetherian ring R. Let X denote a complex of R-modules. Then we have the following long exact sequences:*

(a) $\ldots \to H^i_{(\mathfrak{a}+\mathfrak{b})}(X) \to H^i_{\mathfrak{a}}(X) \oplus H^i_{\mathfrak{b}}(X) \to H^i_{\mathfrak{a}\cap\mathfrak{b}}(X) \to \ldots,$

(b) $\ldots \to \Lambda_i^{\mathfrak{a}\cap\mathfrak{b}}(X) \to \Lambda_i^{\mathfrak{a}}(X) \oplus \Lambda_i^{\mathfrak{b}}(X) \to \Lambda_i^{(\mathfrak{a}+\mathfrak{b})}(X) \to \ldots.$

Proof We start with the proof of (a). To this end let $X \to I$ be a DG-injective resolution of X. Then we apply the functor $\operatorname{Hom}_R(\cdot, I)$ to the above inverse system of short exact sequences. Because I is a complex of injective R-modules it gives a direct system of short exact sequences of complexes

$$0 \to \operatorname{Hom}_R(R/(\mathfrak{a}^t + \mathfrak{b}^t), I) \to \operatorname{Hom}_R(R/\mathfrak{a}^t, I) \oplus \operatorname{Hom}_R(R/\mathfrak{b}^t, I) \\ \to \operatorname{Hom}_R(R/\mathfrak{a}^t \cap \mathfrak{b}^t, I) \to 0$$

for all $t \geq 1$. Then, by passing to the direct limit and taking cohomology yields the proof of the claim in (a).

For the proof of (b) take a DG-flat resolution $F \to X$ of X. Then tensoring the inverse system of short exact sequences by F provides an inverse system of short exact sequences of complexes

$$0 \to F/(\mathfrak{a}^t \cap \mathfrak{b}^t)F \to F/\mathfrak{a}^t F \oplus F/\mathfrak{b}^t F \to F/(\mathfrak{a}^t + \mathfrak{b}^t)F \to 0$$

for all $t \geq 1$. Since these inverse systems are degree-wise surjective it provides a short exact sequence of complexes by passing to the inverse limit. Taking homology proves (b). $\square$

Next we present long exact sequences of Mayer–Vietoris type concerning Čech homology and cohomology. This is one of the few occasions when we can derive properties of Čech (co)-homology from those of local (co)-homology, most often one goes in the opposite direction. Note that the Mayer–Vietoris sequence for Čech cohomology was also obtained by Tête in a completely different way, independent of the classical Mayer–Vietoris sequence in local cohomology (see [84]).

Theorem 9.4.3 *Let $\underline{x} = x_1, \ldots, x_k$ and $\underline{y} = y_1, \ldots, y_l$ be two sequences in a commutative ring R. We form the sequence $\underline{x}, \underline{y}$ and denote by $\underline{z}$ the sequence formed by the $x_i y_j$ in any order. Let X denote an R-complex. Then there are two long exact sequences:*

(a)

$$\ldots \to H^i(\check{C}_{\underline{x},\underline{y}} \otimes_R X) \to H^i(\check{C}_{\underline{x}} \otimes_R X) \oplus H^i(\check{C}_{\underline{y}} \otimes_R X) \\ \to H^i(\check{C}_{\underline{z}} \otimes_R X) \to H^{i+1}(\check{C}_{\underline{x},\underline{y}} \otimes_R X) \to \ldots$$

(b)

$$\ldots \to H_i(\operatorname{RHom}_R(\check{C}_{\underline{z}}, X)) \to H_i(\operatorname{RHom}_R(\check{C}_{\underline{x}}, X)) \oplus H_i(\operatorname{RHom}_R(\check{C}_{\underline{y}}, X)) \\ \to H_i(\operatorname{RHom}_R(\check{C}_{\underline{x},\underline{y}}, X)) \to H_{i-1}(\operatorname{RHom}_R(\check{C}_{\underline{z}}, X)) \to \ldots .$$

Proof Let $X_1, \ldots, X_k, Y_1, \ldots, Y_l$ be indeterminates. We form the polynomial ring $S = \mathbb{Z}[X_1, \ldots, X_k, Y_1, \ldots, Y_l]$ and define a ring homomorphism by $\varphi : S \to R : X_i \mapsto x_i, Y_j \mapsto y_j$. We view the R-complex X as an S-complex via φ. We need only prove the statements for the sequences $\underline{X} := X_1, \ldots, X_k$, $\underline{Y} := Y_1, \ldots, Y_l$ and the sequence $\underline{Z}$ formed by the $X_i Y_j$ (see 6.3.5). The ring S is Noetherian. Hence the three sequences $\underline{X}$, $\underline{Y}$ and $\underline{Z}$ are weakly pro-regular and the associated Čech homology and cohomology with respect to these sequences coincide with the associated local homology and cohomology with respect to the ideals $\underline{X}S$, $\underline{Y}S$ and $\underline{Z}S = \underline{X}S \cdot \underline{Y}S$ (see 7.4.4 and 7.5.12). Note that the ideals $\underline{Z}S$ and $\underline{X}S \cap \underline{Y}S$ are topologically equivalent because S is Noetherian. Hence

$$H^i_{\underline{Z}S}(X) = H^i_{(\underline{X}S \cap \underline{Y}S)}(X) \text{ and } \Lambda_i^{\underline{Z}S}(X) = \Lambda_i^{(\underline{X}S \cap \underline{Y}S)}(X).$$

Now the statements follows from Theorem 9.4.2. □

The above considerations have unexpected consequences.

Corollary 9.4.4 *Let $\underline{x} = x_1, \ldots, x_k$ and $\underline{y} = y_1, \ldots, y_l$ be two weakly pro-regular sequences in a commutative ring R. Assume that the sequence $\underline{x}, \underline{y}$ is also weakly pro-regular. Then the sequence $\underline{z}$ formed by the $x_i y_j$ is also weakly pro-regular.*

Proof For all injective R-modules I and all $i \geq 1$ we have the following

$$H^i(\check{C}_{\underline{x}} \otimes_R I) \cong H^i_{(\underline{x}R)}(I) = 0,$$
$$H^i(\check{C}_{\underline{y}} \otimes_R I) \cong H^i_{(\underline{y}R)}(I) = 0 \text{ and}$$
$$H^i(\check{C}_{\underline{x},\underline{y}} \otimes_R I) \cong H^i_{((\underline{x},\underline{y})R)}(I) = 0$$

(see 7.4.5 for the isomorphisms). It follows that $H^i(\check{C}_{\underline{z}} \otimes_R I) = 0$ for all $i \geq 1$ by the first long exact sequence in Theorem 9.4.3. So the result follows by 7.3.3. □

Corollary 9.4.5 *Let $\mathfrak{a}, \mathfrak{b}$ two ideals of a commutative ring R, generated respectively by the weakly pro-regular sequences $\underline{x} = x_1, \ldots, x_k$ and $\underline{y} = y_1, \ldots, y_l$. Assume that the sequence $\underline{x}, \underline{y}$ is also weakly pro-regular. Let X denote a complex of R-modules. Then we have the following long exact sequences:*

(a) $\ldots \to H^i_{(\mathfrak{a}+\mathfrak{b})}(X) \to H^i_{\mathfrak{a}}(X) \oplus H^i_{\mathfrak{b}}(X) \to H^i_{\mathfrak{a}\cdot\mathfrak{b}}(X) \to \ldots,$

(b) $\ldots \to \Lambda_i^{\mathfrak{a}\cdot\mathfrak{b}}(X) \to \Lambda_i^{\mathfrak{a}}(X) \oplus \Lambda_i^{\mathfrak{b}}(X) \to \Lambda_i^{(\mathfrak{a}+\mathfrak{b})}(X) \to \ldots.$

Proof As the sequence $\underline{z}$ formed by the $x_i y_j$ is also weakly pro-regular and generates the ideal $\mathfrak{a} \cdot \mathfrak{b}$, and in view of 7.4.5 and 7.5.12, the sequences are a reformulation of those in Theorem 9.4.3. □

We end this section with short exact sequences of another type. They concern the enlargement of ideals and the corresponding local (co)-homology functors. We start with the case of the addition of a singly generated ideal.

Proposition 9.4.6 *Let $\underline{x} = x_1, \ldots, x_k$ denote a weakly pro-regular sequence in a commutative ring R and put $\mathfrak{a} = \underline{x}R$. Suppose that R is of bounded yR-torsion and such that $\underline{x}, y$ is a weakly pro-regular sequence too. Let X denote a complex of R-modules. Then there are short exact sequences*

(a) $0 \to H^1_{yR}(H^{i-1}_{\mathfrak{a}}(X)) \to H^i_{(\mathfrak{a}+yR)}(X) \to H^0_{yR}(H^i_{\mathfrak{a}}(X)) \to 0$ *and*
(b) $0 \to \Lambda^{yR}_0(\Lambda^{\mathfrak{a}}_i(X)) \to \Lambda^{(\mathfrak{a}+yR)}_i(X) \to \Lambda^{yR}_1(\Lambda^{\mathfrak{a}}_{i-1}(X)) \to 0$

for all $i \in \mathbb{Z}$.

Proof We start with a general remark. Let $\underline{z} = z_1, \ldots, z_l$ denote a weakly pro-regular sequence of elements in R and let $\mathfrak{b} = \underline{z}R$. Then $H^i_{\mathfrak{b}}(X) \cong H^i(\check{C}_{\underline{z}} \otimes_R X)$ (see 7.4.4) and $\Lambda^{\mathfrak{b}}_i(X) \cong H_i(\operatorname{Hom}_R(L_{\underline{z}}, X))$ (see 7.5.12) for all $i \in \mathbb{Z}$. By the assumptions about the length one sequence y and the sequences $\underline{x}$ and $\underline{x}, y$, we may apply the previous results to any of the three sequences and the ideals generated by them.

For the proof of (a) we refer to 6.1.11. The statement in (b) follows by 6.2.5. □

9.5 On the use of the classes $\mathcal{C}_{\mathfrak{a}}$ and $\mathcal{B}_{\mathfrak{a}}$

We observed in 2.5.12 that the local homology of an R-module M with respect to an ideal $\mathfrak{a}$ may be computed with a left resolution of M by means of modules in the class $\mathcal{C}_{\mathfrak{a}}$. When the ideal $\mathfrak{a}$ is generated by a weakly pro-regular sequence we shall see that this also holds for the local homology of any unbounded complex. We shall also obtain an analogous statement in local cohomology. We end the section with more results about relatively-$\mathfrak{a}$-flat and relatively-$\mathfrak{a}$-injective modules.

Lemma 9.5.1 *Let* $\mathfrak{a}$ *be an ideal of a commutative ring* R *generated by a weakly pro-regular sequence* $\underline{x} = x_1, \ldots, x_k$ *and let* Y *be an unbounded exact* R*-complex.*

(a) *If* Y *is a complex of modules in the class* $\mathcal{C}_{\mathfrak{a}}$ *(see 2.5.12), then* $\hat{Y}^{\mathfrak{a}}$ *is also exact.*
(b) *If* Y *is a complex of modules in the class* $\mathcal{B}_{\mathfrak{a}}$ *(see 2.7.6), then* $\Gamma_{\mathfrak{a}}(Y)$ *is also exact.*

Proof (a): We view Y as a descending complex, pick any integer i and put $M_i = \operatorname{Im}(d_i)$. The truncated complex $\ldots \to Y_{i+1} \to Y_i \to 0$ gives a resolution of the module M_i (up to a shift) and may be used to compute the local homology of M_i as observed in 2.5.12. In particular, we have that $\Lambda^{\mathfrak{a}}_j(M_i) \cong H_{i+j}(\hat{Y}^{\mathfrak{a}})$ for all $j \geq 1$. But $\Lambda^{\mathfrak{a}}_j(M_i) = 0$ for all $j > k$ by the bound in 7.5.7. It follows that $H_t(\hat{Y}^{\mathfrak{a}}) = 0$ for all $t > i + k$. As this holds for all $i \in \mathbb{Z}$ the conclusion follows.

(b): The proof of this is similar to that of (a), using the remarks in 2.7.6 in place of the ones in 2.5.12, and the bound in 7.4.6 in place of those in 7.5.7. □

When the ideal $\mathfrak{a}$ of a commutative ring R is generated by a weakly pro-regular sequence we have seen that the local homology of an unbounded complex may be computed with flat resolutions and that the local cohomology may be computed with injective resolutions. Here we shall see that they may be computed by using some other "suitable" resolutions.

Proposition 9.5.2 *Let* $\mathfrak{a}$ *be an ideal of a commutative ring* R *generated by a weakly pro-regular sequence* $\underline{x} = x_1, \ldots, x_k$ *and let* X *be any unbounded* R*-complex.*

(a) *Let $Y \xrightarrow{\sim} X$ be a left resolution of X such that all the modules Y_i are in $\mathcal{C}_\mathfrak{a}$ (e.g. such that the Y_i are relatively-$\mathfrak{a}$-flat). Then $\mathrm{L}\Lambda^\mathfrak{a}(X)$ is represented by $\Lambda^\mathfrak{a}(Y)$ and the natural map*

$$\alpha_X^\mathfrak{a} : X \to \mathrm{L}\Lambda^\mathfrak{a}(X)$$

is represented by the natural morphism $\tau_Y^\mathfrak{a} : Y \to \Lambda^\mathfrak{a}(Y)$. In particular, $\Lambda_i^\mathfrak{a}(X) \cong H_i(\Lambda^\mathfrak{a}(Y))$ for all $i \in \mathbb{Z}$.

(b) *Let $X \xrightarrow{\sim} Z$ be a right resolution of X such that all the modules Z^i are in $\mathcal{B}_\mathfrak{a}$ (e.g. such that the Z^i are relatively-$\mathfrak{a}$-injective). Then $\mathrm{R}\Gamma_\mathfrak{a}(X)$ is represented by $\Gamma_\mathfrak{a}(Z)$ and the natural map*

$$\iota_\mathfrak{a}^X : \mathrm{R}\Gamma_\mathfrak{a}(X) \to X$$

is represented by the natural injection $\Gamma_\mathfrak{a}(Z) \hookrightarrow Z$. In particular, $H_\mathfrak{a}^i(X) \cong H^i(\Gamma_\mathfrak{a}(Z))$ for all $i \in \mathbb{Z}$.

Proof We only prove (a), the proof of (b) is similar. To this end we take a flat resolution $h : F \xrightarrow{\sim} Y$ of Y and consider the cone $C(h)$. This cone is an exact complex of modules in $\mathcal{C}_\mathfrak{a}$. Hence the complex $\Lambda^\mathfrak{a}(C(h))$ is exact by the above lemma. But the functor $\Lambda^\mathfrak{a}$ commutes with the formation of cones as easily seen: we have $\Lambda^\mathfrak{a}(C(h)) \cong C(\Lambda^\mathfrak{a}(h))$. Hence $\Lambda^\mathfrak{a}(h)$ is a quasi-isomorphism. So there is a commutative square

$$\begin{tikzcd} F \arrow[r, "\sim"] \arrow[d, "\tau_F^\mathfrak{a}"'] & Y \arrow[d, "\tau_Y^\mathfrak{a}"] \\ \Lambda^\mathfrak{a}(F) \arrow[r, "\sim"] & \Lambda^\mathfrak{a}(Y). \end{tikzcd}$$

To conclude we recall that $\mathrm{L}\Lambda^\mathfrak{a}(X)$ is represented by $\Lambda^\mathfrak{a}(F)$ and that the map $\alpha_X^\mathfrak{a}$ is represented by the morphism $\tau_F^\mathfrak{a} : F \to \Lambda^\mathfrak{a}(F)$ (see 7.5.5). □

Remark 9.5.3 When the ideal $\mathfrak{a}$ is generated by a weakly pro-regular sequence we have seen in 7.5.13 that $\mathfrak{a}$-complete modules are in $\mathcal{C}_\mathfrak{a}$ and in 7.4.7 that $\mathfrak{a}$-torsion modules are in $\mathcal{B}_\mathfrak{a}$. Hence the above proposition provides yet another proof of the fact that the functors $\mathrm{L}\Lambda^\mathfrak{a}$ and $\mathrm{R}\Gamma_\mathfrak{a}$ are idempotent in that case.

Remark 9.5.4 We note that the definitions of the classes $\mathcal{B}_\mathfrak{a}$ and $\mathcal{C}_\mathfrak{a}$ are similar. When the ideal $\mathfrak{a}$ of the commutative ring R is generated by a weakly pro-regular sequence these classes are also related through the general Matlis duality. Note that

$$M \in \mathcal{B}_\mathfrak{a} \Leftrightarrow M^\vee \in \mathcal{C}_\mathfrak{a},$$

as follows from the remarks in 9.2.6.

In view of the rôle played by the classes $\mathcal{C}_\mathfrak{a}$ and $\mathcal{B}_\mathfrak{a}$ in the study of local homology and cohomology the following criteria may be of some interest.

Proposition 9.5.5 *Let $\mathfrak{a}$ be an ideal of a commutative ring R generated by a weakly pro-regular sequence $\underline{x}$ and let N denote an R-module. Then $N \in \mathcal{C}_{\mathfrak{a}}$ if and only if the natural morphism $M(\underline{x}; N) \to M(\underline{x}; \hat{N}^{\mathfrak{a}})$ induced by the natural homomorphism $\tau_N^{\mathfrak{a}} : N \to \hat{N}^{\mathfrak{a}}$ is a quasi-isomorphism.*

Proof Recall that the natural morphism $N \to M(\underline{x}; N)$ represents the natural map $\alpha_N^{\mathfrak{a}} : N \to \mathrm{L}\Lambda^{\mathfrak{a}}(N)$ in the derived category and induces the natural homomorphism $\eta_N^{\mathfrak{a}} : N \to \Lambda_0^{\mathfrak{a}}(N)$ in homological degree 0 (see 7.5.5 and 7.5.8). Recall also that $\hat{N}^{\mathfrak{a}} \in \mathcal{C}_{\mathfrak{a}}$ (see 7.5.13). All this together proves the claim. □

Proposition 9.5.6 *Let $\mathfrak{a}$ be an ideal of a commutative ring R generated by a weakly pro-regular sequence $\underline{x}$ and let M denote an R-module. Then $M \in \mathcal{B}_{\mathfrak{a}}$ if and only if the natural morphism $\check{C}_{\underline{x}} \otimes_R \Gamma_{\mathfrak{a}}(M) \to \check{C}_{\underline{x}} \otimes_R M$ induced by the inclusion $\Gamma_{\mathfrak{a}}(M) \subset M$ is a quasi-isomorphism.*

Proof By 7.4.7 we know that the natural morphism $\check{C}_{\underline{x}} \otimes_R \Gamma_{\mathfrak{a}}(M) \to \Gamma_{\mathfrak{a}}(M)$ is a quasi-isomorphism. As always $H^0(\check{C}_{\underline{x}} \otimes_R M) = \Gamma_{\mathfrak{a}}(M)$ the conclusion follows. □

We end the section with the announced extensions of 2.6.11 and of 2.7.10.

Proposition 9.5.7 *Let $\mathfrak{a}$ be an ideal of a commutative ring R generated by the weakly pro-regular sequence $\underline{x} = x_1, \ldots, x_k$. Let M denote an R-module. Assume that M is relatively-$\mathfrak{a}$-flat. Then $\Lambda^{\mathfrak{a}}(M)$ is relatively-$\mathfrak{a}$-flat. Viewed as an $\hat{R}^{\mathfrak{a}}$-module $\Lambda^{\mathfrak{a}}(M)$ is also relatively-$\mathfrak{a}\hat{R}^{\mathfrak{a}}$-flat.*

If, moreover, the ring R is Noetherian then $\Lambda^{\mathfrak{a}}(M)$ is $\hat{R}^{\mathfrak{a}}$-flat.

Proof The first assertion is a direct consequence of 2.6.11 and 7.5.13. For the second consider first the case of a flat R-module F. As $\Lambda^{\mathfrak{a}}(F) = \Lambda^{\mathfrak{a}\hat{R}^{\mathfrak{a}}}(\hat{R}^{\mathfrak{a}} \otimes_R F)$ (see 2.2.5) and because the sequence $\underline{x}$ is also weakly pro-regular on $\hat{R}^{\mathfrak{a}}$ (see 7.5.14) we first note that $\Lambda^{\mathfrak{a}}(F)$ is relatively-$\mathfrak{a}\hat{R}^{\mathfrak{a}}$-flat. For the general case we take a free resolution $L \xrightarrow{\sim} M$ of the R-module M. By the hypotheses on M we note that $L/\mathfrak{a}^t L$ provides a left resolution of $M/\mathfrak{a}^t M$. Taking inverse limits, and in view of 1.2.8, we obtain that $\Lambda^{\mathfrak{a}}(L)$ provides a left resolution of $\Lambda^{\mathfrak{a}}(M)$. As the $\Lambda^{\mathfrak{a}}(L_i)$'s are relatively-$\mathfrak{a}\hat{R}^{\mathfrak{a}}$-flat this left resolution may be used to compute the $\mathrm{Tor}_i^{\hat{R}^{\mathfrak{a}}}(\hat{R}^{\mathfrak{a}}/\mathfrak{a}^t\hat{R}^{\mathfrak{a}}, \Lambda^{\mathfrak{a}}(M))$: $\mathrm{Tor}_i^{\hat{R}^{\mathfrak{a}}}(\hat{R}^{\mathfrak{a}}/\mathfrak{a}^t\hat{R}^{\mathfrak{a}}, \Lambda^{\mathfrak{a}}(M)) \cong H_i(\hat{R}^{\mathfrak{a}}/\mathfrak{a}^t\hat{R}^{\mathfrak{a}} \otimes_{\hat{R}^{\mathfrak{a}}} \Lambda^{\mathfrak{a}}(L))$, (see the standard homological argument in 1.5.3). But we have

$$\hat{R}^{\mathfrak{a}}/\mathfrak{a}^t\hat{R}^{\mathfrak{a}} \otimes_{\hat{R}^{\mathfrak{a}}} \Lambda^{\mathfrak{a}}(L) \cong \Lambda^{\mathfrak{a}}(L)/\mathfrak{a}^t\Lambda^{\mathfrak{a}}(L) \cong L/\mathfrak{a}^t L$$

(see 2.2.2) and the complex $L/\mathfrak{a}^t L$ is exact in positive degree. It follows that $\mathrm{Tor}_i^{\hat{R}^{\mathfrak{a}}}(\hat{R}^{\mathfrak{a}}/\mathfrak{a}^t\hat{R}^{\mathfrak{a}}, \Lambda^{\mathfrak{a}}(M)) = 0$ for all $i > 0$. The conclusion follows from the criterion in 2.6.3 because we also have that $\Lambda^{\mathfrak{a}}(M)/\mathfrak{a}^t\Lambda^{\mathfrak{a}}(M) \cong M/\mathfrak{a}^t M$ is flat over $R/\mathfrak{a}^t R \cong \hat{R}^{\mathfrak{a}}/\mathfrak{a}^t\hat{R}^{\mathfrak{a}}$.

Now the last assertion follows by 2.6.3. □

Proposition 9.5.8 *Let $\mathfrak{a}$ be an ideal of a commutative ring R generated by the weakly pro-regular sequence $\underline{x} = x_1, \ldots, x_k$. Let M denote an R-module. Assume that M is relatively-$\mathfrak{a}$-injective. Then $\Gamma_{\mathfrak{a}}(M)$ is relatively-$\mathfrak{a}$-injective. Viewed as an $\hat{R}^{\mathfrak{a}}$-module $\Gamma_{\mathfrak{a}}(M)$ is also relatively-$\mathfrak{a}\hat{R}^{\mathfrak{a}}$-injective.*

If, moreover, the ring R is Noetherian then $\Gamma_{\mathfrak{a}}(M)$ is $\hat{R}^{\mathfrak{a}}$-injective.

Proof The proof is dual to that of 9.5.7. The first assertion is a direct consequence of 2.7.10 and 7.4.7. For the second we first consider the case of an injective R-module J. As $\Gamma_{\mathfrak{a}}(J) \cong \Gamma_{\mathfrak{a}\hat{R}^{\mathfrak{a}}}(\operatorname{Hom}_R(\hat{R}^{\mathfrak{a}}, J))$ (see 2.2.5), as $\operatorname{Hom}_R(\hat{R}^{\mathfrak{a}}, J)$ is $\hat{R}^{\mathfrak{a}}$-injective and because the sequence $\underline{x}$ is also weakly pro-regular on $\hat{R}^{\mathfrak{a}}$ (see 7.5.14) we first note that $\Gamma_{\mathfrak{a}}(J)$ is relatively-$\mathfrak{a}\hat{R}^{\mathfrak{a}}$-injective. For the general case we take an injective resolution $M \xrightarrow{\sim} I$ of the R-module M. We note the quasi-isomorphism $\operatorname{Hom}_R(R/\mathfrak{a}^t, M) \xrightarrow{\sim} \operatorname{Hom}_R(R/\mathfrak{a}^t, I)$. By taking direct limits, and because the direct limit functor is exact, we obtain that $\Gamma_{\mathfrak{a}}(I)$ provides a right resolution of $\Gamma_{\mathfrak{a}}(M)$ by means of relatively-$\mathfrak{a}\hat{R}^{\mathfrak{a}}$-injective module. This right resolution may be used to compute the $\operatorname{Ext}^i_{\hat{R}^{\mathfrak{a}}}(\hat{R}^{\mathfrak{a}}/\mathfrak{a}^t\hat{R}^{\mathfrak{a}}, \Gamma_{\mathfrak{a}}(M))$. Recall that $\Gamma_{\mathfrak{a}}(M)$ has the natural structure of an $\hat{R}^{\mathfrak{a}}$-module. Hence we have

$$\begin{aligned}\operatorname{Ext}^i_{\hat{R}^{\mathfrak{a}}}(\hat{R}^{\mathfrak{a}}/\mathfrak{a}^t\hat{R}^{\mathfrak{a}}, \Gamma_{\mathfrak{a}}(M)) &\cong H^i(\operatorname{Hom}_{\hat{R}^{\mathfrak{a}}}(\hat{R}^{\mathfrak{a}}/\mathfrak{a}^t\hat{R}^{\mathfrak{a}}, \Gamma_{\mathfrak{a}}(I))) \\ &\cong H^i(\operatorname{Hom}_R(R/\mathfrak{a}^t, \Gamma_{\mathfrak{a}}(I))) = 0 \text{ for } i > 0.\end{aligned}$$

The conclusion follows from the criterion in 2.7.2 as we also have that

$$\operatorname{Hom}_{\hat{R}^{\mathfrak{a}}}(\hat{R}^{\mathfrak{a}}/\mathfrak{a}^t\hat{R}^{\mathfrak{a}}, \Gamma_{\mathfrak{a}}(M)) \cong \operatorname{Hom}_R(R/\mathfrak{a}^t, \Gamma_{\mathfrak{a}}(M)) \cong \operatorname{Hom}_R(R/\mathfrak{a}^t, M)$$

is injective over $R/\mathfrak{a}^t R \cong \hat{R}^{\mathfrak{a}}/\mathfrak{a}^t\hat{R}^{\mathfrak{a}}$. Now the last assertion follows by 2.7.5. $\square$

9.6 Homologically Complete and Cohomologically Torsion Complexes

This section may be viewed as a prolongation of the first section in Chap. 3, which contained completeness and pseudo-completeness criteria for modules. We provide complex versions of these criteria and their counterparts in local cohomology, including those already obtained in [64] and [87]. The results obtained in Chap. 6 for some classes of complexes will play an important rôle here.

Definition 9.6.1 Let $\mathfrak{a}$ be an arbitrary ideal of a commutative ring R. Following more or less Yekutieli we say that an R-complex is homologically-$\mathfrak{a}$-complete if the natural map $\alpha^{\mathfrak{a}}_X; X \to \mathrm{L}\Lambda^{\mathfrak{a}}(X)$ is an isomorphism in the derived category. We denote their class by $\mathcal{G}^{\Lambda}_{\mathfrak{a}}$.

Dually we say that an R-complex is cohomologically-$\mathfrak{a}$-torsion if the natural map $\iota_{\mathfrak{a}}^X; \mathrm{R}\Gamma_{\mathfrak{a}}(X) \to X$ is an isomorphism in the derived category. We denote their class by $\mathcal{G}_{\mathfrak{a}}^{\Gamma}$.

The first results about these classes are already in [64] and in [87]. Some others are also in the previous Theorems 7.2.4, 7.2.7 and 7.2.8. In this section we provide a systematic study of these classes with many criteria and add some more results.

These classes have nice properties.

Lemma 9.6.2 *Let $\mathfrak{a}$ be any ideal of a commutative ring R and*

$$0 \to X \to Y \to Z \to 0$$

be an exact sequence of R-complexes. Then

(a) if two of the three complexes X, Y, Z are in $\mathcal{G}_{\mathfrak{a}}^{\Lambda}$, then the three complexes are in $\mathcal{G}_{\mathfrak{a}}^{\Lambda}$ and
(b) if two of the three complexes X, Y, Z are in $\mathcal{G}_{\mathfrak{a}}^{\Gamma}$, then the three complexes are in $\mathcal{G}_{\mathfrak{a}}^{\Gamma}$.

Proof We recall the inverse system $\mathfrak{L}^{\mathfrak{a}}$ as it was introduced in 7.1.1. Since the functor $\mathrm{Mic}(\mathfrak{L}^{\mathfrak{a}} \otimes_R \cdot)$ is exact (see 4.2.8) there is a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Mic}(\mathfrak{L}^{\mathfrak{a}} \otimes_R X) \arrow[r] & \mathrm{Mic}(\mathfrak{L}^{\mathfrak{a}} \otimes_R Y) \arrow[r] & \mathrm{Mic}(\mathfrak{L}^{\mathfrak{a}} \otimes_R Z) \arrow[r] & 0
\end{tikzcd}$$

where the vertical maps represent the natural maps $\alpha_X^{\mathfrak{a}}, \alpha_Y^{\mathfrak{a}}, \alpha_Z^{\mathfrak{a}}$, respectively (see 7.1.6). If two of these three maps are quasi-isomorphisms, then the third one is also a quasi-isomorphism. This proves (a).

The statement in (b) is proved similarly, noting that the functor $\mathrm{Tel}(\mathrm{Hom}_R(\mathfrak{L}^{\mathfrak{a}}, \cdot))$ is also exact by 4.3.9, and using 7.1.15 in place of 7.1.6. $\square$

The following is rather obvious.

Lemma 9.6.3 *Let $\mathfrak{a}$ be any ideal of a commutative ring R.*

(a) If $X \in \mathcal{G}_{\mathfrak{a}}^{\Lambda}$ and if Y is quasi-isomorphic to X, then $Y \in \mathcal{G}_{\mathfrak{a}}^{\Lambda}$. If $X \in \mathcal{G}_{\mathfrak{a}}^{\Gamma}$ and if Y is quasi-isomorphic to X, then $Y \in \mathcal{G}_{\mathfrak{a}}^{\Gamma}$.
(b) If the ideals $\mathfrak{a}$ and $\mathfrak{b}$ are topologically equivalent, then $\mathcal{G}_{\mathfrak{a}}^{\Lambda} = \mathcal{G}_{\mathfrak{b}}^{\Lambda}$ and $\mathcal{G}_{\mathfrak{a}}^{\Gamma} = \mathcal{G}_{\mathfrak{b}}^{\Gamma}$.

Remark 9.6.4 By 9.6.3 the classes $\mathcal{G}_{\mathfrak{a}}^{\Lambda}$ and $\mathcal{G}_{\mathfrak{a}}^{\Gamma}$ are well defined in the derived category. In general, modules in $\mathcal{G}_{\mathfrak{a}}^{\Lambda}$ are $\mathfrak{a}$-pseudo-complete (see 7.1.9) and clearly modules in $\mathcal{G}_{\mathfrak{a}}^{\Gamma}$ are $\mathfrak{a}$-torsion.

Remark 9.6.5 Assume now that the ideal $\mathfrak{a}$ of the commutative ring R is generated by a weakly pro-regular sequence $\underline{x} = x_1, \ldots, x_k$. Then

(a) An R-module M is in $\mathcal{G}_{\mathfrak{a}}^{\Lambda}$ if and only if it is $\mathfrak{a}$-pseudo-complete (see 7.5.13).
(b) An R-module M is in $\mathcal{G}_{\mathfrak{a}}^{\Gamma}$ if and only if it is $\mathfrak{a}$-torsion (see 7.4.7).
(c) The class $\mathcal{G}_{\mathfrak{a}}^{\Lambda}$ coincides with the class $\mathcal{G}_{\underline{x}}^{L\check{C}}$ introduced in 6.4.3. This is because $\mathrm{RHom}_R(\check{C}_{\underline{x}}, X)$ and $\mathrm{L}\Lambda^{\mathfrak{a}}(X)$ are both represented by $\mathrm{Hom}_R(T(\underline{x}), X) \cong M(\underline{x}; X)$ (see 7.5.12 and 7.5.9). For examples of complexes in $\mathcal{G}_{\mathfrak{a}}^{\Lambda} = \mathcal{G}_{\underline{x}}^{L\check{C}}$ we refer to 6.4.4.
(d) The class $\mathcal{G}_{\mathfrak{a}}^{\Gamma}$ coincides with the class $\mathcal{G}_{\underline{x}}^{R\check{C}}$ introduced in 6.4.6. This is because $\check{C}_{\underline{x}} \otimes_R^L X$ and $\mathrm{R}\Gamma_{\mathfrak{a}}(X)$ are both represented by $\check{C}_{\underline{x}} \otimes_R X$ (see 7.4.4). For examples of complexes in $\mathcal{G}_{\mathfrak{a}}^{\Gamma} = \mathcal{G}_{\underline{x}}^{R\check{C}}$ we refer to 6.4.7.

We also mention the following, related to Proposition 6.4.9.

Proposition 9.6.6 *Let $\underline{x} = x_1, \ldots, x_k$ be a sequence of elements of a commutative ring R and write $\mathfrak{a} = \underline{x}R$. Let X denote an R-complex with* $\operatorname{Supp} H^i(X) \subseteq V(\mathfrak{a})$ *for all $i \in \mathbb{Z}$ and let I denote a K-injective complex. Then the natural morphism*

$$\mathrm{Hom}_R(X, I) \to \mathrm{Hom}_R(\check{L}_{\underline{x}}, \mathrm{Hom}_R(X, I))$$

is a quasi-isomorphism. That is, $\mathrm{Hom}_R(X, I) \in \mathcal{G}_{\underline{x}}^{L\check{C}}$.

Moreover, if the sequence $\underline{x}$ is weakly pro-regular then $\mathrm{Hom}_R(X, I) \in \mathcal{G}_{\mathfrak{a}}^{\Lambda}$ *and there are isomorphisms*

$$H_i(\mathrm{Hom}_R(X, I)) \cong \Lambda_i^{\mathfrak{a}}(\mathrm{Hom}_R(X, I))$$

for all $i \in \mathbb{Z}$.

Proof We have $X \in \mathcal{G}_{\underline{x}}^{R\check{C}}$ by Theorem 6.4.6. Hence $\mathrm{Hom}_R(X, I) \in \mathcal{G}_{\underline{x}}^{L\check{C}}$ by 6.4.9 and the quasi-isomorphism is obtained by 6.4.3. The last claims follow by 9.6.5 (c) and the definition of the class $\mathcal{G}_{\mathfrak{a}}^{\Lambda}$. □

Here is the MGM equivalence. In the following the classes of complexes $\mathcal{G}_{\mathfrak{a}}^{\Lambda}$ and $\mathcal{G}_{\mathfrak{a}}^{\Gamma}$ are viewed as full subcategories of the derived category.

Theorem 9.6.7 (see *[1]* or *[64]*). *Let R denote a commutative ring and $\mathfrak{a}$ an ideal generated by a weakly pro-regular sequence. Then the full subcategories $\mathcal{G}_{\mathfrak{a}}^{\Lambda}$ and $\mathcal{G}_{\mathfrak{a}}^{\Gamma}$ of the derived category are equivalent.*

Proof This is a particular case of Theorem 9.1.3. □

For further information about the MGM-equivalence we refer to [72].

Now we summarize and translate the various criteria given in Chap. 6, section 4. The following may be compared to the completeness criterion given in 3.1.6. It also contains a generalization and a complex version of the pseudo-completeness criterion given in 3.1.11.

Theorem 9.6.8 *Let $\mathfrak{a}$ be an ideal of a commutative ring R generated by the weakly pro-regular sequence $\underline{x} = x_1, \ldots, x_k$. For any R-complex X the following conditions are equivalent:*

(i) $X \in \mathcal{G}_{\mathfrak{a}}^{\Lambda}$,
(ii) *the natural morphism* $X \to \operatorname{Hom}_R(\check{L}_{\underline{x}}, X)$ *is a quasi-isomorphism,*
(iii) *the natural morphism* $X \to \operatorname{Hom}_R(T(\underline{x}), X)$ *is a quasi-isomorphism,*
(iv) *the natural morphism* $X \to M(\underline{x}; X)$ *is a quasi-isomorphism,*
(v) $\operatorname{RHom}_R(T, X) \simeq 0$ *for all* $T \in \mathcal{T}_{\mathfrak{a}}$,
(vi) $\operatorname{RHom}_R(R_z, X) \simeq 0$ *for all* $z \in \operatorname{Rad}(\mathfrak{a})$.
(vii) $\operatorname{RHom}_R(\oplus_{i=1}^{k} R_{x_i}, X) \simeq 0$,
(viii) $H_i(X) \in \mathcal{G}_{\mathfrak{a}}^{\Lambda}$ *for all* $i \in \mathbb{Z}$,
(ix) $H_i(X)$ *is* $\mathfrak{a}$*-pseudo-complete for all* $i \in \mathbb{Z}$,
(x) *the natural morphism* $F \to \Lambda^{\mathfrak{a}}(F)$ *is a quasi-isomorphism for any* K*-flat resolution* $F \xrightarrow{\sim} X$ *of* X.

Proof Note first that the implication (i) $\Rightarrow$ (v) and the implication (i) $\Rightarrow$ (ix) hold for any finitely generated ideal (see 7.2.4 and 2.5.7). But here $\mathcal{G}_{\mathfrak{a}}^{\Lambda} = \mathcal{G}_{\underline{x}}^{L\check{C}}$. Hence the equivalence of (i), (ii), (v), (vi) and (vii) is given in 6.4.3. The equivalence of (i), (iii), (iv) and (x) follows by 7.5.12. The equivalence of (i) and (viii) follows from 6.4.10. The equivalence of (viii) and (ix) was already observed in 7.5.13. □

In the following, statement (c) provides a complex version of (2.5.9 (b)) in our particular case and may also be compared with (2.2.9 (b)).

Corollary 9.6.9 *Let* $\mathfrak{a}$ *be an ideal of a commutative ring* R *generated by the weakly pro-regular sequence* $\underline{x} = x_1, \ldots, x_k$ *and let* X *denote an* R*-complex.*

(a) *If* X *is a complex of* $\mathfrak{a}$*-complete or* $\mathfrak{a}$*-pseudo-complete* R*-modules, then* $X \in \mathcal{G}_{\mathfrak{a}}^{\Lambda}$.
(b) $\Lambda^{\mathfrak{a}}(X) \in \mathcal{G}_{\mathfrak{a}}^{\Lambda}$ *and the image of* X *by the functor* $\Lambda_0^{\mathfrak{a}}$ *also is in* $\mathcal{G}_{\mathfrak{a}}^{\Lambda}$.
(c) *The natural map* $\alpha_{\mathrm{L}\Lambda^{\mathfrak{a}}(X)}^{\mathfrak{a}} : \mathrm{L}\Lambda^{\mathfrak{a}}(X) \to \mathrm{L}\Lambda^{\mathfrak{a}}(\mathrm{L}\Lambda^{\mathfrak{a}}(X))$ *is an isomorphism in the derived category, that is* $\mathrm{L}\Lambda^{\mathfrak{a}}(X) \in \mathcal{G}_{\mathfrak{a}}^{\Lambda}$ *(this provides another proof of the fact the functor* $\mathrm{L}\Lambda^{\mathfrak{a}}$ *is idempotent).*
(d) *Assume that* $\mathfrak{b}$ *is another ideal such that* $\mathfrak{a} \subseteq \operatorname{Rad}(\mathfrak{b})$, *assume also that* $X \in \mathcal{G}_{\mathfrak{b}}^{\Lambda}$, *then* $X \in \mathcal{G}_{\mathfrak{a}}^{\Lambda}$.

Proof We shall deduce these statements from (9.6.8, (i) $\Leftrightarrow$ (ix)). For a complex X as in (a) note that its homology modules $H_i(X)$ are $\mathfrak{a}$-pseudo-complete (see 2.5.7 and 7.5.13). Note that the image of the complex X by the functor $\Lambda_0^{\mathfrak{a}}$ is a complex of $\mathfrak{a}$-pseudo-complete modules (see 2.5.7 again). Statement (b) follows as does (c) since $\Lambda^{\mathfrak{a}}(X)$ is represented by a complex of $\mathfrak{a}$-complete modules, namely by $\hat{F}^{\mathfrak{a}}$, where $F \xrightarrow{\sim} X$ is a flat or K-flat resolution. Now we prove (d). Note that the hypothesis $\mathfrak{a} \subseteq \operatorname{Rad}(\mathfrak{b})$ implies that the ideal $\mathfrak{b}$ is $\mathfrak{a}$-open. Hence the natural morphism $\tau_F^{\mathfrak{b}} : F \to \Lambda^{\mathfrak{b}}(F)$ factors through $\Lambda^{\mathfrak{a}}(F)$. If $\tau_F^{\mathfrak{b}}$ is a quasi-isomorphism, then $H_i(X) \cong H_i(F)$ is a direct summand of $H_i(\Lambda^{\mathfrak{a}}(F))$ which is $\mathfrak{a}$-pseudo-complete. Whence $H_i(X)$ is also $\mathfrak{a}$-pseudo-complete and the conclusion follows. □

Remark 9.6.10 For an ideal generated by a weakly pro-regular sequence the above theorem also implies that a complex X with $\mathfrak{a}$-complete homology modules is homologically-$\mathfrak{a}$-complete (because $\mathfrak{a}$-complete modules are also $\mathfrak{a}$-pseudo-complete (see 2.5.7)). Moreover, a homologically-$\mathfrak{a}$-complete complex with $\mathfrak{a}$-separated homology modules has $\mathfrak{a}$-complete homology modules (because $\mathfrak{a}$-pseudo-complete modules which are $\mathfrak{a}$-separated are $\mathfrak{a}$-complete (see 2.5.8)). This was also observed by Yekutieli in the particular case of a homologically bounded complex over a Noetherian ring [87, Theorem 2].

However, note that this particular case is also a consequence of the previously known 2.2.11, 2.5.13 and 2.5.15. Indeed, a homologically bounded complex has a right-bounded free resolution L. If $L \to \hat{L}^{\mathfrak{a}}$ is a quasi-isomorphism, then $H_i(L) \cong H_i(\hat{L}^{\mathfrak{a}})$ is $\mathfrak{a}$-quasi-complete for all i (see 2.2.11), i.e. $\mathfrak{a}$-complete when $\mathfrak{a}$-separated. Conversely, if $H_i(L)$ is $\mathfrak{a}$-complete for all i and if R is Noetherian, then $L \to \hat{L}^{\mathfrak{a}}$ is a quasi-isomorphism by 2.5.13 since free modules and $\mathfrak{a}$-complete modules are in $\mathcal{C}_{\mathfrak{a}}$ (see 2.5.15).

We now turn to the local cohomology side.

Theorem 9.6.11 *Let $\mathfrak{a}$ be an ideal of a commutative ring R generated by the weakly pro-regular sequence $\underline{x} = x_1, \ldots, x_k$. For any complex X the following conditions are equivalent:*

(i) $X \in \mathcal{G}_{\mathfrak{a}}^{\Gamma}$,
(ii) *the natural morphism $T(\underline{x}; X) \to X$ is a quasi-isomorphism,*
(iii) $T \otimes_R^{\mathrm{L}} X \simeq 0$ *for all* $T \in \mathcal{T}_{\mathfrak{a}}$,
(iv) $R_z \otimes_R X$ *is exact for all* $z \in \mathrm{Rad}(\mathfrak{a})$,
(v) $\oplus_{i=1}^{k} R_{x_i} \otimes_R X$ *is exact,*
(vi) $\mathrm{Supp}_R\, H^i(X) \subseteq V(\mathfrak{a})$ *for all* $i \in \mathbb{Z}$,
(vii) $H^i(X) \in \mathcal{G}_{\mathfrak{a}}^{\Gamma}$ *for all* $i \in \mathbb{Z}$,
(viii) $H^i(X)$ *is $\mathfrak{a}$-torsion for all* $i \in \mathbb{Z}$,
(ix) $X^{\vee} \in \mathcal{G}_{\mathfrak{a}}^{\Lambda}$, *where $X^{\vee}$ is the general Matlis dual of X,*
(x) *the natural morphism $\Gamma_{\mathfrak{a}}(I) \to I$ is a quasi-isomorphism for all injective or K-injective resolutions $X \xrightarrow{\sim} I$ of X.*

Proof Note first that the implication (i) ⇒ (iii) holds for any finitely generated ideal (see 7.2.8). As $\mathcal{G}_{\mathfrak{a}}^{\Gamma} = \mathcal{G}_{\underline{x}}^{R\check{C}}$ the equivalence of (i), (iii), (iv), (v) and (vi) is in 6.4.6. The equivalence of (i) and (ii) follows by 7.4.11. The equivalence of (i) and (vii) is in 6.4.10. The equivalence of (vi) and (viii) was observed in 2.1.13. The equivalence of (i) and (ix) is shown in 6.4.9. The equivalence of (i) and (x) follows by 7.4.4. □

Corollary 9.6.12 *Let $\mathfrak{a}$ be an ideal of a commutative ring R generated by the weakly pro-regular sequence $\underline{x} = x_1, \ldots, x_k$ and let X denote an R-complex.*

(a) *If X is a complex of $\mathfrak{a}$-torsion R-modules, then $X \in \mathcal{G}_{\mathfrak{a}}^{\Gamma}$.*
(b) $\Gamma_{\mathfrak{a}}(X) \in \mathcal{G}_{\mathfrak{a}}^{\Gamma}$.
(c) *The natural map $\iota_{\mathfrak{a}}^{\mathrm{R}\Gamma_{\mathfrak{a}}(X)} : \mathrm{R}\Gamma_{\mathfrak{a}}(\mathrm{R}\Gamma_{\mathfrak{a}}(X)) \to \mathrm{R}\Gamma_{\mathfrak{a}}(X)$ is an isomorphism in the derived category, that is, $\mathrm{R}\Gamma_{\mathfrak{a}}(X) \in \mathcal{G}_{\mathfrak{a}}^{\Gamma}$. (This provides another proof of the fact that the functor $\mathrm{R}\Gamma_{\mathfrak{a}}$ is idempotent).*
(d) *Assume that $\mathfrak{b}$ is another ideal such that $\mathfrak{a} \subseteq \mathrm{Rad}(\mathfrak{b})$, assume also that $X \in \mathcal{G}_{\mathfrak{b}}^{\Gamma}$, then $X \in \mathcal{G}_{\mathfrak{a}}^{\Gamma}$.*

Proof We shall deduce these statements from (9.6.11, (i) $\Leftrightarrow$ (xiii)). For a complex as in (a) its cohomology modules are $\mathfrak{a}$-torsion, as is easily seen. Statement (b) follows as does (c) since $\mathrm{R}\Gamma_{\mathfrak{a}}(X)$ is represented by $\Gamma_{\mathfrak{a}}(I)$, where $X \xrightarrow{\sim} I$ is an injective or K-injective resolution. For (d) note the inclusion $\Gamma_{\mathfrak{b}}(I) \subseteq \Gamma_{\mathfrak{a}}(I) \subseteq I$. If the map $\iota_{\mathfrak{b}}^{X}$, which is represented by the inclusion $\Gamma_{\mathfrak{b}}(I) \subseteq I$, is an isomorphism in the derived category, then $H^i(X) \cong H^i(I)$ is a direct summand of $H^i(\Gamma_{\mathfrak{a}}(I))$, hence $\mathfrak{a}$-torsion. $\square$

Corollary 9.6.13 *Let $(R, \mathfrak{m}, \Bbbk)$ denote a Noetherian local ring and let $\mathfrak{a} \subset R$ be an ideal. Let X be an R-complex and assume that X has finitely generated cohomology modules. As before, denote by $X^{\vee}$ the Matlis dual complex $\mathrm{Hom}_R(X, E_R(\Bbbk))$. Then*

$$X^{\vee} \in \mathcal{G}_{\mathfrak{a}}^{\Gamma}.$$

In other words, there is an isomorphism $\mathrm{R}\Gamma_{\mathfrak{a}}(X^{\vee}) \simeq X^{\vee}$ in the derived category and a quasi-isomorphism $\check{C}_{\underline{x}} \otimes_R X^{\vee} \xrightarrow{\sim} X^{\vee}$ for any sequence $\underline{x} = x_1, \ldots, x_k$ generating the ideal $\mathfrak{a}$.

Proof The modules $H^i(X^{\vee}) \cong H_i(X)^{\vee}$ are Artinian, hence $\mathfrak{m}$-torsion and also $\mathfrak{a}$-torsion. Recall also that any sequence $\underline{x}$ generating the ideal $\mathfrak{a}$ is weakly pro-regular because R is Noetherian (see 7.3.2 or the more general A.2.3). Therefore the first statements are a direct consequence of Theorem 9.6.11. For the last one recall that the map $\iota_{\mathfrak{a}}^{X^{\vee}}$, which is an isomorphism in the derived category by the above, is also represented by the natural morphism $\check{C}_{\underline{x}} \otimes_R X^{\vee} \to X^{\vee}$ (see 7.4.4). $\square$

For further information on complexes as in 9.6.13, see the forthcoming 9.8.4. Here are some more comparison formulas, they contain a far reaching generalization of Proposition 2.5.21 (b).

Proposition 9.6.14 *Let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals of a commutative ring R, generated by the weakly pro-regular sequences $\underline{x} = x_1, \ldots, x_k$ and $\underline{y} = y_1, \ldots, y_l$ respectively. Assume that the sequence $\underline{x}, \underline{y}$ is also weakly pro-regular. Then*

$$\mathcal{G}_{(\mathfrak{a}+\mathfrak{b})}^{\Lambda} = \mathcal{G}_{\mathfrak{a}}^{\Lambda} \cap \mathcal{G}_{\mathfrak{b}}^{\Lambda} \text{ and } \mathcal{G}_{(\mathfrak{a}+\mathfrak{b})}^{\Gamma} = \mathcal{G}_{\mathfrak{a}}^{\Gamma} \cap \mathcal{G}_{\mathfrak{b}}^{\Gamma}.$$

Proof The statements are a reinterpretation of those in 6.4.14 in our particular case. $\square$

Here is a further criterion.

Theorem 9.6.15 *Let $\mathfrak{a}$ be an ideal of a commutative ring R generated by the weakly pro-regular sequence $\underline{x} = x_1, \ldots, x_k$. For any complex X the following conditions are equivalent:*

(i) $X \in \mathcal{G}_{\mathfrak{a}}^{\Gamma}$,
(ii) $\mathrm{RHom}_R(X, T) \simeq 0$ *for all* $T \in \mathcal{T}_{\mathfrak{a}}$.

Proof The implication (i) $\Rightarrow$ (ii) follows by 7.2.7: note that $T \in \mathcal{T}_{\mathfrak{a}}$ if and only if $\text{E-dp}(\mathfrak{a}, X) = \infty$ because $\mathfrak{a}$ is finitely generated (see 5.3.5).

For the reverse implication let E be an injective cogenerator as described in 1.4.8 and note that the injective module $R_{x_i}^{\vee} = \operatorname{Hom}_R(R_{x_i}, E) \in \mathcal{T}_{\mathfrak{a}}$ by 5.3.6 because $\operatorname{E-dp}(\mathfrak{a}, R_{x_i}^{\vee}) = \operatorname{T-codp}(\mathfrak{a}, R_{x_i}) = \infty$. When condition (ii) is satisfied we thus have that the complex $\operatorname{Hom}_R(X, \operatorname{Hom}_R(R_{x_i}, E))$ is exact. Note that the module $\operatorname{Hom}_R(R_{x_i}, E)$ is injective. But

$$\operatorname{Hom}_R(X, \operatorname{Hom}_R(R_{x_i}, E)) \cong \operatorname{Hom}_R(X \otimes_R R_{x_i}, E).$$

Hence the complex $X \otimes_R R_{x_i}$ is also exact and condition (i) is satisfied in view of Theorem 9.6.11. $\square$

9.7 Homological Completeness and Cosupport

When an ideal $\mathfrak{a}$ of a commutative ring R is generated by a weakly pro-regular sequence we have seen in 9.6.11 that the natural map $\mathrm{R}\Gamma_{\mathfrak{a}}(X) \to X$ is an isomorphism in the derived category if and only $\operatorname{Supp}_R(X) \subseteq V(\mathfrak{a})$. Here we present a counterpart on the local homology side when the ring is Noetherian, as already observed by Sather-Wagstaff and Wicklein in [73].

9.7.1 *Co-Support and small co-support.* Let R be a commutative ring and X an R-complex with one of its K-injective resolutions $X \xrightarrow{\sim} I$. Following [73] we define the (big) Co-Support of X by

$$\operatorname{Co-Support}_R(X) = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid \operatorname{Hom}_R(R_{\mathfrak{p}}, I) \text{ is not exact}\}$$

and the small co-support of X by

$$\operatorname{co-support}_R(X) = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid \operatorname{Hom}_R(\Bbbk(\mathfrak{p}), I) \text{ is not exact}\}$$

where $\Bbbk(\mathfrak{p})$ denotes as usual the residue field $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ of the local ring $R_{\mathfrak{p}}$.

In other words

$$\operatorname{Co-Support}_R(X) = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid \operatorname{RHom}_R(R_{\mathfrak{p}}, X) \neq 0\} \text{ and}$$
$$\operatorname{co-support}_R(X) = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid \operatorname{RHom}_R(\Bbbk(\mathfrak{p}), X) \neq 0\}.$$

Note that the big Co-Support in the case of a module was first invented in [60] and mainly studied for Artinian modules. Note that $\operatorname{Ext}_R^i(R_{\mathfrak{p}}, M) = 0$ for all $i \neq 0$ when M is an Artinan module over a Noetherian ring. This follows since the functor $\operatorname{Hom}_R(R_{\mathfrak{p}}, \cdot)$ is exact on the category of Artinian modules over a Noetherian ring as shown in [60].

Observations 9.7.2 Let again R be a commutative ring and X an R-complex with one of its K-injective resolutions $X \xrightarrow{\sim} I$.

(a) We first note that the Co-Support and the small co-support of the K-injective R-complex I behave well under *co-localization* (see also [60]). Namely, let S be a multiplicatively closed subset of R ($1 \in S$) and let us denote as usual by R_S the ring of fractions r/s, $r \in R$, $s \in S$. We noted in 4.4.21 the change of ring formula: for any R_S-complex Y we have

$$\mathrm{Hom}_R(Y, I) \cong \mathrm{Hom}_{R_S}(Y, \mathrm{Hom}_R(R_S, I)).$$

Note also that

$$\mathrm{Hom}_{R_S}(Y, \mathrm{Hom}_R(R_S, I)) \cong \mathrm{Hom}_R(Y, \mathrm{Hom}_R(R_S, I)).$$

We also recall that $\mathrm{Hom}_R(R_S, I)$, viewed as an R-complex or as an R_S-complex, is K-injective (and even DG-injective if I is DG-injective). Then recall that every prime ideal of R_S is of the form $\mathfrak{p}R_S$ for a unique prime $\mathfrak{p} \in \mathrm{Spec}(R)$ such that $\mathfrak{p} \cap S = \emptyset$, so that $(R_S)_{\mathfrak{p}R_S} = R_{\mathfrak{p}}$ and $\Bbbk(\mathfrak{p}R_S) = \Bbbk(\mathfrak{p})$. It follows that

$$\begin{aligned} \text{Co-Support}_R(I) \cap \mathrm{Spec}(R_S) &= \text{Co-Support}_{R_S}(\mathrm{Hom}_R(R_S, I)) \text{ and} \\ \text{co-support}_R(I) \cap \mathrm{Spec}(R_S) &= \text{co-support}_{R_S}(\mathrm{Hom}_R(R_S, I)). \end{aligned}$$

(However, note that the morphism $\mathrm{Hom}_R(S^{-1}R, X) \to \mathrm{Hom}_R(S^{-1}R, I)$ is not necessarily a quasi-isomorphism.)

(b) If $\mathfrak{p} \subset \mathfrak{q}$ are two prime ideals of R and if $\mathfrak{p} \in \text{Co-Support}_R(X)$ then $\mathfrak{q} \in \text{Co-Support}_R(X)$. (If $\mathfrak{q} \notin \text{Co-Support}_R(X)$ the complex $\mathrm{Hom}_R(R_{\mathfrak{q}}, I)$ is exact. As this exact R-complex is K-injective it is contractible and the complex $\mathrm{Hom}_R(R_{\mathfrak{p}}, \mathrm{Hom}_R(R_{\mathfrak{q}}, I))$ is exact. But $R_{\mathfrak{p}} \otimes_R R_{\mathfrak{q}} \cong R_{\mathfrak{p}}$ and $\mathrm{Hom}_R(R_{\mathfrak{p}}, \mathrm{Hom}_R(R_{\mathfrak{q}}, I)) \cong \mathrm{Hom}_R(R_{\mathfrak{p}}, I)$. Hence $\mathfrak{p} \notin \text{Co-Support}_R(X)$.)

(c) Also note in accordance with (a) that

$$\begin{aligned} \mathrm{Hom}_R(\Bbbk(\mathfrak{p}), I) &\cong \mathrm{Hom}_{R_{\mathfrak{p}}}(\Bbbk(\mathfrak{p}), \mathrm{Hom}_R(R_{\mathfrak{p}}, I)) \\ &\cong \mathrm{Hom}_R(\Bbbk(\mathfrak{p}), \mathrm{Hom}_R(R_{\mathfrak{p}}, I)) \end{aligned}$$

because $\Bbbk(\mathfrak{p})$ is an $R_{\mathfrak{p}}$-module.

(d) By (c) it follows that $\text{co-support}_R(X) \subseteq \text{Co-Support}_R(X)$. (If the K injective complex $\mathrm{Hom}_R(R_{\mathfrak{p}}, I)$ is exact it is contractible and so is $\mathrm{Hom}_R(\Bbbk(\mathfrak{p}), I)$.)

(e) When R is Noetherian note that

$$\mathrm{Hom}_R(\Bbbk(\mathfrak{p}), I) \cong \mathrm{Hom}_R(\Bbbk(\mathfrak{p}), \Gamma_{\mathfrak{p}}(I))$$

because $\Bbbk(\mathfrak{p})$ is $\mathfrak{p}$-torsion. Note also that the subcomplex $\Gamma_{\mathfrak{p}}(I)$ of I is injective if I is DG-injective (see 4.5.9).

(f) The Co-Support and the Support are related through the general Matlis duality: by adjointness we have

$$\mathrm{Supp}_R(X) = \text{Co-Support}_R(X^{\vee}).$$

Similarly the small co-support and Foxby small support are also related through the general Matlis duality. Recall that the small support of a complex X was defined by

$$\operatorname{supp}_R(X) = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid \Bbbk(\mathfrak{p}) \otimes^{\mathrm{L}}_R X \neq 0\}.$$

We also have

$$\operatorname{supp}_R(X) = \text{co-support}_R(X^\vee).$$

For more information on these notions we refer to [73] and to [12], which also contains another description of the small co-support.

The following result was stated in [12]. For sake of completeness, and because the description of the small co-support in [12] is rather different (though equivalent, see [73]) from the one we use here, we provide another proof of this important fact.

Theorem 9.7.3 *Let R be a Noetherian ring and X a non-exact R-complex. Then* $\text{co-support}_R(X) \neq \emptyset$.

Proof Let $X \xrightarrow{\sim} I$ be a minimal DG-injective resolution (such a resolution exists in view of 4.5.6). Then I is not trivial and the modules I^j are direct sums of modules of the form $E(R/\mathfrak{p})$, $\mathfrak{p} \in \operatorname{Spec}(R)$. Let $\mathfrak{q}$ be a prime ideal maximal among the primes occurring in the complex I. We only need to prove that the complex $\operatorname{Hom}_R(\Bbbk(\mathfrak{q}), I)$ is not exact. To this end, let r be some integer such that $E(R/\mathfrak{q})$ appears as a direct summand of I^r. Then $\operatorname{Hom}_R(\Bbbk(\mathfrak{q}), I^r) \neq 0$. But $\Bbbk(\mathfrak{q})$ is a $\mathfrak{q}$-torsion module, hence

$$\operatorname{Hom}_R(\Bbbk(\mathfrak{q}), I) \cong \operatorname{Hom}_R(\Bbbk(\mathfrak{q}), \Gamma_{\mathfrak{q}}(I)).$$

Then note that $E(R/\mathfrak{q})$ is the only indecomposable module occurring in the injective complex $\Gamma_{\mathfrak{q}}(I)$ by the maximality of $\mathfrak{q}$. Hence $\Gamma_{\mathfrak{q}}(I)$ is a non-zero injective $R_{\mathfrak{q}}$-complex, minimal as a subcomplex of a minimal complex, and

$$\operatorname{Hom}_R(\Bbbk(\mathfrak{q}), \Gamma_{\mathfrak{q}}(I)) \cong \operatorname{Hom}_{R_{\mathfrak{q}}}(\Bbbk(\mathfrak{q}), \Gamma_{\mathfrak{q}}(I)).$$

Recall now that the complex $\operatorname{Hom}_{R_{\mathfrak{q}}}(\Bbbk(\mathfrak{q}), \Gamma_{\mathfrak{q}}(I))$ has zero differentials (see 4.5.7). It follows that $H^r(\operatorname{Hom}_R(\Bbbk(\mathfrak{q}), I)) \neq 0$ and that $\mathfrak{q} \in \text{co-support}_R(X)$. □

Remark 9.7.4 Let X be a non-exact complex over a Noetherian ring R. Then $\operatorname{supp}_R(X) \neq \emptyset$. This fact follows from Theorem 9.7.3 because $\operatorname{supp}_R(X) = \text{co-support}_R(X^\vee)$.

(See also [11] for a different proof and note that this was first proved in [31] in the case when X is right-bounded.)

We are ready for the result we had in mind. The following appears in [73] with a different proof.

Theorem 9.7.5 *Let X be a non-exact complex over the Noetherian ring R and let $\mathfrak{a}$ be an ideal of R. The following conditions are equivalent:*

(i) *the natural map* $X \to \mathrm{L}\Lambda^{\mathfrak{a}}(X)$ *is an isomorphism in the derived category, that is,* $X \in \mathcal{G}^{\Lambda}_{\mathfrak{a}}$,
(ii) $\text{Co-Support}_R(X) \subseteq V(\mathfrak{a})$,
(iii) $\text{co-support}_R(X) \subseteq V(\mathfrak{a})$.

Proof Let $\underline{x} = x_1, \dots, x_k$ be a sequence of elements of R generating the ideal $\mathfrak{a}$. This sequence is weakly pro-regular and the natural map $X \to \mathrm{L}\Lambda^{\mathfrak{a}}(X)$ is represented by the natural morphism $I \to \operatorname{Hom}_R(\check{C}_{\underline{x}}, I)$, where $X \xrightarrow{\sim} I$ is a DG-injective resolution of X (see 7.5.12).

Assume first that this morphism is a quasi-isomorphism. Then so is the induced morphism

$$\operatorname{Hom}_R(R_{\mathfrak{p}}, I) \to \operatorname{Hom}_R(R_{\mathfrak{p}}, \operatorname{Hom}_R(\check{C}_{\underline{x}}, I)) \cong \operatorname{Hom}_R(R_{\mathfrak{p}} \otimes_R \check{C}_{\underline{x}}, I)$$

for all $\mathfrak{p} \in \operatorname{Spec}(R)$ because $\operatorname{Hom}_R(\check{C}_{\underline{x}}, I)$ is also DG-injective. If $\mathfrak{a} \not\subseteq \mathfrak{p}$ the complex $R_{\mathfrak{p}} \otimes_R \check{C}_{\underline{x}}$ is exact and so is the complex $\operatorname{Hom}_R(R_{\mathfrak{p}} \otimes_R \check{C}_{\underline{x}}, I) \simeq \operatorname{Hom}_R(R_{\mathfrak{p}}, I)$. Hence $\mathfrak{p} \notin \text{Co-support}_R(X)$.

The implication (ii) $\Rightarrow$ (iii) is obvious.

Assume now condition (iii) and let z be any element in $\mathfrak{a}$. In view of 9.6.8 it is enough to prove that the complex $\operatorname{Hom}_R(R_z, I)$ is exact. Note that this complex is a DG-injective R_z-complex. If it is not exact then

$$\text{co-support}_{R_z}(\operatorname{Hom}_R(R_z, I)) \neq \emptyset$$

in view of 9.7.3. But

$$\begin{aligned}\text{co-support}_{R_z}(\operatorname{Hom}_R(R_z, I)) &= \text{co-support}_R(I) \cap \operatorname{Spec}(R_z)\\ &= \text{co-support}_R(X) \cap \operatorname{Spec}(R_z)\end{aligned}$$

(see 9.7.2 (a)). It follows that $\text{co-support}_R(X) \not\subseteq V(\mathfrak{a})$. This contradiction finishes the proof. □

We end the section with some more remarks involving both the small co-support and the small support.

Lemma 9.7.6 *Let* $\mathfrak{m}$ *be any maximal ideal of a commutative ring* R *and assume that* $\mathfrak{m}$ *is finitely generated. Then the following conditions are equivalent*

(i) $\mathfrak{m} \in \text{co-support}_R(X)$,
(ii) $\text{E-dp}(\mathfrak{m}, X) < \infty$,
(iii) $\text{T-codp}(\mathfrak{m}, X) < \infty$,
(iv) $\mathfrak{m} \in \text{support}_R(X)$,

for any R*-complex* X.

Proof Because $\Bbbk(\mathfrak{m}) = R/\mathfrak{m}$ the equivalences (i) $\Leftrightarrow$ (ii) and (iii) $\Leftrightarrow$ (iv) follow from the definitions. The equivalence (ii) $\Leftrightarrow$ (iii) was shown in 5.3.5. □

9.8 Change of Rings

In this section we shall investigate the behaviour of local homology and cohomology under change of rings. Surprisingly they behave well under base change, with certain mild assumptions.

Proposition 9.8.1 *Let $\varphi : R \to B$ be a homomorphism of commutative rings. Let $\mathfrak{a}$ be an ideal of R generated by a sequence $\underline{x} = x_1, \ldots, x_k$. Write $\mathfrak{a}^e$ for the ideal of B generated by $\varphi(\mathfrak{a})$. Let X be a B-complex which may also be viewed as an R-complex via φ. Then*

(a) $T(\underline{x}) \otimes_R B \cong T(\varphi(\underline{x}))$,
(b) $T(\underline{x}; X) \cong T(\varphi(\underline{x}); X)$ *and*
(c) $\operatorname{Hom}_R(T(\underline{x}), X) \cong \operatorname{Hom}_B(T(\varphi(\underline{x}), X)$.

Let $\underline{x}$ be a weakly pro-regular sequence. Assume that the sequence $\varphi(\underline{x})$ is weakly pro-regular on B. Then $H^i_{\mathfrak{a}}(X) \cong H^i_{\mathfrak{a}^e}(X)$ and $\Lambda^{\mathfrak{a}}_i(X) \cong \Lambda^{\mathfrak{a}^e}_i(X)$ for all $i \in \mathbb{Z}$.

Proof The isomorphisms involving $T(\underline{x})$ are rather obvious. Note that $K^\bullet(\underline{x}^t) \otimes_R B \cong K^\bullet(\varphi(\underline{x})^t)$. Then the isomorphisms involving local homology and cohomology follow by 7.5.12 and 7.4.11. □

To go the other way we need some additional hypotheses.

Proposition 9.8.2 *In the situation of 9.8.1 assume that B, viewed as an R-module via φ, is flat. Let Y be an R-complex.*

(a) $(\check{C}_{\underline{x}} \otimes_R Y) \otimes_R B \cong \check{C}_{\varphi(\underline{x})} \otimes_B (B \otimes_R Y)$ *and therefore* $H^i_{\mathfrak{a}}(Y) \otimes_R B \cong H^i_{\mathfrak{a}^e}(B \otimes_R Y)$ *for all $i \in \mathbb{Z}$ provided $\underline{x}$ and $\phi(\underline{x})$ are weakly pro-regular on R and B resp.*
(b) *Assume moreover that $R/\mathfrak{b} \otimes_R \varphi : R/\mathfrak{b} \to B/\mathfrak{b}^e$ is an isomorphism for all $\mathfrak{a}$-open ideals $\mathfrak{b}$ of R. Then there is an isomorphism*

$$\mathrm{L}\Lambda^{\mathfrak{a}}(Y) \simeq \mathrm{L}\Lambda^{\mathfrak{a}}(B \otimes_R Y)$$

in the derived category. Moreover, $\mathrm{L}\Lambda^{\mathfrak{a}}(Y)$ and $\mathrm{L}\Lambda^{\mathfrak{a}^e}(B \otimes_R Y)$ are both represented by the complex $\Lambda^{\mathfrak{a}}(F)$, where $F \xrightarrow{\sim} Y$ is a K-flat resolution of Y. On the homology level we have isomorphisms

$$\Lambda^{\mathfrak{a}}_i(Y) \cong \Lambda^{\mathfrak{a}}_i(B \otimes_R Y) \cong \Lambda^{\mathfrak{a}^e}_i(B \otimes_R Y)$$

for all $i \in \mathbb{Z}$.

Proof (a) The isomorphism involving $\check{C}_{\underline{x}}$ is quite obvious and was already observed in 6.3.6. The isomorphism involving local cohomology follows now by 7.4.4.

(b) There is a quasi-isomorphism $B \otimes_R F \xrightarrow{\sim} B \otimes_R Y$ which provides a K-flat resolution of $B \otimes_R Y$ viewed as an R-complex or as a B-complex, as is easily seen. Now we have

$$R/\mathfrak{a}^t \otimes_R (B \otimes_R F) \cong B/\mathfrak{a}^t B \otimes_R F \cong R/\mathfrak{a}^t \otimes_R F.$$

Taking inverse limits we obtain the isomorphism of complexes $\Lambda^{\mathfrak{a}}(B \otimes_R F) \cong \Lambda^{\mathfrak{a}}(F)$. The isomorphism in the derived category follows. Note also that

$$B/(\mathfrak{a}^e)^t \otimes_B (B \otimes_R F) \cong R/\mathfrak{a}^t \otimes_B (B \otimes_R F) \cong R/\mathfrak{a}^t \otimes_R F$$

by hypothesis. Hence $\Lambda^{\mathfrak{a}^e}(B \otimes_R F) \cong \Lambda^{\mathfrak{a}}(F)$. Then the remaining statements are clear. □

Corollary 9.8.3 *Let R be a Noetherian ring, $\mathfrak{a}$ an ideal of R and X an R-complex. Then there is an isomorphism*

$$\mathrm{L}\Lambda^{\mathfrak{a}}(X) \simeq \mathrm{L}\Lambda^{\mathfrak{a}}(\hat{R}^{\mathfrak{a}} \otimes_R X)$$

in the derived category. Moreover, $\mathrm{L}\Lambda^{\mathfrak{a}}(\hat{R}^{\mathfrak{a}} \otimes_R X)$ *and* $\mathrm{L}\Lambda^{\mathfrak{a}\hat{R}^{\mathfrak{a}}}(\hat{R}^{\mathfrak{a}} \otimes_R X))$ *are both represented by $\Lambda^{\mathfrak{a}}(F)$, where $F \xrightarrow{\sim} X$ is a K-flat resolution of X. At the homology level there are isomorphisms*

$$\Lambda_i^{\mathfrak{a}}(X) \cong \Lambda_i^{\mathfrak{a}\hat{R}^{\mathfrak{a}}}(\hat{R}^{\mathfrak{a}} \otimes_R X) \cong \Lambda_i^{\mathfrak{a}}(\hat{R}^{\mathfrak{a}} \otimes_R X)$$

for all $i \in \mathbb{Z}$.

Proof This follows by 9.8.2 (b) applied to the change of rings $R \to \hat{R}^{\mathfrak{a}}$. □

When M is a finitely generated module over a Noetherian ring R we have seen in 2.3.1 that $\hat{M}^{\mathfrak{a}} \cong \hat{R}^{\mathfrak{a}} \otimes_R M$. As $\mathfrak{a}$-complete modules are also $\mathfrak{a}$-pseudo-complete (see 2.5.7) it follows that $\hat{R}^{\mathfrak{a}} \otimes_R M$ belongs to the class $\mathcal{G}_{\mathfrak{a}}^{\Lambda}$ introduced in 9.6.1 (see also Theorem 9.6.8). As a companion result note also the following.

Theorem 9.8.4 *Let $\mathfrak{a}$ be an ideal of a Noetherian ring R and let X denote an R-complex with finitely generated homology modules. Then $\hat{R}^{\mathfrak{a}} \otimes_R X \in \mathcal{G}_{\mathfrak{a}}^{\Lambda}$ and there are isomorphisms*

$$\hat{R}^{\mathfrak{a}} \otimes_R X \simeq \mathrm{L}\Lambda^{\mathfrak{a}}(\hat{R}^{\mathfrak{a}} \otimes_R X) \simeq \mathrm{L}\Lambda^{\mathfrak{a}}(X)$$

in the derived category. At the homology level there are isomorphisms

$$\Lambda_i^{\mathfrak{a}}(X) \cong \Lambda^{\mathfrak{a}}(H_i(X) \cong \hat{R}^{\mathfrak{a}} \otimes_R H_i(X)$$

for all $i \in \mathbb{Z}$.

Proof We note that $\hat{R}^{\mathfrak{a}} \otimes_R H_i(X) \cong H_i(\hat{R}^{\mathfrak{a}} \otimes_R X)$ because $\hat{R}^{\mathfrak{a}}$ is R-flat. We also note that $\hat{R}^{\mathfrak{a}} \otimes_R H_i(X) \cong \Lambda^{\mathfrak{a}}(H_i(X))$ because $H_i(X)$ is finitely generated. Hence the complex $\hat{R}^{\mathfrak{a}} \otimes_R X$ has $\mathfrak{a}$-complete homology. Because $\mathfrak{a}$-complete modules are also $\mathfrak{a}$-pseudo-complete (see 2.5.7) it follows that $\hat{R}^{\mathfrak{a}} \otimes_R X \in \mathcal{G}_{\mathfrak{a}}^{\Lambda}$ (see Theorem

9.6.8). This means that the natural map $\alpha^{\mathfrak{a}}_{\hat{R}^{\mathfrak{a}} \otimes_R X}; \hat{R}^{\mathfrak{a}} \otimes_R X \to \mathrm{L}\Lambda^{\mathfrak{a}}(\hat{R}^{\mathfrak{a}} \otimes_R X)$ is an isomorphism in the derived category. This together with Corollary 9.8.3 implies the last isomorphism in the derived category and the isomorphisms at the homology level. □

On the local cohomology side note also the following.

Theorem 9.8.5 *Let $\mathfrak{a}$ be an ideal of a Noetherian ring R generated by a sequence $\underline{x} = x_1, \ldots, x_k$. Let X denote any R-complex. Then the natural homomorphism $R \to \hat{R}^{\mathfrak{a}}$ induces a quasi-isomorphism*

$$\check{C}_{\underline{x}} \otimes_R X \xrightarrow{\sim} \check{C}_{\underline{x}} \otimes_R X \otimes_R \hat{R}^{\mathfrak{a}}$$

and an isomorphism

$$\mathrm{R}\Gamma_{\mathfrak{a}}(X) \simeq \mathrm{R}\Gamma_{\mathfrak{a}}(X \otimes_R \hat{R}^{\mathfrak{a}})$$

in the derived category.

Proof The last assertion is a consequence of the first one in view of 7.4.4. To prove the first one let $X \xrightarrow{\sim} I$ be a DG-injective resolution of X. Then we have $H^i(\check{C}_{\underline{x}} \otimes_R X) \cong H^i_{\mathfrak{a}}(X) \cong H^i(\Gamma_{\mathfrak{a}}(I))$. Hence the R-modules $H^i(\check{C}_{\underline{x}} \otimes_R X)$ are $\mathfrak{a}$-torsion. It follows that the natural homomorphism $R \to \hat{R}^{\mathfrak{a}}$ induces isomorphisms

$$H^i(\check{C}_{\underline{x}} \otimes_R X) \cong H^i(\check{C}_{\underline{x}} \otimes_R X) \otimes_R \hat{R}^{\mathfrak{a}}$$

(see 2.2.6). As $\hat{R}^{\mathfrak{a}}$ is R-flat we have $H^i(\check{C}_{\underline{x}} \otimes_R X) \otimes_R \hat{R}^{\mathfrak{a}} \cong H^i(\check{C}_{\underline{x}} \otimes_R X \otimes_R \hat{R}^{\mathfrak{a}})$ and the conclusion follows. □

See however the forthcoming 11.5.4 for a more general result.

Remark 9.8.6 Let $\mathfrak{a}$ be an ideal of R generated by a weakly pro-regular sequence $\underline{x} = x_1, \ldots, x_k$. We may consider the change of rings

$$\mathbb{Z}[X_1, \ldots, X_k] \to R : X_i \mapsto x_i$$

where the X_i are indeterminate. In view of 9.8.1 this reduces the study of the local homology and cohomology with respect to $\mathfrak{a}$ to the case where $\mathfrak{a}$ is an ideal of a Noetherian ring generated by a regular sequence and also where the ring is $\mathfrak{a}$-separated. With such a reduction trick we may also provide different proofs of some of the previous results.

Part III
Duality

Chapter 10
Čech and Local Duality

In this chapter we provide some duality formulas for the Čech cohomology of an unbounded complex, which involve the general Matlis dual $\check{C}_{\underline{x}}^{\vee}$ of the Čech complex. When the sequence $\underline{x}$ is a system of parameters of a Noetherian local ring our formulas provide a version of the Grothendieck Local Duality for Cohen–Macaulay or Gorenstein local rings. As a byproduct we obtain new characterizations of Gorenstein local rings in terms of local homology. As another byproduct there are some characterizations of $\mathfrak{m}$-torsion and $\mathfrak{m}$-pseudo complete modules over a Gorenstein local ring. When the sequence $\underline{x}$ is a system of parameters of a complete Noetherian local ring, it turns out that the complex $\check{C}_{\underline{x}}^{\vee}$ is a bounded complex of injective modules with finitely generated cohomology. For that reason we start the chapter with an investigation of such complexes.

10.1 Bounded Injective Complexes with Finitely Generated Cohomology

Let $\underline{x} = x_1, \ldots, x_d$ be a system of parameters of a Noetherian local ring $(R, \mathfrak{m}, \Bbbk)$. Soon we shall see that the complex $\check{C}_{\underline{x}}^{\vee} := \operatorname{Hom}_R(\check{C}_{\underline{x}}, E_R(\Bbbk))$, viewed as an $\hat{R}^{\mathfrak{m}}$-complex, is a bounded injective complex with finitely generated cohomology modules (see the forthcoming 10.2.6). Another example of such a complex is the minimal injective resolution of a finitely generated R-module M, provided of course M has finite injective dimension. Bounded injective complexes with finitely generated cohomology are well documented. In this section we recall some of their main properties.

Lemma 10.1.1 *Let $(R, \mathfrak{m}, \Bbbk)$ denote a Noetherian local ring. Let X denote a non-exact homologically left-bounded R-complex with finitely generated cohomology*

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modules and let $X \xrightarrow{\sim} I$ *be its minimal injective resolution. Then the following conditions are equivalent*

(a) I *is bounded,*
(b) there is an integer t *such that* $\operatorname{Ext}^i_R(\Bbbk, X) = 0$ *for all* $i > t$.

When these conditions are satisfied $n = \sup\{i \mid \operatorname{Ext}^i_R(\Bbbk, X) \neq 0\}$ *is finite,* $I^j = 0$ *for all* $j > n$, $I^n \cong E_R(\Bbbk)^{\mu_n(\mathfrak{m}, X)} \neq 0$ *and* $\dim \operatorname{Ext}^n_R(\Bbbk, X) = \mu_n(\mathfrak{m}, X)$.

Proof Note first that I is left-bounded, more precisely recall that

$$\inf\{j \mid I^j \neq 0\} = \inf\{j \mid H^i(X) \neq 0\}.$$

Hence I is DG-injective and $\operatorname{Ext}^i(\Bbbk, X) \cong H^i(\operatorname{Hom}_R(\Bbbk, I))$ for all $i \in \mathbb{Z}$. Whence the implication (a) $\Rightarrow$ (b) follows.

Assume now that condition (b) is satisfied. If $I^j \neq 0$ for some $j \in \mathbb{Z}$, then $\mu_j(\mathfrak{p}, X) \neq 0$ for some prime $\mathfrak{p} \in \operatorname{Spec}(R)$ (see the description of I^j in 4.5.18). By an iterative use of 4.5.19 it follows that $\mu_{j+s}(\mathfrak{m}, X) \neq 0$ where $s = \dim(R/\mathfrak{p})$. But $\mu_t(\mathfrak{m}, X) = \dim_{\Bbbk} \operatorname{Ext}^t_R(\Bbbk, X)$ by definition (see 4.5.16). Hence $\sup\{i \mid \operatorname{Ext}^i_R(\Bbbk, X) \neq 0\}$ is finite. Again by 4.5.19 it also follows that $\mu_j(\mathfrak{p}, X) = 0$ for all prime ideals $\mathfrak{p} \neq \mathfrak{m}$ and all $j \geq \sup\{i \mid \operatorname{Ext}^i_R(\Bbbk, X) \neq 0\} = n$. Hence $I^j = 0$ for all $j > n$.

The last assertions about I^n follow from its description in 4.5.18. Recall that the complex $\operatorname{Hom}_R(\Bbbk, I)$ has zero differentials (see 4.5.7). □

Remark 10.1.2 Let M be a finitely generated module over the Noetherian local $(R, \mathfrak{m}, \Bbbk)$. By the above we recover the following well-known fact: $\operatorname{id}_R(M) = \sup\{i \mid \operatorname{Ext}^i_R(\Bbbk, M) \neq 0\}$.

The following seems to be not known. It is related to [63, Theorem 4.15].

Proposition 10.1.3 *Let* $(R, \mathfrak{m}, \Bbbk)$ *be a Noetherian local ring and let*

$$I : I^0 \to I^1 \to \ldots \to I^n \to 0$$

be a non-exact minimal injective complex such that its cohomology modules $H^j(I)$ *are finitely generated. Let* M *be any finitely generated* R*-module. Then*

$$\operatorname{depth}(M) = n - \sup\{i \mid \operatorname{Ext}^i_R(M, I) \neq 0\}.$$

Proof We proceed by an induction on $\operatorname{depth}_R M$. If $\operatorname{depth}_R M = 0$, there is an injection $\Bbbk \hookrightarrow M$. It yields the exact sequence $\operatorname{Ext}^n_R(M, I) \to \operatorname{Ext}^n_R(\Bbbk, I) \to 0$ because $\operatorname{Ext}^{n+1}_R(N, I) = 0$ for any R-module N. As $\operatorname{Ext}^n_R(\Bbbk, I) \neq 0$ by 10.1.1 we also have that $\operatorname{Ext}^n_R(M, I) \neq 0$.

Assume now that $\operatorname{depth}(M) = r > 0$ and the assertion is proved for modules of depth smaller than r. We choose an element $x \in \mathfrak{m}$ regular on M. Then consider the short exact sequence

$$0 \to M \xrightarrow{x} M \to M/xM \to 0$$

and the associated long exact sequence of the $\operatorname{Ext}^i_R(\cdot, I)$. For all $i \geq n - r + 1$ we have the exact sequence

$$\operatorname{Ext}^i_R(M, I) \xrightarrow{x} \operatorname{Ext}^i_R(M, I) \to 0$$

because $\operatorname{Ext}^{i+1}_R(M/xM, I) = 0$ by the inductive hypothesis. As the $\operatorname{Ext}^j_R(M, I)$ are all finitely generated (see 4.5.14) it follows by Nakayama's Lemma that $\operatorname{Ext}^i_R(M, I) = 0$ for all $i \geq n - r + 1$. It remains to show that $\operatorname{Ext}^{n-r}_R(M, I) \neq 0$. By the above we also have the exact sequence

$$\operatorname{Ext}^{n-r}_R(M, I) \to \operatorname{Ext}^{n-r+1}_R(M/xM, I) \to 0$$

and $\operatorname{Ext}^{n-r+1}_R(M/xM, I) \neq 0$ by the induction hypothesis. The conclusion follows. □

10.1.4 The particular case of 10.1.3 where the complex I is a minimal injective resolution of a finitely generated module M of finite injective dimension has been proved by Peskine and Szpiro in [63]. Note that the above also provides another proof of a result of Bass in [9]:

$$\operatorname{id}_R M = \operatorname{depth}_R R \ \text{ provided } \ \operatorname{id}_R M < \infty.$$

In [9] Bass conjectured that, for a Noetherian local ring, the existence of a non-trivial finitely generated module of finite injective dimension implies that the ring is Cohen–Macaulay. This has been proved by Peskine and Szpiro in [63], provided the ring is equicharacteristic. Their proof used their Intersection Conjecture, which they proved for equicharacteristic local rings (see also Hochster's investigations [45] on the homological conjectures).

Later an equivalent form of the Intersection Conjecture, called the New Intersection Conjecture, was established. This was proved in full generality by Paul Roberts in [70] (see Theorem 10.1.6 below).

Roberts had previously proposed in [69] another conjecture concerning bounded injective complexes with finitely generated cohomology, the one we are interested in, and he claimed that it was equivalent to the New Intersection Conjecture.

To provide a proof of Roberts' statement on bounded injective complexes with finitely generated cohomology we need some reductions tricks.

Proposition 10.1.5 *Let $(S, \mathfrak{n}, \mathbb{l})$ and $(R, \mathfrak{m}, \Bbbk)$ be two Noetherian local rings with their maximal ideal and residue field resp. Let $f : (S, \mathfrak{n}, \mathbb{l}) \to (R, \mathfrak{m}, \Bbbk)$ be a local flat homomorphism such that $\mathfrak{m} = \mathfrak{n}R$. Let I be a bounded injective S-complex with finitely generated cohomology modules. We consider the R-complex $I \otimes_S R$.*

(a) There are isomorphisms

$$\operatorname{Ext}^i_R(\Bbbk, I \otimes_S R) \cong \operatorname{Ext}^i_S(\mathbb{l}, I) \otimes_S R \textit{ for all } i \in \mathbb{Z}.$$

(b) Let $I \otimes_S R \xrightarrow{\sim} J$ *be the minimal injective resolution of the* R*-complex* $I \otimes_S R$*. Then the* R*-complex* J *also has finitely generated cohomology and is bounded. If, moreover,* I *is minimal, then* $\operatorname{length}(I) = \operatorname{length}(J)$.

Proof Because $\mathfrak{m} = \mathfrak{n}R$ it follows that $R/\mathfrak{m} \cong S/\mathfrak{n} \otimes_S R$. Hence the first assertion is a direct consequence of 4.4.21.

As f is flat we have that $H^i(J) \cong H^i(I) \otimes_S R$ is a finitely generated R-module. In view of 10.1.1 and the first assertion it follows that J is bounded and that $\sup\{i \mid J^i \neq 0\} = \sup\{i \mid I^i \neq 0\}$.

By the hypothesis on the homomorphism f we note that f is faithfully flat. The assertion on the lengths of I and J follows. Recall that $\inf\{j \mid I^j \neq 0\} = \inf\{j \mid H^j(I) \neq 0\}$ when I is minimal and likewise for J. □

Note that the above may be applied to the change of rings $R \to \hat{R}^{\mathfrak{a}}$, where $\mathfrak{a}$ is any ideal of the Noetherian local ring R. In the following we shall derive a consequence of Roberts' New Intersection Theorem.

Theorem 10.1.6 (see [70], New Intersection Theorem.) *Let* $(R, \mathfrak{m}, \Bbbk)$ *be a Noetherian local ring and let*

$$0 \to L_n \to \cdots \to L_1 \to L_0 \to 0$$

be a non-exact complex of finitely generated free R*-modules. If the homology modules* $H_i(L)$ *are of finite length for all* i*, then* $n \geq \dim(R)$.

Theorem 10.1.7 (see [69]) *Let* $(R, \mathfrak{m}, \Bbbk)$ *be a Noetherian local ring. Let*

$$I : I^0 \to I^1 \to \cdots \to I^n \to 0$$

be a non-exact bounded complex of injective modules. Assume that its cohomology modules $H^j(I)$ *are finitely generated. Then* $n \geq \dim(R)$.

Proof We may and do assume that I is minimal. We start with a reduction to the case of a catenary local domain. To this end consider the changes of rings

$$R \to \hat{R}^{\mathfrak{m}} \to B$$

where $B = \hat{R}^{\mathfrak{m}}/\mathfrak{p}$ for some prime ideal $\mathfrak{p}$ of $\hat{R}^{\mathfrak{m}}$ such that $\dim(B) = \dim(\hat{R}^{\mathfrak{m}})$. We take the minimal injective resolution $I \otimes_R \hat{R}^{\mathfrak{m}} \xrightarrow{\sim} J$ of the $\hat{R}^{\mathfrak{m}}$-complex $I \otimes_R \hat{R}^{\mathfrak{m}}$. Then consider the B-complex $\operatorname{Hom}_{\hat{R}^{\mathfrak{m}}}(B, J)$. We first note that the $\hat{R}^{\mathfrak{m}}$-complex J has finitely generated cohomology with $\operatorname{length}(J) = \operatorname{length}(I)$ (see 10.1.5). We then note that the B-complex $\operatorname{Hom}_{\hat{R}^{\mathfrak{m}}}(B, J)$, which is a complex of injective B-modules, also has finitely generated cohomology (see 4.5.14). But $\dim(B) = \dim(\hat{R}^{\mathfrak{m}}) = \dim(R)$ and obviously we have $\operatorname{length}(\operatorname{Hom}_{\hat{R}^{\mathfrak{m}}}(B, J)) \leq \operatorname{length}(J)$. Hence it is enough to prove that $\operatorname{length}(\operatorname{Hom}_{\hat{R}^{\mathfrak{m}}}(B, J)) \geq \dim(B)$. That is, we are reduced to the case where the ring is a catenary local Noetherian domain.

We proceed by induction on $\dim_R \operatorname{Supp}_R(I)$. If $\dim_R \operatorname{Supp}_R(I) = 0$, then $\mu_j(\mathfrak{p}, I) = 0$ for all prime ideals $\mathfrak{p} \neq \mathfrak{m}$ and all j. Hence $I^j \cong E_R(\Bbbk)^{\mu_j(\mathfrak{m},I)}$ and note that these $\mu_j(\mathfrak{m}, I) = \dim_{\Bbbk}(\operatorname{Ext}^j_R(\Bbbk, I))$ are finite (see 4.5.16). Then the Matlis dual complex $I^\vee := \operatorname{Hom}_R(I, E_R(\Bbbk))$ is an $\hat{R}^{\mathfrak{m}}$-complex of finitely generated modules with Artinian homology because the R-modules $H^j(I)$ are finitely generated. But the $\hat{R}^{\mathfrak{m}}$-modules $H_i(I^\vee)$ are also finitely generated. Hence they are of finite length and the result follows by 10.1.6 applied to the ring $\hat{R}^{\mathfrak{m}}$.

Assume now that $\dim_R \operatorname{Supp}_R(I) > 0$. We pick a prime ideal $\mathfrak{p} \in \operatorname{Supp}_R(I)$, $\mathfrak{p} \neq \mathfrak{m}$. We note that the $R_\mathfrak{p}$-complex $I_\mathfrak{p}$ is minimal injective because I is minimal injective (see 4.5.9). Also note that $\dim_{R_\mathfrak{p}} \operatorname{Supp}_{R_\mathfrak{p}}(I_\mathfrak{p}) < \dim_R \operatorname{Supp}_R(I)$. By the induction hypothesis we have that $\operatorname{length}(I_\mathfrak{p}) \geq \dim(R_\mathfrak{p})$. In particular, we have

$$m = \sup\{j \mid I^j_\mathfrak{p} \neq 0\} \geq \operatorname{length}(I_\mathfrak{p}) \geq \dim(R_\mathfrak{p}).$$

Applying 10.1.1 to the $R_\mathfrak{p}$-complex $I_\mathfrak{p}$ we have $\mu_m(\mathfrak{p}R_\mathfrak{p}, I_\mathfrak{p}) \neq 0$. But $\mu_m(\mathfrak{p}R_\mathfrak{p}, I_\mathfrak{p}) = \mu_m(\mathfrak{p}, I)$ by definition. By an iterative use of 4.5.19 we obtain that $\mu_{m+s}(\mathfrak{m}, I) \neq 0$ for $s = \dim(R/\mathfrak{p})$. It follows that

$$\operatorname{length}(I) \geq m + s \geq \dim(R_\mathfrak{p}) + \dim(R/\mathfrak{p}).$$

But $\dim(R_\mathfrak{p}) + \dim(R/\mathfrak{p}) = \dim R$ because R is a catenary local domain. This finishes the proof. $\square$

Corollary 10.1.8 *Let $(R, \mathfrak{m}, \Bbbk)$ be a Noetherian local ring and assume there is a finitely generated R-module $M \neq 0$ of finite injective dimension. Then R is Cohen–Macaulay and* $\operatorname{id}_R M = \dim R$.

10.2 Čech Cohomology and Duality

Here we show how the general Matlis dual of the Čech complex provides information on the Čech cohomology. Some particular cases will retain our attention, in particular the case where the sequence $\underline{x}$ is completely secant, or the case where $\underline{x}$ is a system of parameters of a Noetherian local ring. As before, E denotes the injective cogenerator of the category of R-modules as described in 1.4.8 and the general Matlis duality functor $\operatorname{Hom}_R(\cdot, E)$ is denoted by $(\cdot)^\vee$.

10.2.1 Recall first that we already have information on the complex $\check{C}^\vee_{\underline{x}}$: it is a bounded complex of injective modules, hence DG-injective, and its cohomology modules are $\mathfrak{a}$-complete, where $\mathfrak{a} = \underline{x}R$. Moreover, we have $\operatorname{Hom}_R(R/\mathfrak{a}, \check{C}^\vee_{\underline{x}}) \cong (R/\mathfrak{a})^\vee$ (see 6.1.5).

10.2.2 *Čech cohomology and duality.* Let $\underline{x} = x_1, \ldots, x_k$ be a sequence of elements in a commutative ring R. Recall that the Čech cohomology of an R-complex X with respect to this sequence is given by

$$H^i(\check{C}_{\underline{x}} \otimes_R X) = \mathrm{Tor}^R_{-i}(X, \check{C}_{\underline{x}}) = \mathrm{Tor}^R_{k-i}(X, \check{C}_{\underline{x}})_{[k]}.$$

Therefore we have the Čech duality formulas

$$(\check{C}_{\underline{x}} \otimes_R X)^\vee \cong \mathrm{Hom}_R(X, \check{C}^\vee_{\underline{x}}),$$
$$\mathrm{Ext}^{-i}_R(X, \check{C}^\vee_{\underline{x}}) \cong \mathrm{Ext}^{k-i}_R(X, \check{C}^{\vee[k]}_{\underline{x}}) \cong (H^i(\check{C}_{\underline{x}} \otimes_R X))^\vee$$

obtained by adjointness. The shift in the above formulas are motivated by the following observations.

10.2.3 *Assume now that the sequence $\underline{x} = x_1, \ldots, x_k$ of a commutative ring R is completely secant and that $R \neq \underline{x}R$.* This amounts to saying that $\text{E-dp}(\underline{x}R, R) = k$ and that the complexes $K^\bullet(\underline{x}^t)$ have their cohomology concentrated in degree k (see 5.4.2). Then the complex $\check{C}_{\underline{x}}$ also has its cohomology concentrated in degree k and provides, up to a shift, a flat resolution of the module $H^k(\check{C}_{\underline{x}})$. More precisely, viewing the complex $\check{C}_{\underline{x}}$ as a descending one, we have the quasi-isomorphism

$$(\check{C}_{\underline{x}})_{[k]} \xrightarrow{\sim} H^k(\check{C}_{\underline{x}})$$

which is a bounded flat resolution of the module $H^k(\check{C}_{\underline{x}})$. By duality we have the quasi-isomorphism

$$(H^k(\check{C}_{\underline{x}}))^\vee \xrightarrow{\sim} \check{C}^{\vee[k]}_{\underline{x}}$$

which is a bounded injective resolution of the module $(H^k(\check{C}_{\underline{x}}))^\vee$. In this situation the module $H^k(\check{C}_{\underline{x}})$ also plays a rôle in the study of the Čech cohomology, we have

$$H^i(\check{C}_{\underline{x}} \otimes_R X) \cong \mathrm{Tor}^R_{k-i}(X, \check{C}_{\underline{x}[k]}) \cong \mathrm{Tor}^R_{k-i}(X, H^k(\check{C}_{\underline{x}})) \text{ and}$$
$$(H^i(\check{C}_{\underline{x}} \otimes_R X))^\vee \cong \mathrm{Ext}^{k-i}_R(X, (H^k(\check{C}_{\underline{x}}))^\vee).$$

Note that

$$\mathrm{fd}_R(H^k(\check{C}_{\underline{x}})) = k = \mathrm{id}_R((H^k(\check{C}_{\underline{x}}))^\vee).$$

Because the descending complex $R/\underline{x}R \otimes_R \check{C}_{\underline{x}[k]}$ has its homology concentrated in degree k we also have

$$\text{T-codp}(\underline{x}R, H^k(\check{C}_{\underline{x}})) = k = \text{E-dp}(\underline{x}R, (H^k(\check{C}_{\underline{x}}))^\vee).$$

Hence the sequence $\underline{x}$ is completely secant on the module $(H^k(\check{C}_{\underline{x}}))^\vee$ (see 5.4.2 again).

In the above the particular case when the module $H^k(\check{C}_{\underline{x}})$ is itself a general Matlis dual also presents some interest, mainly when the ring is Noetherian. In that case recall that the Čech cohomology with respect to a sequence $\underline{x}$ coincides with the local cohomology with respect to the ideal $\mathfrak{a} = \underline{x}R$ (see 7.4.4 and 7.4.5).

Proposition 10.2.4 *Let $\underline{x} = x_1, \ldots, x_k$ be a regular sequence in a Noetherian ring R. Assume there is an R-module $K_{\underline{x}}$ such that $H^k(\check{C}_{\underline{x}}) \cong K_{\underline{x}}^{\vee}$ and let $\mathfrak{a}$ be any ideal such that* $\operatorname{Rad}(\mathfrak{a}) = \operatorname{Rad}(\underline{x}R)$. *Then*

(a) $\operatorname{id}_R(K_{\underline{x}}) = k$,
(b) $H^i_{\mathfrak{a}}(X) \cong (\operatorname{Ext}^{k-i}_R(X, K_{\underline{x}}))^{\vee}$ *for any R-complex X with finitely generated homology modules.*

Proof The first assertion is clear: we have $\operatorname{id}_R(K_{\underline{x}}) = \operatorname{fd}_R(K_{\underline{x}}^{\vee}) = k$ (see 1.4.10 and 10.2.3).

By 7.4.5 and 10.2.3 it follows that

$$H^i_{\mathfrak{a}}(X) \cong \operatorname{Tor}^R_{-i}(X, \check{C}_{\underline{x}}) \cong \operatorname{Tor}^R_{k-i}(X, H^k(\check{C}_{\underline{x}})).$$

We note that the homology modules $\operatorname{Tor}^R_{k-i}(X, H^k(\check{C}_{\underline{x}}))$ and $\operatorname{Ext}^{k-i}_R(X, K_{\underline{x}})$ only depend on a soft truncation $\cdots \to X_{n+1} \to X_n \to \operatorname{Ker}(d_{n-1}) \to 0$ for n large enough because the module $H^k(\check{C}_{\underline{x}})$ has finite flat dimension and the module $K_{\underline{x}}$ has finite injective dimension. So we may and do assume that X is right-bounded. In that case there is a DG-projective resolution $L \xrightarrow{\sim} X$ where L is a right-bounded complex of finitely generated free modules (see 1.1.12). Then

$$L \otimes_R H^k(\check{C}_{\underline{x}}) \cong \operatorname{Hom}_R(\operatorname{Hom}_R(L, R), H^k(\check{C}_{\underline{x}}))$$

(see 1.1.9). By adjointness, and because $H^k(\check{C}_{\underline{x}}) \cong K_{\underline{x}}^{\vee}$, we also have

$$\begin{aligned}\operatorname{Hom}_R(\operatorname{Hom}_R(L, R), H^k(\check{C}_{\underline{x}})) &\cong \operatorname{Hom}_R(\operatorname{Hom}_R(L, R) \otimes_R K_{\underline{x}}, E)\\ &\cong (\operatorname{Hom}_R(L, K_{\underline{x}}))^{\vee}.\end{aligned}$$

The claim is obtained by taking homology. □

We continue the investigation of $\check{C}_{\underline{x}}^{\vee} = \operatorname{Hom}_R(\check{C}_{\underline{x}}, E)$.

Lemma 10.2.5 *Let $\underline{x} = x_1, \ldots, x_k$ be a sequence in a commutative ring R. There is a quasi-isomorphism*

$$\check{C}_{\underline{x}} \otimes_R E \xrightarrow{\sim} \check{C}_{\underline{x}} \otimes_R \check{C}_{\underline{x}}^{\vee}.$$

If, moreover, the sequence $\underline{x}$ is weakly pro-regular, there is also a quasi-isomorphism $\Gamma_{\mathfrak{a}}(E) \xrightarrow{\sim} \check{C}_{\underline{x}} \otimes_R E$, where $\mathfrak{a}$ denotes the ideal generated by the sequence $\underline{x}$.

Proof As usual write $\underline{x}^t = x_1^t, \ldots, x_k^t$. By adjointness we have for all $t > 0$ the isomorphisms

$$K^\bullet(\underline{x}^t) \otimes_R \check{C}_{\underline{x}}^\vee \cong \operatorname{Hom}_R(K_\bullet(\underline{x}^t), \check{C}_{\underline{x}}^\vee) \cong (K_\bullet(\underline{x}^t) \otimes_R \check{C}_{\underline{x}})^\vee.$$

We also have a quasi-isomorphism $K_\bullet(\underline{x}^t) \otimes_R \check{C}_{\underline{x}} \xrightarrow{\sim} K_\bullet(\underline{x}^t)$. This follows by 6.1.8 because $\operatorname{Supp}_R(K_\bullet(\underline{x}^t)) \subseteq V(\mathfrak{a})$. By duality we have

$$K^\bullet(\underline{x}^t) \otimes_R E \cong K_\bullet(\underline{x}^t)^\vee \xrightarrow{\sim} (K_\bullet(\underline{x}^t) \otimes_R \check{C}_{\underline{x}})^\vee \cong K^\bullet(\underline{x}^t) \otimes_R \check{C}_{\underline{x}}^\vee.$$

We take direct limits and obtain the first quasi-isomorphism.

For the second one note that $H^i(\check{C}_{\underline{x}} \otimes_R E) = 0$ for all $i > 0$ by Lemma 7.3.3. Note also that $H^0(\check{C}_{\underline{x}} \otimes_R E) \cong \Gamma_\mathfrak{a}(E)$ as easily seen. The claim follows. □

In the case when $(R, \mathfrak{m}, \Bbbk)$ is a Noetherian local ring with its maximal ideal and residue field $\Bbbk$ we can say more. In that case, recall that $E = E_R(\Bbbk)$ is the injective hull of the residue field (see the description of the injective cogenerator given in 1.4.8). Recall also that E has the structure of an $\hat{R}^\mathfrak{m}$-module and that, with this structure, $E = E_{\hat{R}^\mathfrak{m}}(\Bbbk)$ (see 2.3.2).

Theorem 10.2.6 *Let $(R, \mathfrak{m}, \Bbbk)$ be a Noetherian local ring with its maximal ideal and residue field. Let also $\underline{x} = x_1, \ldots, x_k$ be a sequence of elements in R, $R \neq \underline{x}R$. Then:*

(a) $\check{C}_{\underline{x}}^\vee = \operatorname{Hom}_R(\check{C}_{\underline{x}}, E)$ *is a bounded complex of injective R-modules.*
(b) There is an isomorphism $\Bbbk \cong \operatorname{Hom}_R(\Bbbk, \check{C}_{\underline{x}}^\vee)$.
(c) There is a quasi-isomorphism $\operatorname{Hom}_R(\check{C}_{\underline{x}}^\vee, \check{C}_{\underline{x}}^\vee) \xrightarrow{\sim} \hat{R}^\mathfrak{m}$. *Its composite with the natural morphism* $R \to \operatorname{Hom}_R(\check{C}_{\underline{x}}^\vee, \check{C}_{\underline{x}}^\vee)$ *is the natural homomorphism* $R \to \hat{R}^\mathfrak{m}$.
(d) $\check{C}_{\underline{x}}^\vee \cong \operatorname{Hom}_{\hat{R}^\mathfrak{m}}(\check{C}_{\underline{x}} \otimes_R \hat{R}^\mathfrak{m}, E)$ *is a bounded complex of injective $\hat{R}^\mathfrak{m}$-modules and there is an isomorphism* $\Bbbk \cong \operatorname{Hom}_{\hat{R}^\mathfrak{m}}(\Bbbk, \check{C}_{\underline{x}}^\vee)$.
(e) The natural morphism

$$\hat{R}^\mathfrak{m} \to \operatorname{Hom}_{\hat{R}^\mathfrak{m}}(\check{C}_{\underline{x}}^\vee, \check{C}_{\underline{x}}^\vee)$$

is a quasi-isomorphism.

Proof The statement in (a) has already been noticed and holds in general (see 10.2.1).

For (b) note that we have $\operatorname{Hom}_R(\Bbbk, \check{C}_{\underline{x}}^\vee) \cong (\check{C}_{\underline{x}} \otimes_R \Bbbk)^\vee$ by adjointness. Moreover, we have the isomorphisms $\check{C}_{\underline{x}} \otimes_R \Bbbk \cong \Bbbk$ and $(\Bbbk)^\vee \cong \Bbbk$.

Now we prove (c). By adjointness and commutativity we have

$$\operatorname{Hom}_R(\check{C}_{\underline{x}}^\vee, \check{C}_{\underline{x}}^\vee) \cong (\check{C}_{\underline{x}} \otimes_R \check{C}_{\underline{x}}^\vee)^\vee.$$

By 10.2.5 there is a quasi-isomorphism $\check{C}_{\underline{x}} \otimes_R E \xrightarrow{\sim} \check{C}_{\underline{x}} \otimes_R \check{C}_{\underline{x}}^\vee$. But $\check{C}_{\underline{x}} \otimes_R E \cong E$ because E is an $\mathfrak{m}$-torsion module, which implies $\check{C}_{\underline{x}}^i \otimes_R E = 0$ for all $i > 0$. Putting all this together we obtain a quasi-isomorphism

$$\operatorname{Hom}_R(\check{C}_{\underline{x}}^{\vee}, \check{C}_{\underline{x}}^{\vee}) \xrightarrow{\sim} \hat{R}^{\mathfrak{m}}.$$

To this end recall that $\operatorname{Hom}_R(E, E) \cong \hat{R}^{\mathfrak{m}}$. The last assertion in (c) follows.

For the last statements note first that we have the isomorphism

$$\check{C}_{\underline{x}}^{\vee} \cong \operatorname{Hom}_{\hat{R}^{\mathfrak{m}}}(\check{C}_{\underline{x}} \otimes_R \hat{R}^{\mathfrak{m}}, E)$$

because E has the structure of an $\hat{R}^{\mathfrak{m}}$-module. Hence $\check{C}_{\underline{x}}^{\vee}$ is the Matlis dual over the ring $\hat{R}^{\mathfrak{m}}$ of the Čech complex built on the sequence $\underline{x}\hat{R}^{\mathfrak{m}}$ of $\hat{R}^{\mathfrak{m}}$. The statements in (d) and (e) now follow from those in (a), (b) and (c). □

For technical reasons the complex $\check{L}_{\underline{x}}^{\vee}$, which is the Matlis dual of a bounded DG-projective complex, could be more useful than $\check{C}_{\underline{x}}^{\vee}$. To this end we note the following.

Corollary 10.2.7 *In the situation of 10.2.6 we also have a quasi-isomorphism*

$$\operatorname{Hom}_R(\check{L}_{\underline{x}}^{\vee}, \check{L}_{\underline{x}}^{\vee}) \xrightarrow{\sim} \hat{R}^{\mathfrak{m}}.$$

If, moreover, R is $\mathfrak{m}$-complete, then the natural morphism $R \to \operatorname{Hom}_R(\check{L}_{\underline{x}}^{\vee}, \check{L}_{\underline{x}}^{\vee})$ is also a quasi-isomorphism.

Proof We have a bounded free resolution $\check{L}_{\underline{x}} \xrightarrow{\sim} \check{C}_{\underline{x}}$ of $\check{C}_{\underline{x}}$ (see 6.2.3). It induces a quasi-isomorphism $\check{C}_{\underline{x}}^{\vee} \xrightarrow{\sim} \check{L}_{\underline{x}}^{\vee}$ of bounded complexes of injective modules. Hence there are quasi-isomorphisms $\check{C}_{\underline{x}}^{\vee} \xrightarrow{\sim} \check{L}_{\underline{x}}^{\vee} \xrightarrow{\sim} \check{C}_{\underline{x}}^{\vee}$, the composite of which is homotopic to the identity on $\check{C}_{\underline{x}}^{\vee}$ (see 4.4.6). In view of 10.2.6 (c) the conclusion follows. □

The previous Theorem 10.2.6 is the motivation for further investigation. In particular, it would be of some interest to have an R-complex D with finitely generated cohomology satisfying the properties (a) and (b) above. This is the concept of a dualizing complex, as described later.

In this direction note already the following.

Theorem 10.2.8 *Let $(R, \mathfrak{m}, \Bbbk)$ be a Noetherian local ring with its maximal ideal and residue field $\Bbbk$. Let also $\underline{x} = x_1, \ldots, x_k$ be a sequence generating an $\mathfrak{m}$-primary ideal. Then the modules $H^i(\check{C}_{\underline{x}}^{\vee})$ are finitely generated over $\hat{R}^{\mathfrak{m}}$.*

Proof In our situation recall that $H^i(\check{C}_{\underline{x}}) \cong H^i_{\mathfrak{m}}(R)$ (see 7.4.5), that the R-modules $H^i(\check{C}_{\underline{x}}) \cong H^i_{\mathfrak{m}}(R)$ are Artinian. It follows that the modules $H^i(\check{C}_{\underline{x}}^{\vee}) \cong (H^{-i}(\check{C}_{\underline{x}}))^{\vee}$ are finitely generated over $\hat{R}^{\mathfrak{m}}$ (see 2.3.2). □

10.3 Canonical Modules

In this section we introduce the notion of a canonical module for a finitely generated module over a Noetherian local ring $(R, \mathfrak{m}, \Bbbk)$, with particular attention on the canonical module of a Cohen–Macaulay local ring, when it exists. For Cohen–Macaulay rings and modules we refer to [16]. The canonical module is a matter of local cohomology and Matlis duality. Here $E := E_R(\Bbbk)$ is the injective hull of the residue field and the Matlis duality functor $\operatorname{Hom}_R(\cdot, E)$ is denoted – as before – by $(\cdot)^\vee$. It will be convenient to have the following recalls at hand.

Recalls 10.3.1 Let $(R, \mathfrak{m}, \Bbbk)$ be a Noetherian local ring and M a finitely generated R-module. At first note that $M \otimes_R \hat{R}^{\mathfrak{m}} \cong \hat{M}^{\mathfrak{m}}$. Because the local cohomology modules $H^i_{\mathfrak{m}}(M)$ are $\mathfrak{m}$-torsion and because $\hat{R}^{\mathfrak{m}}$ is R-flat there are isomorphisms

$$H^i_{\mathfrak{m}}(M) \cong H^i_{\mathfrak{m}}(M) \otimes_R \hat{R}^{\mathfrak{m}} \cong H^i_{\mathfrak{m}\hat{R}^{\mathfrak{m}}}(M \otimes_R \hat{R}^{\mathfrak{m}}) \cong H^i_{\mathfrak{m}\hat{R}^{\mathfrak{m}}}(\hat{M}^{\mathfrak{m}})$$

(see 2.2.6 and 9.8.2). Since $E_R(\Bbbk) \cong E_{\hat{R}^{\mathfrak{m}}}(\Bbbk)$ admits the structure of an $\hat{R}^{\mathfrak{m}}$-module there are also isomorphisms

$$(H^i_{\mathfrak{m}}(M))^\vee \cong \operatorname{Hom}_{\hat{R}^{\mathfrak{m}}}(H^i_{\mathfrak{m}}(M) \otimes_R \hat{R}^{\mathfrak{m}}, E) \cong \operatorname{Hom}_{\hat{R}^{\mathfrak{m}}}(H^i_{\mathfrak{m}\hat{R}^{\mathfrak{m}}}(\hat{M}^{\mathfrak{m}}), E)).$$

Recall also Grothendieck's Theorem: $H^i_m(M)$ is Artinian for all $i \geq 0$ and $\dim M = \sup\{i \mid H^i_{\mathfrak{m}}(M) \neq 0$ (see A.1.1).

Theorem and Definition 10.3.2 *Let $(R, \mathfrak{m}, \Bbbk)$ be a Noetherian local ring and let M be a non-zero finitely generated R-module of dimension n.*

(a) For an R-module C the following conditions are equivalent:

 (i) $C^\vee \cong H^n_{\mathfrak{m}}(M)$,
 (ii) C is a finitely generated R-module and $C \otimes_R \hat{R}^{\mathfrak{m}} \cong \operatorname{Hom}_{\hat{R}^{\mathfrak{m}}}(H^n_{\mathfrak{m}\hat{R}^{\mathfrak{m}}}(\hat{M}^{\mathfrak{m}}), E)$.

(b) If there is an R-module satisfying one of the above equivalent conditions, then it is unique up to isomorphism. It is called the canonical module of M and is denoted by K_M.

Proof Assume that C satisfies condition (i). Because the R-module $H^n_{\mathfrak{m}}(M)$ is a non-zero Artinian module we first have that C is a non-zero finitely generated R-module in view of 1.4.9. Then we have the isomorphisms $C \otimes_R \hat{R}^{\mathfrak{m}} \cong C^{\vee\vee} \cong (H^n_{\mathfrak{m}}(M))^\vee$ (see 2.3.2 (d) for the first one). This together with the recalls in 10.3.1 shows that C satisfies condition (ii).

In general we have $C^\vee \cong \operatorname{Hom}_{\hat{R}^{\mathfrak{m}}}(C \otimes_R \hat{R}^{\mathfrak{m}}, E)$. When C satisfies condition (ii) it follows that $C^\vee \cong \operatorname{Hom}_{\hat{R}^{\mathfrak{m}}}(\operatorname{Hom}_{\hat{R}^{\mathfrak{m}}}(H^n_{\mathfrak{m}\hat{R}^{\mathfrak{m}}}(\hat{M}^{\mathfrak{m}}), E), E)$. But the $\hat{R}^{\mathfrak{m}}$-module $H^n_{\mathfrak{m}\hat{R}^{\mathfrak{m}}}(\hat{M}^{\mathfrak{m}})$ is Artinian, hence $C^\vee \cong H^n_{\mathfrak{m}\hat{R}^{\mathfrak{m}}}(\hat{M}^{\mathfrak{m}})$ and C satisfies condition (i) in view of the recalls in 10.3.1.

Assume now there are two modules C and C' satisfying condition (ii). Then $C \otimes_R \hat{R}^{\mathfrak{m}}$ and $C' \otimes_R \hat{R}^{\mathfrak{m}}$ are both isomorphic to $(H^n_{\mathfrak{m}}(M))^\vee$. It follows that C and C' are isomorphic by the following well-known lemma. $\square$

Lemma 10.3.3 *Let M and M' be two finitely generated modules over the Noetherian local ring $(R, \mathfrak{m}, \Bbbk)$. If the $\hat{R}^{\mathfrak{m}}$-modules $M \otimes_R \hat{R}^{\mathfrak{m}}$ and $M' \otimes_R \hat{R}^{\mathfrak{m}}$ are isomorphic, then the R-modules M and M' are isomorphic.*

Proof Because $\hat{R}^{\mathfrak{m}}$ is R-flat we get the isomorphism

$$\operatorname{Hom}_R(M, M') \otimes_R \hat{R}^{\mathfrak{m}} \cong \operatorname{Hom}_{\hat{R}^{\mathfrak{m}}}(M \otimes_R \hat{R}^{\mathfrak{m}}, M' \otimes_R \hat{R}^{\mathfrak{m}}).$$

We note that an isomorphism $h \in \operatorname{Hom}_{\hat{R}^{\mathfrak{m}}}(M \otimes_R \hat{R}^{\mathfrak{m}}, M' \otimes_R \hat{R}^{\mathfrak{m}})$ is a minimal generator of this finitely generated $\hat{R}^{\mathfrak{m}}$-module. We pick in the finitely generated R-module $\operatorname{Hom}_R(M, M')$ a minimal generator f such that $f \otimes_R \operatorname{id}_{\hat{R}^{\mathfrak{m}}}$ maps onto h modulo $\mathfrak{m}\hat{R}^{\mathfrak{m}}$ by the above isomorphism. It is easy to see that $f \otimes_R \operatorname{id}_{\hat{R}^{\mathfrak{m}}}$ is also an isomorphism. Then f itself is an isomorphism because $\hat{R}^{\mathfrak{m}}$ is R-faithfully-flat. $\square$

Remarks 10.3.4 Let $(R, \mathfrak{m}, \Bbbk)$ be a Noetherian local ring and let M be a non-zero finitely generated R-module of dimension n.

(a) If R is $\mathfrak{m}$-complete then M has a canonical module, namely

$$K_M \cong (H^n_{\mathfrak{m}}(M))^\vee.$$

(Because $H^n_{\mathfrak{m}}(M)$ is Artinian and R is $\mathfrak{m}$-complete we have $(H^n_{\mathfrak{m}}(M))^{\vee\vee} \cong H^n_{\mathfrak{m}}(M)$ so that $(H^n_{\mathfrak{m}}(M))^\vee$ satisfies condition (i) of 10.3.2.)

(b) In general, $(H^n_{\mathfrak{m}}(M))^\vee$ is the canonical module of $\hat{M}^{\mathfrak{m}}$:

$$(H^n_{\mathfrak{m}}(M))^\vee \cong K_{\hat{M}^{\mathfrak{m}}}.$$

(In view of the isomorphism $(H^n_{\mathfrak{m}}(M))^\vee \cong \operatorname{Hom}_{\hat{R}^{\mathfrak{m}}}(H^n_{\mathfrak{m}\hat{R}^{\mathfrak{m}}}(\hat{M}^{\mathfrak{m}}), E)$ this follows by (a).)

(c) If M admits a canonical module K_M, then $K_M \otimes_R \hat{R}^{\mathfrak{m}} \cong K_{\hat{M}^{\mathfrak{m}}}$. (Since K_M satisfies condition (ii) of 10.3.2 this also follows by (a).)

(d) More precisely, let C be denote an R-module. Then $C \cong K_M$ if and only if $C \otimes_R \hat{R}^{\mathfrak{m}} \cong K_{\hat{M}^{\mathfrak{m}}}$. (Note that $\operatorname{Hom}_R(C, E) \cong \operatorname{Hom}_{\hat{R}^{\mathfrak{m}}}(C \otimes_R \hat{R}^{\mathfrak{m}}, E)$. If $C \otimes_R \hat{R}^{\mathfrak{m}} \cong K_{\hat{M}^{\mathfrak{m}}}$, then

$$\operatorname{Hom}_R(C, E) \cong \operatorname{Hom}_{\hat{R}^{\mathfrak{m}}}(K_{\hat{M}^{\mathfrak{m}}}, E) \cong H^n_{\mathfrak{m}\hat{R}^{\mathfrak{m}}}(\hat{M}^{\mathfrak{m}}) \cong H^n_{\mathfrak{m}}(M).$$

Hence C satisfies condition (i) of 10.3.2 and $C \cong K_M$. This together with (c) proves the claim.)

Remark 10.3.5 Let $(R, \mathfrak{m}, \Bbbk)$ be a Noetherian local ring of dimension d and M a finitely generated R-module of dimension n. We have seen in 9.2.5 that $(H^n_{\mathfrak{m}}(M))^\vee \cong \Lambda^{\mathfrak{m}}_n(M^\vee)$. Hence $\Lambda^{\mathfrak{m}}_n(M^\vee) \cong K_{\hat{M}^{\mathfrak{m}}}$ (in view of 10.3.4 (b)).

In particular, $(H^d_{\mathfrak{m}}(R))^\vee \cong \Lambda^{\mathfrak{m}}_d(E) \cong K_{\hat{R}^{\mathfrak{m}}}$.

Remark 10.3.6 For a more detailed study and basic properties of canonical modules we refer to [74]. In particular, we have $\operatorname{Ass}_R K_M \subseteq \operatorname{Ass}_R M$ and $\dim_R M = \dim_R K_M$. If $\dim_R M > 0$ then also $\mathfrak{m} \notin \operatorname{Ass}_R K_M$ (see A.1.1).

When R is a homomorphic image of a Gorenstein local ring, note also that every finitely generated R-module M has a canonical module, an explicit description of which will be given in 10.6.3. In that case, K_M satisfies the Serre condition S_2.

(See also the forthcoming 12.2.4 for another explicit description of K_M when R admits a dualizing complex.)

We now turn to the canonical module of a Cohen–Macaulay local ring, when it exists. For the next result see also [16, Section 3.3].

Proposition 10.3.7 *Let $(R, \mathfrak{m}, \Bbbk)$ be a Cohen–Macaulay local ring of dimension d and assume that R admits the canonical module K_R. Let also $\underline{x} = x_1, \ldots, x_d$ be a parameter system of R. Then K_R has the following properties:*

(a) $(K_R)^\vee \cong H^d(\check{C}_{\underline{x}})$ *and K_R is a finitely generated R-module,*
(b) $\operatorname{id}_R(K_R) = d$,
(c) $\operatorname{Ext}^d_R(\Bbbk, K_R) \cong \Bbbk$ *and* $\operatorname{Ext}^i_R(\Bbbk, K_R) = 0$ *for all $i \neq d$,*
(d) K_R is a maximal Cohen–Macaulay R-module, that is every system of parameters of R forms a regular sequence on K_R,
(e) $\operatorname{Hom}_R(K_R, K_R) \cong R$,
(f) $\operatorname{Ext}^i_R(K_R, K_R) = 0$ *for all $i \neq 0$,*
(g) $H^d_{\mathfrak{m}}(K_R) \cong E_R(\Bbbk)$.

Proof For (a) recall that $H^d(\check{C}_{\underline{x}}) \cong H^d_{\mathfrak{m}}(R)$ (see 7.4.5).

We obtain the statements in (b), (c) and (d) via the observations in 10.2.3. More precisely, we have $\operatorname{id}_R(K_R) = \operatorname{fd}_R(K_R^\vee) = d$ and $\operatorname{E-dp}(\mathfrak{m}, K_R) = \operatorname{T-codp}(\mathfrak{m}, K_R^\vee) = d$ in view of the observations in 10.2.3 (recall that the injective and flat dimensions, the Ext-depth and Tor-codepth, are interchanged under Matlis duality). Because K_R is finitely generated with $\operatorname{E-dp}(\mathfrak{m}, K_R) = d$. It also follows that K_R is a maximal Cohen–Macaulay R-module.

We first prove (e) in the case R is $\mathfrak{m}$-complete. Then we have the isomorphisms and a bounded injective resolution

$$K_R \cong (K_R)^{\vee\vee} \cong (H^d(\check{C}_{\underline{x}}))^\vee \xrightarrow{\sim} \check{C}_{\underline{x}}^{\vee[d]}$$

(see 10.2.3). It follows that

$$\operatorname{Hom}_R(K_R, K_R) \cong H^0(\operatorname{Hom}_R(K_R, \check{C}_{\underline{x}}^{\vee[d]})) \cong H^0(\operatorname{Hom}_R(\check{C}_{\underline{x}}^{\vee[d]}, \check{C}_{\underline{x}}^{\vee[d]}))$$

and the result follows by Theorem 10.2.6.

In general, and because $K_R \otimes_R \hat{R}^{\mathfrak{m}} \cong K_{\hat{R}^{\mathfrak{m}}}$ (see 10.3.4), we have the isomorphisms

$$\begin{aligned}\operatorname{Hom}_R(K_R, K_R) \otimes_R \hat{R}^{\mathfrak{m}} &\cong \operatorname{Hom}_{\hat{R}^{\mathfrak{m}}}(K_R \otimes_R \hat{R}^{\mathfrak{m}}, K_R \otimes_R \hat{R}^{\mathfrak{m}}) \\ &\cong \operatorname{Hom}_{\hat{R}^{\mathfrak{m}}}(K_{\hat{R}^{\mathfrak{m}}}, K_{\hat{R}^{\mathfrak{m}}}).\end{aligned}$$

As we already know that $\operatorname{Hom}_R(K_{\hat{R}^{\mathfrak{m}}}, K_{\hat{R}^{\mathfrak{m}}}) \cong \hat{R}^{\mathfrak{m}}$ the statement in (e) for an arbitrary Cohen–Macaulay local ring now follows by Lemma 10.3.3.

Statement (f) is a consequence of (d) and Proposition 10.1.3.

Statement (g) follows from (c). Recall that $\dim_k(\operatorname{Ext}^i_R(\Bbbk, K_R)) = \mu_i(\mathfrak{m}, K_R)$ (see 4.5.16) and the description of the minimal injective resolution of K_R given in Theorem 4.5.18. □

In the next chapter we shall see that properties (b), (c) and (d) of the above proposition provides a characterization of the canonical module of a Cohen–Macaulay local ring (see the forthcoming 12.2.6).

In the next section, we shall see that the canonical module of a Cohen–Macaulay local ring, when it exists, has nice dualizing properties (see the forthcoming Theorem 10.4.3).

10.4 Local Duality over Cohen–Macaulay Local Rings

Here we apply the results and observations of the previous sections to the local homology over a Cohen–Macaulay local ring.

Recalls 10.4.1 Let $\underline{x} = x_1, \ldots, x_d$ be any system of parameters of a Noetherian local ring $(R, \mathfrak{m}, \Bbbk)$, $d = \dim R$.

(a) Recall that $H^i_{\mathfrak{m}}(X) \cong H^i(\check{C}_{\underline{x}} \otimes_R X)$ for all R-complexes X (see 7.4.5). In particular, we have $H^d(\check{C}_{\underline{x}}) \cong H^d_{\mathfrak{m}}(R)$.

(b) If R is Cohen–Macaulay, then the sequence $\underline{x}$ is regular, hence completely secant, and $\operatorname{fd}_R(H^d_{\mathfrak{m}}(R)) = d = \operatorname{id}_R(H^d_{\mathfrak{m}}(R)^\vee)$ (see 10.2.3). Note that here $(\cdot)^\vee = \operatorname{Hom}_R(\cdot, E_R(\Bbbk))$ denotes, as before, the Matlis duality functor.

By the above recalls together with the other observations in 10.2.3 we obtain a first local duality result for Cohen–Macaulay local rings.

Theorem 10.4.2 *Let $(R, \mathfrak{m}, \Bbbk)$ be a Cohen–Macaulay local ring, $d = \dim R$. Then*

$$H^i_{\mathfrak{m}}(X) \cong \operatorname{Tor}^R_{d-i}(X, H^d_{\mathfrak{m}}(R)) \textit{ and } (H^i_{\mathfrak{m}}(X))^\vee \cong \operatorname{Ext}^{d-i}_R(X, (H^d_{\mathfrak{m}}(R))^\vee)$$

for all R-complexes X and all $i \in \mathbb{Z}$.

We now turn to the case where R has a canonical module K_R, that is a finitely generated R-module the Matlis dual of which is isomorphic to $H^d_{\mathfrak{m}}(R)$ (see 10.3.2). By the recalls in 10.4.1 (a) and Proposition 10.2.4 we obtain a second local duality for Cohen–Macaulay local rings.

Theorem 10.4.3 *Let* $(R, \mathfrak{m}, \Bbbk)$ *be a Cohen–Macaulay local ring,* $d = \dim R$*. Suppose that* R *admits the canonical module* K_R*. Then*

$$H^i_{\mathfrak{m}}(X) \cong (\operatorname{Ext}^{d-i}_R(X, K_R))^{\vee}$$

for any R*-complex* X *with finitely generated homology modules.*

The previous results are particular cases of duality theorems investigated in detail later. In the next section we will continue with the study of Gorenstein rings. In addition to various well-known assertions we use our previous results to shed some new light on the ubiquity of Gorenstein rings.

10.5 On Gorenstein Local Rings and Duality

Gorenstein rings play a central role in local duality and form an important class of rings in various fields of commutative algebra and algebraic geometry. For that reason we shall add here a few results about them. First some recalls. In his paper (see [9]) Bass gave a proof of the following characterization of Gorenstein local rings.

Theorem 10.5.1 (see [9]) *Let* $(R, \mathfrak{m}, \Bbbk)$ *denote a Noetherian local ring and* $d = \dim R$*. Let* $E = E_R(\Bbbk)$ *be the injective hull of the residue field. Then the following conditions are equivalent:*

- *(i)* R *is a Gorenstein local ring.*
- *(ii)* $\operatorname{Ext}^i_R(\Bbbk, R) = 0$ *for all* $i \neq d$ *and* $\operatorname{Ext}^d_R(\Bbbk, R) \cong \Bbbk$.
- *(iii)* R *is a Cohen–Macaulay local ring and* $\dim_{\Bbbk} \operatorname{Ext}^d_R(\Bbbk, R) = 1$.
- *(iv)* R *is a Cohen–Macaulay local ring and* $\dim_{\Bbbk} \operatorname{Hom}_R(\Bbbk, R/\underline{x}R) = 1$ *for any system of parameters* $\underline{x} = x_1, \ldots, x_d$.
- *(v)* $H^i_{\mathfrak{m}}(R) = 0$ *for all* $i \neq d$ *and* $H^d_{\mathfrak{m}}(R) \cong E$.

Remarks 10.5.2 Let $(R, \mathfrak{m}, \Bbbk)$ be a Noetherian local ring.

(a) If R is Gorenstein then $\operatorname{id}_R R = \operatorname{ext}^+_R(\Bbbk, R) = \dim R$ in view of 10.1.1. Moreover, R is its canonical module as defined in 10.3.2 because $H^d_{\mathfrak{m}}(R) \cong E \cong R^{\vee}$.

(b) Note that $\operatorname{Ext}^i_R(\Bbbk, R) \cong \operatorname{Ext}^i_{\hat{R}^{\mathfrak{m}}}(\Bbbk, \hat{R}^{\mathfrak{m}})$. It follows that R is Gorenstein if and only if $\hat{R}^{\mathfrak{m}}$ is Gorenstein.

(c) Let $\underline{x} = x_1, \ldots, x_k$ be a regular sequence. We note also with [9] that R is Gorenstein if and only if $R/\underline{x}R$ is Gorenstein.

In the following we shall translate the characterization of Gorenstein rings given in 10.5.1 in terms of the injective hull of the residue field.

Proposition 10.5.3 *Let* $(R, \mathfrak{m}, \Bbbk)$ *denote a Noetherian local ring,* $d = \dim R$*, and write* E *for the injective hull of the residue field. Then the following conditions are equivalent:*

(i) *R is a Gorenstein ring.*
(ii) $\mathrm{id}_R\, R < \infty$.
(iii) $\mathrm{fd}_R\, E < \infty$.
(iv) $\mathrm{pd}_R\, E < \infty$.

Proof The equivalence of (i) and (ii) was proved in [9]. The equivalence of (ii) and (iii) is clear, recall that flat dimensions and injective dimensions are interchanged under Matlis duality. Since (iv) $\Longrightarrow$ (iii) holds trivially it remains to show that (i) $\Longrightarrow$ (iv).

Suppose R is Gorenstein and let $\underline{x} = x_1, \ldots, x_d$ be a system of parameters. Then $\check{C}_{\underline{x}}$ provides, up to a shift, a flat resolution of $H^d_{\mathfrak{m}}(R)$ (see 10.2.3) and $\check{L}^{[-d]}_{\underline{x}} \xrightarrow{\sim} H^d_{\mathfrak{m}}(R)$ is a projective resolution of $E \cong H^d_{\mathfrak{m}}(R)$. This finishes the proof. □

10.5.4 From the above characterization it follows that $R_{\mathfrak{p}}$ is Gorenstein for all $\mathfrak{p} \in \operatorname{Spec}(R)$ when R is local Gorenstein. Moreover, if $R \xrightarrow{\sim} I$ is a minimal injective resolution, it also follows that $I^r \cong \oplus_{\{\mathfrak{p}\in \operatorname{Spec} R | \operatorname{height}(\mathfrak{p})=r\}} E_R(R/\mathfrak{p})$.

Here is a first way to compute local cohomology over a Gorenstein local ring.

Theorem 10.5.5 *Let $(R, \mathfrak{m}, \Bbbk)$ be a Gorenstein local ring, $d = \dim R$, and write E for the injective hull of $\Bbbk$. Then*

$$H^i_{\mathfrak{m}}(X) \cong \operatorname{Tor}^R_{d-i}(X, E) \text{ and } (H^i_{\mathfrak{m}}(X))^{\vee} \cong \operatorname{Ext}^{d-i}_R(X, \hat{R}^{\mathfrak{m}})$$

for any R-complex X and all $i \in \mathbb{Z}$.

Proof Because $H^d_{\mathfrak{m}}(R) \cong E$ this is a direct consequence of 10.4.2. Recall also that $E^{\vee} \cong \hat{R}^{\mathfrak{m}}$. □

This has some interesting consequences concerning special types of R-modules.

Corollary 10.5.6 *Let $(R, \mathfrak{m}, \Bbbk)$ be a Gorenstein local ring, $d = \dim R$ and $E = E_R(\Bbbk)$. Let M denote an R-module. We have the following:*

(a) *M is $\mathfrak{m}$-torsion if and only if $\operatorname{Tor}^R_d(E, M) \cong M$. If the R-module M is $\mathfrak{m}$-torsion, then also $\operatorname{Tor}^R_i(E, M) = 0$ for all $i \neq d$.*
(b) *If $M = I$ is an injective R-module then $\operatorname{Tor}^R_i(E, I) = 0$ for $i \neq d$ and $\operatorname{Tor}^R_d(E, I) \cong \Gamma_{\mathfrak{m}}(I)$.*
(c) *If $M = F$ is a flat R-module then $H^i_{\mathfrak{m}}(F) = 0$ for $i \neq d$ and $H^d_{\mathfrak{m}}(F) \cong E \otimes_R F$.*

Proof This is clear in view of the isomorphism $H^i_{\mathfrak{m}}(M) \cong \operatorname{Tor}^R_{d-i}(M, E)$ obtained in 10.5.5. Note also if the R-module M is $\mathfrak{m}$-torsion we of course have $M = \Gamma_{\mathfrak{m}}(M) \cong H^0_{\mathfrak{m}}(M)$ but we also have $H^i_{\mathfrak{m}}(M) = 0$ for all $i \neq 0$ (see 7.4.7). □

Here is a Grothendieck local duality theorem for Gorenstein local rings and unbounded complexes.

Theorem 10.5.7 *Let $(R, \mathfrak{m}, \Bbbk)$ be a Gorenstein local ring, $d = \dim R$. Then*

$$H^i_{\mathfrak{m}}(X)) \cong (\operatorname{Ext}^{d-i}_R(X, R))^\vee$$

for all $i \in \mathbb{Z}$ and any R-complex X with finitely generated cohomology modules $H_i(X)$, $i \in \mathbb{Z}$.

Proof Because $H^d_{\mathfrak{m}}(R) \cong E \cong R^\vee$, so that $R \cong K_R$, the statement is a reformulation of 10.4.3 in this case. □

In the following criteria we also have information about the Betti and Bass numbers of R and E, respectively.

Proposition 10.5.8 *Let $(R, \mathfrak{m}, \Bbbk)$ denote a Noetherian local ring and write E for the injective hull of the residue field. Let $d = \dim R$. Then the following conditions are equivalent:*

(i) *R is a Gorenstein ring.*
(ii) $\operatorname{Ext}^i_R(\Bbbk, R) = 0$ *for all $i \neq d$ and* $\operatorname{Ext}^d_R(\Bbbk, R) \cong \Bbbk$.
(iii) $\operatorname{Tor}^R_i(\Bbbk, E) = 0$ *for all $i \neq d$ and* $\operatorname{Tor}^R_d(\Bbbk, E) \cong \Bbbk$.
(iv) $\operatorname{Ext}^i_R(E, \Bbbk) = 0$ *for all $i \neq d$ and* $\operatorname{Ext}^d_R(E, \Bbbk) \cong \Bbbk$.

Proof The equivalence of (i) and (ii) has been shown by H. Bass (see 10.5.1).

By the Ext-Tor duality we have $\operatorname{Tor}^R_i(\Bbbk, E) = \operatorname{Tor}^R_i(\Bbbk, R^\vee) \cong \operatorname{Ext}^i_R(\Bbbk, R)^\vee$ (see 1.4.8). As $\Bbbk^\vee \cong \Bbbk$, the equivalence of (ii) and (iii) follows.

We also have $\operatorname{Ext}^i_R(E, \Bbbk) \cong \operatorname{Ext}^i_R(E, \Bbbk^\vee) \cong \operatorname{Tor}^R_i(E, \Bbbk)^\vee$ (see 1.4.8 again). Then the equivalence of (iii) and (iv) follows. □

As a consequence of our previous investigations there is another characterization of Gorenstein local rings in terms of local homology.

Theorem 10.5.9 *Let $(R, \mathfrak{m}, \Bbbk)$ be a Noetherian local ring, $d = \dim R$ and $E = E_R(\Bbbk)$. The following conditions are equivalent*

(i) *R is Gorenstein,*
(ii) $\Lambda^{\mathfrak{m}}_i(X) \cong \operatorname{Ext}^{d-i}_R(E, X)$ *for all $i \in \mathbb{Z}$ and any R-complex X,*
(iii) $\Lambda^{\mathfrak{m}}_i(E) = 0$ *for all $i \neq d$ and* $\Lambda^{\mathfrak{m}}_d(E) \cong \hat{R}^{\mathfrak{m}}$.

Proof Let $\underline{x} = x_1, \ldots, x_d$ be a parameter system of R. In 7.5.12 we have seen that $\mathrm{L}\Lambda^{\mathfrak{m}}(\mathrm{X})$ and $\operatorname{RHom}_R(\check{C}_{\underline{x}}, X)$ are isomorphic in the derived category. Hence $\Lambda^{\mathfrak{m}}_i(X) \cong \operatorname{Ext}^{-i}_R(\check{C}_{\underline{x}}, X)$. When R is Gorenstein we have a quasi-isomorphism and isomorphisms $(\check{C}_{\underline{x}})_{[d]} \xrightarrow{\sim} H^d(\check{C}_{\underline{x}}) \cong H^d_{\mathfrak{m}}(R) \cong E$ (see 10.2.3 and 7.4.5). It follows that $\operatorname{Ext}^{-i}_R(\check{C}_{\underline{x}}, X) \cong \operatorname{Ext}^{d-i}_R(E, X)$ by the definition of Ext. Hence condition (ii) is satisfied.

Assume condition (ii) is satisfied and apply it to $\Bbbk$. We obtain $\Lambda^{\mathfrak{m}}_i(\Bbbk) \cong \operatorname{Ext}^{d-i}_R(E, \Bbbk)$. But $\Lambda^{\mathfrak{m}}_i(\Bbbk) = 0$ for all $i \neq 0$ and $\Lambda^{\mathfrak{m}}_0(\Bbbk) = \Bbbk$ (this is a very particular case of 2.5.15). The above together with 10.5.8 implies that R is Gorenstein.

In general, we have $\Lambda_i^{\mathfrak{m}}(E) = \Lambda_i^{\mathfrak{m}}(R^\vee) \cong (H^i_{\mathfrak{m}}(R))^\vee)$ (see 9.2.5). When R is Gorenstein it follows that conditions (iii) are satisfied. When conditions (iii) are satisfied we have that $H^i_{\mathfrak{m}}(R) = 0$ for $i \neq d$ and that $(H^d_{\mathfrak{m}}(R))^\vee \cong \hat{R}^{\mathfrak{m}}$. But $(H^i_{\mathfrak{m}}(R))^\vee = \operatorname{Hom}_R(H^i_{\mathfrak{m}}(R), E) \cong \operatorname{Hom}_{\hat{R}^{\mathfrak{m}}}(H^i_{\mathfrak{m}\hat{R}^{\mathfrak{m}}}(\hat{R}^{\mathfrak{m}}), E))$. We take the Matlis dual over $\hat{R}^{\mathfrak{m}}$ and obtain that $H^d_{\mathfrak{m}\hat{R}^{\mathfrak{m}}}(\hat{R}^{\mathfrak{m}}) = E$ because $H^d_{\mathfrak{m}\hat{R}^{\mathfrak{m}}}(\hat{R}^{\mathfrak{m}})$ is Artinian. Hence the conditions in (iii) imply that $\hat{R}^{\mathfrak{m}}$ is Gorenstein and so is R. □

As above there are several consequences of Theorem 10.5.9 for modules of particular types.

Corollary 10.5.10 *Let* $(R, \mathfrak{m}, \Bbbk)$ *be a Gorenstein local ring,* $d = \dim R$ *and* E *the injective hull of the residue field* $\Bbbk$*. Let* M *denote an* R*-module.*

(a) M *is* $\mathfrak{m}$*-pseudo-complete if and only if* $\operatorname{Ext}^i_R(E, M) = 0$ *for* $i \neq d$ *and* $\operatorname{Ext}^d_R(E, M) \cong M$.
(b) If M *is finitely generated or* $\mathfrak{m}$*-complete then* $\operatorname{Ext}^i_R(E, M) = 0$ *for* $i \neq d$ *and* $\operatorname{Ext}^d_R(E, M) \cong \hat{M}^{\mathfrak{m}}$.
(c) If $M = F$ *is flat then* $\operatorname{Ext}^i_R(E, F) = 0$ *for* $i \neq d$ *and* $\operatorname{Ext}^d_R(E, F) \cong \hat{F}^{\mathfrak{m}}$.
(d) If $M = I$ *is injective then* $\Lambda_i^{\mathfrak{m}}(I) = 0$ *for* $i \neq d$ *and* $\Lambda_d^{\mathfrak{m}}(I) \cong \operatorname{Hom}_R(E, I)$.

Proof The statements are true in view of the isomorphisms $\Lambda_i^{\mathfrak{m}}(M) \cong \operatorname{Ext}^{d-i}_R(E, M)$ obtained in 10.5.9. For (a) we also recall 2.5.15 and 2.5.16. For (b) and (c) remember that if M is flat or finitely generated or $\mathfrak{m}$-complete then $\Lambda_i^{\mathfrak{m}}(M) = 0$ for all $i \neq 0$ and $\Lambda_0^{\mathfrak{m}}(M) \cong \hat{M}^{\mathfrak{m}}$ (see 2.5.12, 2.5.14 and 2.5.15 respectively). Statement (d) is a direct consequence of the above isomorphisms. □

Remark 10.5.11 Let R denote a commutative ring and $\mathfrak{a} \subset R$ an ideal generated by a weakly pro-regular sequence $\underline{x} = x_1, \ldots, x_k$. Let X be an R-complex. By 9.2.4 there is an isomorphism

$$\mathrm{L}\Lambda^{\mathfrak{a}}(X) \simeq \mathrm{RHom}_R(\mathrm{R}\Gamma_{\mathfrak{a}}(R), X)$$

in the derived category.

If $(R, \mathfrak{m}, \Bbbk)$ is a d-dimensional Gorenstein ring and if $\mathfrak{a}$ is an $\mathfrak{m}$-primary ideal we note that $\mathrm{R}\Gamma_{\mathfrak{a}}(R) = \mathrm{R}\Gamma_{\mathfrak{m}}(R)$ is represented by $E^{[-d]}$. Hence the isomorphisms in 10.5.9 (ii) may also be obtained by taking homology on each side of the above isomorphism.

There are several other consequences of the above isomorphism, e.g. for a d-dimensional Cohen–Macaulay local ring $\mathrm{R}\Gamma_{\mathfrak{m}}(R)$ is represented by $H^d_{\mathfrak{m}}(R)^{[-d]}$.

Remark 10.5.12 Here is a slight generalization of (10.5.9 (i) ⇒ (ii)).

Let $f : (S, \mathfrak{n}, \mathbb{l}) \to (R, \mathfrak{m}, \Bbbk)$ be a local homomorphism of Noetherian local rings, where S is Gorenstein of dimension d and such that $\mathfrak{m} = \operatorname{Rad}(f(\mathfrak{n}))$. Let

X denote an R-complex which may also be viewed as an S-complex via f. Then $\Lambda_i^{\mathfrak{m}}(X) \cong \Lambda_i^{\mathfrak{n}}(X)$ for all $i \in \mathbb{Z}$ (see 9.8.1). In view of 10.5.9 we now have

$$\Lambda_i^{\mathfrak{m}}(X) \cong \operatorname{Ext}_S^{d-i}(E(S/\mathfrak{n}), X)$$

for all $i \in \mathbb{Z}$. In this case we may also compute the $\mathfrak{m}$-adic local homology by certain Ext-modules.

We may also consider non-local Gorenstein rings. Recall that a Noetherian ring R is called Gorenstein if its localizations $R_{\mathfrak{p}}$ are local Gorenstein for all $\mathfrak{p} \in \operatorname{Spec}(R)$ and note the following easy lemma.

Lemma 10.5.13 *Let R be a Noetherian ring of finite Krull dimension. Then* $\operatorname{id}_R R < \infty$ *if and only if R is Gorenstein.*

Proof Let $R \xrightarrow{\sim} I$ be the minimal injective resolution of R. Assume first that R is Gorenstein and assume $I^s \neq 0$ for some $s \in \mathbb{N}$. Then there is a prime ideal $\mathfrak{p}$ such that $E_R(R/\mathfrak{p})$ occurs as a direct summand of I^s, that is, such that $\mu_s(\mathfrak{p}, I) \neq 0$ (see 4.5.18). Let $\mathfrak{m}$ be a maximal ideal containing $\mathfrak{p}$. Then $\mu_{s+t}(\mathfrak{m}, I) \neq 0$ where $t = \operatorname{height}(\mathfrak{m}/\mathfrak{p})$ by an iterative use of 4.5.19. But $R_{\mathfrak{m}}$ is local Gorenstein, hence $s + t = \dim R_{\mathfrak{m}}$ (see 10.5.4) and $s \leq \dim R_{\mathfrak{m}} \leq \dim R$. It follows that $\operatorname{id}_R R < \infty$.

The converse is a direct consequence of Proposition 10.5.3 since $R_{\mathfrak{p}} \xrightarrow{\sim} I_{\mathfrak{p}}$ is an injective resolution of $R_{\mathfrak{p}}$ for all $\mathfrak{p} \in \operatorname{Spec}(R)$. $\square$

10.6 Local Cohomology over Finite Local Gorenstein Algebras

Local cohomology may also be computed in terms of Ext over a finite local Gorenstein algebra. First a general lemma.

Lemma 10.6.1 *Let $f : (S, \mathfrak{n}, \mathbb{l}) \to (R, \mathfrak{m}, \mathbb{k})$ be a local homomorphism of Noetherian local rings such that R is a finitely generated S-module. There is an isomorphism*

$$E_R(R/\mathfrak{m}) \cong \operatorname{Hom}_S(R, E_S(S/\mathfrak{n}))$$

for the injective hulls $E_S(S/\mathfrak{n})$ and $E_R(R/\mathfrak{m})$.

Proof Clearly $\operatorname{Hom}_S(R, E_S(S/\mathfrak{n}))$ is an injective R-module. Since R is finitely generated as an S-module $\operatorname{Hom}_S(R, E_S(S/\mathfrak{n}))$ is Artinian as an R-module and therefore isomorphic to $E_R(R/\mathfrak{m})^n$ for some $n \geq 1$ by Matlis' Structure Theory (see [56]). Moreover, we have the isomorphisms

$$\begin{aligned} \mathbb{k}^n &\cong \operatorname{Hom}_R(\mathbb{k}, \operatorname{Hom}_S(R, E_S(\mathbb{l}))) \cong \operatorname{Hom}_S(\mathbb{k}, E_S(\mathbb{l})) \\ &\cong \operatorname{Hom}_{\mathbb{l}}(\mathbb{k}, \operatorname{Hom}_S(\mathbb{l}, E_S(\mathbb{l}))). \end{aligned}$$

The last module in this sequence is isomorphic to $\mathrm{Hom}_{\mathbb{l}}(\mathbb{k}, \mathbb{l}) \cong \mathbb{k}$. That is, $n = 1$, as required. □

The following is an extension of Theorem 10.5.7.

Theorem 10.6.2 *Let $f : (S, \mathfrak{n}, \mathbb{l}) \to (R, \mathfrak{m}, \mathbb{k})$ be a local homomorphism of Noetherian local rings such that R is a finitely generated S-module and S is Gorenstein. Then, for all $i \geq 0$ there are natural homomorphisms*

$$H^i_{\mathfrak{m}}(X) \cong \mathrm{Hom}_R(\mathrm{Ext}^{d-i}_S(X, S), E_R(\mathbb{k})), \; d = \dim S,$$

for any R-complex X with finitely generated cohomology modules.

Proof We note that $f(\mathfrak{n})$ is an $\mathfrak{m}$-primary ideal. In view of 9.8.1 there are isomorphisms $H^i_{\mathfrak{m}}(X) \cong H^i_{f(\mathfrak{n})}(X) \cong H^i_{\mathfrak{n}}(X)$. By 10.5.7 we therefore have

$$\begin{aligned} H^i_{\mathfrak{m}}(X) &\cong \mathrm{Hom}_S(\mathrm{Ext}^{d-i}_S(X, S), E_S(\mathbb{l})) \\ &\cong \mathrm{Hom}_R(\mathrm{Ext}^{d-i}_S(X, S), \mathrm{Hom}_S(R, E_S(\mathbb{l}))). \end{aligned}$$

The result now follows by 10.6.1. □

Corollary 10.6.3 *If the Noetherian local ring R is a homomorphic image of a Gorenstein local ring $(S, \mathfrak{n})$, then every finitely generated R-module M has a canonical module, namely*

$$K_M \cong \mathrm{Ext}^{d-n}_S(M, S), \quad \text{where } d = \dim S \text{ and } n = \dim_R M.$$

If $\underline{x} = x_1, \ldots, x_g$ is a maximal regular sequence in $\mathrm{Ann}_S M$ then $K_M \cong \mathrm{Hom}_S(M, S/\underline{x}S)$ and K_M satisfies the Serre condition S_2. In particular, if $n = d$, then $K_M \cong \mathrm{Hom}_S(M, S)$.

Proof For the first assertion note that $\mathrm{Ext}^{d-n}_S(M, S)$ satisfies condition (i) of 10.3.2. Note also that $d - n = \mathrm{ext}^-_S(M, S)$ because $H^i_{\mathfrak{n}}(M) = 0$ for $i > n$. Then the last assertions follow. Note that one might also consider M as an $S/\underline{x}S$-module. □

Remark 10.6.4 If the Cohen–Macaulay local ring R is a homomorphic image of a Gorenstein local ring S, it follows by the previous corollary that R has a canonical module, namely $K_R = \mathrm{Ext}^{d-n}_S(R, S)$, where $d = \dim S$ and $n = \dim R$.

Conversely, if the Cohen–Macaulay local ring $(R, \mathfrak{m})$ has the canonical module K_R, it is known that R is a homomorphic image of a Gorenstein local ring (see [29], [28] and [68]): the trivial extension $R \ltimes K_R$ does the trick (see 2.8.5 for the definition).

Namely, if R is Cohen–Macaulay, then K_R is also Cohen–Macaulay of the same dimension. Moreover, $R \ltimes K_R$ is a Noetherian local ring with maximal ideal $\mathfrak{m} \ltimes K_R$ and of dimension $\dim R \ltimes K_R = \dim R =: d$. Then it follows that a system of parameters $\underline{x} = x_1, \ldots, x_d$ of R is $R \ltimes K_R$-regular and

$$(R \ltimes K_R)/\underline{x}(R \ltimes K_R) \cong (R/\underline{x}R) \ltimes (K_R/\underline{x}K_R).$$

Moreover, recall that $K_R/\underline{x}K_R \cong K_{R/\underline{x}R}$.

Therefore $R \ltimes K_R$ is a d-dimensional local Cohen–Macaulay ring. In order to show that it is a Gorenstein ring it will be enough to show that $A \ltimes K_A$, $A = R/\underline{x}R$, has a one-dimensional socle. To this end first note that $K_A \cong E_A(\Bbbk) =: E$ and $\operatorname{Ann}_A E = 0$. Now let $(a, e) \in A \ltimes E$. Then $(a, e) \in \operatorname{socle}(A \ltimes E)$ if and only if $aE = 0$ (and therefore $a = 0$) and $e \in \operatorname{socle} E$. Because $\dim_{\Bbbk} \operatorname{socle}(E) = 1$ this proves that $R \ltimes K_R$ is a Gorenstein ring. Moreover, $R \cong (R \ltimes K_R)/(0 \ltimes K_R)$ for the ideal $0 \ltimes K_R \subset R \ltimes K_R$.

Furthermore, we note the following extension of 10.5.5

Theorem 10.6.5 *Let $f : (S, \mathfrak{n}, \mathbb{l}) \to (R, \mathfrak{m}, \Bbbk)$ be a local homomorphism of Noetherian local rings such that R is a finitely generated S-module and S is Gorenstein. Then, for all $i \geq 0$ there are natural homomorphisms*

$$\operatorname{Hom}_R(H^i_{\mathfrak{m}}(X), E_R(\Bbbk)) \cong \operatorname{Ext}_S^{d-i}(X, \hat{S}^{\mathfrak{n}}),\ d = \dim S,$$

for any R-complex X.

Chapter 11
Dualizing Complexes

In this chapter we provide a short and self-contained approach to the notion of a dualizing complex for Noetherian rings, to be used in the next chapter. Most of the results are not new, but some proofs are. In particular, we provide a proof of the existence of a dualizing complex for a complete Noetherian local ring independent of the Cohen structure theorem. This is part of an interesting interaction between the notion of a dualizing complex for a Noetherian ring and the notion of a Čech complex. This interaction also appears when we consider a change of rings of the form $R \to \hat{R}^{\mathfrak{a}}$. In that case, given a dualizing complex of a Noetherian ring R, we provide an explicit construction of a dualizing complex for $\hat{R}^{\mathfrak{a}}$ involving the Čech complex built on a generating set of $\mathfrak{a}$. In the last section we provide some new properties of dualizing complexes related to the completion functor, a recurrent theme in this monograph.

We start with some recalls on evaluation morphisms, which play an important rôle in the subject. For some further aspects, see also the recent textbook [46].

11.1 Evaluation Morphisms of Complexes

In the following we recall a few canonical isomorphisms of complexes. They are useful and used in several places. To this end we recall the definition of $\operatorname{Hom}_R(X, Y)$ and $X \otimes_R Y$ for two R-complexes as given in 1.1.5.

11.1.1 *Tensor evaluation.* Let X, Y, Z denote three R-modules. Then we have the tensor evaluation

$$\operatorname{Hom}_R(X, Y) \otimes_R Z \to \operatorname{Hom}_R(X, Y \otimes_R Z), \quad \phi \otimes z \mapsto [x \mapsto \phi(x) \otimes z],$$

P. Schenzel and A.-M. Simon, *Completion, Čech and Local Homology and Cohomology*, Springer Monographs in Mathematics,
https://doi.org/10.1007/978-3-319-96517-8_11

natural in each of the variables. If R is Noetherian, X finitely generated and Z flat, we note that this homomorphism is an isomorphism (because it is so for $X = R^n$). This extends to a corresponding result for complexes.

Lemma 11.1.2 *Let X, Y, Z be three R-complexes. Then there is a morphism*

$$\mathrm{Hom}_R(X, Y) \otimes_R Z \to \mathrm{Hom}_R(X, Y \otimes_R Z)$$

which is natural in X, Y, and Z. This tensor evaluation is an isomorphism under each of the following conditions:

(a) X is right-bounded and degree-wise finitely generated, Y is left-bounded and Z is bounded, provided X is a complex of projective R-modules, or Z is a complex of flat R-modules and R is Noetherian.
(b) R is Noetherian and X is bounded degree-wise finitely generated, provided X is a complex of projective R-modules, or Z is a complex of flat R-modules.

Proof This follows by 1.1.9 and the remark in 11.1.1. With the boundedness conditions in (a) or in (b) note also that $\mathrm{Hom}_R(X, Y)^n = \prod_{i+j=n} \mathrm{Hom}_R(X_i, Y^j)$ is a finite product of the $\mathrm{Hom}_R(X_i, Y^j)$'s and that all the products involved in the description of $(\mathrm{Hom}_R(X, Y \otimes_R Z))^n$ are also finite. For more details, we refer to [30, Proposition 1.1] or to the more detailed exposition in [32]. □

11.1.3 *Homomorphism evaluation.* For three R-modules X, Y, Z we have the homomorphism evaluation

$$X \otimes_R \mathrm{Hom}_R(Y, Z) \to \mathrm{Hom}_R(\mathrm{Hom}_R(X, Y), Z), \quad x \otimes \psi \mapsto [\theta \mapsto (\psi \circ \theta)(x)],$$

natural in each of the variables. If R is Noetherian, X finitely generated and Z injective, we note that this homomorphism is an isomorphism (see 1.4.1). There is an extension to the case of complexes.

Lemma 11.1.4 *Let X, Y, Z denote three R-complexes. Then there is a morphism*

$$X \otimes_R \mathrm{Hom}_R(Y, Z) \to \mathrm{Hom}_R(\mathrm{Hom}_R(X, Y), Z)$$

which is natural in X, Y, and Z. This homomorphism evaluation is an isomorphism under each of the following conditions:

(a) X is right-bounded and degree-wise finitely generated, Y is left-bounded and Z is bounded, provided X is a complex of projective R-modules, or Z is a complex of injective R-modules and R is Noetherian.
(b) R is Noetherian and X is bounded degree-wise finitely generated, provided X is a complex of projective R-modules, or Z is a complex of injective R-modules.

Proof This is a consequence of 1.1.9 and the remark in 11.1.3. With the boundedness conditions in (a) note also that $\mathrm{Hom}_R(Y, Z)_n = \prod_{i+j=n} \mathrm{Hom}_R(Y^i, Z_j)$ is a finite product of the $\mathrm{Hom}_R(Y^i, Z_j)$ and that all the products involved in the description of $\mathrm{Hom}_R(\mathrm{Hom}_R(X, Y), Z)_i$ are also finite. With the conditions in (b) recall that the $(X_i \otimes_R \cdot)$'s commute with products. For more details, we refer to [30, Proposition 1.1] or to the more detailed exposition in [32]. $\square$

The draft [32] is not yet available and – maybe – will not be completed. Another reference for the proofs of Lemmas 11.1.2 and 11.1.4 is [21, Appendix A.2].

As a first application of Lemma 11.1.4 note the following.

Proposition 11.1.5 *Let R be a Noetherian ring, X an R-complex with finitely generated cohomology modules and I, J two R-complexes of injective R-modules. Assume that I and J are bounded. Then the evaluation morphism*

$$X \otimes_R \mathrm{Hom}_R(I, J) \to \mathrm{Hom}_R(\mathrm{Hom}_R(X, I), J)$$

is a quasi-isomorphism.

Proof Assume first that X is right-bounded. Then X has a free resolution $L \xrightarrow{\sim} X$ where L is a right-bounded complex of finitely generated free R-modules. We have a commutative square

$$\begin{array}{ccc} L \otimes_R \mathrm{Hom}_R(I, J) & \longrightarrow & \mathrm{Hom}_R(\mathrm{Hom}_R(L, I), J) \\ \downarrow & & \downarrow \\ X \otimes_R \mathrm{Hom}_R(I, J) & \longrightarrow & \mathrm{Hom}_R(\mathrm{Hom}_R(X, I), J). \end{array}$$

In this square the vertical morphism on the left is a quasi-isomorphism because $\mathrm{Hom}_R(I, J)$ is a bounded complex of flat-R-modules and the vertical morphism on the right is a quasi-isomorphism because I and J are bounded complexes of injective-R-modules. As the top horizontal morphism is an isomorphism by 11.1.4 the conclusion follows.

For an unbounded complex X with finitely generated cohomology modules and a fixed $i \in \mathbb{Z}$ note that both modules $(X \otimes_R \mathrm{Hom}_R(I, J))^i$ and $(\mathrm{Hom}_R(\mathrm{Hom}_R(X, I), J))^i$ depend only on a soft truncation $\ldots \to X^n \to \mathrm{Ker}\, d_X^{n+1} \to 0$ for $n \gg 0$. $\square$

Remark 11.1.6 Let $(R, \mathfrak{m}, \Bbbk)$ be a Noetherian local ring, complete in its $\mathfrak{m}$-adic topology. Let X be an R-complex with finitely generated cohomology modules and let $\underline{x} = x_1, \ldots, x_k$ be a sequence in R. We may apply the above to the R-complex $\check{C}_{\underline{x}}^{\vee}$ in place of I ($\check{C}_{\underline{x}}^{\vee}$ is a bounded complex of injective R-modules). Recall the quasi-isomorphism $R \xrightarrow{\sim} \mathrm{Hom}_R(\check{C}_{\underline{x}}^{\vee}, \check{C}_{\underline{x}}^{\vee})$ obtained in Theorem 10.2.6. We note that it is a quasi-isomorphism between bounded flat complexes, hence induces a quasi-isomorphism $X \xrightarrow{\sim} X \otimes_R \mathrm{Hom}_R(\check{C}_{\underline{x}}^{\vee}, \check{C}_{\underline{x}}^{\vee})$. Combining this with Proposition 11.1.5 we obtain that the natural morphism

$$X \to \operatorname{Hom}_R(\operatorname{Hom}_R(X, \check{C}_{\underline{x}}^{\vee}), \check{C}_{\underline{x}}^{\vee})$$

is a quasi-isomorphism. In the case when $\underline{x}$ denotes the empty set we have $\check{C}_{\underline{x}} = R$ and $(\check{C}_{\underline{x}})^{\vee} = E_R(\Bbbk)$. Whence the above quasi-isomorphism is just Matlis duality.

11.2 Definition of Dualizing Complexes for Noetherian Rings

At the beginning let us recall the duality for finite-dimensional $\Bbbk$-vector spaces: the natural homomorphism

$$V \to \operatorname{Hom}_{\Bbbk}(\operatorname{Hom}_{\Bbbk}(V, \Bbbk), \Bbbk)$$

is an isomorphism for any finite-dimensional $\Bbbk$-vector space V. The basic idea of this section is a far reaching extension of this concept.

Let R be a commutative ring and Y an arbitrary R-complex. There is a natural morphism, the homothety morphism,

$$R \to \operatorname{Hom}_R(Y, Y), \ r \mapsto \mu_r,$$

where $\mu_r = \mu_r(Y)$ is the multiplication by r on Y. Let X be a second R-complex. Combining this with the evaluation morphism

$$X \otimes_R \operatorname{Hom}_R(Y, Y) \to \operatorname{Hom}_R(\operatorname{Hom}_R(X, Y), Y)$$

it provides a natural morphism of complexes

$$X \to \operatorname{Hom}_R(\operatorname{Hom}_R(X, Y), Y).$$

This leads to the following definition, motivated by the work of Grothendieck (see [43]):

Definition 11.2.1 Let R denote a Noetherian ring. A bounded complex of injective R-modules D with finitely generated cohomology modules is called a dualizing complex if the natural morphism

$$X \to \operatorname{Hom}_R(\operatorname{Hom}_R(X, D), D)$$

is a quasi-isomorphism for any R-complex X with finitely generated cohomology modules.

Remark 11.2.2 By 4.5.14 note that $\operatorname{Hom}_R(X, D)$ has finitely generated cohomology modules when X is an R-complex with finitely generated homology modules and D is a dualizing complex of a Noetherian ring R.

An interpretation of $\operatorname{Hom}_R(\operatorname{Hom}_R(X, D), D)$ for an arbitrary R-complex and a complete Noetherian local ring will be given in the forthcoming 12.3.5.

In a certain sense the above definition is related to the Matlis Duality over a Noetherian local ring $(R, \mathfrak{m}, \Bbbk)$ with respect to the injective hull $E_R(\Bbbk)$ of the residue field $\Bbbk$. The injective hull $E_R(\Bbbk)$ is finitely generated if and only if $(R, \mathfrak{m})$ is an Artinian ring. In this case $E_R(\Bbbk)$ is a dualizing complex since the natural homomorphism $M \to \operatorname{Hom}_R(\operatorname{Hom}_R(M, E_R(\Bbbk)), E_R(\Bbbk))$ is an isomorphism for any finitely generated R-module M. That is, the concept of a dualizing complex is an extension of these ideas to arbitrary rings.

The following result is an easy consequence of the definition 11.2.1.

Proposition 11.2.3 *Let R denote a Noetherian ring and let D denote a bounded complex of injective R-modules with finitely generated cohomology modules. Then the complex D is a dualizing complex if and only if the homothety morphism $R \to \operatorname{Hom}_R(D, D)$ is a quasi-isomorphism.*

Proof If D is a dualizing complex put $X = R$ in the definition. For the converse we have the composition of morphisms

$$X \to X \otimes_R \operatorname{Hom}_R(D, D) \to \operatorname{Hom}_R(\operatorname{Hom}_R(X, D), D).$$

We have to show that they are quasi-isomorphisms when X has finitely generated cohomology modules. The first one is clear since by assumption $R \to \operatorname{Hom}_R(D, D)$ is a quasi-isomorphism of bounded complexes of flat R-modules. The second one is a quasi-isomorphism by Proposition 11.1.5. □

Here is a "generic" example.

Lemma 11.2.4 *Let R be a commutative Noetherian ring with a finite injective resolution I. Then I is a dualizing complex.*

In particular, a Gorenstein ring of finite Krull dimension has a dualizing complex.

Proof The injective resolution I of R is by assumption a bounded complex of injective R-modules with finitely generated cohomology. It induces a commutative diagram

$$\begin{tikzcd} R \arrow[r] \arrow[d] & \operatorname{Hom}_R(I, I) \arrow[d] \\ I \arrow[r] & \operatorname{Hom}_R(R, I). \end{tikzcd}$$

The vertical maps are quasi-isomorphisms. The lower horizontal map is an isomorphism. Whence the homothety morphism is a quasi-isomorphism.

Then note that a Gorenstein ring of finite Krull dimension has finite injective dimension (see 10.5.13). □

We have already met other examples in the case of local rings.

Example 11.2.5 Let $(R, \mathfrak{m}, \Bbbk)$ be a Noetherian local ring and let $\underline{x} = x_1, \ldots, x_k$ be a sequence in R generating an $\mathfrak{m}$-primary ideal. Then the complex $\check{C}_{\underline{x}}^{\vee}$ is a dualizing complex for $\hat{R}^{\mathfrak{m}}$ and so is the complex $\check{L}_{\underline{x}}^{\vee}$. (This is a direct consequence of Theorems 10.2.6 and 10.2.8.)

In particular, it follows that a Noetherian local ring, complete in its maximal-adic topology, always has a dualizing complex.

A further example of local rings possessing a dualizing complex is the following:

Example 11.2.6 Let $(R, \mathfrak{m}, \Bbbk)$ be a Cohen–Macaulay local ring of dimension d. Assume that R has the canonical module K_R. Then the minimal injective resolution I of K_R is a normalized dualizing complex of R as follows by 10.3.7.

Note already that dualizing complexes also provide a duality between local homology and cohomology. The following may be compared with 9.2.5.

Proposition 11.2.7 *Let R denote a Noetherian ring possessing a dualizing complex D, let $\mathfrak{a} \subset R$ be an ideal and X any R-complex. Then we have an isomorphism*

$$\mathrm{RHom}_R(\mathrm{R}\Gamma_{\mathfrak{a}}(X), D) \simeq \mathrm{L}\Lambda^{\mathfrak{a}}(\mathrm{Hom}_R(X, D))$$

in the derived category.

Proof This is a direct consequence of 9.2.4. □

Another result about the complex $\mathrm{Hom}_R(X, D)$ requires stronger hypotheses.

Proposition 11.2.8 *Let $\mathfrak{a} \subset R$ denote an ideal of a Noetherian ring R possessing a dualizing complex D. Then in the derived category there is an isomorphism*

$$\mathrm{R}\Gamma_{\mathfrak{a}}(\mathrm{Hom}_R(X, D)) \simeq \mathrm{RHom}_R(X, \Gamma_{\mathfrak{a}}(D))$$

for any right-bounded R-complex X with finitely generated homology.

Proof This is a direct consequence of the more general 9.1.8. □

In the case when $X = R$ note also the following.

Proposition 11.2.9 *Let R denote a Noetherian ring possessing a dualizing complex D. Then there is a quasi-isomorphism and an isomorphism*

$$\hat{R}^{\mathfrak{a}} \xrightarrow{\sim} \mathrm{Hom}_R(\Gamma_{\mathfrak{a}}(D), D) \cong \mathrm{Hom}_R(\Gamma_{\mathfrak{a}}(D), \Gamma_{\mathfrak{a}}(D)).$$

Proof The inverse system of quasi-isomorphisms

$$R/\mathfrak{a}^t \xrightarrow{\sim} \mathrm{Hom}_R(\mathrm{Hom}_R(R/\mathfrak{a}^t, D), D)$$

induces, by passing to the inverse limit, the quasi-isomorphism of the statement because both systems are degree-wise surjective (see 1.2.9). The isomorphism is obvious. □

11.3 First Change of Rings

In this section we provide some change of ring arguments for the existence of dualizing complexes. The first one shows that dualizing complexes localize.

Proposition 11.3.1 *Let R denote a Noetherian ring and let $T \subset R$ denote a multiplicatively closed subset. Assume there is a bounded complex D of injective R-modules with finitely generated cohomology.*

(a) If D is a dualizing complex for R, then $D_T := D \otimes_R R_T$ is a dualizing complex of R_T.
(b) If $D_\mathfrak{m}$ is a dualizing complex for $R_\mathfrak{m}$ for all maximal ideals $\mathfrak{m}$ of R, then D is a dualizing complex for R.

Proof Note first that the R_T-complex D_T is a bounded complex of injective R_T-modules with finitely generated cohomology modules over R_T. The homothety morphism $h : R \to \mathrm{Hom}_R(D, D)$ is a morphism of bounded flat R-complexes. We tensor it by R_T then compose with the quasi-isomorphism $\mathrm{Hom}_R(D, D) \otimes_R R_T \xrightarrow{\sim} \mathrm{Hom}_{R_T}(D_T, D_T)$ obtained in 4.5.13.

If D is a dualizing complex for R, then h and $h \otimes_R R_T$ are quasi-isomorphisms and so is the composite

$$R_T \xrightarrow{\sim} \mathrm{Hom}_R(D, D) \otimes_R R_T \xrightarrow{\sim} \mathrm{Hom}_{R_T}(D_T, D_T).$$

Hence D_T is a dualizing complex of R_T (see 11.2.3).

Assume now that the condition in (b) is satisfied. Then the composite

$$R_\mathfrak{m} \xrightarrow{h_\mathfrak{m}} \mathrm{Hom}_R(D, D) \otimes_R R_\mathfrak{m} \xrightarrow{\sim} \mathrm{Hom}_{R_\mathfrak{m}}(D_\mathfrak{m}, D_\mathfrak{m})$$

is a quasi-isomorphism for all maximal ideals $\mathfrak{m}$ of R and so is $h_\mathfrak{m}$. It follows that $\mathrm{Hom}_R(D, D)$ is exact in degree $\neq 0$ and that the induced homomorphism $R \to H^0(\mathrm{Hom}_R(D, D))$ is an isomorphism. Hence D is a dualizing complex for R (see 11.2.3 again). □

Corollary 11.3.2 *If a Noetherian ring R has a dualizing complex D, then* $\mathrm{Supp}_R(D) = \mathrm{Spec}(R)$ *and R has finite Krull dimension.*

Proof As a dualizing complex is obviously non-exact, and in view of the preceding, the first assertion is clear. The second assertion follows by an iterative use of 4.5.19. □

In the following, given a homomorphism $\varphi : S \to R$ of commutative rings and an S complex Y we denote $\operatorname{Hom}_S(R, Y)$ by $Y^{(\varphi)}$ or Y^R when there is no doubt about the homomorphism φ. Recall the change of rings formula: for any R-complex X and any S-complex Y we have the isomorphism $\operatorname{Hom}_R(X, Y^R) \cong \operatorname{Hom}_S(X, Y)$.

Proposition 11.3.3 *Let $S \to R$ be a homomorphism of Noetherian rings such that R is a finitely generated S-module. Suppose that S admits a dualizing complex D. Then $D^R := \operatorname{Hom}_S(R, D)$ is a dualizing complex of R.*

Proof By the change of rings formula we have that $D^R := \operatorname{Hom}_S(R, D)$ is a bounded complex of injective R-modules. We know that its cohomology modules are finitely generated (see 4.5.14).

In order to finish the proof we have to show that the homothety morphism

$$R \to \operatorname{Hom}_R(D^R, D^R)$$

is a quasi-isomorphism. This morphism is the composition of the following morphisms

$$R \to \operatorname{Hom}_S(\operatorname{Hom}_S(R, D), D)) = \operatorname{Hom}_S(D^R, D) \cong \operatorname{Hom}_R(D^R, D^R).$$

The left morphism is a quasi-isomorphism because D is dualizing for S and R is a finitely generated S-module. The isomorphism on the right is given by the change of rings results. □

11.3.4 The previous result together with Lemma 11.2.4 provides that any factor ring of a Gorenstein ring of finite Krull dimension admits a dualizing complex. The converse is also true. It was shown by T. Kawasaki (see [50]) that a Noetherian local ring that possesses a dualizing complex is a factor ring of a Gorenstein ring.

By the Cohen Structure Theorem it also follows that a complete Noetherian local ring admits a dualizing complex. Note however that we have already exhibited a dualizing complex for a complete local ring without the use of Cohen's Structure Theorem (see Example 11.2.5).

Here is an extension of Proposition 11.3.3.

Theorem 11.3.5 *Let R denote a Noetherian ring possessing a dualizing complex D. Let B denote an R-algebra of finite type. Then B admits a dualizing complex.*

Proof By virtue of Proposition 11.3.3 and induction it will be enough to prove the claim for $B = R[T]$, the polynomial ring in one variable T over R. First let I denote an injective R-module and write $I[T] = I \otimes_R R[T]$. Then $I[T] \cong I^{(\mathbb{N})}$ is an injective R-module and therefore $\operatorname{Hom}_R(R[T], I[T])$ is an injective $R[T]$-module. The short exact sequence $0 \to R[T] \xrightarrow{\mu_T} R[T] \to R \to 0$ induces a short exact sequence of complexes

$$0 \to D[T] \to \mathrm{Hom}_R(R[T], D[T]) \xrightarrow{\mu_T} \mathrm{Hom}_R(R[T], D[T]) \to 0$$

since $D[T]$ is a bounded complex of injective R-modules. Here μ_T denotes the multiplication by T on the $R[T]$-complex $\mathrm{Hom}_R(R[T], D[T])$. Whence there is a quasi-isomorphism

$$D[T] \xrightarrow{\sim} F(\mu_T),$$

where $F(\mu_T)$ is the fibre of the morphism μ_T (see 1.5.6). Therefore $F(\mu_T)$, which is a bounded $R[T]$-complex of injective $R[T]$-modules, has finitely generated cohomology modules over $R[T]$, in fact it is an injective resolution of $D[T]$.

To complete the proof we will show that the natural morphism

$$R[T] \to \mathrm{Hom}_{R[T]}(F(\mu_T), F(\mu_T))$$

is a quasi-isomorphism. Let $L \xrightarrow{\sim} D$ be a free resolution of D, where L is a right-bounded R-complex of finitely generated free R-modules. Then $L[T] \xrightarrow{\sim} D[T]$ is a free resolution of $D[T]$ as an $R[T]$-complex and there are quasi-isomorphisms

$$\begin{aligned} \mathrm{Hom}_{R[T]}(F(\mu_T), F(\mu_T)) &\xrightarrow{\sim} \mathrm{Hom}_{R[T]}(D[T], F(\mu_T)) \\ &\xrightarrow{\sim} \mathrm{Hom}_{R[T]}(L[T], F(\mu_T)) \end{aligned}$$

because $F(\mu_T)$ is an $R[T]$-injective complex. But now we have the quasi-isomorphisms

$$\begin{aligned} \mathrm{Hom}_{R[T]}(L[T], F(\mu_T)) &\xrightarrow{\sim} F(\mathrm{Hom}_{R[T]}(L[T], \mu_T)) \\ &\xleftarrow{\sim} \mathrm{Hom}_{R[T]}(L[T], D[T]). \end{aligned}$$

For the first one, see 1.5.4. For the last one note that $\mathrm{Hom}_{R[T]}(L[T], \cdot)$ applied to the above short exact sequence of complexes provides again a short exact sequence of complexes since $L[T]$ is a complex of free $R[T]$-modules, then apply 1.5.6 again.

Moreover, $\mathrm{Hom}_R(L, D)[T] \cong \mathrm{Hom}_{R[T]}(L[T], D[T])$ and

$$R \xrightarrow{\sim} \mathrm{Hom}_R(D, D) \xrightarrow{\sim} \mathrm{Hom}_R(L, D).$$

So there is a quasi-isomorphism $R[T] \xrightarrow{\sim} \mathrm{Hom}_{R[T]}(L[T], D[T])$. This finally proves the claim. $\square$

11.4 Characterization and Uniqueness

For the characterization of dualizing complexes over a Noetherian local ring we need a preliminary technical statement.

Proposition 11.4.1 *Let $(R, \mathfrak{m}, \Bbbk)$ be a Noetherian local ring and $\mathcal{P}$ a property of R-modules such that the following holds:*

(a) the residue field $\Bbbk$ of R satisfies property $\mathcal{P}$,
(b) if $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of finitely generated R-modules and M' and M'' satisfy property $\mathcal{P}$, then M satisfies property $\mathcal{P}$,
(c) if M is finitely generated, if x is an M-regular element and if M/xM satisfies property $\mathcal{P}$, then so does M.

Then any finitely generated R-module M has $\mathcal{P}$.

Proof We prove the statement by induction on $\dim M$. If $\dim M = 0$, that is if M has finite length, we obtain that M has $\mathcal{P}$ by (a) and an iterative use of (b) by induction on the length.

If $\dim M \geq 0$ we have already proved that the finite length module $H^0_{\mathfrak{m}}(M)$ has $\mathcal{P}$. Hence it is enough to show that $M/H^0_{\mathfrak{m}}(M)$ has $\mathcal{P}$ by (b). Put $M/H^0_{\mathfrak{m}}(M) = M_1$. As M_1 has positive depth we have an M_1-regular element x. As $\dim M_1/xM_1 = \dim M_1 - 1$ we have that M_1 has $\mathcal{P}$ by the inductive hypothesis and the result follows by (c). □

Theorem 11.4.2 *Let $(R, \mathfrak{m}, \Bbbk)$ be a Noetherian local ring. Let D denote a bounded complex of injective R-modules with finitely generated cohomology modules. Then D is a dualizing complex for R if and only if there is an integer t such that $H^n(\operatorname{Hom}_R(\Bbbk, D)) = 0$ for all $n \neq t$ and $H^t(\operatorname{Hom}_R(\Bbbk, D)) \cong \Bbbk$, more precisely such that $\operatorname{Hom}_R(\Bbbk, D)$ is quasi-isomorphic to $\Bbbk^{[-t]}$.*

Proof Suppose that D is a dualizing complex. Then $\operatorname{Hom}_R(\Bbbk, D)$ is a dualizing complex for the residue field $\Bbbk = R/\mathfrak{m}$ (see 11.3.3). Now it is easy to see that any dualizing complex of the field $\Bbbk$ is quasi-isomorphic to $\Bbbk^{[t]}$ for some $t \in \mathbb{Z}$. That is, $\operatorname{Hom}_R(\Bbbk, D)$ has the required property.

In order to prove the converse it will be enough to show that the natural map

$$M \to \operatorname{Hom}_R(\operatorname{Hom}_R(M, D), D)$$

induces an isomorphism in cohomology for any finitely generated R-module M (in view of Proposition 11.2.3). We proceed by Proposition 11.4.1, where $\mathcal{P}$ denotes the property that the previous morphism induces an isomorphism in cohomology. By the assumption (a) holds. In order to prove (b) note that $\operatorname{Hom}_R(\operatorname{Hom}_R(\cdot, D), D)$ transforms short exact sequences of modules into short exact sequences of complexes since D is a bounded complex of injective R-modules.

To prove (c) let M be a finitely generated R-module with an M-regular element x such that M/xM satisfies property $\mathcal{P}$. We have the exact sequence

$$0 \to \operatorname{Hom}_R(\operatorname{Hom}_R(M, D), D) \xrightarrow{x} \operatorname{Hom}_R(\operatorname{Hom}_R(M, D), D)$$
$$\to \operatorname{Hom}_R(\operatorname{Hom}_R(M/xM, D), D) \to 0$$

because the functor $\operatorname{Hom}_R(\operatorname{Hom}_R(\cdot, D), D)$ is exact. Now look at the associated long exact sequence in cohomology. As

$$H^i(\operatorname{Hom}_R(\operatorname{Hom}_R(M/xM, D), D)) = 0 \text{ for all } i \neq 0$$

by assumption and because the modules $H^i(\operatorname{Hom}_R(\operatorname{Hom}_R(M, D), D))$ are finitely generated (see 4.5.14) we first obtain by Nakayama's Lemma that

$$H^i(\operatorname{Hom}_R(\operatorname{Hom}_R(M, D), D)) = 0 \text{ for all } i \neq 0.$$

Now put $T(\cdot) = H^0(\operatorname{Hom}_R(\operatorname{Hom}_R(\cdot, D), D))$. It remains to show that the homothety morphism induces an isomorphism $M \to T(M)$. We have a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "x"] \arrow[d] & M \arrow[r] \arrow[d] & M/xM \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & T(M) \arrow[r, "x"] & T(M) \arrow[r] & T(M/xM) \arrow[r] & 0.
\end{tikzcd}$$

Now the snake lemma and Nakayama's Lemma provides that $M \to T(M)$ is an isomorphism. □

In the non-local case we then have the following.

Corollary 11.4.3 *Let R denote a Noetherian ring. Let D denote a bounded complex of injective R-modules with finitely generated cohomology modules. Then D is a dualizing complex for R if and only if for all maximal ideals $\mathfrak{m}$ of R the complex* $\operatorname{Hom}_R(R/\mathfrak{m}, D)$ *is quasi-isomorphic to $(R/\mathfrak{m})^{[t]}$ for some $t \in \mathbb{Z}$ depending on $\mathfrak{m}$.*

Proof Because $R/\mathfrak{m}$ and $\operatorname{Hom}_R(R/\mathfrak{m}, D)$ have the structure of an $R_\mathfrak{m}$-complex and in view of 4.5.12 we have the isomorphisms

$$\operatorname{Hom}_R(R/\mathfrak{m}, D) \cong \operatorname{Hom}_R(R/\mathfrak{m}, D) \otimes_R R_\mathfrak{m} \cong \operatorname{Hom}_{R_\mathfrak{m}}(R/\mathfrak{m}, D_\mathfrak{m}).$$

Then the conclusion follows by the above Theorem 11.4.2 together with Proposition 11.3.1. □

In the proof of Theorem 11.4.2 we noticed that the dualizing complex of a field is unique up to a shift and a quasi-isomorphism. We now extend this uniqueness result to an arbitrary Noetherian local ring. For a further generalization to non-local Noetherian rings we refer to [43, Theorem 3.1, p. 266] of Hartshorne's Lecture Notes.

Theorem 11.4.4 *Let $(R, \mathfrak{m}, \Bbbk)$ denote a Noetherian local ring. Suppose D and D' are two dualizing complexes of R. Then there is an integer t such that $D' \xrightarrow{\sim} D^{[t]}$.*

Proof Since D and D' are bounded complexes of injective R-modules the complex $\operatorname{Hom}_R(D', D)$, which is a bounded complex of flat R-modules, has finitely generated cohomology modules (see 4.5.14 or 11.2.2). Then $\operatorname{Hom}_R(D, D')$ has a minimal free resolution, i.e., there is a quasi-isomorphism

$$F \xrightarrow{\sim} \operatorname{Hom}_R(D', D),$$

where F is a right bounded complex of finitely generated free R-modules such that $d_n^F(F_n) \subseteq \mathfrak{m}F_{n-1}$ for all $n \in \mathbb{Z}$. This induces a quasi-isomorphism

$$k \otimes_R F \xrightarrow{\sim} k \otimes_R \operatorname{Hom}_R(D', D).$$

By virtue of 11.1.4 there is an isomorphism

$$k \otimes_R \operatorname{Hom}_R(D', D) \cong \operatorname{Hom}_R(\operatorname{Hom}_R(k, D'), D).$$

Because D and D' are dualizing complexes it follows (by Theorem 11.4.2) that

$$\operatorname{Hom}_R(\operatorname{Hom}_R(k, D'), D) \xrightarrow{\sim} k^{[-t]}$$

for a certain integer t. Hence $k \otimes_R F \xrightarrow{\sim} k^{[-t]}$. But this together with the minimality of F implies that $F \cong R^{[-t]}$. That is, there is a quasi-isomorphism

$$R^{[-t]} \xrightarrow{\sim} \operatorname{Hom}_R(D', D).$$

By applying $\operatorname{Hom}_R(\cdot, D)$ it provides a quasi-isomorphism $D' \xrightarrow{\sim} D^{[t]}$ as follows from definition 11.2.1 because D is a dualizing complex. □

Note that Theorem 11.4.2 also provides a characterization of dualizing complexes in the derived category (see [43, V, Sect. 3.5]).

Proposition 11.4.5 *Let* $(R, \mathfrak{m}, \Bbbk)$ *denote a Noetherian local ring. Let* X *denote a homologically left-bounded* R*-complex with finitely generated cohomology modules. Then the following conditions are equivalent*

(i) *X is quasi-isomorphic to a dualizing complex,*
(ii) *there is an integer t such that* $\operatorname{Ext}_R^i(\Bbbk, X) = 0$ *for all $i \neq t$ and* $\operatorname{Ext}_R^t(\Bbbk, X) \cong \Bbbk$.

When these conditions are satisfied the minimal injective resolution of X is a dualizing complex.

Proof Suppose first that X is quasi-isomorphic to a dualizing complex D. Then $\operatorname{Ext}_R^i(\Bbbk, X) \cong H^i(\operatorname{Hom}_R(\Bbbk, D))$ for all $i \in \mathbb{Z}$ and condition (ii) follows by Theorem 11.4.2.

Assume now that condition (ii) is satisfied. It remains to show that the minimal injective resolution I of X is a dualizing complex. We first note that I is bounded by 10.1.1. Then it follows that I is a dualizing complex by Theorem 11.4.2 again. □

11.4.6 *Normalized dualizing complexes.* Let $(R, \mathfrak{m}, \Bbbk)$ denote a Noetherian local ring. If R admits a dualizing complex D, clearly any shift of D is again a dualizing complex. That is, by a shift and in view of Theorem 11.4.2 we might realize that $H^d(\operatorname{Hom}_R(\Bbbk, D)) \cong \Bbbk$ for $d = \dim R$ and $H^i(\operatorname{Hom}_R(\Bbbk, D)) = 0$ for $i \neq d$. In this case we call D a *normalized dualizing complex*. In this case there is also a quasi-isomorphism $f : \Bbbk^{[-d]} \xrightarrow{\sim} \operatorname{Hom}_R(\Bbbk, D)$ (defined by $f^i = 0$ for all $i \neq d$ and $f^d(1) = c$ where c is a cocycle of degree d in $\operatorname{Hom}_R(\Bbbk, D)$) the image of which in $H^d(\operatorname{Hom}_R(\Bbbk, D))$ is 1.

When R admits a dualizing complex we may always assume that it is normalized.

Example 11.4.7 Let $(R, \mathfrak{m}, \Bbbk)$ be a Noetherian local ring of dimension d and let $\underline{x} = x_1, \ldots, x_d$ be a system of parameters. We have seen in 11.2.5 that the Matlis dual

$$\check{C}_{\underline{x}}^{\vee} := \operatorname{Hom}_R(\check{C}_{\underline{x}}, E_R(\Bbbk))$$

of the Čech complex $\check{C}_{\underline{x}}$ is a dualizing complex for the ring $\hat{R}^{\mathfrak{m}}$. Here we note that the shifted complex $\check{C}_{\underline{x}}^{\vee[-d]}$ is a normalized dualizing complex for $\hat{R}^{\mathfrak{m}}$ and so is the complex $\check{L}_{\underline{x}}^{\vee[-d]}$. This is because $\operatorname{Hom}_{\hat{R}^{\mathfrak{m}}}(\Bbbk, \check{C}_{\underline{x}}^{\vee}) \cong \operatorname{Hom}_R(\Bbbk, \check{C}_{\underline{x}}^{\vee}) \cong (\Bbbk \otimes_R \check{C}_{\underline{x}})^{\vee} \cong \Bbbk^{\vee} \cong \Bbbk$.

Example 11.4.8 Let $(R, \mathfrak{m}, \Bbbk)$ be a Gorenstein local ring. Then the minimal injective resolution of R is a normalized dualizing complex.

More generally, let $(R, \mathfrak{m}, \Bbbk)$ denote a Cohen–Macaulay local ring and assume that R has the canonical module K_R. Then the minimal injective resolution of K_R is a normalized dualizing complex for R. This follows from Proposition 10.3.7.

Here is a first result on the dualizing complex of a Noetherian local Ring.

Lemma 11.4.9 *Let $(R, \mathfrak{m}, \Bbbk)$ denote a Noetherian local ring possessing a normalized dualizing complex D. Then there is a quasi-isomorphism $\Gamma_{\mathfrak{m}}(D) \xrightarrow{\sim} E_R(R/\mathfrak{m})^{[-d]}$, where $d = \dim R$.*

Proof First of all note that $\Gamma_{\mathfrak{m}}(D) \cong \varinjlim \operatorname{Hom}_R(R/\mathfrak{m}^n, D)$. By virtue of Proposition 11.3.3 $\operatorname{Hom}_R(R/\mathfrak{m}^n, D)$ is a dualizing complex for $R/\mathfrak{m}^n$ for all $n \geq 1$. But the Artinian ring $R/\mathfrak{m}^n$ has the normalized dualizing complex $E_{R/\mathfrak{m}^n}(R/\mathfrak{m}) \cong \operatorname{Hom}_R(R/\mathfrak{m}^n, E_R(R/\mathfrak{m}))$ (the isomorphism is a particular case of Lemma 10.6.1). Hence there is a quasi-isomorphism

$$\operatorname{Hom}_R(R/\mathfrak{m}^n, D)^{[d]} \xrightarrow{\sim} \operatorname{Hom}_R(R/\mathfrak{m}^n, E_R(R/\mathfrak{m}))$$

as follows by Theorem 11.4.4. By passing to the direct limit we get a quasi-isomorphism

$$\Gamma_{\mathfrak{m}}(D)^{[d]} \xrightarrow{\sim} \varinjlim \operatorname{Hom}_R(R/\mathfrak{m}^n, E_R(R/\mathfrak{m})) \cong E_R(R/\mathfrak{m}),$$

which finishes the proof. Note that direct limits preserve quasi-isomorphisms.

(One may also argue as follows. In view of 4.5.5 one may assume that D is minimal. Then the conclusion follows by the description of D given below.) □

The following gives some more precise information on the structure of a normalized dualizing complex.

Remark 11.4.10 Let $(R, \mathfrak{m}, \Bbbk)$ be a Noetherian local ring of dimension d admitting a normalized dualizing complex D. Then $D \otimes_R R_\mathfrak{p}, \mathfrak{p} \in \operatorname{Spec} R$, is a dualizing complex for $R_\mathfrak{p}$ (see Proposition 11.3.1). In view of Theorem 11.4.2 there is a unique integer $r(\mathfrak{p})$ such that $H^i(\operatorname{Hom}_{R_\mathfrak{p}}(k(\mathfrak{p}), D \otimes_R R_\mathfrak{p})) = 0$ for all $i \neq r(\mathfrak{p})$, defining a function $r : \operatorname{Spec} R \to \mathbb{Z}$, the codimension function for D. Note that $r(\mathfrak{q}) = r(\mathfrak{p}) + 1$ for any pair of prime ideals $\mathfrak{p} \subset \mathfrak{q}$ with height $\mathfrak{q}/\mathfrak{p} = 1$ (see 4.5.19). As $r(\mathfrak{m}) = d$ since D is normalized we thus have $r(\mathfrak{p}) = d - \dim R/\mathfrak{p}$. It follows that $D \otimes_R R_\mathfrak{p}$ is a normalized dualizing complex for $R_\mathfrak{p}$ when $\dim R_\mathfrak{p} + \dim R/\mathfrak{p} = \dim R$. It also follows that the ring R is catenerian. Then note that the minimal injective resolution of D is also a normalized dualizing complex (this may be seen by 4.5.5). Assume now that D itself is minimal. Then D has the following form

$$D^i \cong \oplus_{\{\mathfrak{p} \in \operatorname{Spec} R \mid \dim R/\mathfrak{p} = d-i\}} E_R(R/\mathfrak{p})$$

for all $i \in \mathbb{Z}$ because $D_\mathfrak{p}$ is also minimal by 4.5.10.

Note also that $\operatorname{depth}(R) = d - \sup\{i \mid H^i(D) \neq 0\}$ in view of 10.1.3. For more precise information on the cohomology of D, see the forthcoming 12.2.3 and 12.4.4.

(For these and related things the reader may also refer to Hartshorne's Lecture Notes [43].)

11.5 Flat Change of Rings, Dualizing Complexes and Completion

After a flat change of rings result we show that a ring of the form $\hat{R}^\mathfrak{a}$ has a dualizing complex, provided R is a Noetherian ring admitting a dualizing complex. More precisely, we shall give an explicit construction of a dualizing complex for $\hat{R}^\mathfrak{a}$.

Proposition 11.5.1 *Let $(S, \mathfrak{n}, \mathbb{l})$ be a Noetherian local ring possessing a dualizing complex D. Let $(R, \mathfrak{m}, \Bbbk)$ be another Noetherian local ring and assume there is local flat homomorphism $f : (S, \mathfrak{n}, \mathbb{l}) \to (R, \mathfrak{m}, \Bbbk)$ such that $\mathfrak{m} = \mathfrak{n}R$. Then the minimal injective resolution I of $D \otimes_S R$ over R is a dualizing complex of R. If, moreover, D is normalized, then so is I.*

Proof This is a direct consequence of Proposition 10.1.5 applied to the S-complex D together with Theorem 11.4.2. □

The previous result applies to the case of the completion of a local ring.

Corollary 11.5.2 *Let $(R, \mathfrak{m}, \Bbbk)$ denote a Noetherian local ring and $\mathfrak{a}$ a proper ideal of R. Suppose that R admits a dualizing complex D. Then the minimal injective resolution I of $D \otimes_R \hat{R}^{\mathfrak{a}}$ over $\hat{R}^{\mathfrak{a}}$ is a dualizing complex of $\hat{R}^{\mathfrak{a}}$. In the case D is normalized, then I is a normalized dualizing complex of $\hat{R}^{\mathfrak{a}}$.*

Proof First note that $R \to \hat{R}^{\mathfrak{a}}$ is a local flat homomorphism of rings. Moreover, $\mathfrak{m}\hat{R}^{\mathfrak{a}}$ is the maximal ideal of $\hat{R}^{\mathfrak{a}}$ so that 11.5.1 applies. □

One of the problems with the above Corollary is that while the $\hat{R}^{\mathfrak{a}}$-complex $\hat{R}^{\mathfrak{a}} \otimes_R D$ has finitely generated cohomology $\hat{R}^{\mathfrak{a}}$-modules it does not consist of injective $\hat{R}^{\mathfrak{a}}$-modules. Another problem is that Corollary 11.5.2 only concerns Noetherian local rings. In the following, given a dualizing complex for a Noetherian R, local or not, we shall give an explicit construction of a dualizing complex of $\hat{R}^{\mathfrak{a}}$. Our construction involves the Čech complex built over a generating set of the ideal $\mathfrak{a}$ and its free resolution $\check{L}_{\underline{x}} \xrightarrow{\sim} \check{C}_{\underline{x}}$ obtained in 6.2.3. We first investigate the complexes $\operatorname{Hom}_R(\check{C}_{\underline{x}}, D)$ and $\operatorname{Hom}_R(\check{L}_{\underline{x}}, D)$.

Proposition 11.5.3 *Let R denote a Noetherian ring possessing a dualizing complex D. Let $\mathfrak{a}$ be a proper ideal generated by the sequence $\underline{x} = x_1, \ldots, x_k$.*

(a) There are quasi-isomorphisms $\operatorname{Hom}_R(\check{C}_{\underline{x}}, D) \xrightarrow{\sim} \operatorname{Hom}_R(\check{L}_{\underline{x}}, D) \xrightarrow{\sim} \operatorname{Hom}_R(\check{C}_{\underline{x}}, D)$ of bounded complexes of injective R-modules, these complexes are quasi-isomorphic to $D \otimes_R \hat{R}^{\mathfrak{a}}$ and their cohomology modules are finitely generated $\hat{R}^{\mathfrak{a}}$-modules.
(b) Let X be an R-complex with finitely generated cohomology modules. Then both complexes

$$\operatorname{Hom}_R(\operatorname{Hom}_R(X, \operatorname{Hom}_R(\check{L}_{\underline{x}}, D)), \operatorname{Hom}_R(\check{L}_{\underline{x}}, D)) \text{ and}$$
$$\operatorname{Hom}_R(\operatorname{Hom}_R(X, \operatorname{Hom}_R(\check{C}_{\underline{x}}, D)), \operatorname{Hom}_R(\check{C}_{\underline{x}}, D))$$

are quasi-isomorphic to $X \otimes_R \hat{R}^{\mathfrak{a}}$.
(c) There are quasi-isomorphisms

$$\Gamma_{\mathfrak{a}}(D) \xrightarrow{\sim} \Gamma_{\mathfrak{a}}(\operatorname{Hom}_R(\check{C}_{\underline{x}}, D)) \xrightarrow{\sim} \Gamma_{\mathfrak{a}}(\operatorname{Hom}_R(\check{L}_{\underline{x}}, D)).$$

(d) If $(R, \mathfrak{m}, \Bbbk)$ is local of dimension d and D is normalized then there are quasi-isomorphisms

$$\Bbbk^{[-d]} \xrightarrow{\sim} \operatorname{Hom}_R(\Bbbk, \operatorname{Hom}_R(\check{C}_{\underline{x}}, D)) \xrightarrow{\sim} \operatorname{Hom}_R(\Bbbk, \operatorname{Hom}_R(\check{L}_{\underline{x}}, D)).$$

Proof By the definition of $\check{L}_{\underline{x}}$ there is a quasi-isomorphism $\check{L}_{\underline{x}} \xrightarrow{\sim} \check{C}_{\underline{x}}$. It induces a quasi-isomorphism $\operatorname{Hom}_R(\check{C}_{\underline{x}}, D) \xrightarrow{\sim} \operatorname{Hom}_R(\check{L}_{\underline{x}}, D)$ and it is clear that both complexes $\operatorname{Hom}_R(\check{C}_{\underline{x}}, D)$ and $\operatorname{Hom}_R(\check{L}_{\underline{x}}, D)$ are bounded complexes of injective R-modules (see 1.4.1). They both represent $\mathrm{L}\Lambda^{\mathfrak{a}}(D)$ in the derived category (see 7.5.12), which is also represented by $\hat{R}^{\mathfrak{a}} \otimes_R D$ (see 9.8.4). Hence the cohomology modules $H^i(\operatorname{Hom}_R(\check{C}_{\underline{x}}, D)) \cong H^i(\operatorname{Hom}_R(\check{L}_{\underline{x}}, D))$ are finitely generated over $\hat{R}^{\mathfrak{a}}$. Moreover, there is also a quasi-isomorphism $\operatorname{Hom}_R(\check{L}_{\underline{x}}, D) \xrightarrow{\sim} \operatorname{Hom}_R(\check{C}_{\underline{x}}, D)$ in view of 4.4.8.

We now prove (b). In view of (a) it is enough the prove the $\check{L}_{\underline{x}}$ part of the statement. We first consider the case where $X = R$. We recall the quasi-isomorphism

$$\check{L}_{\underline{x}} \otimes_R D \xrightarrow{\sim} \check{L}_{\underline{x}} \otimes_R (\operatorname{Hom}_R(\check{L}_{\underline{x}}, D))$$

obtained in 6.5.4. In view of this there are quasi-isomorphisms and isomorphisms

$$\begin{aligned} &\operatorname{Hom}_R(\operatorname{Hom}_R(\check{L}_{\underline{x}}, D), \operatorname{Hom}_R(\check{L}_{\underline{x}}, D)) \cong \operatorname{Hom}_R(\operatorname{Hom}_R(\check{L}_{\underline{x}}, D) \otimes_R \check{L}_{\underline{x}}, D) \\ &\xrightarrow{\sim} \operatorname{Hom}_R(\check{L}_{\underline{x}} \otimes_R D, D) \cong \operatorname{Hom}_R(\check{L}_{\underline{x}}, \operatorname{Hom}_R(D, D)) \xleftarrow{\sim} \operatorname{Hom}_R(\check{L}_{\underline{x}}, R), \end{aligned}$$

recall also the quasi-isomorphism $R \xrightarrow{\sim} \operatorname{Hom}_R(D, D)$ from the definition of a dualizing complex. Since we have the quasi-isomorphism $\operatorname{Hom}_R(\check{L}_{\underline{x}}, R) \xrightarrow{\sim} \hat{R}^{\mathfrak{a}}$ (see 7.5.16) this proves the case $X = R$ of the $\check{L}_{\underline{x}}$ part of the statement. Then we note that all the above quasi-isomorphisms are quasi-isomorphisms between bounded complexes of flat R-modules. It follows that $X \otimes_R \hat{R}^{\mathfrak{a}}$ is quasi-isomorphic to $X \otimes_R \operatorname{Hom}_R(\operatorname{Hom}_R(\check{L}_{\underline{x}}, D), \operatorname{Hom}_R(\check{L}_{\underline{x}}, D))$ (see 4.4.11) and the result follows by Proposition 11.1.5.

Now we prove (c). For each $t \geq 1$ there are quasi-isomorphisms

$$\check{L}_{\underline{x}} \otimes_R R/\mathfrak{a}^t \xrightarrow{\sim} \check{C}_{\underline{x}} \otimes_R R/\mathfrak{a}^t \xrightarrow{\sim} R/\mathfrak{a}^t.$$

By applying $\operatorname{Hom}_R(\cdot, D)$ this preserves the quasi-isomorphisms. Then the result in (c) follows by adjointness and passing to the direct limit.

Statement (d) follows by (a) and adjointness, note that $\check{C}_{\underline{x}} \otimes_R \Bbbk \cong \Bbbk$ and see also 11.4.6. ☐

In view of the above the complexes $\operatorname{Hom}_R(\check{L}_{\underline{x}}, D)$ and $\operatorname{Hom}_R(\check{C}_{\underline{x}}, D)$ are close to a dualizing complex of $\hat{R}^{\mathfrak{a}}$. But they do not have the structure of an $\hat{R}^{\mathfrak{a}}$-complex. To remedy this situation some auxiliary results are needed.

Lemma 11.5.4 *Let R denote a commutative ring and $\underline{x} = x_1, \ldots, x_k$ a weakly pro-regular sequence generating the proper ideal $\mathfrak{a}$.*

(a) There is a commutative diagram of quasi-isomorphisms

$$\begin{array}{ccccc} \check{L}_{\underline{x}} & \xrightarrow{\sim} & \check{L}_{\underline{x}} \otimes_R \operatorname{Hom}_R(\check{L}_{\underline{x}}, R) & \xrightarrow{\sim} & \check{L}_{\underline{x}} \otimes_R \hat{R}^{\mathfrak{a}} \\ \downarrow & & \downarrow & & \downarrow \\ \check{C}_{\underline{x}} & \xrightarrow{\sim} & \check{C}_{\underline{x}} \otimes_R \operatorname{Hom}_R(\check{L}_{\underline{x}}, R) & \xrightarrow{\sim} & \check{C}_{\underline{x}} \otimes_R \hat{R}^{\mathfrak{a}} \end{array}$$

(b) There are quasi-isomorphisms

$$\operatorname{Hom}_R(\hat{R}^{\mathfrak{a}}, \operatorname{Hom}_R(\check{L}_{\underline{x}}, I)) \xrightarrow{\sim} \operatorname{Hom}_R(\check{L}_{\underline{x}}, I) \text{ and}$$
$$\operatorname{Hom}_R(\hat{R}^{\mathfrak{a}}, \operatorname{Hom}_R(\check{C}_{\underline{x}}, I)) \xrightarrow{\sim} \operatorname{Hom}_R(\check{C}_{\underline{x}}, I)$$

for any K-injective R-complex I.

Proof We first prove (a). Since $\check{L}_{\underline{x}} \to \check{C}_{\underline{x}}$ is a quasi-isomorphism (see 6.2.3) of bounded flat R-complexes the vertical morphisms in the diagram are quasi-isomorphisms (4.4.11). The first horizontal quasi-isomorphisms above and below are shown in 6.5.4. The second horizontal quasi-isomorphisms above and below follow since $\operatorname{Hom}_R(\check{L}_{\underline{x}}, R) \xrightarrow{\sim} \hat{R}^{\mathfrak{a}}$ is a flat resolution of $\hat{R}^{\mathfrak{a}}$ (see 7.5.16).

Statement (b) follows by (a) and adjointness. □

Now we are ready to present an explicit description of the dualizing complex of $\hat{R}^{\mathfrak{a}}$.

Theorem 11.5.5 *Let R denote a Noetherian ring possessing a dualizing complex D. Let $\underline{x} = x_1, \ldots, x_k$ denote a system of elements generating the proper ideal $\mathfrak{a}$. Then there is a quasi-isomorphism*

$$\operatorname{Hom}_R(\hat{R}^{\mathfrak{a}}, \operatorname{Hom}_R(\check{C}_{\underline{x}}, D)) \xrightarrow{\sim} \operatorname{Hom}_R(\hat{R}^{\mathfrak{a}}, \operatorname{Hom}_R(\check{L}_{\underline{x}}, D))$$

and both complexes are dualizing complexes for $\hat{R}^{\mathfrak{a}}$.

Moreover, there is a quasi-isomorphism

$$\Gamma_{\mathfrak{a}}(D) \xrightarrow{\sim} \Gamma_{\mathfrak{a}}(\operatorname{Hom}_R(\hat{R}^{\mathfrak{a}}, \operatorname{Hom}_R(\check{C}_{\underline{x}}, D))).$$

Proof Since $\Gamma_{\mathfrak{a}}(\operatorname{Hom}_R(\hat{R}^{\mathfrak{a}}, \operatorname{Hom}_R(\check{C}_{\underline{x}}, D))) \cong \Gamma_{\mathfrak{a}}(\operatorname{Hom}_R(\check{C}_{\underline{x}}, D))$, as is easily seen, the last statement is a direct consequence of 11.5.3 (c).

Let us prove the first statement. The quasi-isomorphism $\check{L}_{\underline{x}} \to \check{C}_{\underline{x}}$ induces a quasi-isomorphism $\operatorname{Hom}_R(\check{C}_{\underline{x}}, D) \xrightarrow{\sim} \operatorname{Hom}_R(\check{L}_{\underline{x}}, D)$ of bounded complexes of injective R-modules. By applying the functor $\operatorname{Hom}_R(\hat{R}^{\mathfrak{a}}, \cdot)$ it yields the first quasi-isomorphism of the statement (see 4.4.11). Clearly both complexes $\operatorname{Hom}_R(\hat{R}^{\mathfrak{a}}, \operatorname{Hom}_R(\check{C}_{\underline{x}}, D))$ and $\operatorname{Hom}_R(\hat{R}^{\mathfrak{a}}, \operatorname{Hom}_R(\check{L}_{\underline{x}}, D))$ are bounded complexes of injective $\hat{R}^{\mathfrak{a}}$-modules. In view of Lemma 11.5.4 we also have the quasi-isomorphism

$$\operatorname{Hom}_R(\hat{R}^{\mathfrak{a}}, \operatorname{Hom}_R(\check{C}_{\underline{x}}, D)) \xrightarrow{\sim} \operatorname{Hom}_R(\check{C}_{\underline{x}}, D).$$

Hence the $\hat{R}^{\mathfrak{a}}$-complex $\operatorname{Hom}_R(\hat{R}^{\mathfrak{a}}, \operatorname{Hom}_R(\check{C}_{\underline{x}}, D))$ has finitely generated cohomology by Proposition 11.5.3.

It remains to show that the homothety morphism

$$\hat{R}^{\mathfrak{a}} \to \operatorname{Hom}_{\hat{R}^{\mathfrak{a}}}(\operatorname{Hom}_R(\hat{R}^{\mathfrak{a}}, \operatorname{Hom}_R(\check{C}_{\underline{x}}, D)), \operatorname{Hom}_R(\hat{R}^{\mathfrak{a}}, \operatorname{Hom}_R(\check{C}_{\underline{x}}, D)))$$

is a quasi-isomorphism. By a change of rings formula (see 4.4.21) and in view of the above we have an isomorphism and a quasi-isomorphism

$$\begin{aligned}
&\operatorname{Hom}_{\hat{R}^{\mathfrak{a}}}(\operatorname{Hom}_R(\hat{R}^{\mathfrak{a}}, \operatorname{Hom}_R(\check{C}_{\underline{x}}, D)), \operatorname{Hom}_R(\hat{R}^{\mathfrak{a}}, \operatorname{Hom}_R(\check{C}_{\underline{x}}, D)))\\
&\cong \operatorname{Hom}_R(\operatorname{Hom}_R(\hat{R}^{\mathfrak{a}}, \operatorname{Hom}_R(\check{C}_{\underline{x}}, D)), \operatorname{Hom}_R(\check{C}_{\underline{x}}, D))\\
&\xleftarrow{\sim} \operatorname{Hom}_R(\operatorname{Hom}_R(\check{C}_{\underline{x}}, D), \operatorname{Hom}_R(\check{C}_{\underline{x}}, D)).
\end{aligned}$$

But the complex $\operatorname{Hom}_R(\operatorname{Hom}_R(\check{C}_{\underline{x}}, D), \operatorname{Hom}_R(\check{C}_{\underline{x}}, D))$ is quasi-isomorphic to $\hat{R}^{\mathfrak{a}}$ (see Proposition 11.5.3). The conclusion follows (see 1.1.7). □

Note that the complex $\operatorname{Hom}_R(\check{C}_{\underline{x}}, D)$ also has some dualizing properties.

Corollary 11.5.6 *In the situation of 11.5.5 let X denote an R-complex with finitely generated modules. Then there is a quasi-isomorphism*

$$X \otimes_R \hat{R}^{\mathfrak{a}} \xrightarrow{\sim} \operatorname{Hom}_R(\operatorname{Hom}_R(X \otimes_R \hat{R}^{\mathfrak{a}}, \operatorname{Hom}_R(\check{C}_{\underline{x}}, D)), \operatorname{Hom}_R(\check{C}_{\underline{x}}, D)).$$

Proof For simplicity we put $\mathcal{D} = \operatorname{Hom}_R(\check{C}_{\underline{x}}, D)$. In view of 11.5.5 there is a quasi-isomorphism and isomorphisms

$$\begin{aligned}
X \otimes_R \hat{R}^{\mathfrak{a}} &\xrightarrow{\sim} \operatorname{Hom}_{\hat{R}^{\mathfrak{a}}}(\operatorname{Hom}_{\hat{R}^{\mathfrak{a}}}(X \otimes_R \hat{R}^{\mathfrak{a}}, \operatorname{Hom}_R(\hat{R}^{\mathfrak{a}}, \mathcal{D})), \operatorname{Hom}_R(\hat{R}^{\mathfrak{a}}, \mathcal{D}))\\
&\cong \operatorname{Hom}_R(\operatorname{Hom}_R(X, \operatorname{Hom}_R(\hat{R}^{\mathfrak{a}}, \mathcal{D})), \mathcal{D}) \cong \operatorname{Hom}_R(\operatorname{Hom}_R(X \otimes_R \hat{R}^{\mathfrak{a}}, \mathcal{D}), \mathcal{D}),
\end{aligned}$$

where adjointness is used. □

Another bounded complex of injective $\hat{R}^{\mathfrak{a}}$-modules is the complex $\operatorname{Hom}_R(\hat{R}^{\mathfrak{a}}, D)$. However, in general it seems that its cohomology is not finitely generated over $\hat{R}^{\mathfrak{a}}$.

Nevertheless we shall have a look at it.

Proposition 11.5.7 *Let R be a Noetherian ring possessing a dualizing complex D. Let $\underline{x} = x_1, \ldots, x_k$ a system of elements generating a proper ideal $\mathfrak{a}$. Then $\operatorname{Hom}_R(\hat{R}^{\mathfrak{a}}, D)$ is a bounded complex of injective $\hat{R}^{\mathfrak{a}}$-modules with the following properties:*

(a) $\operatorname{Hom}_R(\check{L}_{\underline{x}}, \operatorname{Hom}_R(\hat{R}^{\mathfrak{a}}, D)) \xrightarrow{\sim} \operatorname{Hom}_R(\check{L}_{\underline{x}}, D)$.
(b) $\Gamma_{\mathfrak{a}}(\operatorname{Hom}_R(\hat{R}^{\mathfrak{a}}, D)) \cong \Gamma_{\mathfrak{a}}(D)$.
(c) If $(R, \mathfrak{m}, \Bbbk)$ is a Noetherian local ring of dimension d and D is normalized, then there is a quasi-isomorphism $\Bbbk^{[-d]} \xrightarrow{\sim} \operatorname{Hom}_{\hat{R}^{\mathfrak{a}}}(\Bbbk, \operatorname{Hom}_R(\hat{R}^{\mathfrak{a}}, D))$.

Proof By adjointness we have the isomorphism

$$\operatorname{Hom}_R(\check{L}_{\underline{x}}, \operatorname{Hom}_R(\hat{R}^{\mathfrak{a}}, D)) \cong \operatorname{Hom}_R(\check{L}_{\underline{x}} \otimes_R \hat{R}^{\mathfrak{a}}, D).$$

Hence (a) follows by 11.5.4.

For the proof of (b) recall that

$$\Gamma_{\mathfrak{a}}(\operatorname{Hom}_R(\hat{R}^{\mathfrak{a}}, D)) \cong \varinjlim \operatorname{Hom}_R(R/\mathfrak{a}^t, \operatorname{Hom}_R(\hat{R}^{\mathfrak{a}}, D)).$$

By adjointness and because $R/\mathfrak{a}^t \cong R/\mathfrak{a}^t \otimes_R \hat{R}^{\mathfrak{a}}$ for all $t \geq 1$ the claim in (b) follows.

For the proof of (c) we note that $\Bbbk$ has the structure of an $\hat{R}^{\mathfrak{a}}$-module and we use a change of ring formula (see 4.4.21) in order to get the following isomorphism and quasi-isomorphism

$$\operatorname{Hom}_{\hat{R}^{\mathfrak{a}}}(\Bbbk, \operatorname{Hom}_R(\hat{R}^{\mathfrak{a}}, D)) \cong \operatorname{Hom}_R(\Bbbk, D) \xleftarrow{\sim} \Bbbk^{[-d]},$$

where the last quasi-isomorphism holds since D is a normalized dualizing complex. $\square$

We now investigate the endomorphism complex of $\operatorname{Hom}_R(\hat{R}^{\mathfrak{a}}, D)$. The following lemma will be useful.

Lemma 11.5.8 *Let $\mathfrak{a}$ be a finitely generated ideal of a commutative ring R and I a complex of injective R-modules. Assume that R is $\mathfrak{a}$-complete and that $\Lambda^{\mathfrak{a}}(\operatorname{Hom}_R(I, I)$ is quasi-isomorphic to R. Then the homothety morphism $h : R \to \operatorname{Hom}_R(I, I)$ induces, by completion, a quasi-isomorphism*

$$\Lambda^{\mathfrak{a}}(h) : R \xrightarrow{\sim} \Lambda^{\mathfrak{a}}(\operatorname{Hom}_R(I, I)).$$

Proof There is an isomorphism $\Lambda^{\mathfrak{a}}(\operatorname{Hom}_R(I, I)) \cong \operatorname{Hom}_R(\Gamma_{\mathfrak{a}}(I), \Gamma_{\mathfrak{a}}(I))$ (see 9.2.7). The image of 1 by the composite

$$\Lambda^{\mathfrak{a}}(h) : R \to \Lambda^{\mathfrak{a}}(\operatorname{Hom}_R(I, I)) \cong \operatorname{Hom}_R(\Gamma_{\mathfrak{a}}(I), \Gamma_{\mathfrak{a}}(I))$$

is $\mathrm{id}_{\Gamma_{\mathfrak{a}}(I)}$. Hence this composite is the homothety morphism of $\Gamma_{\mathfrak{a}}(I)$. It is a quasi-isomorphism by 1.1.7 and the conclusion follows. $\square$

Proposition 11.5.9 *Let R, D, and $\mathfrak{a}$ be as in 11.5.7. Then the $\hat{R}^{\mathfrak{a}}$-complex*

$$\operatorname{Hom}_{\hat{R}^{\mathfrak{a}}}(\operatorname{Hom}_R(\hat{R}^{\mathfrak{a}}, D), \operatorname{Hom}_R(\hat{R}^{\mathfrak{a}}, D))$$

is a bounded complex of flat $\hat{R}^{\mathfrak{a}}$-modules and the homothety morphism

$$h : \hat{R}^{\mathfrak{a}} \to \operatorname{Hom}_{\hat{R}^{\mathfrak{a}}}(\operatorname{Hom}_R(\hat{R}^{\mathfrak{a}}, D), \operatorname{Hom}_R(\hat{R}^{\mathfrak{a}}, D))$$

induces, by completion, a quasi-isomorphism

$$\Lambda^{\mathfrak{a}}(h) : \hat{R}^{\mathfrak{a}} \xrightarrow{\sim} \Lambda^{\mathfrak{a}}(\operatorname{Hom}_{\hat{R}^{\mathfrak{a}}}(\operatorname{Hom}_R(\hat{R}^{\mathfrak{a}}, D), \operatorname{Hom}_R(\hat{R}^{\mathfrak{a}}, D))).$$

Proof The first assertion is clear. Let us prove the second. Because $\operatorname{Hom}_R(\hat{R}^{\mathfrak{a}}, D)$ is an $\hat{R}^{\mathfrak{a}}$-complex of injective modules we only need to prove that the $\hat{R}^{\mathfrak{a}}$-complex

$$\Lambda^{\mathfrak{a}}(\operatorname{Hom}_{\hat{R}^{\mathfrak{a}}}(\operatorname{Hom}_R(\hat{R}^{\mathfrak{a}}, D), \operatorname{Hom}_R(\hat{R}^{\mathfrak{a}}, D)))$$

is quasi-isomorphic to $\hat{R}^{\mathfrak{a}}$ (see 11.5.8 and note that there is an isomorphism $\Lambda^{\mathfrak{a}}(X) \cong \Lambda^{\mathfrak{a}\hat{R}^{\mathfrak{a}}}(X)$ for any $\hat{R}^{\mathfrak{a}}$-complex X). Then note that

$$\operatorname{Hom}_{\hat{R}^{\mathfrak{a}}}(\operatorname{Hom}_R(\hat{R}^{\mathfrak{a}}, D), \operatorname{Hom}_R(\hat{R}^{\mathfrak{a}}, D)) \cong \operatorname{Hom}_R(\operatorname{Hom}_R(\hat{R}^{\mathfrak{a}}, D), D)$$

as follows by a change of ring formula. Then note the quasi-isomorphisms and isomorphisms

$$\begin{gathered} R/\mathfrak{a}^t \otimes_R \operatorname{Hom}_R(\operatorname{Hom}_R(\hat{R}^{\mathfrak{a}}, D), D) \xrightarrow{\sim} \\ \operatorname{Hom}_R(\operatorname{Hom}_R(R/\mathfrak{a}^t, \operatorname{Hom}_R(\hat{R}^{\mathfrak{a}}, D)), D) \cong \operatorname{Hom}_R(\operatorname{Hom}_R(R/\mathfrak{a}^t, D), D) \xleftarrow{\sim} R/\mathfrak{a}^t \end{gathered}$$

for all $t \geq 1$. Here the first quasi-isomorphism is obtained by 11.1.5, the isomorphism is obtained by adjointness and the last quasi-isomorphism is given by the definition of a dualizing complex. We note that the corresponding inverse systems are degree-wise surjective. Hence by passing to the inverse limits it yields quasi-isomorphisms (see 1.2.9) and the conclusion follows. □

Corollary 11.5.10 *Let R, D, and $\mathfrak{a}$ be as in 11.5.9. Then the natural homomorphism*

$$H_0(h) : \hat{R}^{\mathfrak{a}} \to H_0(\operatorname{Hom}_{\hat{R}^{\mathfrak{a}}}(\operatorname{Hom}_R(\hat{R}^{\mathfrak{a}}, D), \operatorname{Hom}_R(\hat{R}^{\mathfrak{a}}, D)))$$

is split-injective and

$$\Lambda_i^{\mathfrak{a}}(\operatorname{Hom}_{\hat{R}^{\mathfrak{a}}}(\operatorname{Hom}_R(\hat{R}^{\mathfrak{a}}, D), \operatorname{Hom}_R(\hat{R}^{\mathfrak{a}}, D))) = 0$$

for all $i \neq 0$

Proof There is a commutative square

$$\begin{array}{ccc} \hat{R}^{\mathfrak{a}} & \xrightarrow{h} & \operatorname{Hom}_{\hat{R}^{\mathfrak{a}}}(\operatorname{Hom}_R(\hat{R}^{\mathfrak{a}}, D), \operatorname{Hom}_R(\hat{R}^{\mathfrak{a}}, D)) \\ \| & & \downarrow{\scriptstyle \tau^{\mathfrak{a}}} \\ \hat{R}^{\mathfrak{a}} & \xrightarrow{\Lambda^{\mathfrak{a}}(h)} & \Lambda^{\mathfrak{a}}(\operatorname{Hom}_{\hat{R}^{\mathfrak{a}}}(\operatorname{Hom}_R(\hat{R}^{\mathfrak{a}}, D), \operatorname{Hom}_R(\hat{R}^{\mathfrak{a}}, D))) \end{array}$$

where the upper horizontal morphism is the homothety morphism and where the others are given by the completion process. The bottom horizontal morphism is a quasi-isomorphism by 11.5.9. Taking homology we obtain a commutative square

$$\begin{tikzcd} \hat{R}^{\mathfrak{a}} \arrow[r, "H_0(h)"] \arrow[d, equal] & H_0(\mathrm{Hom}_{\hat{R}^{\mathfrak{a}}}(\mathrm{Hom}_R(\hat{R}^{\mathfrak{a}}, D), \mathrm{Hom}_R(\hat{R}^{\mathfrak{a}}, D))) \arrow[d, "\tau_0^{\mathfrak{a}}"] \\ \hat{R}^{\mathfrak{a}} \arrow[r, equal, "\sim"] & H_0(\Lambda^{\mathfrak{a}}(\mathrm{Hom}_{\hat{R}^{\mathfrak{a}}}(\mathrm{Hom}_R(\hat{R}^{\mathfrak{a}}, D), \mathrm{Hom}_R(\hat{R}^{\mathfrak{a}}, D)))). \end{tikzcd}$$

It follows that the homomorphism $H_0(h)$ is split-injective. The second statement is a consequence of 11.5.9. Note that

$$\begin{aligned} &H_i(\Lambda^{\mathfrak{a}}(\mathrm{Hom}_{\hat{R}^{\mathfrak{a}}}(\mathrm{Hom}_R(\hat{R}^{\mathfrak{a}}, D), \mathrm{Hom}_R(\hat{R}^{\mathfrak{a}}, D)))) \\ &= \Lambda_i^{\mathfrak{a}}(\mathrm{Hom}_{\hat{R}^{\mathfrak{a}}}(\mathrm{Hom}_R(\hat{R}^{\mathfrak{a}}, D), \mathrm{Hom}_R(\hat{R}^{\mathfrak{a}}, D))) \end{aligned}$$

because $\mathrm{Hom}_{\hat{R}^{\mathfrak{a}}}(\mathrm{Hom}_R(\hat{R}^{\mathfrak{a}}, D), \mathrm{Hom}_R(\hat{R}^{\mathfrak{a}}, D))$ is a complex of flat modules. □

Remark 11.5.11 Because of the obvious isomorphism

$$\mathrm{Hom}_{\hat{R}^{\mathfrak{a}}}(\mathrm{Hom}_R(\hat{R}^{\mathfrak{a}}, D), \mathrm{Hom}_R(\hat{R}^{\mathfrak{a}}, D)) \cong \mathrm{Hom}_R(\mathrm{Hom}_R(\hat{R}^{\mathfrak{a}}, D), D)$$

it follows (see 11.5.10) that $\Lambda_i^{\mathfrak{a}}(\mathrm{Hom}_R(\mathrm{Hom}_R(\hat{R}^{\mathfrak{a}}, D), D)) = 0$ for all $i \neq 0$ and $\hat{R}^{\mathfrak{a}}$ is a direct summand of $H_0(\mathrm{Hom}_R(\mathrm{Hom}_R(\hat{R}^{\mathfrak{a}}, D), D))$. We do not know what its co-summand is.

11.6 Further Properties of Dualizing Complexes

For a d-dimensional Gorenstein ring $(R, \mathfrak{m}, \Bbbk)$, a system of parameters $\underline{x} = x_1, \ldots, x_d$ and $E := E_R(\Bbbk)$ the injective hull of the residue field, there are isomorphisms

$$\mathrm{L}\Lambda^{\mathfrak{m}}(X) \simeq \mathrm{Hom}_R(\check{L}_{\underline{x}}, X) \simeq \mathrm{RHom}_R(E, X)^{[d]}$$

in the derived category for any R-complex X. This follows since the composite $\check{L}_{\underline{x}} \xrightarrow{\sim} \check{C}_{\underline{x}} \xrightarrow{\sim} E^{[-d]}$ is a projective resolution. In the following we want to extend this result to the situation of any Noetherian local $(R, \mathfrak{m}, \Bbbk)$ possessing a normalized dualizing complex D.

Proposition 11.6.1 *Let* $(R, \mathfrak{m}, \Bbbk)$ *denote a* d*-dimensional Noetherian local ring possessing a normalized dualizing complex* D*. Let* X *be any* R*-complex. Then there is an isomorphism*

$$\mathrm{L}\Lambda^{\mathfrak{m}}(\mathrm{RHom}_R(D, X)) \simeq \mathrm{RHom}_R(E, X)^{[d]}$$

in the derived category. In particular, there are isomorphisms

$$\Lambda_i^{\mathfrak{m}}(\mathrm{RHom}_R(D, X)) \cong \mathrm{Ext}_R^{d-i}(E, X).$$

Here E denotes as usual the injective hull of the residue field.

Proof By virtue of 9.2.4 there is an isomorphism

$$\mathrm{L}\Lambda^{\mathfrak{m}}(\mathrm{RHom}_R(D, X)) \simeq \mathrm{RHom}_R(\mathrm{R}\Gamma_{\mathfrak{m}}(D), X)$$

in the derived category for any R-complex X. Since $\mathrm{R}\Gamma_{\mathfrak{m}}(D) \simeq \Gamma_{\mathfrak{m}}(D) \cong E^{[-d]}$ (see 11.4.9) this proves the statement. $\square$

The previous result has an interesting consequence for finitely generated R-modules.

Corollary 11.6.2 *In the situation of 11.6.1 let M denote a finitely generated R-module. Then $\mathrm{Ext}^i_R(D, M)$ is finitely generated and there are isomorphisms*

$$\Lambda^{\mathfrak{m}}(\mathrm{Ext}^i_R(D, M)) \cong \mathrm{Ext}^{d+i}_R(E, M)$$

for all $i \in \mathbb{Z}$.

Proof Let $L \xrightarrow{\sim} D$ denote a free resolution of D by finitely generated free R-modules. Then $\mathrm{Hom}_R(L, M)$ is a representative of $\mathrm{RHom}_R(D, M)$ by a left-bounded complex of finitely generated R-modules and $\mathrm{Ext}^i(D, M) = H^i(\mathrm{Hom}_R(L, M))$ is finitely generated. Hence $\Lambda^{\mathfrak{m}}(\mathrm{Ext}^i_R(D, M)) \cong \Lambda^{\mathfrak{m}}_0(\mathrm{Ext}^i_R(D, M))$ and $\Lambda^{\mathfrak{m}}_j(\mathrm{Ext}^i_R(D, M)) \cong 0$ for all $j \in \mathbb{Z}$.

Let $\underline{x} = x_1, \ldots, x_d$ denote a system of parameters of R. Then $\mathrm{L}\Lambda^{\mathfrak{m}}(\mathrm{RHom}_R(D, M))$ is represented by $\mathrm{Hom}_R(\check{L}_{\underline{x}}, \mathrm{Hom}_R(L, M))$. For the computation of the cohomology of this last complex we use the spectral sequence

$$E_2^{-i,j} = \Lambda^{\mathfrak{m}}_i(H^j(\mathrm{Hom}_R(L, M))) \Longrightarrow H^{j-i}(\mathrm{Hom}_R(\check{L}_{\underline{x}}, \mathrm{Hom}_R(L, M))).$$

Since all the $H^j(\mathrm{Hom}_R(L, M)))$ are finitely generated R-modules it follows that $E_2^{-i,j} = 0$ for $i \neq 0$ and $E_2^{0,j} \cong \Lambda^{\mathfrak{m}}(H^j(\mathrm{Hom}_R(L, M)))$. That is, the spectral sequence degenerates to the isomorphisms

$$\Lambda^{\mathfrak{m}}_{-j}(\mathrm{RHom}_R(D, M)) \cong \Lambda^{\mathfrak{m}}(\mathrm{Ext}^j_R(D, M)).$$

But $\Lambda^{\mathfrak{m}}_{-j}(\mathrm{RHom}_R(D, M)) \cong \mathrm{Ext}^{d+j}_R(E, M)$ as follows by 11.6.1. $\square$

For those who do not like spectral sequences we provide a second proof of 11.6.2.

11.6.3 *Second proof of* 11.6.2. Let $L \xrightarrow{\sim} D$ denote a free resolution of D by finitely generated free R-modules. Then $\mathrm{Hom}_R(L, M)$ is a representative of $\mathrm{RHom}_R(D, M)$ by a left-bounded complex of finitely generated R-modules and $\mathrm{Ext}^i(D, M) = H^i(\mathrm{Hom}_R(L, M)$ is finitely generated. It follows that

$$\mathrm{L}\Lambda^{\mathfrak{m}}(\mathrm{RHom}_R(D, M)) \simeq \mathrm{L}\Lambda^{\mathfrak{m}}(\mathrm{Hom}_R(L, M)) \simeq \mathrm{Hom}_R(L, M) \otimes_R \hat{R}^{\mathfrak{m}}$$

(see 9.8.4 for the last isomorphism in the derived category). We take cohomology and obtain

$$\Lambda_{-i}^{\mathfrak{m}}(\mathrm{RHom}_R(D, M)) \cong \mathrm{Ext}_R^i(D, M) \otimes_R \hat{R}^{\mathfrak{m}} \cong \Lambda^{\mathfrak{m}}(\mathrm{Ext}_R^i(D, M)).$$

Since $\Lambda_{-i}^{\mathfrak{m}}(\mathrm{RHom}_R(D, M)) \cong \mathrm{Ext}_R^{d+i}(E, M)$ in view of 11.6.1 this finishes the proof. □

As a particular case of 11.6.2 we look at the case of a Cohen–Macaulay ring $(R, \mathfrak{m}, \Bbbk)$ with a canonical module K_R.

Corollary 11.6.4 *Let $(R, \mathfrak{m}, \Bbbk)$ denote a d-dimensional Cohen–Macaulay ring with a canonical module K_R. Then there are isomorphisms*

$$\Lambda^{\mathfrak{m}}(\mathrm{Ext}_R^i(K_R, M)) \cong \mathrm{Ext}_R^{d+i}(E, M)$$

for all $i \in \mathbb{Z}$ and any finitely generated R-module M.

Proof By 11.4.8 the minimal injective resolution of K_R is a normalized dualizing complex. Then the claim is a consequence of 11.6.2. □

Observe that the previous corollaries are an extension of Corollary 10.5.10 concerning Gorenstein local rings to the case of a Noetherian local ring or a Cohen–Macaulay local ring.

In particular, let R denote a Cohen–Macaulay local ring that admits the canonical module K_R. Since $\mathrm{Ext}_R^i(K_R, K_R) = 0$ for $i > 0$ and $\mathrm{Hom}_R(K_R, K_R) \cong R$ (see 10.3.7) we get $\mathrm{Ext}_R^i(E, K_R) = 0$ for $i \neq d$ and $\mathrm{Ext}_R^d(E, K_R) \cong \hat{R}^{\mathfrak{m}}$. Further results related to those of Sect. 9.5 are easily derived.

As a general result, for an arbitrary local ring we note a counterpart of 11.6.1 in the following.

Theorem 11.6.5 *Let $(R, \mathfrak{m}, \Bbbk)$ denote a d-dimensional Noetherian local ring. Let $\underline{x} = x_1, \ldots, x_d$ be a system of parameters of R. Then*

(a) $\mathrm{R}\Gamma_{\mathfrak{m}}(\check{C}_{\underline{x}}^{\vee})$ is represented by $E := E_R(\Bbbk)$,
(b) for any R-complex X there is an isomorphism

$$\mathrm{L}\Lambda^{\mathfrak{m}}(\mathrm{RHom}_R(\check{C}_{\underline{x}}^{\vee}, X)) \simeq \mathrm{RHom}_R(E, X)$$

in the derived category.

Proof By virtue of 9.2.4 there is an isomorphism

$$\mathrm{L}\Lambda^{\mathfrak{m}}(\mathrm{RHom}_R(\check{C}_{\underline{x}}^{\vee}, X)) \simeq \mathrm{RHom}_R(\mathrm{R}\Gamma_{\mathfrak{m}}(\check{C}_{\underline{x}}^{\vee}), X)$$

in the derived category. Moreover, $\mathrm{R}\Gamma_{\mathfrak{m}}(\check{C}_{\underline{x}}^{\vee})$ is represented by the complexes

$$\check{L}_{\underline{x}} \otimes_R \operatorname{Hom}_R(\check{L}_{\underline{x}}, E) \xleftarrow{\sim} \check{L}_{\underline{x}} \otimes_R \operatorname{Hom}_R(\check{C}_{\underline{x}}, E).$$

By 6.5.4 there is a quasi-isomorphism $\check{L}_{\underline{x}} \otimes_R E \xrightarrow{\sim} \check{L}_{\underline{x}} \otimes_R \operatorname{Hom}_R(\check{L}_{\underline{x}}, E)$. Because $\check{L}_{\underline{x}} \xrightarrow{\sim} \check{C}_{\underline{x}}$ is a projective resolution it induces a quasi-isomorphism

$$\check{L}_{\underline{x}} \otimes_R E \xrightarrow{\sim} \check{C}_{\underline{x}} \otimes_R E \cong E.$$

The claims are proved by putting these data together. $\square$

Chapter 12
Local Duality with Dualizing Complexes and Other Dualities

In this chapter we present a version of Grothendieck local duality for a Noetherian local ring admitting a dualizing complex. We derive it from a more general result involving the local cohomology with respect to an arbitrary ideal of a Noetherian ring admitting a dualizing complex, originally proved by Hartshorne for a regular ring of finite Krull dimension. We also extend Hartshorne's affine duality stated for regular rings of finite Krull dimension to any Noetherian ring with a dualizing complex and provide a counterpart in local homology. Among other things we provide duality results involving both local homology and local cohomology, a recurrent theme in this monograph. We also investigate the local homology of a complex with Artinian homology, more generally with mini-max homology. We end the chapter with a short approach to Greenlees' Warwick duality.

12.1 General Dualities and Hartshorne's Affine Duality

In his paper (see [44]) R. Hartshorne proved a certain generalization of the local duality theorem for an arbitrary ideal of a regular ring R of finite Krull dimension. For the same ring R he then proved what he calls an affine duality theorem. In this section we shall generalize Hartshorne's results to any Noetherian ring possessing a dualizing complex. We end the section with a duality formula involving both local homology and cohomology.

First we need a technical result.

Lemma 12.1.1 *Let R be a Noetherian ring, I and J two bounded complexes of injective modules and $\underline{x} = x_1, \ldots, x_k$ a sequence of elements in R. Assume that I has finitely generated cohomology modules. Then the evaluation morphisms*

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$$\operatorname{Hom}_R(I, J) \otimes_R \check{C}_{\underline{x}} \to \operatorname{Hom}_R(I, J \otimes_R \check{C}_{\underline{x}}) \quad \textit{and}$$
$$\operatorname{Hom}_R(I, J) \otimes_R \check{L}_{\underline{x}} \to \operatorname{Hom}_R(I, J \otimes_R \check{L}_{\underline{x}})$$

are quasi-isomorphisms of bounded complexes of flat R-modules.

Proof First of all note that $J \otimes_R \check{C}_{\underline{x}}$ and $J \otimes_R \check{L}_{\underline{x}}$ are bounded complexes of injective R-modules (that is because R is Noetherian). Then observe that the complexes in the statement are R-flat and bounded.

There is a resolution $L \xrightarrow{\sim} I$, where L is a right-bounded complex of finitely generated free R-modules. This yields a commutative square

$$\begin{tikzcd}
\operatorname{Hom}_R(I, J) \otimes_R \check{C}_{\underline{x}} \arrow[r] \arrow[d] & \operatorname{Hom}_R(I, J \otimes_R \check{C}_{\underline{x}}) \arrow[d] \\
\operatorname{Hom}_R(L, J) \otimes_R \check{C}_{\underline{x}} \arrow[r] & \operatorname{Hom}_R(L, J \otimes_R \check{C}_{\underline{x}})
\end{tikzcd}$$

where the horizontal morphisms are given by evaluation. In this square the vertical morphisms are quasi-isomorphisms because J and $J \otimes_R \check{C}_{\underline{x}}$ are bounded complexes of injective R-modules. As the bottom horizontal morphism is an isomorphism by 11.1.2 the conclusion follows for the $\check{C}_{\underline{x}}$ part of the statement. The proof of the $\check{L}_{\underline{x}}$ part is analogous. □

The following result is an extension of Hartshorne's result (see [44, Theorem 2.1]) originally proved for a regular ring of finite Krull dimension and a left-bounded complex X. (Hartshorne reduced it to the case $X = R$. Then he used "way-out" techniques for the case of a left-bounded complex.)

Theorem 12.1.2 *Let R denote a Noetherian ring admitting a dualizing complex D. Let $\underline{x} = x_1, \ldots, x_k$ denote a sequence of elements of R and write $\mathfrak{a} = \underline{x}R$. Let X denote an R-complex. Then there is a natural morphism*

$$\check{C}_{\underline{x}} \otimes_R X \to \operatorname{Hom}_R(\operatorname{Hom}_R(X, D), \check{C}_{\underline{x}} \otimes_R D).$$

It is a quasi-isomorphism if X has finitely generated cohomology modules. In that case it induces an isomorphism

$$\mathrm{R}\Gamma_{\mathfrak{a}}(X) \simeq \mathrm{RHom}_R(\mathrm{RHom}_R(X, D), \mathrm{R}\Gamma_{\mathfrak{a}}(D))$$

in the derived category. More precisely, $\mathrm{R}\Gamma_{\mathfrak{a}}(X)$ *is represented by the complex* $\operatorname{Hom}_R(\operatorname{Hom}_R(X, D), \Gamma_{\mathfrak{a}}(D))$.

Proof By the definition of a dualizing complex the natural morphism $R \to \operatorname{Hom}_R(D, D)$ is a quasi-isomorphism between bounded complexes of flat modules. This together with Lemma 12.1.1 induces quasi-isomorphisms of bounded flat complexes

$$\check{C}_{\underline{x}} \xrightarrow{\sim} \operatorname{Hom}_R(D, D) \otimes_R \check{C}_{\underline{x}} \xrightarrow{\sim} \operatorname{Hom}_R(D, D \otimes_R \check{C}_{\underline{x}}).$$

We apply the tensor product by X and obtain the quasi-isomorphism

$$X \otimes_R \check{C}_{\underline{x}} \xrightarrow{\sim} X \otimes_R \operatorname{Hom}_R(D, D \otimes_R \check{C}_{\underline{x}}).$$

We also have the evaluation morphism

$$X \otimes_R \operatorname{Hom}_R(D, D \otimes_R \check{C}_{\underline{x}}) \to \operatorname{Hom}_R(\operatorname{Hom}_R(X, D), D \otimes_R \check{C}_{\underline{x}}).$$

It is a quasi-isomorphism when X has finitely generated cohomology modules by Proposition 11.1.5. Recall that the complex $D \otimes_R \check{C}_{\underline{x}}$ is a bounded complex of injective R-modules. The first statement is now obtained by composition.

Since $(\check{C}_{\underline{x}} \otimes_R \cdot)$ represents $\mathrm{R}\Gamma_{\mathfrak{a}}(\cdot)$ in the derived category (see 7.4.4) and since $\check{C}_{\underline{x}} \otimes_R D$ and $\Gamma_{\mathfrak{a}}(D)$ are bounded complexes of injective R-modules, the last statements follow by the first. □

12.1.3 Let R, $\mathfrak{a}$ and D be as in Theorem 12.1.2. Here is another proof – in some sense more direct – of the isomorphism $\mathrm{R}\Gamma_{\mathfrak{a}}(X) \simeq \mathrm{RHom}_R(\mathrm{RHom}_R(X, D), \mathrm{R}\Gamma_{\mathfrak{a}}(D))$ in the case when X is homologically left-bounded with finitely generated cohomology modules. In that case X has a left-bounded injective resolution $X \xrightarrow{\sim} I$. The complex $\operatorname{Hom}_R(I, D)$ is right-bounded and has finitely generated homology modules (see 4.5.14 or 11.2.2). Whence it has a free resolution $L \xrightarrow{\sim} \operatorname{Hom}_R(I, D)$, where L is a right-bounded R-complex of finitely generated free modules. Then we have quasi-isomorphisms

$$I \xrightarrow{\sim} \operatorname{Hom}_R(\operatorname{Hom}_R(I, D), D) \xrightarrow{\sim} \operatorname{Hom}_R(L, D).$$

These are quasi-isomorphisms between left-bounded complexes of injective R-modules. Therefore they induce direct systems of quasi-isomorphisms and isomorphisms

$$\begin{aligned} \operatorname{Hom}_R(R/\mathfrak{a}^t, I) &\xrightarrow{\sim} \operatorname{Hom}_R(R/\mathfrak{a}^t, \operatorname{Hom}_R(L, D)) \\ &\cong \operatorname{Hom}_R(L, \operatorname{Hom}_R(R/\mathfrak{a}^t, D)). \end{aligned}$$

We take direct limits. Because the direct limit functor is exact and in view of Lemma 1.3.4, we obtain a quasi-isomorphism and an isomorphism

$$\Gamma_{\mathfrak{a}}(I) \xrightarrow{\sim} \varinjlim(\operatorname{Hom}_R(L, \operatorname{Hom}_R(\operatorname{Hom}_R(R/\mathfrak{a}^t, D))) \cong \operatorname{Hom}_R(L, \Gamma_{\mathfrak{a}}(D)).$$

There is also a quasi-isomorphism

$$\operatorname{Hom}_R(L, \Gamma_{\mathfrak{a}}(D)) \xleftarrow{\sim} \operatorname{Hom}_R(\operatorname{Hom}_R(I, D), \Gamma_{\mathfrak{a}}(D))$$

because the complex $\Gamma_{\mathfrak{a}}(D)$ is a bounded complex of injective R-modules. Hence the conclusion follows.

A dual statement to Theorem 12.1.2 for local homology will be proved in the following.

Theorem 12.1.4 *Let R denote a Noetherian ring admitting a dualizing complex D. Let $\underline{x} = x_1, \ldots, x_k$ denote a sequence of elements of R and $\mathfrak{a} = \underline{x}R$. Let X denote an R-complex. Then there is a natural morphism*

$$\operatorname{Hom}_R(\check{L}_{\underline{x}}, X) \to \operatorname{Hom}_R(\operatorname{Hom}_R(X, D), \operatorname{Hom}_R(\check{L}_{\underline{x}}, D)).$$

It is a quasi-isomorphism if X has finitely generated cohomology modules.

In that case it induces an isomorphism

$$\mathrm{L}\Lambda^{\mathfrak{a}}(X) \simeq \operatorname{RHom}_R(\operatorname{RHom}_R(X, D), \mathrm{L}\Lambda^{\mathfrak{a}}(D))$$

in the derived category.

Proof By definition (see 11.2.1) there is a morphism

$$X \to \operatorname{Hom}_R(\operatorname{Hom}_R(X, D), D)$$

that is a quasi-isomorphism for an R-complex X with finitely generated cohomology. Because $\check{L}_{\underline{x}}$ is a bounded complex of free R-modules it induces a morphism

$$\operatorname{Hom}_R(\check{L}_{\underline{x}}, X) \to \operatorname{Hom}_R(\check{L}_{\underline{x}}, \operatorname{Hom}_R(\operatorname{Hom}_R(X, D), D))$$

which is a quasi-isomorphism provided X has finitely generated cohomology. By adjointness and commutativity the second complex is isomorphic to

$$\operatorname{Hom}_R(\operatorname{Hom}_R(X, D), \operatorname{Hom}_R(\check{L}_{\underline{x}}, D)),$$

which proves the first part of the claim.

Since $\mathrm{L}\Lambda^{\mathfrak{a}}(\cdot)$ is represented by $\operatorname{Hom}_R(\check{L}_{\underline{x}}, \cdot)$ (see 7.5.12) and since $\operatorname{Hom}_R(\check{L}_{\underline{x}}, D)$ is a bounded complex of injective R-modules this also proves the second part. □

Let R denote a regular ring of finite Krull dimension, complete with respect to the $\mathfrak{a}$-adic topology. Let I denote an injective resolution of R. There is a natural morphism

$$X \to \operatorname{Hom}_R(\operatorname{Hom}_R(X, \Gamma_{\mathfrak{a}}(I)), \Gamma_{\mathfrak{a}}(I)).$$

In the case when X is a complex with finitely generated cohomology R. Hartshorne proved that it is a quasi-isomorphism (see [44, Theorem 4.1]). It is called affine duality. In the following we shall generalize the affine duality to any Noetherian ring possessing a dualizing complex. Note that the injective resolution of a regular ring of finite Krull dimension is a dualizing complex.

Theorem 12.1.5 *Let R denote a Noetherian ring possessing a dualizing complex D. Let $\mathfrak{a} \subset R$ denote an ideal. Then there are natural quasi-isomorphisms*

$$X \otimes_R \hat{R}^{\mathfrak{a}} \xrightarrow{\sim} \operatorname{Hom}_R(\operatorname{Hom}_R(X, \Gamma_{\mathfrak{a}}(D)), \Gamma_{\mathfrak{a}}(D)) \text{ and}$$
$$X \otimes_R \hat{R}^{\mathfrak{a}} \xrightarrow{\sim} \operatorname{Hom}_R(\operatorname{Hom}_R(X, \Gamma_{\mathfrak{a}}(D)), D)$$

for any R-complex X with finitely generated cohomology modules.

Proof We first prove the statement for $X = R$. By the definition of a dualizing complex there is an inverse system of quasi-isomorphisms

$$R/\mathfrak{a}^t \xrightarrow{\sim} \operatorname{Hom}_R(\operatorname{Hom}_R(R/\mathfrak{a}^t, D), D).$$

Both inverse system are degree-wise surjective. By passing to the inverse limit this induces a quasi-isomorphism

$$\hat{R}^{\mathfrak{a}} \xrightarrow{\sim} \operatorname{Hom}_R(\Gamma_{\mathfrak{a}}(D), D),$$

(see 1.2.9) taking into account that

$$\begin{aligned}\operatorname{Hom}_R(\Gamma_{\mathfrak{a}}(D), D) &= \operatorname{Hom}_R(\varinjlim(\operatorname{Hom}_R(R/\mathfrak{a}^t, D)), D)\\ &\cong \varprojlim(\operatorname{Hom}_R(\operatorname{Hom}_R(R/\mathfrak{a}^t, D), D)).\end{aligned}$$

There is also an isomorphism

$$\operatorname{Hom}_R(\Gamma_{\mathfrak{a}}(D)), D) \cong \operatorname{Hom}_R(\Gamma_{\mathfrak{a}}(D)), \Gamma_{\mathfrak{a}}(D)),$$

as is easily seen. This settles the case $X = R$.

For the general case we first note that the above quasi-isomorphisms induce quasi-isomorphisms

$$X \otimes_R \hat{R}^{\mathfrak{a}} \xrightarrow{\sim} X \otimes_R \operatorname{Hom}_R(\Gamma_{\mathfrak{a}}(D), D) \text{ and}$$
$$X \otimes_R \hat{R}^{\mathfrak{a}} \xrightarrow{\sim} X \otimes_R \operatorname{Hom}_R(\Gamma_{\mathfrak{a}}(D), \Gamma_{\mathfrak{a}}(D))$$

because $\hat{R}^{\mathfrak{a}}$, $\operatorname{Hom}_R(\Gamma_{\mathfrak{a}}(D), D)$ and $\operatorname{Hom}_R(\Gamma_{\mathfrak{a}}(D), \Gamma_{\mathfrak{a}}(D))$ are bounded complexes of flat R-modules. Then we compose with the quasi-isomorphisms

$$X \otimes_R \operatorname{Hom}_R(\Gamma_{\mathfrak{a}}(D), D) \xrightarrow{\sim} \operatorname{Hom}_R(\operatorname{Hom}_R(X, \Gamma_{\mathfrak{a}}(D)), D) \text{ and}$$
$$X \otimes_R \operatorname{Hom}_R(\Gamma_{\mathfrak{a}}(D), \Gamma_{\mathfrak{a}}(D)) \xrightarrow{\sim} \operatorname{Hom}_R(\operatorname{Hom}_R(X, \Gamma_{\mathfrak{a}}(D)), \Gamma_{\mathfrak{a}}(D))$$

obtained by Proposition 11.1.5. Recall that D and $\Gamma_{\mathfrak{a}}(D)$ are bounded complexes of injective R-modules. $\square$

There is also a kind of affine duality in local homology.

Theorem 12.1.6 *Let R denote a Noetherian ring possessing a dualizing complex D. Let $\underline{x}$ denote a sequence of elements of R and write $\underline{x}R = \mathfrak{a}$. Let X be an R-complex. Then the natural morphism*

$$X \otimes_R \hat{R}^{\mathfrak{a}} \to \operatorname{Hom}_R(\operatorname{Hom}_R(X \otimes_R \hat{R}^{\mathfrak{a}}, \operatorname{Hom}_R(\check{C}_{\underline{x}}, D)), \operatorname{Hom}_R(\check{C}_{\underline{x}}, D))$$

is a quasi-isomorphism provided X has finitely generated cohomology.

In that case, it induces an isomorphism

$$X \otimes^{\mathrm{L}}_R \hat{R}^{\mathfrak{a}} \simeq \operatorname{RHom}_R(\operatorname{RHom}_R(X \otimes^{\mathrm{L}}_R \hat{R}^{\mathfrak{a}}, \mathrm{L}\Lambda^{\mathfrak{a}}(D)), \mathrm{L}\Lambda^{\mathfrak{a}}(D)))$$

in the derived category.

Proof First recall that $D_{\hat{R}^{\mathfrak{a}}} := \operatorname{Hom}_R(\hat{R}^{\mathfrak{a}}, \operatorname{Hom}_R(\check{C}_{\underline{x}}, D))$ is a dualizing complex of $\hat{R}^{\mathfrak{a}}$ (see Theorem 11.5.5). Hence the natural morphism

$$X \otimes_R \hat{R}^{\mathfrak{a}} \to \operatorname{Hom}_{\hat{R}^{\mathfrak{a}}}(\operatorname{Hom}_{\hat{R}^{\mathfrak{a}}}(X \otimes_R \hat{R}^{\mathfrak{a}}, D_{\hat{R}^{\mathfrak{a}}}), D_{\hat{R}^{\mathfrak{a}}})$$

is a quasi-isomorphism provided X has finitely generated cohomology (in that case the $\hat{R}^{\mathfrak{a}}$-complex $X \otimes_R \hat{R}^{\mathfrak{a}}$ has finitely generated cohomology over $\hat{R}^{\mathfrak{a}}$). Then note that the complex on the right is isomorphic to

$$\operatorname{Hom}_R(\operatorname{Hom}_{\hat{R}^{\mathfrak{a}}}(X \otimes_R \hat{R}^{\mathfrak{a}}, D_{\hat{R}^{\mathfrak{a}}}), \operatorname{Hom}_R(\check{C}_{\underline{x}}, D))$$

and that

$$\operatorname{Hom}_{\hat{R}^{\mathfrak{a}}}(X \otimes_R \hat{R}^{\mathfrak{a}}, D_{\hat{R}^{\mathfrak{a}}}) \cong \operatorname{Hom}_R(X \otimes_R \hat{R}^{\mathfrak{a}}, \operatorname{Hom}_R(\check{C}_{\underline{x}}, D))$$

as follows by a change of rings formula. This gives the first part of the statement.

Since $\operatorname{Hom}_R(\check{C}_{\underline{x}}, D)$ is a bounded complex of injective R-modules representing $\mathrm{L}\Lambda^{\mathfrak{a}}(D)$ (see Theorem 7.5.12) the second part follows. □

In the previous theorem note the difference of the duality with respect to a dualizing complex. Recall that $\operatorname{Hom}_R(\check{C}_{\underline{x}}, D)$ is not necessarily a dualizing complex of $\hat{R}^{\mathfrak{a}}$.

The last duality result of this section is a mixed one.

Proposition 12.1.7 *Let $\mathfrak{a}$ denote an ideal of a Noetherian ring R possessing a dualizing complex D. Let X be an R-complex with finitely generated homology modules. Then there are natural isomorphisms*

$$\hat{R}^{\mathfrak{a}} \otimes^{\mathrm{L}} X \simeq \mathrm{L}\Lambda^{\mathfrak{a}}(X) \simeq \operatorname{RHom}_R(\mathrm{R}\Gamma_{\mathfrak{a}}(\operatorname{RHom}_R(X, D)), D))$$

in the derived category.

Proof For the quasi-isomorphism on the left we refer to 9.8.4. For the one on the right let $F \xrightarrow{\sim} X$ be a DG-flat resolution. Then $\mathrm{L}\Lambda^{\mathfrak{a}}(X)$ is represented by $\Lambda^{\mathfrak{a}}(F)$. On the other hand the quasi-isomorphism $\mathrm{Hom}_R(X, D) \xrightarrow{\sim} \mathrm{Hom}_R(F, D)$ is a DG-injective resolution. Hence $\mathrm{RHom}_R(\mathrm{R}\Gamma_{\mathfrak{a}}(\mathrm{RHom}_R(X, D)), D))$ is represented by $\mathrm{Hom}_R(\Gamma_{\mathfrak{a}}(\mathrm{Hom}_R(F, D)), D))$. We first describe the natural morphism $\Lambda^{\mathfrak{a}}(F) \to \mathrm{Hom}_R(\Gamma_{\mathfrak{a}}(\mathrm{Hom}_R(F, D)), D)$ which will represent a natural map

$$\mathrm{L}\Lambda^{\mathfrak{a}}(X) \to \mathrm{RHom}_R(\mathrm{R}\Gamma_{\mathfrak{a}}(\mathrm{RHom}_R(X, D)), D))$$

in the derived category.

By the definition of a dualizing complex there is a quasi-isomorphism $u : F \xrightarrow{\sim} \mathrm{Hom}_R(\mathrm{Hom}_R(F, D), D)$. It induces morphisms of inverse systems

$$\begin{aligned} R/\mathfrak{a}^t \otimes_R F &\to R/\mathfrak{a}^t \otimes_R \mathrm{Hom}_R(\mathrm{Hom}_R(F, D), D) \\ &\cong \mathrm{Hom}_R(\mathrm{Hom}_R(R/\mathfrak{a}^t, \mathrm{Hom}_R(F, D)), D), \end{aligned}$$

(see also 1.4.6). By passing to the limit they induce a morphism and an isomorphism

$$\Lambda^{\mathfrak{a}}(F) \to \varprojlim(\mathrm{Hom}_R(\mathrm{Hom}_R(R/\mathfrak{a}^t, \mathrm{Hom}_R(F, D)), D)) \cong \mathrm{Hom}_R(\Gamma_{\mathfrak{a}}(\mathrm{Hom}_R(F, D)), D).$$

It remains to show that the above morphism is a quasi-isomorphism.

Consider first the case of a right-bounded complex X. In this case we have a right-bounded DG-flat resolution $F \xrightarrow{\sim} X$. Then $\mathrm{Hom}_R(F, D)$ is a left-bounded complex of injective R-modules and $\mathrm{Hom}_R(\mathrm{Hom}_R(F, D), D)$ is a right-bounded complex of flat R-modules, hence DG-flat. It follows that the morphisms $R/\mathfrak{a}^t \otimes_R u$ are quasi-isomorphisms (see 4.4.11) and that we have an inverse system of quasi-isomorphisms

$$R/\mathfrak{a}^t \otimes_R F \xrightarrow{\sim} \mathrm{Hom}_R(\mathrm{Hom}_R(R/\mathfrak{a}^t, \mathrm{Hom}_R(F, D)), D).$$

Since these inverse systems are degree-wise surjective it induces – by passing to the limit – a quasi-isomorphism

$$\begin{aligned} \Lambda^{\mathfrak{a}}(F) &\xrightarrow{\sim} \varprojlim(\mathrm{Hom}_R(\mathrm{Hom}_R(R/\mathfrak{a}^t, \mathrm{Hom}_R(F, D)), D)) \\ &\cong \mathrm{Hom}_R(\Gamma_{\mathfrak{a}}(\mathrm{Hom}_R(F, D)), D), \end{aligned}$$

(see 1.2.9). This settles the case where X is right-bounded.

For the general case let $\underline{x} = x_1, \ldots, x_k$ be a sequence generating the ideal $\mathfrak{a}$. Then $\mathrm{L}\Lambda^{\mathfrak{a}}(X)$ is also represented by the complex $\mathrm{Hom}_R(\check{L}_{\underline{x}}, X)$ (see 7.5.12) while $\mathrm{RHom}_R(\mathrm{R}\Gamma_{\mathfrak{a}}(\mathrm{RHom}_R(X, D)), D))$ is represented by the complex

$$\mathrm{Hom}_R(\check{C}_{\underline{x}} \otimes_R \mathrm{Hom}_R(X, D), D)$$

(see 7.4.4). It follows that the homology modules

$$H_i(\mathrm{L}\Lambda^{\mathfrak{a}}(X)) \text{ and } H_i(\mathrm{RHom}_R(\mathrm{R}\Gamma_{\mathfrak{a}}(\mathrm{RHom}_R(X, D)), D)))$$

depend only on a soft truncation $\cdots \to X_{n+1} \to X_n \to \mathrm{Ker}(d_n^X) \to 0$ for $n \gg 0$. This reduces the general case to the case of a right-bounded complex and finishes the proof. □

12.2 Local Duality with Dualizing Complexes

In this section we will give another version of Grothendieck's Local Duality Theorem for unbounded complexes. Then we draw some of its consequences.

Theorem 12.2.1 *Let $(R, \mathfrak{m}, \Bbbk)$ denote a Noetherian local ring of dimension d with a normalized dualizing complex D. Let X be any R-complex with finitely generated cohomology modules. Then there is a natural isomorphism*

$$\mathrm{R}\Gamma_{\mathfrak{m}}(X) \simeq \mathrm{RHom}_R(\mathrm{RHom}_R(X, D), E_R(\Bbbk)^{[-d]})$$

in the derived category. In particular, there are natural isomorphisms

$$H^i_{\mathfrak{m}}(X) \cong \mathrm{Hom}_R(\mathrm{Ext}^{d-i}_R(X, D), E_R(\Bbbk))$$

for all $i \in \mathbb{Z}$.

Proof We put $\mathfrak{a} = \mathfrak{m}$ in Theorem 12.1.2 and recall the quasi-isomorphism $\Gamma_{\mathfrak{m}}(D) \xrightarrow{\sim} E_R(\Bbbk)^{[-d]}$ (see 11.4.9). This provides the first part of the statement. The second one follows by taking cohomology. □

Remark 12.2.2 Theorem 12.2.1 was already proved in Hartshorne's Lecture Notes for a left-bounded complex (see [43, Theorem 6.2, p. 278]) with a rather different proof. Hartshorne reduced it to the case when X is a finitely generated module. Then he used "way-out" techniques for the case of a left-bounded complex.

Note that we have already seen another version of Grothendieck's Local Duality Theorem in 10.6.2 without the assumption of the existence of a dualizing complex. The present approach is more intrinsic and also, in some sense, more functorial.

If R is a Gorenstein local ring, then we may take for D the minimal injective resolution of R and we recover Grothendieck's Local Duality as stated in 10.5.7.

Put $X = R$ in Theorem 12.2.1 and note that $\mathrm{Ext}^i_R(R, D) \cong H^i(D)$. Then we also obtain information on the cohomology of a dualizing complex.

Corollary 12.2.3 *Let $(R, \mathfrak{m}, \Bbbk)$ denote a Noetherian local ring with a normalized dualizing complex D and $d = \dim R$. Then*

$$\mathrm{Hom}_R(H^i(D), E_R(\Bbbk)) \cong H^{d-i}_{\mathfrak{m}}(R).$$

The existence of a dualizing complex also implies the existence of canonical modules.

Corollary 12.2.4 *Let $(R, \mathfrak{m}, \Bbbk)$ denote a Noetherian local ring of dimension d. Assume that R admits a normalized dualizing complex D. Let M be a finitely generated R-module of dimension n. Then M admits the canonical module*

$$K_M := \operatorname{Ext}_R^{d-n}(M, D).$$

In particular, $K_R \cong H^0(D)$.

Proof By 12.2.1 the module $\operatorname{Ext}_R^{d-n}(M, D)$ satisfies condition (i) of 10.3.2. $\square$

As an application we describe the localization of a canonical module in the case when the underlying ring possesses a dualizing complex.

Corollary 12.2.5 *Let $(R, \mathfrak{m})$ denote a Noetherian local ring possessing a normalized dualizing complex D. Let K_R denote its canonical module. Suppose $\mathfrak{p} \in \operatorname{Spec} R$ is a prime ideal such that $\dim R_{\mathfrak{p}} + \dim R/\mathfrak{p} = \dim R$. Then $(K_R)_{\mathfrak{p}}$ is the canonical module $K_{R_{\mathfrak{p}}}$ of $R_{\mathfrak{p}}$.*

Proof When $\dim R_{\mathfrak{p}} + \dim R/\mathfrak{p} = \dim R$ note that $D \otimes_R R_{\mathfrak{p}}$ is a normalized dualizing complex of $R_{\mathfrak{p}}$ in view of 11.4.10. Hence the statement follows by 12.2.4 and localization. $\square$

Note that the previous statement is false if $\dim R_{\mathfrak{p}} + \dim R/\mathfrak{p} < \dim R$, as follows by easy examples. For a Cohen–Macaulay ring R the equality holds for any prime ideal $\mathfrak{p}$.

We also obtain another characterization of the canonical module of a Cohen–Macaulay local ring.

Proposition 12.2.6 *Let $(R, \mathfrak{m}, \Bbbk)$ be a Cohen–Macaulay local ring of dimension d and C a finitely generated module with its minimal injective resolution $C \xrightarrow{\sim} I$. The following conditions are equivalent:*

(i) *$C \cong K_R$ is the canonical module of R,*
(ii) *C is a maximal Cohen–Macaulay module of finite injective dimension such that $\operatorname{Ext}_R^d(\Bbbk, C) \cong \Bbbk$, and*
(iii) *I is a normalized dualizing complex for R.*

Proof The implication (i) $\Rightarrow$ (ii) is shown in 10.3.7.

When conditions (ii) is satisfied we have $\operatorname{id}_R(C) = \operatorname{depth}(R) = d$ (see 10.1.4) and $\operatorname{depth}(C) = d = \operatorname{ext}_R^-(\Bbbk, C)$. In view of Theorem 11.4.2 and Remark 11.4.6 this together with the condition $\operatorname{Ext}_R^d(\Bbbk, C) \cong \Bbbk$ implies that the minimal injection resolution I of C is a normalized dualizing complex for R.

In general, note that $C = H^0(I) = \operatorname{Ext}_R^0(R, I)$. When condition (iii) is satisfied we have $C^\vee \cong (\operatorname{Ext}_R^0(R, I))^\vee \cong H_{\mathfrak{m}}^d(R)$ in view of Theorem 12.2.1. Hence C is the canonical module of R by definition (see 10.3.2). $\square$

12.3 Extensions of Dualities, Complexes with Mini-Max Homology

Let $(R, \mathfrak{m}, \Bbbk)$ denote a Noetherian local ring with $E := E_R(\Bbbk)$ the injective hull of the residue field. Let $\underline{x} = x_1, \ldots, x_k$ denote a sequence of elements of R and put $\underline{x}R = \mathfrak{a}$. We have a quasi-isomorphism $\operatorname{Hom}_R(\check{C}_{\underline{x}}, E) \xrightarrow{\sim} \operatorname{Hom}_R(\check{L}_{\underline{x}}, E)$ of bounded complexes of injective modules since $\check{L}_{\underline{x}} \xrightarrow{\sim} \check{C}_{\underline{x}}$ is a bounded free resolution of $\check{C}_{\underline{x}}$. In the case when $\operatorname{Rad}(\underline{x}R) = \mathfrak{m}$ it was shown in 11.2.5 that $\check{L}_{\underline{x}}^{\vee} := \operatorname{Hom}_R(\check{L}_{\underline{x}}, E)$ is a dualizing complex of $\hat{R}^{\mathfrak{m}}$. That is, there is a quasi-isomorphism

$$X \xrightarrow{\sim} \operatorname{Hom}_{\hat{R}^{\mathfrak{m}}}(\operatorname{Hom}_{\hat{R}^{\mathfrak{m}}}(X, \check{L}_{\underline{x}}^{\vee}), \check{L}_{\underline{x}}^{\vee})$$

for any $\hat{R}^{\mathfrak{m}}$-complex X with finitely generated cohomology modules. In the following we discuss this phenomenon in the more general situation of any proper ideal $\mathfrak{a}$. The most interesting feature of this section is an application to complexes with Artinian homology modules, more generally with "mini-max" homology modules. However, our first results in this direction concern complexes with finitely generated cohomology modules.

Theorem 12.3.1 *Let $(R, \mathfrak{m}, \Bbbk)$ denote a Noetherian local ring and let $\underline{x} = x_1, \ldots, x_k$ be a sequence of elements in R. Then there are quasi-isomorphisms*

$$X \otimes_R \hat{R}^{\mathfrak{m}} \xleftarrow{\sim} X \otimes_R \operatorname{Hom}_R(\check{L}_{\underline{x}}^{\vee}, \check{L}_{\underline{x}}^{\vee}) \xrightarrow{\sim} \operatorname{Hom}_R(\operatorname{Hom}_R(X, \check{L}_{\underline{x}}^{\vee}), \check{L}_{\underline{x}}^{\vee})$$

for any R-complex X with finitely generated cohomology modules.

Proof The quasi-isomorphism on the right is given by evaluation (see 11.1.5). For the one on the left we recall the quasi-isomorphism $\operatorname{Hom}_R(\check{L}_{\underline{x}}^{\vee}, \check{L}_{\underline{x}}^{\vee}) \xrightarrow{\sim} \hat{R}^{\mathfrak{m}}$ obtained in 10.2.7. This is a quasi-isomorphism between bounded complexes of flat modules. We tensor it by X, this provides the quasi-isomorphism on the left. □

Corollary 12.3.2 *In the situation of 12.3.1 assume, moreover, that R is $\mathfrak{m}$-complete. Then the natural morphism*

$$X \to \operatorname{Hom}_R(\operatorname{Hom}_R(X, \check{L}_{\underline{x}}^{\vee}), \check{L}_{\underline{x}}^{\vee})$$

is a quasi-isomorphism for any R-complex X with finitely generated cohomology modules.

Proof When R is $\mathfrak{m}$-complete the natural morphism $R \to \operatorname{Hom}_R(\check{L}_{\underline{x}}^{\vee}, \check{L}_{\underline{x}}^{\vee})$ is a quasi-isomorphism (see 10.2.7) between bounded flat R-complexes. Hence it induces a quasi-isomorphism $X \to X \otimes_R \operatorname{Hom}_R(\check{L}_{\underline{x}}^{\vee}, \check{L}_{\underline{x}}^{\vee})$. □

The previous result is remarkable in the sense that the left side of the quasi-isomorphism does not depend upon the sequence $\underline{x}$. In the case when $\underline{x}$ is the empty set we have that $\check{L}_{\underline{x}} = R$, so that 12.3.2 is nothing else but the Matlis duality. In the case of Rad $\underline{x}R = \mathfrak{m}$ and a complete Noetherian local ring $(R, \mathfrak{m})$ we get the duality of the dualizing complex.

Our next results are valid over any commutative ring R. We follow the notations introduced in 1.4.8, that is, E is the injective cogenerator described in 1.4.8 and the general Matlis duality functor is defined by $(\cdot)^\vee := \operatorname{Hom}_R(\cdot, E)$. In this generality we note that the complex $\check{L}_{\underline{x}}^\vee$ is still DG-injective. When R is local with maximal ideal $\mathfrak{m}$ then $E = E_R(R/\mathfrak{m})$.

Theorem 12.3.3 *Let R denote any commutative ring and let $\underline{x} = x_1, \ldots, x_k$ be any sequence in R. There is a quasi-isomorphism*

$$\operatorname{Hom}_R(\operatorname{Hom}_R(X, \check{L}_{\underline{x}}^\vee), \check{L}_{\underline{x}}^\vee) \xrightarrow{\sim} \operatorname{Hom}_R(\check{L}_{\underline{x}}, X^{\vee\vee})$$

for any R-complex X.

Proof By adjointness and commutativity there is an isomorphism

$$\operatorname{Hom}_R(\operatorname{Hom}_R(X, \check{L}_{\underline{x}}^\vee), \check{L}_{\underline{x}}^\vee) \cong \operatorname{Hom}_R(\check{L}_{\underline{x}} \otimes_R \operatorname{Hom}_R(\check{L}_{\underline{x}}, X^\vee), E).$$

By 6.5.4 (d) there is a quasi-isomorphism

$$\check{L}_{\underline{x}} \otimes_R X^\vee \xrightarrow{\sim} \check{L}_{\underline{x}} \otimes_R \operatorname{Hom}_R(\check{L}_{\underline{x}}, X^\vee).$$

By applying $\operatorname{Hom}_R(\cdot, E)$ and using adjointness this proves the statement. $\square$

The previous statement provides an interesting relation between a variant of the duality with respect to the complex $\check{L}_{\underline{x}}^\vee$ and the Matlis duality.

Corollary 12.3.4 *In the situation of 12.3.3 assume, moreover, that the sequence $\underline{x}$ is weakly pro-regular and let $\mathfrak{a}$ be a finitely generated ideal such that* $\operatorname{Rad}(\mathfrak{a}) = \operatorname{Rad}(\underline{x}R)$. *There is an isomorphism in the derived category*

$$\operatorname{RHom}_R(\operatorname{RHom}_R(X, \check{L}_{\underline{x}}^\vee), \check{L}_{\underline{x}}^\vee) \simeq \mathrm{L}\Lambda^{\mathfrak{a}}(X^{\vee\vee}) \simeq (\mathrm{R}\Gamma_{\mathfrak{a}}(X^\vee))^\vee.$$

Proof This is a consequence of 12.3.3 because $\mathrm{L}\Lambda^{\mathfrak{a}}((X)^{\vee\vee})$ is represented by $\operatorname{Hom}_R(\check{L}_{\underline{x}}, X^{\vee\vee})$ in the derived category (see 7.5.12). See also 9.2.5 for the last isomorphism. $\square$

Corollary 12.3.5 *Let $(R, \mathfrak{m}, \Bbbk)$ be a Noetherian local ring, complete in its $\mathfrak{m}$-adic topology, and let D denote a dualizing complex for R. Then* $\operatorname{Hom}_R(\operatorname{Hom}_R(X, D), D)$ *represents* $(\mathrm{R}\Gamma_{\mathfrak{m}}(X^\vee))^\vee \simeq \mathrm{L}\Lambda^{\mathfrak{m}}(X^{\vee\vee})$ *in the derived category.*

Proof This follows by 12.3.4 applied to a system of parameters $\underline{x}$ of R. That is because for such an $\underline{x}$ the complex $\check{L}_{\underline{x}}^\vee$ is a dualizing complex for R (see 11.2.5), hence quasi-isomorphic up to shift to D. $\square$

We now apply the above to complexes with Artinian homology modules, more generally with "mini-max" homology modules.

Recall 12.3.6 Let $(R, \mathfrak{m}, \mathbb{k})$ denote a Noetherian local ring, complete in its $\mathfrak{m}$-adic topology, and write as before $E = E_R(\mathbb{k})$ for the injective hull of the residue field. Let M denote an R-module. It is known that the natural homomorphism

$$M \to \operatorname{Hom}_R(\operatorname{Hom}_R(M, E), E)$$

is an isomorphism if and only if there is a finitely generated submodule $N \subset M$ such that M/N is an Artinian module (see [90] or [10]). In that case one says that M is a "mini-max" R-module. More generally, let X denote an R-complex with "mini-max" homology modules. By the exactness of the functor $\operatorname{Hom}_R(\operatorname{Hom}_R(\cdot, E), E)$ and a degree-wise inspection in homology it follows that the natural morphism

$$X \to \operatorname{Hom}_R(\operatorname{Hom}_R(X, E), E)$$

is a quasi-isomorphism.

We now apply 12.3.3 and 12.3.4 to R-complexes with "mini-max" homology modules.

Theorem 12.3.7 *Let $(R, \mathfrak{m}, \mathbb{k})$ denote a Noetherian local ring, complete in its $\mathfrak{m}$-adic topology. Let $\underline{x} = x_1, \ldots, x_k$ be any sequence of elements and let $\mathfrak{a}$ be an ideal with* $\operatorname{Rad}(\underline{x}R) = \operatorname{Rad}(\mathfrak{a})$. *Then there are quasi-isomorphisms*

$$\operatorname{Hom}_R(\operatorname{Hom}_R(X, \check{L}_{\underline{x}}^{\vee}), \check{L}_{\underline{x}}^{\vee}) \xrightarrow{\sim} \operatorname{Hom}_R(\check{L}_{\underline{x}}, X^{\vee\vee}) \xleftarrow{\sim} \operatorname{Hom}_R(\check{L}_{\underline{x}}, X)$$

for any R-complex X with "mini-max" homology modules.

For such an R-complex there is an isomorphism

$$\mathrm{L}\Lambda^{\mathfrak{a}}(X) \simeq \operatorname{RHom}_R(\operatorname{RHom}_R(X, \check{L}_{\underline{x}}^{\vee}), \check{L}_{\underline{x}}^{\vee})$$

in the derived category. In particular, there are isomorphisms

$$\Lambda_i^{\mathfrak{a}}(X) \cong H_i(\operatorname{Hom}_R(\operatorname{Hom}_R(X, \check{L}_{\underline{x}}^{\vee}), \check{L}_{\underline{x}}^{\vee}))$$

for all $i \in \mathbb{Z}$.

Proof The natural morphism $X \to X^{\vee\vee}$ is a quasi-isomorphism for any R-complex X with "mini-max" cohomology modules (see 12.3.6). Since $\check{L}_{\underline{x}}$ is a bounded complex of free R-modules it induces a quasi-isomorphism

$$\operatorname{Hom}_R(\check{L}_{\underline{x}}, X) \xrightarrow{\sim} \operatorname{Hom}_R(\check{L}_{\underline{x}}, X^{\vee\vee}).$$

This together with 12.3.3 proves the quasi-isomorphisms. The other statements follow since $\operatorname{Hom}_R(\check{L}_{\underline{x}}, X)$ represents $\mathrm{L}\Lambda^{\mathfrak{a}}(X)$ in the derived category (see 7.5.12) and since $\check{L}_{\underline{x}}^{\vee}$ is DG-injective. □

Corollary 12.3.8 *Let $(R, \mathfrak{m}, \Bbbk)$ denote an $\mathfrak{m}$-complete Noetherian local ring and let D denote a dualizing complex for R. For any R-complex X with "mini-max" homology modules there is an isomorphism*

$$\mathrm{L}\Lambda^{\mathfrak{m}}(X) \simeq \operatorname{RHom}_R(\operatorname{RHom}_R(X, D), D)$$

in the derived category. In particular, there are isomorphisms

$$\Lambda_i^{\mathfrak{m}}(X) \cong H_i(\operatorname{Hom}_R(\operatorname{Hom}_R(X, D), D))$$

for all $i \in \mathbb{Z}$.

Proof Let $\underline{x}$ be a sequence in R such that $\operatorname{Rad}(\underline{x}R) = \mathfrak{m}$. Then $\check{L}_{\underline{x}}^{\vee}$ is a dualizing complex of R (see 11.2.5) and there is up to a shift a quasi-isomorphism $\check{L}_{\underline{x}}^{\vee} \xrightarrow{\sim} D$ by the uniqueness Theorem 11.4.4. Hence the statement follows by 12.3.7. □

If in the above corollary we take for X an R-complex with finitely generated homology modules then $\mathrm{L}\Lambda^{\mathfrak{m}}(X) \simeq X$ in view of 9.8.4 and the corollary provides the property of a dualizing complex. But the above corollary provides yet another property of a dualizing complex in the case of a complete Noetherian local ring. In the following we shall partially extend this to the situation of any Noetherian local ring possessing a dualizing complex.

Proposition 12.3.9 *Let $(R, \mathfrak{m}, \Bbbk)$ denote a Noetherian local ring with a dualizing complex D. Let X be an R-complex with Artinian homology modules. Then there is an isomorphism*

$$\mathrm{L}\Lambda^{\mathfrak{m}}(X) \simeq \operatorname{RHom}_R(\operatorname{RHom}_R(X, D \otimes_R \hat{R}^{\mathfrak{m}}), D \otimes_R \hat{R}^{\mathfrak{m}}))$$

in the derived category.

Proof Recall first that any Artinian R-module M is $\mathfrak{m}$-torsion, hence there is an isomorphism $M \cong \hat{R}^{\mathfrak{m}} \otimes_R M$ (see 2.2.6). Because $\hat{R}^{\mathfrak{m}}$ is R-flat it follows that the natural morphism $X \to \hat{R}^{\mathfrak{m}} \otimes_R X$ is a quasi-isomorphism. Hence the $\hat{R}^{\mathfrak{m}}$-complex $\hat{R}^{\mathfrak{m}} \otimes_R X$ has Artinian homology and there is an isomorphism

$$\mathrm{L}\Lambda^{\mathfrak{m}}(X) \simeq \mathrm{L}\Lambda^{\mathfrak{m}}(\hat{R}^{\mathfrak{m}} \otimes_R X)$$

in the derived category. On the other hand $\mathrm{L}\Lambda^{\mathfrak{m}}(\hat{R}^{\mathfrak{m}} \otimes_R X)$ and $\mathrm{L}\Lambda^{\mathfrak{m}\hat{R}^{\mathfrak{m}}}(\hat{R}^{\mathfrak{m}} \otimes_R X)$ are both represented by the same complex (namely $\Lambda^{\mathfrak{m}}(F)$, where $F \xrightarrow{\sim} X$ is a K-flat resolution of X (see 9.8.3)). Now let $\underline{x} = x_1, \ldots, x_d$ be a system of parameters of R. Then both complexes $\check{L}_{\underline{x}}^{\vee}$ and $\operatorname{Hom}_R(\hat{R}^{\mathfrak{m}}, \operatorname{Hom}_R(\check{L}_{\underline{x}}, D))$ are dualizing complexes for $\hat{R}^{\mathfrak{m}}$ (see 11.2.5 and 11.5.5). By the uniqueness Theorem 11.4.4 there are up to a shift quasi-isomorphisms $\check{L}_{\underline{x}}^{\vee} \xrightarrow{\sim} \operatorname{Hom}_R(\hat{R}^{\mathfrak{m}}, \operatorname{Hom}_R(\check{L}_{\underline{x}}, D)) \xrightarrow{\sim} \check{L}_{\underline{x}}^{\vee}$. In view of the above and Theorem 12.3.7 it follows that $\mathrm{L}\Lambda^{\mathfrak{m}}(X)$ is represented by the complex

$$\begin{aligned} &\operatorname{Hom}_{\hat{R}^{\mathfrak{m}}}(\operatorname{Hom}_{\hat{R}^{\mathfrak{m}}}(X \otimes_R \hat{R}^{\mathfrak{m}}, \operatorname{Hom}_R(\hat{R}^{\mathfrak{m}}, \operatorname{Hom}_R(\check{L}_{\underline{x}}, D))), \operatorname{Hom}_R(\hat{R}^{\mathfrak{m}}, \operatorname{Hom}_R(\check{L}_{\underline{x}}, D))) \\ &\quad \cong \operatorname{Hom}_R(\operatorname{Hom}_R(X, \operatorname{Hom}_R(\hat{R}^{\mathfrak{m}}, \operatorname{Hom}_R(\check{L}_{\underline{x}}, D))), \operatorname{Hom}_R(\check{L}_{\underline{x}}, D)). \end{aligned}$$

On the other hand the minimal injective resolution I of $D \otimes \hat{R}^{\mathfrak{m}}$ is also a dualizing complex for $\hat{R}^{\mathfrak{m}}$ (see 11.5.2) and the complex $\operatorname{Hom}_R(\check{L}_{\underline{x}}, D)$ is quasi-isomorphic to $D \otimes \hat{R}^{\mathfrak{m}}$ (see 11.5.3). The conclusion follows. □

Related to 12.3.6 we want to pose the following question.

Question 12.3.10 It would be of some interest to understand for which R-modules M the morphism $M \to \operatorname{Hom}_R(\operatorname{Hom}_R(M, D), D)$ is a quasi-isomorphism where R is a Noetherian local ring possessing a dualizing complex D.

12.4 Local Duality for Arbitrary Complexes

The dualities of the first two sections of this chapter concerned the dualizing complex of a Noetherian ring and complexes with finitely generated cohomology. In the third section we also considered dualities involving a complex of the form $\check{L}_{\underline{x}}^{\vee}$ and obtained results for complexes with "mini-max" cohomology. In this section we consider arbitrary complexes.

We start with some recalls from the previous chapters.

Recalls 12.4.1 Let $\mathfrak{a}$ be a finitely generated ideal of a commutative ring R and assume there is a weakly pro-regular sequence $\underline{x} = x_1, \ldots, x_k$ such that $\operatorname{Rad}(\mathfrak{a}) = \operatorname{Rad}(\underline{x}R)$. Let I be a K-injective complex. We have the adjointness isomorphism

$$\operatorname{Hom}_R(\check{C}_{\underline{x}} \otimes_R X, I) \simeq \operatorname{Hom}_R(X, \operatorname{Hom}_R(\check{C}_{\underline{x}}, I)).$$

Since $\mathrm{R}\Gamma_{\mathfrak{a}}(X)$ is represented by $\check{C}_{\underline{x}} \otimes_R X$ (see 7.4.4) it induces an isomorphism

$$\mathrm{RHom}_R(\mathrm{R}\Gamma_{\mathfrak{a}}(X), I) \simeq \mathrm{RHom}_R(X, \mathrm{RHom}_R(\check{C}_{\underline{x}}, I))$$

in the derived category. If $Y \xrightarrow{\sim} I$ is a K-injective resolution of an R-complex Y, then $\operatorname{Hom}_R(\check{C}_{\underline{x}}, I)$ represents $\mathrm{L}\Lambda^{\mathfrak{a}}(Y)$ (see 7.5.12) and we recover the adjointness formula

$$\mathrm{RHom}_R(\mathrm{R}\Gamma_{\mathfrak{a}}(X), Y) \simeq \mathrm{RHom}_R(X, \mathrm{L}\Lambda^{\mathfrak{a}}(Y))$$

already obtained in 9.2.4 in a slightly different way.

In the above result we now take for $\underline{x}$ a system of parameters of a Noetherian local ring $(R, \mathfrak{m}, \Bbbk)$ and for I the injective hull of the residue field. We then obtain a Grothendieck local duality formula valid for any R-complex.

Proposition 12.4.2 *Let $(R, \mathfrak{m}, \Bbbk)$ denote a Noetherian local ring of dimension d. Write E for the injective hull of $\Bbbk$ and $D_{\hat{R}^{\mathfrak{m}}}$ for a normalized dualizing complex of the completion $\hat{R}^{\mathfrak{m}}$ of R . Then in the derived category there is an isomorphism*

$$\mathrm{RHom}_R(\mathrm{R}\Gamma_{\mathfrak{m}}(X), E) \simeq \mathrm{RHom}_R(X, D_{\hat{R}^{\mathfrak{m}}}^{[d]})$$

for any R-complex X. In particular, we have isomorphisms

$$\mathrm{Hom}_R(H^i_{\mathfrak{m}}(X), E) \cong \mathrm{Ext}_R^{d-i}(X, D_{\hat{R}^{\mathfrak{m}}})$$

for all $i \in \mathbb{Z}$.

Proof Let $\underline{x} = x_1, \ldots, x_d$ denote a system of parameters of R. In view of 12.4.1 there is an isomorphism $\mathrm{RHom}_R(\mathrm{R}\Gamma_{\mathfrak{m}}(X), E) \simeq \mathrm{RHom}_R(X, \mathrm{RHom}_R(\check{C}_{\underline{x}}, E))$ in the derived category. Moreover, $\mathrm{Hom}_R(\check{C}_{\underline{x}}, E)^{[-d]}$ is a normalized dualizing complex of $\hat{R}^{\mathfrak{m}}$ (see 11.4.7). By the uniqueness Theorem 11.4.4 there is a quasi-isomorphism $\mathrm{Hom}_R(\check{C}_{\underline{x}}, E) \xrightarrow{\sim} D_{\hat{R}^{\mathfrak{m}}}^{[d]}$. The result follows. $\square$

When the Noetherian local ring R itself admits a dualizing complex we obtain the following.

Corollary 12.4.3 *Let $(R, \mathfrak{m}, \Bbbk)$ denote a d-dimensional Noetherian local ring with a normalized dualizing complex D and let X an arbitrary R-complex. Then there is an isomorphism*

$$\mathrm{RHom}_R(\mathrm{R}\Gamma_{\mathfrak{m}}(X), E) \simeq \mathrm{RHom}_R(X, (D \otimes_R \hat{R}^{\mathfrak{m}})^{[d]})$$

in the derived category. In particular, there are isomorphisms

$$\mathrm{Hom}_R(H^i_{\mathfrak{m}}(X), E) \cong \mathrm{Ext}_R^{d-i}(X, D \otimes_R \hat{R}^{\mathfrak{m}})$$

for all $i \in \mathbb{Z}$.

Proof This is because the minimal injective resolution I of the $\hat{R}^{\mathfrak{m}}$-complex $D \otimes_R \hat{R}^{\mathfrak{m}}$ is also a normalized dualizing complex for $\hat{R}^{\mathfrak{m}}$ (see 11.5.2). $\square$

Corollary 12.4.4 *In the situation of 12.4.3 there are isomorphisms*

$$H^{d-i}(D \otimes_R \hat{R}^{\mathfrak{m}}) \cong \Lambda_i^{\mathfrak{m}}(E)$$

for all $i \in \mathbb{Z}$.

Proof This is clear by 12.4.3 for $X = R$. Recall also that $\operatorname{Hom}_R(H^i_{\mathfrak{m}}(R), E) \cong \Lambda_i^{\mathfrak{m}}(E)$ in view of 9.2.5. □

If in Corollary 12.4.4 we take for R a Gorenstein local ring, then we have the quasi-isomorphisms $R \xrightarrow{\sim} D$ and $\hat{R}^{\mathfrak{m}} \xrightarrow{\sim} D \otimes_R \hat{R}^{\mathfrak{m}}$. We recover a property of Gorenstein local rings already mentioned in (10.5.9 (i) $\Rightarrow$ (iii)).

In the following we provide a duality result for the local cohomology with respect to any ideal of a Noetherian ring.

Theorem 12.4.5 *Let $(R, \mathfrak{m}, \Bbbk)$ be a Noetherian local ring possessing a normalized dualizing complex D. Let $\underline{x} = x_1, \ldots, x_k$ denote a sequence of elements of R and write $\mathfrak{a} = \underline{x}R$. Then there is an isomorphism*

$$\operatorname{RHom}_R(\mathrm{R}\Gamma_{\mathfrak{a}}(X), D) \simeq \operatorname{RHom}_R(X, D \otimes_R \hat{R}^{\mathfrak{a}})$$

for any R-complex X.

Proof Recall first that $\operatorname{RHom}_R(\mathrm{R}\Gamma_{\mathfrak{a}}(X), D) \simeq \operatorname{RHom}_R(X, \mathrm{L}\Lambda^{\mathfrak{a}}(D))$. On the other hand $\mathrm{L}\Lambda^{\mathfrak{a}}(D)$ is represented by $D \otimes_R \hat{R}^{\mathfrak{a}}$ (see 9.8.4). The conclusion follows. □

12.5 Greenlees' Warwick Duality

Greenlees' Warwick duality provides another relation between local homology and cohomology. It is also related to Tate's cohomology. Here we present a brief account of these relations.

12.5.1 *The functor $\mathcal{D}_{\mathfrak{a}}$.* Let $\mathfrak{a} \subset R$ denote an ideal of a commutative ring R. The injections $\mathfrak{a}^{t+1} \subset \mathfrak{a}^t$, $t \geq 1$, provide a direct system $\{\operatorname{Hom}_R(\mathfrak{a}^t, X) \mid t \geq 1\}$ for any R-complex X. We define the functor $\mathcal{D}_{\mathfrak{a}}$ by

$$\mathcal{D}_{\mathfrak{a}}(X) = \varinjlim \operatorname{Hom}_R(\mathfrak{a}^t, X).$$

Clearly $\mathcal{D}_{\mathfrak{a}}$ is a left exact additive functor which commutes with shifts. Moreover, it preserves quasi-isomorphisms between K-injective complexes (in view of 4.4.11 and because $\varinjlim$ also preserves quasi-isomorphisms). Hence its right-derived functor $\mathrm{R}\mathcal{D}_{\mathfrak{a}}$ is well defined in the derived category: $\mathrm{R}\mathcal{D}_{\mathfrak{a}}(X)$ is represented by $\mathcal{D}_{\mathfrak{a}}(I)$, where $X \xrightarrow{\sim} I$ denotes a K or DG-injective resolution. We denote the cohomology of $\mathrm{R}\mathcal{D}_{\mathfrak{a}}(X)$ by $\mathrm{R}^i\mathcal{D}_{\mathfrak{a}}(X)$ and note that $\mathrm{R}^i\mathcal{D}_{\mathfrak{a}}(X) := H^i(\mathcal{D}_{\mathfrak{a}}(I)) \cong \varinjlim \operatorname{Ext}^i_R(\mathfrak{a}^t, X)$.

Assume that the resolution $X \xrightarrow{\sim} I$ is DG-injective. Then the short exact sequence $0 \to \mathfrak{a}^t \to R \to R/\mathfrak{a}^t \to 0$ induces an exact sequence

$$0 \to \operatorname{Hom}_R(R/\mathfrak{a}^t, I) \to I \to \operatorname{Hom}_R(\mathfrak{a}^t, I) \to 0.$$

By passing to the direct limit it yields an exact sequence of complexes

$$0 \to \Gamma_{\mathfrak{a}}(I) \to I \to \mathcal{D}_{\mathfrak{a}}(I) \to 0.$$

This induces a long exact cohomology sequence

$$\cdots \to H^i_{\mathfrak{a}}(X) \to H^i(X) \to \mathrm{R}^i\mathcal{D}_{\mathfrak{a}}(X) \to \cdots$$

for all $i \in \mathbb{Z}$.

Note also that $\mathrm{R}\mathcal{D}_{\mathfrak{a}}(X) \simeq 0$ in the derived category if and only if $X \in \mathcal{G}^{\Gamma}_{\mathfrak{a}}$, the class introduced in 9.6.1. When the ideal $\mathfrak{a}$ is generated by a weakly pro-regular sequence Theorem 9.6.11 provides criteria for the vanishing of $\mathrm{R}\mathcal{D}_{\mathfrak{a}}(X)$.

Remark 12.5.2 Assume that the ideal $\mathfrak{a}$ of the commutative ring R is generated by a weakly pro-regular sequence. We have seen in 7.4.2 that the functor $\Gamma_{\mathfrak{a}}$ preserves quasi-isomorphisms between complexes of injective R-modules. It follows that the functor $\mathcal{D}_{\mathfrak{a}}$ also preserves such quasi-isomorphisms and that $\mathrm{R}\mathcal{D}_{\mathfrak{a}}(X)$ may be computed by using an ordinary injective resolution of X.

For the next technical results we refer to the notation in 6.2.6. In particular, for a sequence of elements $\underline{x} = x_1, \ldots, x_k$ of R we recall the short exact sequence $0 \to \check{D}_{\underline{x}} \to \check{C}_{\underline{x}} \to R \to 0$ and the free resolution $\check{L}^{\flat\,[1]}_{\underline{x}} \to \check{D}^{[1]}_{\underline{x}}$ of the global Čech complex $\check{D}^{[1]}_{\underline{x}}$.

The following lemma relates the global Čech complex to $\mathrm{R}\mathcal{D}_{\mathfrak{a}}$.

Lemma 12.5.3 *Let $\underline{x} = x_1, \ldots, x_k$ denote a weakly pro-regular sequence of a commutative ring R and put $\mathfrak{a} = \underline{x}R$. Let J denote a DG-injective complex. Then the complexes $\mathcal{D}_{\mathfrak{a}}(J)$ and $\check{D}^{[1]}_{\underline{x}} \otimes_R J$ are quasi-isomorphic : $\mathcal{D}_{\mathfrak{a}}(J) \simeq \check{D}^{[1]}_{\underline{x}} \otimes_R J$.*

Moreover, let X be an R-complex and $X \xrightarrow{\sim} I$ a K-injective resolution. Then $\mathrm{R}\mathcal{D}_{\mathfrak{a}}(X)$ *is represented by any of the quasi-isomorphic complexes*

$$\check{D}^{[1]}_{\underline{x}} \otimes_R X \xrightarrow{\sim} \check{D}^{[1]}_{\underline{x}} \otimes_R I \xleftarrow{\sim} \check{L}^{\flat\,[1]}_{\underline{x}} \otimes_R I \xleftarrow{\sim} \check{L}^{\flat\,[1]}_{\underline{x}} \otimes_R X.$$

Proof With the previous notation there is a commutative diagram with exact rows

$$\begin{tikzcd}
 & 0 \arrow[r] & \Gamma_{\mathfrak{a}}(J) \arrow[r, "u"] \arrow[d] & J \arrow[r] \arrow[d, equal] & \mathcal{D}_{\mathfrak{a}}(J) \arrow[r] & 0 \\
0 \arrow[r] & \check{D}_{\underline{x}} \otimes_R J \arrow[r] & \check{C}_{\underline{x}} \otimes_R J \arrow[r, "v"] & J \arrow[r] & 0, &
\end{tikzcd}$$

where the first vertical morphism is a quasi-isomorphism (see 7.4.1). The diagram induces quasi-isomorphisms

$$\check{D}^{[1]}_{\underline{x}} \otimes_R J \xrightarrow{\sim} C(v) \xleftarrow{\sim} C(u) \xrightarrow{\sim} \mathcal{D}_{\mathfrak{a}}(J)$$

(see 1.5.5 and 1.5.6). This proves the first part of the claim.

By definition $\mathrm{R}\mathcal{D}_{\mathfrak{a}}(X)$ is represented by $\mathcal{D}_{\mathfrak{a}}(I)$. Let $I \xrightarrow{\sim} J$ be a DG-injective resolution. Then

$$\mathcal{D}_{\mathfrak{a}}(I) \xrightarrow{\sim} \mathcal{D}_{\mathfrak{a}}(J) \simeq \check{D}_{\underline{x}}^{[1]} \otimes_R J$$

in view of 12.5.1 and the first part of the statement. Since $\check{L^{\flat}}_{\underline{x}}^{[1]} \xrightarrow{\sim} \check{D}_{\underline{x}}^{[1]}$ is a quasi-isomorphism of bounded complexes of flat modules (see 6.2.6), hence DG-flat, and in view of 4.4.11 there are also quasi-isomorphisms

$$\check{D}_{\underline{x}}^{[1]} \otimes_R X \xrightarrow{\sim} \check{D}_{\underline{x}}^{[1]} \otimes_R I \xrightarrow{\sim} \check{D}_{\underline{x}}^{[1]} \otimes_R J \xleftarrow{\sim} \check{L^{\flat}}_{\underline{x}}^{[1]} \otimes_R J \xleftarrow{\sim} \check{L^{\flat}}_{\underline{x}}^{[1]} \otimes_R I \xleftarrow{\sim} \check{L^{\flat}}_{\underline{x}}^{[1]} \otimes_R X.$$

The second claim follows. $\square$

The following lemma will be useful.

Lemma 12.5.4 *Let $\underline{x} = x_1, \ldots, x_k$ and $\underline{y} = y_1, \ldots, y_l$ denote two sequences of elements in a commutative ring R, write $\mathfrak{a} = \underline{x}R$ and $\mathfrak{b} = \underline{y}R$. Suppose* $\operatorname{Rad}(\mathfrak{a}) \supseteq \operatorname{Rad}(\mathfrak{b})$. *Then*

(a) The complex $\check{L}_{\underline{y}}^{\flat} \otimes_R \check{L}_{\underline{x}}$ is contractible.

(b) The complex $\operatorname{Hom}_R(\check{L}_{\underline{y}}, \check{L^{\flat}}_{\underline{y}} \otimes_R \operatorname{Hom}_R(\check{L}_{\underline{x}}, X))$ *is exact for any R-complex X.*

Proof Since $\check{L}_{\underline{y}}^{\flat} \otimes_R \check{L}_{\underline{x}}$ is a bounded complex of free R-modules, hence DG-projective, it is sufficient to show that it is exact (see 4.4.4). To this end we tensor the short exact sequence $0 \to \check{L^{\flat}}_{\underline{y}} \to \check{L}_{\underline{y}} \to R \to 0$ of free R-modules by $\check{L}_{\underline{x}}$. Since the induced morphism $\check{L}_{\underline{y}} \otimes_R \check{L}_{\underline{x}} \to \check{L}_{\underline{x}}$ is a quasi-isomorphism (see 6.2.4) this proves the exactness of $L_{\underline{y}}^{\flat} \otimes_R \check{L}_{\underline{x}}$.

For the second statement note first that the telescope $T(\underline{y})$ has the property $T(\underline{y}) \simeq \check{L}_{\underline{y}}$ because both $T(\underline{y})$ and $\check{L}_{\underline{y}}$ are bounded free resolutions of $\check{C}_{\underline{y}}$. Whence it follows that

$$\operatorname{Hom}_R(\check{L}_{\underline{y}}, \check{L^{\flat}}_{\underline{y}} \otimes_R \operatorname{Hom}_R(\check{L}_{\underline{x}}, X)) \simeq$$
$$\operatorname{Hom}_R(T(\underline{y}), \check{L^{\flat}}_{\underline{y}} \otimes_R \operatorname{Hom}_R(\check{L}_{\underline{x}}, X)) \cong M(\underline{y}; \check{L^{\flat}}_{\underline{y}} \otimes_R \operatorname{Hom}_R(\check{L}_{\underline{x}}, X))$$

(see 4.3.6 or 7.5.9 for the isomorphism). Now by definition 7.5.1 it follows that

$$M(\underline{y}; \check{L^{\flat}}_{\underline{y}} \otimes_R \operatorname{Hom}_R(\check{L}_{\underline{x}}, X)) = \operatorname{Mic}(\{K_{\bullet}(\underline{y}^t) \otimes_R \check{L^{\flat}}_{\underline{y}} \otimes_R \operatorname{Hom}_R(\check{L}_{\underline{x}}, X)\}_{t \geq 1}).$$

Moreover, the complexes $K_{\bullet}(\underline{y}^t) \otimes_R \check{L^{\flat}}_{\underline{y}} \xrightarrow{\sim} K_{\bullet}(\underline{y}^t) \otimes_R \check{D}_{\underline{y}}$ are exact for all $t \geq 1$ (see 6.1.8 and recall that $H_i(K_{\bullet}(\underline{y}^t)) \subseteq V(\mathfrak{b})$). Since $K_{\bullet}(\underline{y}^t) \otimes_R \check{L^{\flat}}_{\underline{y}}$ is an exact complex of bounded free R-modules for all $t \geq 1$ the Koszul complexes in the microscope are exact for all $t \geq 1$. It follows that the microscope is exact too (see 4.2.5). This completes the proof. $\square$

Remark 12.5.5 In the situation of 12.5.4 assume that the sequences $\underline{x}$ and $\underline{y}$ are weakly pro-regular. Then $\operatorname{Hom}_R(\check{L}_{\underline{x}}, \cdot)$ represents $\mathrm{L}\Lambda^{\mathfrak{a}}(\cdot)$ and $(\check{L}_{\underline{x}} \otimes_R \cdot) \xrightarrow{\sim} (\check{C}_{\underline{x}} \otimes_R \cdot)$ represents $\mathrm{R}\Gamma_{\mathfrak{a}}(\cdot)$ (see 7.5.12 and 7.4.4). As the shifted complexes

$$\check{L}^{\flat}{}_{\underline{y}}^{[1]} \otimes_R \check{L}_{\underline{x}} \text{ and } \operatorname{Hom}_R(\check{L}_{\underline{y}}, \check{L}^{\flat}{}_{\underline{y}}^{[1]} \otimes_R \operatorname{Hom}_R(\check{L}_{\underline{x}}, X))$$

are exact and in view of 12.5.3 it follows that

$$\mathrm{R}\mathfrak{D}_{\mathfrak{b}}(\mathrm{R}\Gamma_{\mathfrak{a}}(R)) \simeq 0 \simeq \mathrm{L}\Lambda^{\mathfrak{b}}(\mathrm{R}\mathfrak{D}_{\mathfrak{b}}(\mathrm{L}\Lambda^{\mathfrak{a}}(X)))$$

in the derived category.

12.5.6 *Tate's cohomology.* The Tate cohomology is related to the complex

$$\check{L}^{\flat}{}_{\underline{x}}^{[1]} \otimes_R \operatorname{Hom}_R(\check{L}_{\underline{x}}, X)$$

where $\underline{x} = x_1, \ldots, x_k$ is a sequence in the commutative ring R. First note that this complex only depends, up to quasi-isomorphism, on the ideal $\mathfrak{a} := \underline{x}R$. More precisely, let $\underline{y} = y_1, \ldots, y_l$ be another sequence such that $\operatorname{Rad}\ (\underline{y}R) = \operatorname{Rad}\ (\underline{x}R)$. Then there are quasi-isomorphisms $\check{L}_{\underline{x}} \xleftarrow{\sim} \check{L}_{\underline{y}} \xleftarrow{\sim} \check{L}_{\underline{x}}$ (in view of 6.2.4) and these induce a quasi-isomorphism

$$\check{L}^{\flat}{}_{\underline{x}}^{[1]} \otimes_R \operatorname{Hom}_R(\check{L}_{\underline{x}}, X) \xrightarrow{\sim} \check{L}^{\flat}{}_{\underline{y}}^{[1]} \otimes_R \operatorname{Hom}_R(\check{L}_{\underline{y}}, X).$$

Hence we may define the Tate cohomology modules with respect to the ideal $\mathfrak{a} := \underline{x}R$ by

$$\hat{H}^i_{\mathfrak{a}}(X) := \hat{H}^i_{\underline{x}}(X) := H^i(\operatorname{Hom}_R(\check{L}_{\underline{x}}, X) \otimes_R \check{L}^{\flat}{}_{\underline{x}}^{[1]}).$$

For the applications of these Tate cohomology modules in the study of completion theorems and their duals in equivariant topology we refer to [37] and the references therein. Here we only note that the complex $\check{L}^{\flat}{}_{\underline{x}}^{[1]} \otimes_R \operatorname{Hom}_R(\check{L}_{\underline{x}}, X)$ represents $\mathrm{R}\mathfrak{D}_{\mathfrak{a}}(\mathrm{L}\Lambda^{\mathfrak{a}}(X))$ in the derived category when the sequence $\underline{x}$ is weakly pro-regular (in view of 12.5.3 and 7.5.12).

Remark 12.5.7 (a) If we replace the complex $\check{L}_{\underline{x}}$ by the telescope $T(\underline{x})$ (both are quasi-isomorphic complexes of bounded free R-modules) it follows that the complex $\check{L}^{\flat}{}_{\underline{x}}^{[1]} \otimes_R \operatorname{Hom}_R(\check{L}_{\underline{x}}, X)$ is quasi-isomorphic to the complex $T^{\bullet}(X)$ as introduced by J. Greenlees (see [37, Definition 3.1]) for X an R-module.

(b) In his paper (see [37]) Greenlees called the cohomology of

$$\mathrm{R}\mathfrak{D}_{\mathfrak{a}}(X) \simeq \check{L}^{\flat}{}_{\underline{x}}^{[1]} \otimes_R X$$

the Čech cohomology and defined the homology of $\operatorname{Hom}_R(\check{L}^{\flat}{}_{\underline{x}}^{[1]}, X)$ as the Čech homology of X. Greenlees' terminology is different from ours.

Here is the basic result for the so-called Warwick duality. Up to a shift the following Theorem 12.5.8 is the Warwick duality as shown by Greenlees (see [37, Theorem 4.1]).

Theorem 12.5.8 *Let $\underline{x} = x_1, \ldots, x_k$ denote a sequence of elements in a commutative ring R and write $\mathfrak{a} = \underline{x}R$. Then the complexes*

$$\check{L}^{\flat}{}_{\underline{x}}^{[1]} \otimes_R \operatorname{Hom}_R(\check{L}_{\underline{x}}, X) \ \ \text{and} \ \ \operatorname{Hom}_R(\check{L}^{\flat}{}_{\underline{x}}, \check{L}_{\underline{x}} \otimes_R X)$$

are quasi-isomorphic for any R-complex X.

Proof We start with the short exact sequence $0 \to \check{L}^{\flat}{}_{\underline{x}} \to \check{L}_{\underline{x}} \to R \to 0$ of bounded complexes of free R-modules (see 6.2.6) and tensor it by $\operatorname{Hom}_R(\check{L}_{\underline{x}}, X)$. The corresponding sequence is exact. Next we apply the functor $\operatorname{Hom}_R(\check{L}^{\flat}{}_{\underline{x}}, \cdot)$. We get a short exact sequence

$$0 \to \operatorname{Hom}_R(\check{L}^{\flat}{}_{\underline{x}}, \check{L}^{\flat}{}_{\underline{x}} \otimes_R \operatorname{Hom}_R(\check{L}_{\underline{x}}, X)) \xrightarrow{u}$$
$$\operatorname{Hom}_R(\check{L}^{\flat}{}_{\underline{x}}, \check{L}_{\underline{x}} \otimes_R \operatorname{Hom}_R(\check{L}_{\underline{x}}, X)) \to \operatorname{Hom}_R(\check{L}^{\flat}{}_{\underline{x}}, \operatorname{Hom}_R(\check{L}_{\underline{x}}, X)) \to 0.$$

Next we claim that the morphism u is a quasi-isomorphism. To this end it will be enough to show that the third complex in the sequence is exact. By adjointness there is an isomorphism

$$\operatorname{Hom}_R(\check{L}^{\flat}{}_{\underline{x}}, \operatorname{Hom}_R(\check{L}_{\underline{x}}, X)) \cong \operatorname{Hom}_R(\check{L}^{\flat}{}_{\underline{x}} \otimes_R \check{L}_{\underline{x}}, X).$$

Since the complex $\check{L}^{\flat}_{\underline{x}} \otimes_R \check{L}_{\underline{x}}$ is contractible (see 12.5.4) the claim follows.

By 6.5.4 (d) there is a quasi-isomorphism

$$\check{L}_{\underline{x}} \otimes_R X \xrightarrow{\sim} \check{L}_{\underline{x}} \otimes_R \operatorname{Hom}_R(\check{L}_{\underline{x}}, X).$$

Since $\check{L}^{\flat}_{\underline{x}}$ is a bounded complex of free R-modules it provides a quasi-isomorphism

$$\operatorname{Hom}_R(\check{L}^{\flat}{}_{\underline{x}}, \check{L}_{\underline{x}} \otimes_R X) \xrightarrow{\sim} \operatorname{Hom}_R(\check{L}^{\flat}{}_{\underline{x}}, \check{L}_{\underline{x}} \otimes_R \operatorname{Hom}_R(\check{L}_{\underline{x}}, X)).$$

Finally, we have to inspect the complex $\operatorname{Hom}_R(\check{L}^{\flat}{}_{\underline{x}}, \check{L}^{\flat}{}_{\underline{x}} \otimes_R \operatorname{Hom}_R(\check{L}_{\underline{x}}, X))$. To this end we apply the functor $\operatorname{Hom}_R(\cdot, \check{L}^{\flat}{}_{\underline{x}} \otimes_R \operatorname{Hom}_R(\check{L}_{\underline{x}}, X))$ to the short exact sequence $0 \to \check{L}^{\flat}{}_{\underline{x}} \to \check{L}_{\underline{x}} \to R \to 0$ of projective R-modules. It provides a short exact sequence of complexes

$$0 \to \check{L}^{\flat}{}_{\underline{x}} \otimes_R \operatorname{Hom}_R(\check{L}_{\underline{x}}, X) \to \operatorname{Hom}_R(\check{L}_{\underline{x}}, \check{L}^{\flat}{}_{\underline{x}} \otimes_R \operatorname{Hom}_R(\check{L}_{\underline{x}}, X))$$
$$\to \operatorname{Hom}_R(\check{L}^{\flat}{}_{\underline{x}}, \check{L}^{\flat}{}_{\underline{x}} \otimes_R \operatorname{Hom}_R(\check{L}_{\underline{x}}, X)) \to 0.$$

Since the complex in the middle is exact, (see 12.5.4), it follows that

$$\check{L}^{\flat}{}_{\underline{x}}^{[1]} \otimes_R \operatorname{Hom}_R(\check{L}_{\underline{x}}, X) \simeq \operatorname{Hom}_R(\check{L}^{\flat}{}_{\underline{x}}, \check{L}^{\flat}{}_{\underline{x}} \otimes_R \operatorname{Hom}_R(\check{L}_{\underline{x}}, X)).$$

This together with the previous quasi-isomorphisms finishes the proof. □

Next we give a description of the Warwick duality in terms of the derived functors.

Corollary 12.5.9 *Let $\mathfrak{a}$ denote an ideal of a commutative ring R generated by a weakly pro-regular sequence $\underline{x} = x_1, \ldots, x_t$. Then there is an isomorphism*

$$\mathrm{R}\mathcal{D}_{\mathfrak{a}}(\mathrm{L}\Lambda^{\mathfrak{a}}(X)) \simeq \mathrm{RHom}_R(\mathrm{R}\mathcal{D}_{\mathfrak{a}}(R)^{[-1]}, \mathrm{R}\Gamma_{\mathfrak{a}}(X))$$

in the derived category for an R-complex X.

Proof The proof follows in view of the quasi-isomorphism of Theorem 12.5.8 in terms of the derived functors. □

As another application we obtain a spectral sequence for computing the Tate cohomology (see [37, Proposition 4.2]).

Corollary 12.5.10 *In the situation of 12.5.9 we have the spectral sequence*

$$E_2^{i,j} = \operatorname{Ext}_R^i(\mathrm{R}\mathcal{D}_{\mathfrak{a}}(R), H_{\mathfrak{a}}^j(X)) \Longrightarrow E_\infty^{i+j} = \hat{H}_{\mathfrak{a}}^{i+j-1}(X).$$

Proof This follows again by the quasi-isomorphism of Theorem 12.5.8 and the definition of the Tate cohomology. □

For our purposes here the following result provides a relation between local cohomology, derived functors of completion and Tate cohomology. We did not find the result in the literature.

Theorem 12.5.11 *Let $\mathfrak{a} \subset R$ denote an ideal generated by a weakly pro-regular sequence $\underline{x} = x_1, \ldots, x_t$. Then there is a long exact cohomology sequence*

$$\cdots \to H_{\mathfrak{a}}^i(X) \to \Lambda_{-i}^{\mathfrak{a}}(X) \to \check{H}_{\mathfrak{a}}^i(X) \to H_{\mathfrak{a}}^{i+1}(X) \to \cdots$$

for an R-complex X.

Proof We start with the short exact sequence $0 \to \check{L}^{\flat}{}_{\underline{x}} \to \check{L}_{\underline{x}} \to R \to 0$ and tensor it by $\operatorname{Hom}_R(\check{L}_{\underline{x}}, X)$. This yields an exact sequence of complexes

$$0 \to \check{L}^{\flat}{}_{\underline{x}} \otimes_R \operatorname{Hom}_R(\check{L}_{\underline{x}}, X) \to \check{L}_{\underline{x}} \otimes_R \operatorname{Hom}_R(\check{L}_{\underline{x}}, X) \to \operatorname{Hom}_R(\check{L}_{\underline{x}}, X) \to 0.$$

The complex in the middle is quasi-isomorphic to $\check{L}_{\underline{x}} \otimes_R X$ (see 6.5.4) and this complex yields the local cohomology (see 7.4.5). The cohomology of the complex on the right is $\Lambda^{\mathfrak{a}}_{-i}(X))$ (see 7.5.12). Finally, the cohomology of the complex on the left is – up to a shift by one – the Tate cohomology. So the long exact cohomology sequence provides the claim. $\square$

As a particular case let us consider the case of an R-module.

Corollary 12.5.12 *With the notation of 12.5.11 let M denote an R-module. Then there is an exact sequence*

$$\begin{aligned} 0 \to \Lambda^{\mathfrak{a}}_1(M) \to \check{H}^{-1}_{\mathfrak{a}}(M) \to H^0_{\mathfrak{a}}(M) \to \\ \Lambda^{\mathfrak{a}}_0(M) \to \check{H}^0_{\mathfrak{a}}(M) \to H^1_{\mathfrak{a}}(M) \to 0 \end{aligned}$$

and isomorphisms

$$\Lambda^{\mathfrak{a}}_{-i}(M) \cong \check{H}^i_{\mathfrak{a}}(M) \textit{ for } i \le -2 \quad \textit{and} \quad \check{H}^i_{\mathfrak{a}}(M) \cong H^{i+1}_{\mathfrak{a}}(M) \textit{ for } i \ge 1.$$

Proof This is an easy consequence of the long exact sequence shown in 12.5.11. $\square$

It would be of some interest to describe the modules in the exact sequence in more detail. In a certain sense the Tate cohomology of a module combines Čech homology and cohomology.

In the following we shall relate the previous result to the class of complexes $\mathcal{G}^{\Lambda}_{\mathfrak{a}}$ and $\mathcal{G}^{\Gamma}_{\mathfrak{a}}$ introduced in 9.6.1.

Corollary 12.5.13 *In the situation of 12.5.11 let X denote an R-complex.*

(a) Suppose that $X \in \mathcal{G}^{\Gamma}_{\mathfrak{a}}$. Then there is a long exact sequence

$$\cdots \to H^i(X) \to \Lambda^{\mathfrak{a}}_{-i}(X) \to \check{H}^i_{\mathfrak{a}}(X) \to H^{i+1}(X) \to \cdots$$

(b) Suppose that $X \in \mathcal{G}^{\Lambda}_{\mathfrak{a}}$. Then there is a long exact sequence

$$\cdots \to H^i_{\mathfrak{a}}(X) \to H_{-i}(X) \to \check{H}^i_{\mathfrak{a}}(X) \to H^{i+1}_{\mathfrak{a}}(X) \to \cdots$$

Proof The proof for (a) and (b) follows because of the isomorphisms $H^i(X) \cong H^i_{\mathfrak{a}}(X)$ when $X \in \mathcal{G}^{\Gamma}_{\mathfrak{a}}$ and $H_i(X) \cong \Lambda^{\mathfrak{a}}_i(X)$ when $X \in \mathcal{G}^{\Lambda}_{\mathfrak{a}}$. $\square$

Correction to: Completion, Čech and Local Homology and Cohomology

Correction to:
P. Schenzel and A.-M. Simon, *Completion, Čech and Local Homology and Cohomology*, Springer Monographs in Mathematics, https://doi.org/10.1007/978-3-319-96517-8

In the original version of the book, the following corrections have to be incorporated:

In the copyright page of front matter, Anne-Marie Simon's affiliation was published wrongly as "Service de Geometrie differentielle" has been updated as "Service de Géométrie différentielle".

Spacing corrections and corrections in equations and cross-citations were published with errors throughout the book have been corrected.

Incorrect arrow mark has been updated in Appendix (Book Backmatter).

The original version of the book was revised: Author corrections have been incorporated. The correction to the book is available at
https://doi.org/10.1007/978-3-319-96517-8

P. Schenzel and A.-M. Simon, *Completion, Čech and Local Homology and Cohomology*, Springer Monographs in Mathematics,
https://doi.org/10.1007/978-3-319-96517-8_13

Appendix

This appendix covers various aspects of the previous investigations that are important for the necessity of the assumptions we have made. Moreover, it includes some additional results which are important for relations to commutative Noetherian rings. As a sample we prove Grothendieck's non-vanishing theorem in local cohomology as well as a dual version in local homology which seems not to be well-known. Then we study the relation of weakly-pro regular sequences – one of the most important notions in our subject – to the pro-regular sequences as they were introduced by Greenlees and May (see [38]).

The completion functor is neither left nor right exact. The main focus of our investigations is related to the left derived functors of the completion. The right derived functor of the completion was studied for modules in [38]. We extend the Greenlees–May results to the case of unbounded complexes.

In the fourth section we recall the theorems of Bass, Papp and Chase about Noetherian and coherent rings, respectively, as well as Faith's result about self-injective rings. This is the basis for the examples of modules in the last subsection. We present a first example of an injective module such that a localization of it is no longer injective. Note that localizations of injective R-modules are always injective when the ring R is Noetherian. Then we present a ring S such that S^{Λ} is not flat for a certain index set Λ. Note that S^{Λ} is always flat when the ring S is coherent. Finally, we describe a commutative ring S with two injective S-modules I, J such that $\operatorname{Hom}_S(I, J)$ is not S-flat. Recall that $\operatorname{Hom}_S(I, J)$ is always flat when S is coherent and when I and J are injective S-Modules.

A.1 Grothendieck's Non-vanishing Theorem

In this section $(R, \mathfrak{m}, \Bbbk)$ is a Noetherian local ring with its unique maximal ideal $\mathfrak{m}$ and residue field $R/\mathfrak{m} = \Bbbk$. We present a proof of Grothendieck's non-vanishing Theorem in local cohomology (see [41]) and of its counterpart in local homology, independent

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of Cohen's structure theorems for complete Noetherian local rings (for the local cohomology see also [15, 6.1.4]). We first recall that any ideal of R is generated by a weakly pro-regular sequence. We shall make intensive use of Matlis duality, as recalled in 2.3.2. That is, we denote by E the injective hull of $\mathbb{k}$: $E = E_R(\mathbb{k})$, which is an injective cogenerator, and we put $M^\vee = \operatorname{Hom}_R(M, E)$ for the Matlis dual of an R-module M. We simply denote by $\hat{(\cdot)}$ the $\mathfrak{m}$-completion functor and recall that E also has the structure of an $\hat{R}$-module and is also the injective hull of $\mathbb{k}$ as an $\hat{R}$-module.

Theorem A.1.1 *Let* $(R, \mathfrak{m}, \mathbb{k})$ *denote a Noetherian local ring with its maximal ideal and residue field. Let* $M \neq 0$ *be a finitely generated* R*-module and* $d = \dim_R M$*. Then*

(a) $H^i_{\mathfrak{m}}(M)$ *is an Artinian* R*-module for all* $i \in \mathbb{N}$ *and has the structure of an* $\hat{R}$*-module,*
(b) $H^i_{\mathfrak{m}}(M) = 0$ *for all* $i > d$*,*
(c) $H^d_{\mathfrak{m}}(M) \neq 0$ *and* $\mathfrak{m}\hat{R} \notin \operatorname{Ass}_{\hat{R}}(H^d_{\mathfrak{m}}(M))^\vee$ *provided* $d > 0$*.*

Proof Let $M \xrightarrow{\sim} I$ denote a minimal injective resolution of M. Then

$$I^i \cong \oplus_{\mathfrak{p} \in \operatorname{Spec} R} E_R(R/\mathfrak{p})^{\mu_i(\mathfrak{p}, M)}$$

with $\mu_i(\mathfrak{p}, M) = \dim_{k(\mathfrak{p})} \operatorname{Hom}_{R_\mathfrak{p}}(k(\mathfrak{p}), M_\mathfrak{p})) < \infty$ (see e.g. [55] and [9], or 4.5.18). Here $E_R(R/\mathfrak{p})$ denotes the injective hull of $R/\mathfrak{p}$. Because $\Gamma_{\mathfrak{m}}(E_R(R/\mathfrak{p})) = 0$ for each non-maximal prime ideal it follows that $H^i_{\mathfrak{m}}(M)$ is a sub-quotient of $E^{\mu_i(\mathfrak{m}, M)}$ and therefore an Artinian R-module. For the last claim in (a), see 2.3.2.

For the proof of (b) we refer to Corollary 7.6.5. We apply it to M and a system of parameters $\underline{x}$ of M. Note that this is also a system of parameters of $R/\operatorname{Ann}_R(M)$. (We may also refer to the more general Theorem 7.6.7.)

For the proof of (c) recall that $H^i_{\mathfrak{m}}(M) \otimes_R \hat{R} \cong H^i_{\mathfrak{m}\hat{R}}(M \otimes_R \hat{R})$ for all $i \in \mathbb{Z}$, where $\hat{R}$ denotes the $\mathfrak{m}$-completion of R (see 9.8.2). As $\dim_R M = \dim_{\hat{R}}(M \otimes_R \hat{R})$ we may and do assume without loss of generality that R is $\mathfrak{m}$-complete. Then the $(H^i_{\mathfrak{m}}(M))^\vee$ are finitely generated R-modules because the $H^i_{\mathfrak{m}}(M)$ are Artinian (see 2.3.2) so that the $\operatorname{Ass}_R((H^i_{\mathfrak{m}}(M))^\vee)$ are finite sets.

We first show that $\mathfrak{m} \notin \operatorname{Ass}_R(H^d_{\mathfrak{m}}(M))^\vee$ provided $d > 0$. To this end, let $N = \Gamma_{\mathfrak{m}}(M)$ be the largest submodule of finite length of M. Then the short exact sequence $0 \to N \to M \to M/N \to 0$ implies $H^d_{\mathfrak{m}}(M) \cong H^d_{\mathfrak{m}}(M/N)$ as follows because of $d > 0$ and (b). Therefore we may assume $N = 0$ and the existence of an M-regular element $x \in \mathfrak{m}$. So we have the short exact sequence $0 \to M \xrightarrow{x} M \to M/xM \to 0$. By applying the local cohomology functor it induces a surjection $H^d_{\mathfrak{m}}(M) \xrightarrow{x} H^d_{\mathfrak{m}}(M) \to 0$ as follows by (b) and $\dim_R M/xM < d$. Therefore there is an injection $0 \to (H^d_{\mathfrak{m}}(M))^\vee \xrightarrow{x} (H^d_{\mathfrak{m}}(M))^\vee$ which proves the claim.

Now we prove the non-vanishing of $H^d_{\mathfrak{m}}(M)$ by induction on d. The case $d = 0$ is obvious since in this case $0 \neq M = \Gamma_{\mathfrak{m}}(M) = H^0_{\mathfrak{m}}(M)$. For $d > 0$ we may assume (as above) that the largest submodule of finite length N of M vanishes. For $d = 1$ we get $H^1_{\mathfrak{m}}(M) \cong M_x/M \neq 0$. So we may assume that $d \geq 2$. By prime avoidance choose an M-regular element $x \in \mathfrak{m}$ with the additional property that $x \notin \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Ass}_R(H^{d-1}_{\mathfrak{m}}(M))^\vee \setminus \{\mathfrak{m}\}$. Then there is a short exact sequence $0 \to M \xrightarrow{x} M \to M/xM \to 0$ which induces an exact sequence

$$(H^d_{\mathfrak{m}}(M))^\vee \to (H^{d-1}_{\mathfrak{m}}(M/xM))^\vee \to (H^{d-1}_{\mathfrak{m}}(M))^\vee \xrightarrow{x} (H^{d-1}_{\mathfrak{m}}(M))^\vee.$$

By the induction hypothesis $H^{d-1}_{\mathfrak{m}}(M/xM) \neq 0$. Assume now that $H^d_{\mathfrak{m}}(M) = 0$. Then $(H^{d-1}_{\mathfrak{m}}(M/xM))^\vee \cong \operatorname{Hom}_R(R/xR, H^{d-1}_{\mathfrak{m}}(M)^\vee)$ and

$$\varnothing \neq \operatorname{Ass}_R((H^{d-1}_{\mathfrak{m}}(M/xM))^\vee) \subseteq V(xR) \cap \operatorname{Ass}_R(H^{d-1}_{\mathfrak{m}}(M))^\vee = \{\mathfrak{m}\}.$$

This implies $\mathfrak{m} \in \operatorname{Ass}_R(H^{d-1}_{\mathfrak{m}}(M/xM))^\vee$, contradicting the fact that

$$\mathfrak{m} \notin \operatorname{Ass}_R(H^{d-1}_{\mathfrak{m}}(M/xM))^\vee$$

since $d > 1$. Hence $H^d_{\mathfrak{m}}(M) \neq 0$. □

Here is the local homology counterpart of Theorem A.1.1. It concerns Artinian modules and is a rather direct consequence of A.1.1 (in fact, it is equivalent to A.1.1). We do not know if this has already been noticed.

Theorem A.1.2 *Let $(R, \mathfrak{m}, \Bbbk)$ denote a Noetherian local ring. Let $N \neq 0$ be an Artinian R-module and $d = \dim_{\hat{R}}(N^\vee)$. Then*

(a) $\Lambda^{\mathfrak{m}}_i(N)$ is a finitely generated $\hat{R}$-module for all $i \in \mathbb{N}$,
(b) $\Lambda^{\mathfrak{m}}_i(N) = 0$ for all $i > d$, and
(c) $\Lambda^{\mathfrak{m}}_d(N) \neq 0$.

Proof First recall that N has the structure of an $\hat{R}$-module, that $N^\vee := \operatorname{Hom}_R(N, E) \cong \operatorname{Hom}_{\hat{R}}(N, E)$ and that $N^\vee$ is finitely generated over $\hat{R}$ (see 2.3.2). As $\operatorname{Ann}_{\hat{R}}(N^\vee) = \operatorname{Ann}_{\hat{R}}(N)$ we note that $d := \dim_{\hat{R}}(N^\vee) = \dim_{\hat{R}}(\hat{R}/\operatorname{Ann}_{\hat{R}}(N))$. Note also that $\Lambda^{\mathfrak{m}}_i(N) \cong \Lambda^{\mathfrak{m}\hat{R}}_i(N)$ by 9.8.1. All this together reduces the statements to the case of a Noetherian local ring complete in its maximal-adic topology. In this case, the natural embedding $N \hookrightarrow N^{\vee\vee}$ is an isomorphism and

$$\Lambda^{\mathfrak{m}}_i(N) \cong \Lambda^{\mathfrak{m}}_i(N^{\vee\vee}) \cong (H^i_{\mathfrak{m}}(N^\vee))^\vee$$

by 9.2.5. As $N^\vee$ is finitely generated $H^i_{\mathfrak{m}}(N^\vee)$ is Artinian and its Matlis dual is finitely generated (see 2.3.2 again). This proves (a). The statement in (b) follows by Corollary 7.6.5 applied to N, to a parameter system of R in place of $\underline{x}$ and to a parameter system of $R/\operatorname{Ann}_R(N)$ in place of $\underline{y}$. Note that both statements (b) and (c) also follow from the isomorphism

$$\Lambda_i^{\mathfrak{m}}(N) \cong (H_{\mathfrak{m}}^i(N^\vee))^\vee$$

and Theorem A.1.1 applied to the finitely generated module $N^\vee$. □

Let $(R, \mathfrak{m}, \Bbbk)$ be a Noetherian local ring with $E = E_R(\Bbbk)$ the injective hull of the residue field. The finitely generated $\hat{R}$-modules $\Lambda_i^{\mathfrak{m}}(E), i \in \mathbb{Z}$, seem to be interesting objects for further investigation ($\Lambda_i^{\mathfrak{m}}(E) \cong (H_{\mathfrak{m}}^i(R))^\vee$). Some aspects of them were considered in Part III.

Remark A.1.3 Let R denote a commutative ring and N an R-module. In his paper (see [54]) I.G. Macdonald introduced a dual to the primary decomposition, the secondary representation. An R-module M is called secondary if the multiplication map $M \xrightarrow{x} M$ is either surjective or nilpotent. In this case $\mathfrak{p} = \operatorname{Rad} \operatorname{Ann}_R M$ is a prime ideal and M is called $\mathfrak{p}$-secondary. A secondary representation of an R-module M is an expression

$$M = N_1 + \cdots + N_n$$

with a finite sum of $\mathfrak{p}_i$-secondary submodules $N_i, i = 1, \ldots, n$. Such a decomposition can be refined to a minimal one with pairwise different prime ideals $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_m\}$ and $\operatorname{Att}_R M = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_m\}$ is called the set of attached prime ideals of M.

An Artinian R-module possesses a secondary representation. If R is Noetherian and N is an Artinian R-module, then a prime ideal $\mathfrak{p}$ is an attached prime of N if and only if there is an epimorphic image $N \to N' \to 0$ such that $\operatorname{Ann}_R N' = \mathfrak{p}$. We refer to [54] or [58, Appendix to §6] for more details.

Assume now that $(R, \mathfrak{m}, \Bbbk)$ is a Noetherian local ring and simply denote by $\hat{(\cdot)}$ the $\mathfrak{m}$-completion functor. We note that $\mathfrak{m} \in \operatorname{Att}_R(N)$ if and only $\mathfrak{m}\hat{R} \in \operatorname{Ass}_{\hat{R}}(N^\vee)$ for an Artinian R-module N resp. $\mathfrak{m} \in \operatorname{Ass}_R M$ if and only if $\mathfrak{m} \in \operatorname{Att}_R(M^\vee)$ for a finitely generated R-module M.

Related to A.1.1 resp. A.1.2 we have the following:

(a) Let M be a finitely generated R-module. Then $\mathfrak{m} \notin \operatorname{Att}_R H_{\mathfrak{m}}^d(M)$ provided $d = \dim_R M > 0$.
(b) Let N be an Artinian R-module. Then $\mathfrak{m}\hat{R} \notin \operatorname{Ass}_{\hat{R}} \Lambda_d^{\mathfrak{m}}(N)$ for $d = \dim_{\hat{R}}(N^\vee) > 0$.

Note that the proof of (a) is a consequence of A.1.1 (c). For the proof of (b) first recall that $\Lambda_d^{\mathfrak{m}}(N) \cong \Lambda_d^{\mathfrak{m}\hat{R}}(N)$ so we may assume that R is $\mathfrak{m}$-complete. Then we have the isomorphisms

$$\Lambda_d^{\mathfrak{m}}(N) \cong \Lambda_d^{\mathfrak{m}}(N^{\vee\vee}) \cong (H_{\mathfrak{m}}^d(N^\vee))^\vee.$$

As $N^\vee$ is a finitely generated module of dimension d this claim is also a consequence of A.1.1 (c).

Here we have to add a bibliographical remark. Theorem A.1.2 was shown independently in a slightly different form by N.T. Cuong and T.T. Nam (see [22, 4.10]).

Moreover, in that paper the authors also study local homology. For an ideal $\mathfrak{a}$ of a commutative ring R and an R-module M they consider $\varprojlim \operatorname{Tor}_i^R(R/\mathfrak{a}^t, M)$ as the ith local homology of M with respect to $\mathfrak{a}$ (see also [83]). By 7.1.8 resp. by [38] there are short exact sequences

$$0 \to \varprojlim{}^1 \operatorname{Tor}_{i+1}^R(R/\mathfrak{a}^t, M) \to \Lambda_i^{\mathfrak{a}}(M) \to \varprojlim \operatorname{Tor}_i^R(R/\mathfrak{a}^t, M) \to 0.$$

In the case when M is an Artinian R-module over a Noetherian local ring (the situation mainly investigated in [22]) it is shown that $\Lambda_i^{\mathfrak{a}}(M) \cong \varprojlim \operatorname{Tor}_i^R(R/\mathfrak{a}^t, M)$ for all $i \geq 0$. This can be proved as follows: since M is Artinian by Matlis duality there is a finitely generated $\hat{R}$-module N such that $M \cong (N)^\vee$. Then by the definition of the direct limit there is a short exact sequence

$$0 \to \oplus \operatorname{Ext}_R^1(R/\mathfrak{a}^t, N) \to \oplus \operatorname{Ext}_R^1(R/\mathfrak{a}^t, N) \to \varinjlim \operatorname{Ext}_R^1(R/\mathfrak{a}^t, N) \to 0.$$

By applying the Matlis duality functor $\operatorname{Hom}_R(\cdot, E) = (\cdot)^\vee$ and by 1.4.8 we get an exact sequence

$$0 \to \varprojlim \operatorname{Tor}_i^R(R/\mathfrak{a}^t, M) \to \prod \operatorname{Tor}_i^R(R/\mathfrak{a}^t, M) \to \prod \operatorname{Tor}_i^R(R/\mathfrak{a}^t, M) \to 0.$$

By virtue of 1.2.2 this implies $\varprojlim{}^1 \operatorname{Tor}_i^R(R/\mathfrak{a}^t, M) = 0$. In the more general case of an Artinian module over any Notherian ring, note that the $\operatorname{Tor}_i^R(R/\mathfrak{a}^t, M)$'s are Artinian (as sub-quotients of Artinian ones), hence the inverse systems $\{\operatorname{Tor}_i^R(R/\mathfrak{a}^t, M) \mid t > 0\}$ satisfy the Mittag-Leffler conditions and $\varprojlim{}^1(\operatorname{Tor}_i^R(R/\mathfrak{a}^t, M)) = 0$ for all $i \geq 0$.

A.2 Pro-regular or Weakly Pro-regular Sequences, Some Examples

As we have seen, Koszul complexes and weakly pro-regular sequences play a substantial rôle in the study of local homology and cohomology. Note, however, that Greenlees and May in [38], later Lipman et al. in [1], used another notion, slightly stronger as we shall see. In this section we first recall this notion. Then we will investigate several examples which illustrate some of the assumptions in our previous investigations. They are collected from the references and in some cases slightly modified.

The following notions are due to Greenlees and May (see [38]).

Definitions A.2.1 Let $\underline{x} = x_1, \ldots, x_k$ denote a sequence of elements in a commutative ring R and let M be an R-module. This sequence $\underline{x}$ is called *pro-regular* with respect to M (M-pro-regular for short) if for all $i = 1, \ldots, k$ and each integer $n > 0$ there is an integer $m \geq n$ such that

$$(x_1^m, \ldots, x_{i-1}^m)M :_M x_i^m \subseteq (x_1^n, \ldots, x_{i-1}^n)M :_M x_i^{m-n}.$$

It is called pro-regular if it is pro-regular with respect to R.

For a single element x we say that M is of bounded xR-torsion if the increasing sequence $0 :_M x^m, m > 0$, of submodules of M stabilizes. We say that R is of bounded xR-torsion if $0 :_R x^m$, $m > 0$ stabilizes.

Remarks A.2.2 Let $\underline{x} = x_1, \ldots, x_k$ denote a system of elements of the ring R. Let M denote an R-module.

(a) Any M-regular sequence $\underline{x}$ is also M-pro-regular.

(b) If the R-module M is Noetherian, then any sequence $\underline{x} = x_1, \ldots, x_k$ in R is M-pro-regular. This follows because for each fixed $n \geq 1$ the increasing sequence $(x_1^n, \ldots, x_{i-1}^n)M :_M x_i^{m-n}$, $m \geq n$, of submodules of M stabilizes.

(c) A length one sequence x is M-pro-regular if and only if M is of bounded xR-torsion.

(d) Assume that the sequence $\underline{x} = x_1, \ldots, x_k$ is M-pro-regular. Then so is the sequence $x_1, \ldots, x_i$ for all i, $1 \leq i < k$. Moreover, for all $n \geq 1$ and all i, $1 \leq i < k$, the sequence $x_{i+1}^n, \ldots, x_k^n$ is pro-regular with respect to $M/(x_1^n, \ldots, x_i^n)M$. In particular, for all $n \geq 1$ the increasing sequence $(x_1^n, \ldots, x_{k-1}^n)M :_M x_k^r$, $r > 0$, stabilizes.

The next lemma shows that any pro-regular sequence is weakly pro-regular too (the case $M = R$ of the following was already in [75, Lemma 2.7]).

Lemma A.2.3 *Let $\underline{x} = x_1, \ldots, x_k$ denote a sequence of elements in a commutative ring R. Let M denote an R-module. Assume that the sequence $\underline{x}$ is pro-regular with respect to M. Then it is weakly pro-regular with respect to M.*

Proof For the proof we proceed by induction on k. For $k = 1$ the result is clear (see 7.3.2 (c) and A.2.2 (c)). Put $y = x_{k+1}$. Then for natural numbers $s \geq t$ the Koszul homology provides the following commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & H_0(y^s; H_i(\underline{x}^s; M)) \arrow[r] \arrow[d] & H_i(\underline{x}^s, y^s; M) \arrow[r] \arrow[d] & H_1(y^s; H_{i-1}(\underline{x}^s; M)) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H_0(y^t; H_i(\underline{x}^t; M)) \arrow[r] & H_i(\underline{x}^t, y^t; M) \arrow[r] & H_1(y^t; H_{i-1}(\underline{x}^t; M)) \arrow[r] & 0
\end{tikzcd}$$

for each $i \in \mathbb{N}$ (see 5.2.4). The module $H_0(y^s; H_i(\underline{x}^s; M))$ is a quotient of $H_i(\underline{x}^s; M)$. By virtue of 1.2.4 and the inductive hypothesis it follows that the inverse system $\{H_0(y^s; H_i(\underline{x}^s; M))\}$ on the right is pro-zero for all $i \geq 1$. The module $H_1(y^s; H_{i-1}(\underline{x}^s; M))$ is a submodule of $H_{i-1}(\underline{x}^s; M)$. By the same argument as above the inverse system $\{H_1(y^s; H_{i-1}(\underline{x}^s; M))\}$ on the right is pro-zero for all $i > 1$.

By the assumption the sequence $\underline{x}, y$ is M-pro-regular, so y is pro-regular and also weakly pro-regular with respect to $M/\underline{x}^t M \cong H_0(\underline{x}^t; M)$ (see A.2.2). It follows that the inverse system on the right for $i = 1$ is also pro-zero. Then by 1.2.4 again the diagram above implies that the inverse system $\{H_i(\underline{x}^t, y^t; M)\}$ is pro-zero for each $i \neq 0$, completing the inductive step. $\square$

Example A.2.4 It is noteworthy that weakly pro-regular sequences are not always pro-regular. The following example was kindly communicated by J. Lipman (see [2]). It also provides an example of a non-weakly pro-regular sequence. Let $R = \prod_{n>0} \mathbb{Z}/2^n\mathbb{Z}$ and consider the element $x = (2, 2, 2, \dots) \in R$.

(a) First note that R is not of bounded xR-torsion, as is easily seen. Hence the length one sequence x is neither weakly pro-regular nor pro-regular and the sequences x and $x, 1$ are not pro-regular. But the sequence $x, 1$ is weakly pro-regular since $H_i(\underline{x}^n) = 0$ for all $i \geq 0$ (see 5.2.5). Note, however, that the sequence $1, x$ is obviously pro-regular. This example also shows that pro-regular sequences are not permutable without any additional assumptions.

(We note that the above example is very close to an earlier example by Verdier of a sequence which is not weakly pro-regular (see [42] and A.2.5).)

(b) When the ring is Noetherian it is known that $H^i(\check{C}_{\underline{x}} \otimes_R J) = 0$ for all $i > 0$ and all injective R-modules J (see also the more general 7.3.3). This does not hold in full generality. Take again for R the ring described above and consider the length one sequence x. We first observe that the homomorphisms $H_1(x^t) \to H_1(x)$ are not zero. Let f be the injection of $H_1(x)$ into its injective hull, denoted by J. Then $f \in H^1(x; J)$ and its image in $H^1(x^t; J)$ is not zero. Hence $H^1(\check{C}_x \otimes_R J) \cong \varinjlim H^1(x^t; J) \neq 0$.

The next example provides another more explicit injective R-module such that its Čech cohomology does not vanish in positive dimension. To this end we need the notion of trivial extension recalled in 2.8.5.

Example A.2.5 The following example is a slight simplification of an example of J.-L. Verdier (see also [42]).

(a) For the example, let $\Bbbk$ denote a field and $R = \Bbbk[[x]]$ denote the formal power serious ring in the variable x over $\Bbbk$. Let $M = \oplus_{n\geq 1} R/(x^n)R$ and let $m \geq 1$ denote a natural number. Then M is a submodule of $N = \prod_{n\geq 1} R/(x^n)$. Let $S = R \ltimes M$, the trivial extension of R by M and put $x' = (x, 0) \in S$. We have the following isomorphism of the Koszul homology modules

$$H_i(x'^m; S) \cong H_i(x^m; S) \text{ for all } i \in \mathbb{Z}$$

where S is viewed as an R-module.

For S as an R-module it follows that

$$H_1(x^m; S) \cong H_1(x^m; R) \oplus H_1(x^m; M) \cong H_1(x^m; M)$$

since x^m is an R-regular element for all $m \geq 1$. Clearly we get

$$H_1(x^m, M) \cong (\oplus_{1\leq n\leq m} R/x^n R) \oplus (\oplus_{n>m} x^{n-m}(R/x^n R)).$$

For integers $m' \geq m$ we consider the transition homomorphism

$$\rho_{m,m'} : H_1(x^{m'}, M) \xrightarrow{x^{m'-m}} H_1(x^m; M).$$

Then it follows that the map $\rho_{m,m'}$ is onto in components $n \geq m'$. So that the map $\rho_{m,m'}$ is not zero and the inverse system $\{H_1(x^m; M) \mid m \geq 1\}$ is not pro-zero. Another argument is that S is not of bounded $x'S$-torsion, as is seen by a similar argument.

(b) By virtue of the result in 7.3.3 it follows that there is an injective S-module I such that $H^i(\check{C}_{x'} \otimes_S I) \neq 0$ for $i = 1$. Next we shall construct an explicit example of such an I. To this end let $E = E_R(\Bbbk)$ denote the injective hull of the residue field $\Bbbk = R/xR$. Then $\operatorname{Hom}_R(S, E)$ is an injective S-module. We shall prove that $H^1(\check{C}_{\underline{x}'} \otimes_S \operatorname{Hom}_R(S, E)) \neq 0$.

By the change of rings result we first note that there is an isomorphism

$$\check{C}_{x'} \otimes_S \operatorname{Hom}_R(S, E) \cong \check{C}_x \otimes_R \operatorname{Hom}_R(S, E).$$

This yields the isomorphism

$$H^1(\check{C}_{x'} \otimes_S \operatorname{Hom}_R(S, E)) \cong \varinjlim H^1(x^m; \operatorname{Hom}_R(S, E)).$$

Because $S = R \oplus M$ it turns out that

$$\varinjlim H^1(x^m; \operatorname{Hom}_R(S, E)) \cong \varinjlim H^1(x^m; E) \oplus \varinjlim H^1(x^m; \operatorname{Hom}_R(M, E)).$$

Because x is regular on R and since E is an injective R-module the first direct limit vanishes. It will be enough to show that the second limit does not vanish.

By duality we have that $\operatorname{Hom}_R(R/(x^n), E) \cong R/(x^n)$ for all $n \geq 1$, and therefore $\operatorname{Hom}_R(M, E) \cong N$. We note that

$$H^1(x^m; N) \cong N \otimes_R R/(x^m) \cong \prod_{n \geq 1} R/(x^n, x^m).$$

For all $m' \geq m$ we investigate the natural homomorphism

$$\sigma_{m,m'} : H^1(x^m; N) \cong \prod_{n \geq 1} R/(x^n, x^m) \to H^1(x^{m'}; N) \cong \prod_{n \geq 1} R/(x^n, x^{m'}),$$

which is the multiplication by $x^{m'-m}$. Let $p = (1 + xR)_{n \geq 1} \in H^1(x; N)$. Its image under multiplication by x^{m-1} in $H^1(x^m; N)$ is

$$(\underbrace{0, \ldots, 0}_{1 \leq n < m}, \underbrace{x^{m-1} + x^m R, \ldots}_{n \geq m}) \neq 0.$$

Therefore the image of p in $\varinjlim H^1(x^m; N)$ is not zero and $\varinjlim H^1(x^m; N)$ does not vanish. For the injective S-module $\operatorname{Hom}_R(S, E)$ we get $H^1(\check{C}_{x'} \otimes_S \operatorname{Hom}_R(S, E)) \neq 0$. Namely

$$H^1(\check{C}_{x'} \otimes_S \operatorname{Hom}_R(S, E)) \cong \varinjlim H^1(x^n; N) \neq 0$$

as shown above.

Next we produce a commutative ring S and a sequence of elements $\underline{x}'$ such that $M(\underline{x}'; F)$ has non-vanishing homology in positive degree for a certain flat S-module F. Note that $H_i(M(\underline{x}'; F)) \cong \Lambda_i^{xS}(F) = 0$ for $i > 0$ provided $\underline{x}'$ is a weakly pro-regular sequence (see 7.5.5 or 7.5.12). Hence our sequence will not be weakly pro-regular.

Example A.2.6 Let $R = \Bbbk[[x_1, \ldots, x_n]]$ denote the power series ring in n variables over the field $\Bbbk$. Let $E = E_R(\Bbbk)$ denote the injective hull of the residue field. Then define $S = R \ltimes E$ and $\underline{x}' = ((x_1, 0), \ldots, (x_n, 0))$, a sequence of elements in S. We investigate the microscope $M(\underline{x}'; S)$ in the present example. We have

$$M(\underline{x}'; S) \cong M(\underline{x}; S) \cong \operatorname{Hom}_R(T(\underline{x}), S) \cong$$
$$\operatorname{Hom}_R(T(\underline{x}), R) \oplus \operatorname{Hom}_R(T(\underline{x}), E))$$

where $\underline{x} = (x_1, \ldots, x_n)$ is viewed as a sequence in R (see 7.5.9 for the second isomorphism). Note that

$$H_i(\operatorname{Hom}_R(T(\underline{x}), E)) \cong H_i(\operatorname{Hom}_R(\check{C}_{\underline{x}}, E)) \cong H_i((\check{C}_{\underline{x}})^\vee) \cong (H^i(\check{C}_{\underline{x}}))^\vee.$$

But the sequence $\underline{x}$ is regular on R, a fortiori weakly pro-regular. Hence the complex $\check{C}_{\underline{x}}$ represents $\mathrm{R}\Gamma_{\underline{x}R}(R)$ in the derived category (see 7.4.4) and we know by the Grothendieck non-vanishing Theorem (recalled in A.1.1) that $H^n(\check{C}_{\underline{x}}) \neq 0$. Therefore $M(\underline{x}'; S)$ has non-vanishing homology in degree n.

A.3 The Right-Derived Functors of the Completion Functor

In the previous chapters we investigated the left-derived functors $\mathrm{L}\Lambda^{\mathfrak{a}}(\cdot)$ of the functor $\Lambda^{\mathfrak{a}}(\cdot)$. While $\Lambda^{\mathfrak{a}}(\cdot)$ is neither left nor right exact as discussed earlier it makes sense to look at the right-derived functors of $\Lambda^{\mathfrak{a}}(\cdot)$, though these are much less interesting. At first this was done by Greenlees and May (see [38]) in the case of R-modules. In this section we shall slightly extend their results to finitely generated ideals and complexes.

First we have to prove that $\mathrm{R}\Lambda^{\mathfrak{a}}$ is well defined in the derived category.

Lemma A.3.1 *Let $\mathfrak{a} \subset R$ denote an ideal of a commutative ring R and let $f : I \xrightarrow{\sim} I'$ be a quasi-isomorphism between K-injective R-complexes. Then the induced morphism $\Lambda^{\mathfrak{a}}(f) : \Lambda^{\mathfrak{a}}(I)) \to \Lambda^{\mathfrak{a}}(I')$ is also a quasi-isomorphism.*

Proof There is an inverse system of quasi-isomorphisms

$$R/\mathfrak{a}^t \otimes_R I \xrightarrow{\sim} R/\mathfrak{a}^t \otimes_R I'$$

(see 4.4.11). Passing to the inverse limit we obtain the quasi-isomorphism $\Lambda^{\mathfrak{a}}(f) : \Lambda^{\mathfrak{a}}(I)) \xrightarrow{\sim} \Lambda^{\mathfrak{a}}(I')$ because the inverse systems $\{R/\mathfrak{a}^t \otimes_R I\}$ and $\{R/\mathfrak{a}^t \otimes_R I'\}$ are degree-wise surjective (see 1.2.9). □

Theorem and Definition A.3.2 Let $\mathfrak{a} \subset R$ denote an ideal of a commutative ring R. Then the right-derived functor $\mathrm{R}\Lambda^{\mathfrak{a}}$ is well defined in the derived category.

More precisely, let X be any R-complex with one of its K-injective resolutions $X \xrightarrow{\sim} I$. Then $\mathrm{R}\Lambda^{\mathfrak{a}}(X)$ is represented by $\Lambda^{\mathfrak{a}}(I)$.

Note also the natural map $X \to \mathrm{R}\Lambda^{\mathfrak{a}}(X)$ in the derived category, represented by the composite $X \xrightarrow{\sim} I \to \Lambda^{\mathfrak{a}}(I)$.

Proof In view of the remarks in 4.4.12 this is a direct consequence of Lemma A.3.1. □

Notation A.3.3 In the following the ith right-derived functor of the $\Lambda^{\mathfrak{a}}$, namely the ith cohomology modules $H^i(\mathrm{R}\Lambda^{\mathfrak{a}}(\cdot)$, will be denoted by $\mathrm{R}^i\Lambda^{\mathfrak{a}}(\cdot)$.

Of course, we may compute $\mathrm{R}\Lambda^{\mathfrak{a}}(X)$ and its cohomology by using a DG-injective resolution of X. At first we investigate the completion of any complex of injective modules.

Proposition A.3.4 *Let $\mathfrak{a}$ denote a finitely generated ideal of a commutative ring R. Then there is a natural isomorphism*

$$\Lambda^{\mathfrak{a}}(I) \cong \mathrm{Hom}_R(\Gamma_{\mathfrak{a}}(R), I)$$

for any R-complex I of injective modules.

Proof We prove this degree-wise. That is, we may and do assume that I is an injective R-module.

Because the ideal $\mathfrak{a}$ is finitely generated the ideals $\mathfrak{a}^t$ are also finitely generated for all $t \geq 1$. By 1.4.1 for $M = R/\mathfrak{a}^t$ and $N = R$ there is an isomorphism of inverse systems

$$R/\mathfrak{a}^t \otimes_R I \cong \mathrm{Hom}_R(\mathrm{Hom}_R(R/\mathfrak{a}^t, R), I).$$

By passing to the inverse limit and by using 1.3.3 we get the claim. Recall that $\varinjlim \mathrm{Hom}_R(R/\mathfrak{a}^t, R) \cong \Gamma_{\mathfrak{a}}(R)$. □

Now we prove the main result of this section.

Theorem A.3.5 *Let $\mathfrak{a}$ be a finitely generated ideal of a commutative ring R. Let X denote an R-complex. Then there is an isomorphism*

$$\mathrm{R}\Lambda^{\mathfrak{a}}(X) \simeq \mathrm{RHom}_R(\Gamma_{\mathfrak{a}}(R), X)$$

in the derived category. In particular, there are isomorphisms

$$H^i(\mathrm{R}\Lambda^{\mathfrak{a}}(X)) := \mathrm{R}^i\Lambda^{\mathfrak{a}}(X) \cong \mathrm{Ext}^i_R(\Gamma_{\mathfrak{a}}(R), X)$$

for all $i \in \mathbb{Z}$.

Proof Let $X \xrightarrow{\sim} I$ be a DG-injective resolution. By A.3.4 we have the isomorphism $\Lambda^{\mathfrak{a}}(I) \cong \mathrm{Hom}_R(\Gamma_{\mathfrak{a}}(R), I)$. As $\Lambda^{\mathfrak{a}}(I)$ represents $\mathrm{R}\Lambda^{\mathfrak{a}}(X)$ the statements follow. □

Remarks A.3.6 (a) In the situation of A.3.5, if the ring R does not have $\mathfrak{a}$-torsion, in particular if R is an integral domain, then $\Gamma_{\mathfrak{a}}(R) = 0$ and therefore $\mathrm{R}^i\Lambda^{\mathfrak{a}}(X) = 0$ for all $i \in \mathbb{Z}$.

(b) We do not know what happens to $\mathrm{R}\Lambda^{\mathfrak{a}}(X)$ if the ideal $\mathfrak{a}$ is not finitely generated.

(c) Let $\mathfrak{a}$ be an ideal generated by a weakly pro-regular sequence $\underline{x} = x_1, \ldots, x_r$. Then we know that $\Lambda_i^{\mathfrak{a}}(X) \cong H_i(\mathrm{Hom}_R(\check{C}_{\underline{x}}, I))$ for all $i \in \mathbb{Z}$, where $X \xrightarrow{\sim} I$ is a DG-injective resolution of X (see 7.5.12). There is a spectral sequence

$$E^2_{i,j} = H_i(\mathrm{Hom}_R(H^{-j}_{\mathfrak{a}}(R), I)) \Longrightarrow E^{\infty}_{i+j} = \Lambda^{\mathfrak{a}}_{i+j}(X).$$

We observe that $E^2_{i,0} \cong \mathrm{R}^i\Lambda^{\mathfrak{a}}(X)$ is one of the initial stages for computing $\Lambda_i^{\mathfrak{a}}(X)$.

A.4 Specific Rings

In general our interest is in commutative rings. For several results there was a need for additional assumptions (e.g. finitely generated ideals, weakly proregular sequences, or Noetherian, coherent rings and so on). In this part of the Appendix we shall summarize some more results in this direction, in particular illustrating the need for additional assumptions.

First recall the definition of a coherent ring (see 1.4.2). This definition generalizes the notion of a Noetherian ring. As we have seen it is related to the notion of flatness. Here we recall Chase's theorem (see [19], [13, Exercise 1.2, 12] or [26, Theorem 3.2.24]). To this end we need the following interesting lemma (see [26, Lemma 3.2.21]).

Lemma A.4.1 *Let M denote an R-module. Then the following conditions are equivalent.*

(i) *M is finitely generated.*
(ii) *The natural homomorphism $M \otimes_R \prod_{\lambda\in\Lambda} N_\lambda \to \prod_{\lambda\in\Lambda} M \otimes_R N_\lambda$ is surjective for any family of R-modules $\{N_\lambda\}_{i\in\Lambda}$.*
(iii) *The natural homomorphism $M \otimes_R R^\Lambda \to M^\Lambda$ is surjective for any index set Λ.*
(iv) *The natural homomorphism $M \otimes_R R^M \to M^M$ is surjective.*

Proof (i) $\Longrightarrow$ (ii). Let $0 \to N \to L \to M \to 0$ be a short exact sequence with L a finitely generated free R-module. It induces a commutative diagram with exact rows

$$\begin{tikzcd}
N \otimes_R \prod_{\lambda\in\Lambda} N_\lambda \arrow[r] \arrow[d] & L \otimes_R \prod_{\lambda\in\Lambda} N_\lambda \arrow[r] \arrow[d] & M \otimes_R \prod_{\lambda\in\Lambda} N_\lambda \arrow[r] \arrow[d] & 0 \\
\prod_{\lambda\in\Lambda}(N \otimes_R N_\lambda) \arrow[r] & \prod_{\lambda\in\Lambda}(L \otimes_R N_\lambda) \arrow[r] & \prod_{\lambda\in\Lambda}(M \otimes_R N_\lambda) \arrow[r] & 0.
\end{tikzcd}$$

Since the vertical homomorphism in the middle is an isomorphism the vertical homomorphism on the right is surjective.

(ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv). These implications are trivial.

(iv) $\Longrightarrow$ (i). By the assumption $\mathrm{id}_M \in M^M$ is the image of $\sum_{i=1}^n m_i \otimes f_i$ for some $m_i \in M$ and some $f_i \in R^M$. Note that $m_i \otimes f_i$ maps to the function $M \to M : x \mapsto f_i(x) \cdot m_i$. Therefore $m = \sum_{i=1}^n f_i(m)m_i$ for all $m \in M$. That is, $m_1, \ldots, m_n$ generate M. □

The following result was shown by Chase (see [19]).

Theorem A.4.2 *Let R denote a commutative ring. Then the following conditions are equivalent.*

(i) *R is coherent.*
(ii) *Every product of flat R-modules is flat.*
(iii) *R^Λ is a flat R-module for any set Λ.*
(iv) *Every finitely generated submodule of a finitely presented R-module is finitely presented.*

Proof (i) $\Longrightarrow$ (ii). This was already observed in 1.4.4.

(ii) $\Longrightarrow$ (iii) and (iv) $\Longrightarrow$ (i) are trivial implications.

(iii) $\Longrightarrow$ (iv). Let M denote a finitely presented R-module. By 1.4.4 the natural map $M \otimes_R R^\Lambda \to M^\Lambda$ is an isomorphism for any set Λ.

Now let $N \subset M$ denote a finitely generated submodule of M. We have to show that N is finitely presented.

Since R^Λ is R-flat it induces a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & N \otimes_R R^\Lambda \arrow[r] \arrow[d] & M \otimes_R R^\Lambda \arrow[d] \\
0 \arrow[r] & N^\Lambda \arrow[r] & M^\Lambda,
\end{tikzcd}$$

whence the homomorphism $N \otimes_R R^\Lambda \to N^\Lambda$ is injective. Since N is finitely generated this homomorphism is also surjective (see A.4.1) and therefore an isomorphism.

Now let $0 \to N' \to L \to N \to 0$ be a short exact sequence with L a finitely generated free R-module. It induces a commutative diagram with exact rows

$$\begin{tikzcd}
 & N' \otimes R^\Lambda \arrow[r] \arrow[d] & L \otimes R^\Lambda \arrow[r] \arrow[d] & N \otimes R^\Lambda \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & N'^\Lambda \arrow[r] & L^\Lambda \arrow[r] & N^\Lambda \arrow[r] & 0.
\end{tikzcd}$$

Since both vertical homomorphisms at the right are isomorphisms it follows that the vertical homomorphism on the left is surjective. By A.4.1 N' is finitely generated and therefore N is finitely presented. □

In the search for examples the following lemma may be useful.

Lemma A.4.3 *Let R denote a commutative ring and assume there is a principal ideal $\mathfrak{a} = aR \subset R$ such that the ideal $\mathrm{Ann}_R(\mathfrak{a})$ is not finitely generated. Then $\mathfrak{a}$ is not finitely presented and R^Λ is not flat, where Λ denotes the index set given by the pairwise distinct elements of $\mathrm{Ann}_R(\mathfrak{a})$.*

Proof We shall first prove that the natural map $\mathfrak{a} \otimes_R R^\Lambda \to \mathfrak{a}^\Lambda$ is not injective. This will imply that $\mathfrak{a}$ is not finitely presented (see 1.4.4). To this end note that there is a short exact sequence $0 \to \mathrm{Ann}_R(\mathfrak{a}) \to R \to \mathfrak{a} \to 0$ because $\mathfrak{a}$ is principal. It induces a commutative diagram with exact rows

$$\begin{tikzcd}
 & (\mathrm{Ann}_R(\mathfrak{a})) \otimes_R R^\Lambda \arrow[r] \arrow[d] & R^\Lambda \arrow[r] \arrow[d, equal] & \mathfrak{a} \otimes_R R^\Lambda \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (\mathrm{Ann}_R(\mathfrak{a}))^\Lambda \arrow[r] & R^\Lambda \arrow[r] & \mathfrak{a}^\Lambda \arrow[r] & 0.
\end{tikzcd}$$

In this diagram the vertical homomorphism on the left is not surjective by Lemma A.4.1 (i) $\Leftrightarrow$ (iv). By the snake lemma it follows that the vertical homomorphism on the right is not injective.

Suppose now that R^Λ is flat. Then there is a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{a} \otimes_R R^{\Lambda} & \longrightarrow & R^{\Lambda} & \longrightarrow & R/\mathfrak{a} \otimes_R R^{\Lambda} & \longrightarrow & 0 \\ & & \downarrow & & \| & & \downarrow & & \\ 0 & \longrightarrow & \mathfrak{a}^{\Lambda} & \longrightarrow & R^{\Lambda} & \longrightarrow & (R/\mathfrak{a})^{\Lambda} & \longrightarrow & 0 \end{array}$$

and the vertical homomorphism on the left is injective, in contradiction with the first part of the proof. Hence R^{Λ} is not flat. □

With respect to injective modules and direct sums there is a remarkable theorem due Z. Papp (see [62]) and H. Bass (see [8, Theorem 1.1]. We include the proof here.

Theorem A.4.4 *For a commutative ring R the following conditions are equivalent.*

(i) R is Noetherian.
(ii) Every direct limit of injective R-modules is injective.
(iii) Every direct sum of injective R-modules is injective.

Proof (i) $\Longrightarrow$ (ii): See 1.4.7.

(ii) $\Longrightarrow$ (iii): This is trivial since direct sums are particular cases of direct limits.

(iii) $\Longrightarrow$ (i): Suppose that R is not Noetherian. Then there is an infinite sequence of different ideals

$$\mathfrak{a}_1 \subset \mathfrak{a}_2 \subset \ldots \subset \mathfrak{a}_n \subset \ldots .$$

Let $\mathfrak{a} = \cup_{n \geq 1} \mathfrak{a}_n$. Then $\mathfrak{a}$ is an ideal and $\mathfrak{a}/\mathfrak{a}_n \neq 0$ for all $n \geq 1$. Let f_n denote the composition of the projection $p_n : \mathfrak{a} \to \mathfrak{a}/\mathfrak{a}_n$ and the inclusion $\mathfrak{a}/\mathfrak{a}_n \subseteq E_n$, where E_n denotes the injective hull of $\mathfrak{a}/\mathfrak{a}_n$. Note that f_n is not zero but for all $a \in \mathfrak{a}$ almost all the $f_n(a)$ are null. Hence these f_n define a homomorphism

$$f : \mathfrak{a} \to \oplus_{n \geq 1} E_n, a \mapsto (f_n(a))_{n \geq 1}.$$

Now assume that $\oplus_{n \geq 1} E_n$ is an injective R-module. Then there is a map $g : R \to \oplus_{n \geq 1} E_n$ extending f. Note that there is an index n such that the component of $g(1)$ in $E_{n'}$ is null for all $n' > n$. Now choose an index $m > n$ and an element $a_m \in \mathfrak{a} \setminus \mathfrak{a}_m$. The component $f_m(a_m)$ of $f(a_m)$ in E_m is non-zero by the choice of a. On the other hand the component of $f(a_m) = g(a_m) = a_m g(1)$ in E_m is zero by the choice of m. This contradiction finishes the proof. □

As another sample of specific rings we will consider self-injective rings.

Definition A.4.5 A commutative ring R is called self-injective if R as an R-module is injective.

Note that a Noetherian ring R is self-injective only if it is Artinian. However, the notion is interesting for non-Noetherian rings. In the following we shall investigate when the trivial extension $R \ltimes E$ is self-injective (see 2.8.5 for the notion of trivial extension).

Theorem A.4.6 *Let R denote a commutative ring with E an R-module. Then the following conditions on the trivial extension $S = R \ltimes E$ are equivalent.*

(i) *S is self-injective and* $\operatorname{Ann}_R E = 0$.
(ii) *E is R-injective and $R \cong \operatorname{Hom}_R(E, E)$.*

When these conditions are satisfied there is an isomorphism $S \cong \operatorname{Hom}_R(S, E)$ of S-modules.

Proof Let $\mathfrak{a} = 0 \ltimes E$. Clearly $\mathfrak{a}$ is an ideal of S and $\mathfrak{a}^2 = 0$.

(i) $\Rightarrow$ (ii): Since $\operatorname{Ann}_R E = 0$ we have that

$$\mathfrak{a} = 0 :_S \mathfrak{a} \cong \operatorname{Hom}_S(S/\mathfrak{a}, S)$$

as is easily seen. Because S is self-injective note that $\mathfrak{a} \cong \operatorname{Hom}_S(S/\mathfrak{a}, S)$ is an injective module over the ring $S/\mathfrak{a} \cong R$. Since there is an isomorphism of R-modules $\mathfrak{a} \cong E$ it follows that E is an injective R-module.

Next note that the short exact sequence $0 \to \mathfrak{a} \to S \to S/\mathfrak{a} \to 0$ induces the short exact sequence $0 \to \operatorname{Hom}_S(S/\mathfrak{a}, S) \to S \to \operatorname{Hom}_S(\mathfrak{a}, S) \to 0$ and isomorphisms of R-modules

$$\operatorname{Hom}_S(\mathfrak{a}, S) \cong S/\mathfrak{a} \cong R$$

since S is self-injective and $\operatorname{Hom}_S(S/\mathfrak{a}, S) \cong \mathfrak{a}$.

Finally, note the isomorphisms of R-modules

$$\begin{aligned}\operatorname{Hom}_R(E, E) &\cong \operatorname{Hom}_{S/\mathfrak{a}}(\mathfrak{a}, \mathfrak{a}) \cong \operatorname{Hom}_S(\mathfrak{a}, \operatorname{Hom}_S(S/\mathfrak{a}, S))\\ &\cong \operatorname{Hom}_S(\mathfrak{a}/\mathfrak{a}^2, S) \cong \operatorname{Hom}_S(\mathfrak{a}, S) \cong R.\end{aligned}$$

(ii) $\Rightarrow$ (i) and the last assertion: First note that the isomorphism $R \cong \operatorname{Hom}_R(E, E)$ yields $\operatorname{Ann}_R E = 0$. Put $I := \operatorname{Hom}_R(S, E)$. Then I is an injective S-module and there is an isomorphism θ of R-modules

$$I := \operatorname{Hom}_R(S, E) = \operatorname{Hom}_R(R, E) \oplus \operatorname{Hom}_R(E, E) \xleftarrow{\theta} E \oplus R.$$

Note that both I and $E \oplus R \cong R \oplus E = S$ have the structure of an S-module. It remains to check that θ is also an isomorphism of S-modules. To this end let us first describe the image $\theta_{(e,r)}$ of any element $(e, r) \in E \oplus R$ by θ. It is an R-homomorphism $S \to E$ given by

$$\theta_{(e,r)}(r', e') = r'e + re' \text{ for all } (r', e') \in S.$$

Hence

$$\theta_{(r'',e'')\cdot(e,r)}(r', e') = \theta_{(r''e+re'',rr'')}(r', e') = r'(r''e + re'') + rr''e'.$$

On the other hand, by the structure of $I := \operatorname{Hom}_R(S, E)$ as an S-module, we have

$$((r'', e'') \cdot \theta_{(e,r)})(r', e') = \theta_{(e,r)}((r'', e'') \cdot (r', e')) = \theta_{(e,r)}(r'r'', r'e'' + r''e')$$
$$= r'r''e + r(r'e'' + r''e') \text{ for all } (r', e') \in S.$$

It follows that $(r'', e'') \cdot \theta_{(e,r)} = \theta_{(r'',e'')\cdot(e,r)}$. Therefore θ is an isomorphism of S-modules. ☐

The previous result was shown by C. Faith (see [27]) in a slightly more general setting. Our proof is completely different.

A.5 Some Examples of Modules

For an arbitrary commutative ring R the property of being an injective R-module does not localize. This was investigated by E.C. Dade in his remarkable paper [23]. In fact he constructed commutative rings with the property that there exist an injective R-module I and a multiplicatively closed set $S \subset R$ such that I_S is not an injective R-module, but no concrete module is described.

Before we discuss an example, let us consider localizations of injective modules.

Remark A.5.1 Let R denote a commutative ring and $S \subset R$ a multiplicatively closed set. From the natural homomorphism $R \to R_S$ it follows that $\operatorname{Hom}_R(M, N) \cong \operatorname{Hom}_{R_S}(M, N)$ for two R_S-modules M, N. Therefore an R_S-module which is R-injective is also R_S-injective and conversely an R_S-module which is R_S-injective is also R-injective since R_S is R-flat.

The following well-known statement is usually proved with the aid of Matlis' structure theory of injective modules over a Noetherian ring.

Proposition A.5.2 *Let $S \subset R$ be a multiplicatively closed subset of a Noetherian ring R. Let I denote an injective R-module. Then $I_S = I \otimes_R R_S$ is an injective R-module and equivalently an injective R_S-module.*

Proof In view of 1.4.10 it is enough to prove that $(I_S)^\vee$ is R-flat. But $(I_S)^\vee = (I \otimes_R R_S)^\vee \cong \operatorname{Hom}_R(I, (R_S)^\vee)$ and $(R_S)^\vee$ is an injective R-module since R_S is R-flat (see 1.4.10 again or 1.4.1). Then the result follows by 1.4.5. ☐

The first application concerns the localization with respect to a single element $x \in R$. In the following we denote as usual by $I^{(\mathbb{N})}$ the direct sum of copies of I over $\mathbb{N}$.

Lemma A.5.3 *Let I denote an injective R-module such that $I^{(\mathbb{N})}$ is also an injective R-module. Then $I_x = I \otimes_R R_x$ is an injective R-module for all $x \in R$.*

Proof There is a short exact sequence (see 3.1.3)

$$0 \to R^{(\mathbb{N})} \xrightarrow{\phi} R^{(\mathbb{N})} \to R_x \to 0.$$

Since R_x is R-flat, tensoring by I provides a short exact sequence

$$0 \to I^{(\mathbb{N})} \xrightarrow{\phi'} I^{(\mathbb{N})} \to I \otimes_R R_x \to 0.$$

Recall that the tensor product commutes with direct sums.

Now assume that $I^{(\mathbb{N})}$ is injective. Then the previous sequence is split-exact. Therefore $I \otimes_R R_x$ is (as a direct summand of $I^{(\mathbb{N})}$) an injective R-module. $\square$

As far as we know there is no concrete explicit example of an injective R-module, for a certain ring R and a multiplicatively closed set $S \subset R$, such that I_S is not injective as an R-module or equivalently as an R_S-module. In the following we add such an example, strongly motivated by those of E.C. Dade in [23].

Example A.5.4 (a) Let $\Bbbk$ denote a field and $R = \Bbbk[[x]]$ be the formal power series ring in the variable x over $\Bbbk$. Let $M = R^{(\mathbb{N})}$ be the free R-module $M = \oplus_{n\geq 0} Re_n$ with base ($e_n = (0, \ldots, 0, 1, 0 \ldots)$) (the "1" in the nth position). Let Q denote the quotient field of R. Then Q/R is an injective R-module, in fact it is the injective hull $E_R(\Bbbk)$ of the residue field of R.

Let $S = R \ltimes M$, the trivial extension of R by M (see 2.8.5). Then $I = \operatorname{Hom}_R(S, Q/R)$ is an injective S-module. Let $(x, 0) \in S$. We claim that $I_{(x,0)}$ is not an injective S-module.

First note that $I_{(x,0)} \neq 0$. This is because the map $h \in \operatorname{Hom}_R(S, Q/R) = I$ defined by $h((1, 0)) = 0$ and $h((0, e_n)) = (1/x^n + R)/R$ for $n \in \mathbb{N}$ is not annihilated by a power of $(x, 0)$.

Next we claim that $(0 \ltimes M)I_{(x,0)} = 0$. Let $(0, m) \in 0 \ltimes M$ with $m = \sum_{i=0}^n r_i e_i$ and $f \in I$. Then $(0, m)f = \sum_{i=0}^n (0, r_i e_i)f$ and for all $i \in \mathbb{N}$ the homomorphism $(0, e_i)f \in \operatorname{Hom}_R(S, Q/R)$ is defined by $((0, e_i)f)(r, m') = f(0, re_i) = rf(0, e_i)$. As $f(0, e_i) \in Q/R$ is annihilated by a power of x it follows that $(0, e_i)f$ is annihilated by a power of $(x, 0)$ for $i = 0, \ldots, n$ and therefore $(0, m)(f/(x^l, 0)) = 0$ for some $l \in \mathbb{N}$.

Now consider the singly generated ideal $\mathfrak{a} := 0 \ltimes Re_0 \subset 0 \ltimes M$. It is easily seen that $\operatorname{Ann}_S \mathfrak{a} = 0 \ltimes M$ and therefore $\mathfrak{a} \cong S/(0 \ltimes M)$.

Applying $\operatorname{Hom}_S(\cdot, I_{(x,0)})$ to the short exact sequence $0 \to \mathfrak{a} \to S \to S/\mathfrak{a} \to 0$ yields an exact sequence

$$0 \to \operatorname{Hom}_S(S/\mathfrak{a}, I_{(x,0)}) \to I_{(x,0)} \to \operatorname{Hom}_S(\mathfrak{a}, I_{(x,0)}) \to \operatorname{Ext}^1_S(S/\mathfrak{a}, I_{(x,0)}) \to 0.$$

Since $\mathfrak{a}$ annihilates $I_{(x,0)}$ it follows that $\operatorname{Hom}_S(S/\mathfrak{a}, I_{(x,0)}) = I_{(x,0)}$. Whence the homomorphism in the middle is the zero map and therefore $\operatorname{Hom}_S(\mathfrak{a}, I_{(x,0)}) \cong \operatorname{Ext}^1_S(S/\mathfrak{a}, I_{(x,0)})$. But now

$$\operatorname{Ext}^1_S(S/\mathfrak{a}, I_{(x,0)}) \cong \operatorname{Hom}_S(\mathfrak{a}, I_{(x,0)}) \cong \operatorname{Hom}_S(S/(0 \ltimes M), I_{(x,0)}) = I_{(x,0)} \neq 0.$$

Recall that $0 \ltimes M$ annihilates $I_{(x,0)}$. Whence $I_{(x,0)}$ is not an injective S-module.

(b) Another interesting feature of this example is the following: by A.5.3 it turns out that $I^{(\mathbb{N})}$ is not an injective S-module since $I_{(x,0)}$ is not injective.

As far as we know the example in A.5.4 (b) is the first explicit example of a commutative ring S such that the direct sum of injective S-modules is not always injective. We do not know whether the converse of A.5.3 is true.

By the characterization of coherent rings one might ask the question: Let R denote a non-coherent ring. What is an explicit index set Λ such that R^{Λ} is not R-flat. In the following we shall present an explicit example.

Examples A.5.5 (a) We fix the notation of A.5.4. That is, let $R = \Bbbk[|x|]$ denote the formal power series ring over the field $\Bbbk$ in the variable x. Let $M = R^{(\mathbb{N})}$ denote the free R-module with base $(e_0, e_1, \ldots)$. As above let $S = R \ltimes M$ and let $\mathfrak{a} = 0 \ltimes Re_0$. Then S is a ring which is not coherent, namely $\mathfrak{a}$ is not finitely presented. Moreover, the S-module S^R is not flat. This follows by Lemma A.4.3. Indeed, we have $\operatorname{Ann}_S(\mathfrak{a}) = 0 \ltimes M$ and there are set bijections $0 \ltimes M \leftrightarrow M \leftrightarrow R$ because R is infinite.

(b) Here is a smaller example. Let $\Bbbk$ denote a finite field or the field of rational numbers. Let V denote a $\Bbbk$-vector space of countable dimension and let us form the ring $S = \Bbbk \ltimes V$. Then $S^{\mathbb{N}}$ is not flat. This again follows by lemma A.4.3, note the set bijection $V \leftrightarrow \mathbb{N}$.

This example has an interesting consequence. Namely, $S = \Bbbk \ltimes V$ is a ring such that $S[[y]]$, the formal power series ring in one variable y over S, is not S-flat. This is because there is an isomorphism $S^{\mathbb{N}} \cong S[[y]]$ of S-modules.

(c) Another more sophisticated example will be given in A.5.7.

In the following we shall produce an explicit example of a commutative ring S with two injective S-modules I, J such that $\operatorname{Hom}_S(I, J)$ is not a flat S-module. Recall that it is flat in the case when S is a coherent ring (see 1.4.5). First some general observations.

Observations A.5.6 Let $(R, \mathfrak{m}, \Bbbk)$ denote a Noetherian local ring, complete in its $\mathfrak{m}$-adic topology, and let $E = E_R(\Bbbk)$ denote the injective hull of the residue field. We consider the trivial extension $S = R \ltimes E$ and note the following.

(a) The ring S is self-injective, namely there is an isomorphism $S \cong \operatorname{Hom}_R(S, E)$ of S-modules. (This follows by Theorem A.4.6. Note that $R \cong \operatorname{Hom}_R(E, E)$ since R is $\mathfrak{m}$-complete.)

(b) For all $e \in E$, $\operatorname{Ann}_S(S(0, e)) = (0 :_R e) \ltimes E$, as is easily seen.

(c) If $\dim R > 0$ the ring S is not coherent and the S-module S^{Λ} is not flat, where Λ denotes the index set given by the pairwise different elements of $\operatorname{Ann}_S(S(0, e))$ for some $e \in E \setminus \{0\}$. (This follows by A.4.3. Note that the R-module E is not finitely generated when $\dim R > 0$. Hence for any $e \in E \setminus \{0\}$ the ideal

$$\operatorname{Ann}_S(S(0, e)) = (0 :_R e) \ltimes E$$

of S is not finitely generated.)

We are ready for the promised explicit example.

Example A.5.7 Let $R = \Bbbk[[x]]$ denote the formal power series ring in a variable x over the field $\Bbbk$ with $E = E_R(\Bbbk)$ the injective hull of the residue field. Put $S = R \ltimes E$.

(a) Then the ring S is not coherent and the S-module S^R is not flat.
(b) The S-modules S and S^R are injective but the S-module $\operatorname{Hom}_S(S, S^R)$ is not flat.

To check these claims recall first that $E \cong \varinjlim \operatorname{Hom}_R(R/x^t R, E)$ because E is xR-torsion. Moreover, $\operatorname{Hom}_R(R/x^t R, E)$, viewed as an $R/x^t R$-module, is the injective hull of $\Bbbk$ (see 10.6.1). Since $R/x^t R$ is Gorenstein ring of dimension 0 we have an isomorphism $\operatorname{Hom}_R(R/x^t R, E) \cong R/x^t R$ for all $t > 0$ and there is an element $e \in E$ with $(0 :_R e) = xR$. For this element e we then have $\operatorname{Ann}_S(S(0, e)) = xR \ltimes E$ and we compute its cardinality with some basic set theoretic arguments.

If $\Bbbk$ is infinite there are set bijections

$$\operatorname{Hom}_R(R/x^t R, E) \leftrightarrow \Bbbk \qquad E \leftrightarrow \Bbbk \qquad \operatorname{Ann}_S(S(0, e)) \leftrightarrow R.$$

If $\Bbbk$ is finite then E is countable and there is again a set bijection

$$\operatorname{Ann}_S(S(0, e)) \leftrightarrow R.$$

Then the claims follow by the observations in A.5.6. For the last one note also that $\operatorname{Hom}_S(S, S^R) \cong S^R$ and that the S-module S^R is injective as a product of injective modules.

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