HOMOLOGY OF CELL COMPLEXES

BY
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(Based on Lectures by Norman E. Steenrod)

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Foreword

These are notes based on an introductory course in algebraic topology given by Professor Norman Steenrod in the fall of 1963. The principal aim of these notes is to develop efficient techniques for computing homology groups of complexes. The main object of study is a regular complex: a CW-complex such that the attaching map for each cell is an embedding of the boundary sphere. The structure of a regular complex on a given space requires, in general, far fewer cells than the number of simplices necessary to realize the space as a simplicial complex. And yet the procedure: orientation $\longrightarrow$ chain complex $\longrightarrow$ homology groups is essentially as effective as in the case of a simplicial complex.

In Chapter I we define the notion of CW-complex, due to J. H. C. Whitehead. (The letters CW stand for a) closure finite--the closure of each cell is contained in the union of a finite number of (open) cells-- and b) weak topology--the topology on the underlying topological space is the weak topology with respect to the closed cells of the complex.) We give several examples of complexes, regular and irregular, and complete the chapter with a section on simplicial complexes.

In Chapter II we define orientation of a regular complex, and chain complex and homology groups of an oriented regular complex. The definition of orientation of a regular complex requires certain properties of regular complexes which we call <u>redundant restrictions</u>. We assume that all regular complexes satisfy these restrictions, and we prove in a leter chapter (VIII) that the restrictions are indeed redundant. The main results of the rest of Chapter II are: a proof that different orientations on a given regular complex yield isomorphic homology groups, and a proof of the universal coefficient theorem for regular complexes which have finitely many cells in each dimension.

In Chapter III we define homology groups of spaces which are obtained from regular complexes by making cellular identifications. This technique simplifies the computation of the homology groups of many spaces by reducing the number of cells required. We compute the homology of 2-manifolds, certain 3-manifolds called lens spaces, and real and complex projective spaces.

Chapter IV provides background for the Kunneth theorem on the homology of the product of two regular complexes. Given regular complexes K and L there is an obvious way to define a cell structure on $|K| \times |L|$--simply take products of cells in K and in L. But the product topology on $|K| \times |L|$ is in general too coarse to be the weak topology with respect to closed cells. Thus $K \times L$, with the product topology, is not in general a regular complex. To get around this difficulty we alter the topology on the product. The proper notion is that of a compactly generated topology. In order to provide a proper point of view for this question we include in Chapter IV a discussion of categories and functors.

In Chapter V we prove the Kunneth theorem. We also compute the homology of the join of two complexes, and we complete the chapter with a section on relative homology.

In Chapter VI we prove the invariance theorem, which states that homeomorphic finite regular complexes have isomorphic homology groups. We also state and prove the seven Eilenberg-Steenrod axioms for cellular homology.

In Chapter VII we define singular homology. We state and prove axioms for singular homology theory, and show that if X is the underlying topological space of a regular complex K, then the singular homology groups of X are naturally isomorphic to the cellular homology groups of K.

In Chapter VIII we prove Borsuk's theorem on sets in S^n which separate S^n and Brouwer's theorem (invariance of domain) that R^m and R^n are homeomorphic only if $m = n$. We show that any regular complex satisfies the redundant restrictions stated in Chapter II, and settle a question raised in Chapter I concerning quasi complexes.

In Chapter IX we define skeletal decomposition of a space and homology groups of a skeletal decomposition. We prove that the homology groups of a skeletal decomposition are isomorphic to the singular homology groups of the underlying space. Finally, we use skeletal homology to show that the homology groups we defined in Chapter III of a space X obtained by identification from a regular complex are isomorphic to the singular homology groups of X.

We should mention that we sometimes refer to the "homology" of a space without noting which homology theory we are using. This is because all of the different definitions of the homology groups that we give agree on their common domains of definition.

We also remark that cohomology groups, which we define for a regular complex in Chapter II, are only touched on very lightly throughout these notes. We do not cover the cup or cap products, and we do not define singular cohomology groups.

Finally, we wish to express our gratitude to those who have helped us in the preparation of these notes. First, we thank Martin Arkowitz for his efforts in our behalf--he painstakingly read the first draft and made many helpful suggestions for revision. Secondly, we thank the National Science Foundation for supporting the first-named author during a protion of this work. And finally, we wish to thank Elizabeth Epstein, Patricia Clark, Bonnie Kearns, Barbara Duld, June Clausen, and Joanne Beale for typing and correcting the manuscript.

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June 2, 1967

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INTRODUCTION

One of the ways to proceed from geometric to algebraic topology is to associate with each topological space X a sequence of abelian groups $\{H_q(X)\}_{q=0}^{\infty}$, and to each continuous map f of a space X into a space Y a sequence of homomorphisms

$$f_{q*}\colon\ H_q(X) \longrightarrow H_q(Y)\ ,$$

one for each q. The groups are called *homology groups* of X or of Y, as the case may be, and the morphisms f_{q*} are called the *induced homomorphisms of* f, or simply *induced homomorphisms*. Schematically we have

geometry $\longrightarrow$ algebra

space X $\longrightarrow$ homology groups $H_q(X)$

map $f\colon X \longrightarrow Y$ $\longrightarrow$ induced homomorphisms f_{q*}.

The transition has these properties, among others:

1. If $f\colon X \longrightarrow X$ is the identity map, then $f_{q*}\colon H_q(X) \longrightarrow H_q(X)$ is the identity isomorphism for each q.

2. If $f\colon X \longrightarrow Y$ and $g\colon Y \longrightarrow Z$, then $(gf)_{q*} = g_{q*} \cdot f_{q*}$ for each q. That is, if the diagram

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "gf"'] & & Z \end{tikzcd}$$

of spaces and mappings is commutative, then so is the diagram

$$\begin{tikzcd}
 & H_q(Y) \arrow[dr, "g_{q*}"] & \\
H_q(X) \arrow[ur, "f_{q*}"] \arrow[rr, "(gf)_{q*}"'] & & H_q(Z)
\end{tikzcd}$$

of homology groups and induced homomorphisms, for each q. In modern parlance, homology theory is a functor from the category of spaces and mappings to the category of abelian groups and morphisms.

The correspondence here between geometry and algebra is often crude. Topologically distinct spaces may be made to correspond to the same algebraic objects. For example, a disc and a point have the same homology groups. So do a 1-sphere and a solid torus (the cartesian product of a one-sphere and a disc.) Despite this, the methods of algebraic topology may be applied to a broad class of problems, such as extension problems.

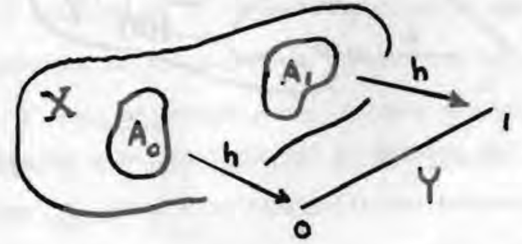
Suppose we are given a space* X, a subspace A of X, and a map h of A into a space Y. If we let i denote the inclusion mapping of A into X, then the extension problem is to decide whether there exists a map f, indicated by the dashed arrow in the diagram below, of X into Y such that $h = fi$.

$$\begin{tikzcd}
 & X \arrow[dr, dashed, "f"] & \\
A \arrow[ur, "i"] \arrow[rr, "h"] & & Y
\end{tikzcd}$$

*Unless otherwise stated, the word <u>space</u> will mean <u>Hausdorff</u> <u>space</u>.

If f exists, we say that f is an extension of h to X, or, equivalently, that h has been extended to a mapping f of X into Y.

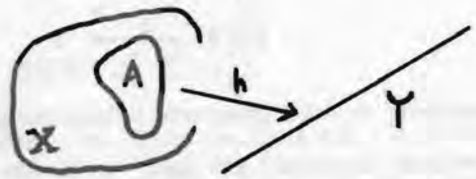
There are famous solutions of this problem in point-set topology. The Urysohn Lemma is one. The hypotheses are: the space X is normal, but otherwise arbitrary; the subspace $A = A_o \cup A_1$, where A_o and A_1 are disjoint, closed subsets of X; the space Y is the closed unit interval $[0,1]$; and the map $h\colon A \longrightarrow Y$ carries A_o to 0 and A_1 to 1.



Under these assumptions the Urysohn Lemma asserts that there exists a mapping $f\colon X \longrightarrow Y$ such that $f = hi$:

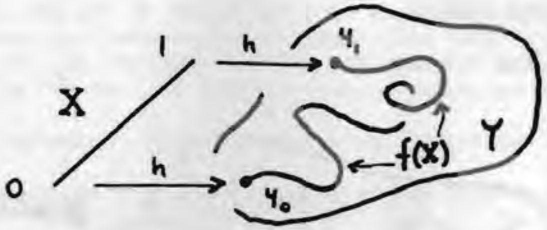
$$\begin{tikzcd}
 & X \arrow[dr, dashed, "f"] & \\
(A_o \cup A_1) \arrow[ur, "i"] \arrow[rr, "h"'] & & {[0,1] = Y}
\end{tikzcd}$$

The Tietze Extension Theorem is another solution. Here, X is normal, A is closed in X, and h is an arbitrary map of A to the set Y of real numbers.



The theorem asserts the existence of an extension $f\colon X \longrightarrow Y$ of h.

Pathwise connectivity can also be described in terms of extensions. Let X be the closed unit interval, and let $A = \{0,1\}$. A space Y is pathwise connected if, given two points y_0 and y_1 of Y , the mapping $h: A \longrightarrow Y$ defined by $h(0) = y_0$ and $h(1) = y_1$, can be extended to a mapping $f: X \longrightarrow Y$.



Furthermore, the problem of the existence of a continuous multiplication with a prescribed two-sided unit in a space Y can be phrased as an extension problem. We seek a map $m: Y \times Y \longrightarrow Y$ and an element 1 in Y such that $m(1,y) = m(y,1) = y$ for each y in Y . The requirement that there be a two-sided unit determines a function h on a subspace $A = (1 \times Y) \cup (Y \times 1) = Y \vee Y$ of $Y \times Y$ into Y . The existence of a continuous multiplication with this two-sided unit is now equivalent to the existence of a map $m: Y \times Y \longrightarrow Y$ such that $mi = h$.

$$\begin{tikzcd} Y \times Y \arrow[dr, dashed, "m"] & \\ Y \vee Y \arrow[u, "i"] \arrow[r, "h"'] & Y \end{tikzcd}$$

Corresponding to each geometric extension problem is a homology extension problem, described for each q by the following diagram:

$$\begin{tikzcd} H_q(X) \arrow[dr, dashed, "\phi"] & \\ H_q(A) \arrow[u, "i_*"] \arrow[r, "h_*"] & H_q(Y) \end{tikzcd}$$

Given the groups, the homomorphisms i_* and h_* , does there exist a homomorphism $\phi: H_q(X) \longrightarrow H_q(Y)$ such that $i_*\phi = h_*$?

If f is a solution of the geometric extension problem, then $\phi = f_{q*}$ is a solution of the algebraic problem for each q . On the other hand, if ϕ does not exist, then f does not exist either. This latter implication is the principal tool in the proofs of many theorems of algebraic topology.

Let A be a subspace of a space X . A map $f: X \longrightarrow A$ is a retraction if $fx = x$ for each point x in A . Under these circumstances, A is called a retract of X . The Tietze Extension Theorem implies that if X is a normal space which contains an arc A , then X can be retracted to A .

Throughout this and subsequent chapters, E^n will denote the closed unit ball in Euclidean n-space R^n , and S^{n-1} will denote the unit (n-1)-sphere in R^n .

THEOREM: S^{n-1} is not a retract of E^n .

Suppose to the contrary that there exists a map $f: E^n \longrightarrow S^{n-1}$ such that $fx = x$ for each x in S^{n-1} . Then f is an extension of the identity map h:

$$\begin{tikzcd} E^n \arrow[dr, dashed, "f"] & \\ S^{n-1} \arrow[u, "i"] \arrow[r, "h"] & S^{n-1} \end{tikzcd}$$

If $n = 1$, the fact that E^1 is connected, while S^0 is not, shows that no such f can exist.

If $n \geq 2$, a different argument is required. Assume that the homology groups of E^n and of S^{n-1} are, as we will later show them to be,

$$H_q(E^n) = \begin{cases} Z & q = 0 \\ 0 & q \geq 1 \end{cases} \qquad H_q(S^{n-1}) = \begin{cases} Z & q = 0, n-1 \\ 0 & 0 < q < n-1 \end{cases}$$

If f exists, then for each q the following diagram is commutative:

$$\begin{tikzcd} H_q(E^n) \arrow[dr, "\phi = f_{q*}"] & \\ H_q(S^{n-1}) \arrow[u] \arrow[r, "h*"', "\substack{\text{identity}\\\text{homomorphism}}"] & H_q(S^{n-1}) \end{tikzcd}$$

For $q = n - 1$, however, the diagram looks like this:

$$\begin{tikzcd} 0 \arrow[dr, "\phi = f_{n-1,*}"] & \\ Z \arrow[u, "i_*"] \arrow[r, "h*"', "\text{identity}"] & Z \end{tikzcd}$$

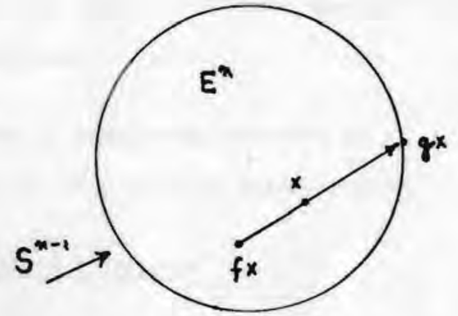
Since the only homomorphism of the zero group into Z is the zero homomorphism, there is no ϕ such that $\phi i_* = h_*$. Hence f cannot exist after all.

This last diagram shows once more that the transition from geometry to algebra may be crude. The inclusion mapping $i: S^{n-1} \longrightarrow E^n$ is a homeomorphism, yet i_* is not one-to-one. Furthermore i_* maps $H_{n-1}(S^{n-1})$ onto $H_{n-1}(E^n)$ even though i does not map S^{n-1} onto E^n.

A <u>fixed point</u> of a mapping $f: X \longrightarrow X$ is a point x in X for which $fx = x$. Using the fact that S^{n-1} is not a retract of E^n we can prove the Brouwer Fixed-point Theorem:

THEOREM: <u>Each mapping</u> $f: E^n \longrightarrow E^n$ <u>has a fixed point</u>.

Suppose to the contrary that there exists a mapping $f: E^n \longrightarrow E^n$ which has no fixed point. Then we can define a retraction $g: E^n \longrightarrow S^{n-1}$ For each point x of E^n we let Rx denote the directed line segment which starts at fx and passes through x. Note that Rx is defined for each x in E^n, since f has no fixed point. Let $gx = Rx \cap (S^{n-1} - \{fx\}$ If x lies in S^{n-1}, then $gx = \{x\} \cap S^{n-1} = x$. The verification that g is continuous is left as an exercise.



Thus g is a retraction of E^n onto S^{n-1}. This contradicts the preceding theorem, and completes the proof of this theorem.

Chapter I

COMPLEXES

1. Complexes

If A is a subspace of X and B a subspace of Y, then a <u>map</u> $f\colon (X,A) \longrightarrow (Y,B)$ <u>of the pair</u> (X,A) to the pair (Y,B) is a map of X into Y that carries A into B. One can compose mappings of pairs. For each pair there is an identity mapping of the pair onto itself. A homeomorphism of (X,A) onto (Y,B) is a homeomorphism of X onto Y which carries A onto B. A more important concept is that of a relative homeomorphism.

A mapping $f\colon (X,A) \longrightarrow (Y,B)$ is a <u>relative homeomorphism</u> if $f|(X-A)$ maps $X-A$ homeomorphically onto $Y-B$. A relative homeomorphism need not map A onto B.

Exercise. Let X be a compact space, and suppose that $f\colon (X,A) \longrightarrow (Y,B)$ is a mapping such that $f|(X-A)$ maps $X-A$ one-to-one onto $Y-B$. Show that if Y is a Hausdorff space then f is a relative homeomorphism.

Exercise. Find an example to show that if Y is not a Hausdorff space then f may not be a relative homeomorphism.

1.1. <u>The definition of a complex</u>.

A complex K consists of a Hausdorff space $|K|$ and a sequence of subspaces, called skeletons, denoted by $|K_n|$ $n = -1, 0, 1, \ldots,$ which satisfy the following conditions.

1. $|K_{-1}|$ is the empty set, and $|K_{-1}| \subset |K_0| \subset |K_1| \ldots \subset |K_n| \subset \ldots$

2. Each $|K_n|$ is closed in $|K|$.

3. $|K| = \cup |K_n|$.

4. For $n \geq 0$, the components $\sigma_1^n, \sigma_2^n, \ldots, \sigma_i^n, \ldots$ of $|K_n| - |K_{n-1}|$ are open sets in the relative topology of $|K_n|$. They are referred to as the n-cells of K.

5. For each n-cell σ_i^n of K let $\bar{\sigma}_i^n$ denote that subspace of $|K|$ which is the closure of σ_i^n in $|K|$ with the relative topology, and let $\dot{\sigma}_i^n = (\bar{\sigma}_i^n - \sigma_i^n)$. For each σ_i^n there exists a relative homeomorphism $f_{n,i}\colon (E^n, S^{n-1}) \longrightarrow (\bar{\sigma}_i^n, \dot{\sigma}_i^n)$ which carries S^{n-1} onto $\dot{\sigma}_i^n$. For convenience, $\bar{\sigma}_i^n$ is called a closed n-cell of K, even though it need not be homeomorphic to E^n.

6. The topology of $|K|$ is the weak topology defined by the closed cells of K: a subset A of $|K|$ is closed if and only if each intersection $A \cap \bar{\sigma}_i^n$ is closed in $\bar{\sigma}_i^n$.

7. The relative topology on $|K_n|$ and the weak topology defined by the closed cells of $|K_n|$ coincide: a subset A of $|K_n|$ is closed in the relative topology of $|K_n|$ if and only if $A \cap \bar{\sigma}_i^q$ is closed in $\bar{\sigma}_i^q$ for each $q \leq n$ and for each i.

1.2. The structure of a complex.

A complex K is said to be a complex on $|K|$, and $|K|$ is called the underlying space, or the geometric realization of K. A finite complex is one with finitely many cells. A complex is called infinite if it is not finite.

If there exists an integer r such that $|K_{q+1}| = |K_q|$ for each $q \geq r$, then K is said to be of finite dimension. The least such r is called the dimension of K.

If σ is a cell of K, we will sometimes write $\sigma \in K$, even though K is not defined as a collection of cells.

The subspace $|K_0|$ is a discrete subspace of $|K|$. Its components, which are points by (5), are also open sets of $|K_0|$, by (4). The cells of $|K_0|$ are called vertices of K.

The relative homeomorphisms $f_{n,i}$ are not part of K. To show that a candidate for a complex satisfies (5) one need only exhibit some set of f's. A complex is to remain unchanged when one set of f's is replaced by another.

By (5), an n-cell σ_i^n of K, with the relative topology, is actually homeomorphic to the n-cell $E^n - S^{n-1}$. Its boundary $\dot{\sigma}_i^n$ need not be homeomorphic to S^{n-1}. If, for each σ_i^n there exists a homeomorphism $f_{n,i}$ of E^n onto $\bar{\sigma}_i^n$, then K is called regular. Otherwise, K is irregular. If σ_i^n is a cell of K and if none of the relative homeomorphisms $f_{n,i}$ is a homeomorphism, then σ_i^n is called irregular.

In order to be sure that the topologies in (6) and (7) are well-defined, one must know that $|K| = \bigcup_{q,i} \bar{\sigma}_i^q$ and that $|K_n| = \bigcup_{i;\, q \leq n} \bar{\sigma}_i^q$. These two facts follow, as they should, from conditions (1) through (5). We leave their verification to the reader.

Exercise. Establish the following elementary facts about the structure of K.

1. $|K| = \bigcup_{n,i} \sigma_i^n = \bigcup_{n,i} \bar{\sigma}_i^n$.

2. If σ^q is a q-cell of K , then $\bar{\sigma}^q \subset |K_q|$.

3. $|K_n| = \bigcup_{i;\, q \leq n} \sigma_i^q = \bigcup_{i;\, q \leq n} \bar{\sigma}_i^q$.

4. Also, although this won't be used immediately, $\dot{\sigma}^q = \bar{\sigma}^q \cap |K_{q-1}|$ for each q-cell σ^q of K . (While the first three parts of this exercise are easily seen to be true, the reader may find this fourth part to require some moments of reflection.)

5. Let X be a space that is the union of finitely many disjoint subsets $\sigma_1, \sigma_2, \ldots, \sigma_m$. Let $\bar{\sigma}_i$ denote the closure of σ_i in X and let $\dot{\sigma}_i = \bar{\sigma}_i - \sigma_i$. Suppose that for each i

   1. There exists a relative homeomorphism of (E^k, S^{k-1}) onto $(\bar{\sigma}_i, \dot{\sigma}_i)$ for some k . (We call σ_i a <u>k-cell</u> and call k the <u>dimension</u> of σ_i .)

   2. $\dot{\sigma}_i$ lies in a union of cells of dimension lower than that of σ_i .

Then the σ_i are the cells of a complex on X .

6. A function f from $|K|$ into a space X is continuous if and only if $f|\bar{\sigma}$ is continuous for each cell σ of K .

7. If K is of finite dimension, then condition (7) implies condition (6).

2. Examples

2.1. <u>A regular complex on</u> S^n .

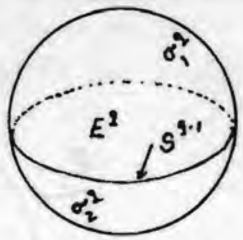
For each k , $0 \leq k \leq n$, identify R^{k+1} with the subspace

$\{(x_1,\ldots,x_{n+1}) | x_{k+2} = \ldots = x_{n+1} = 0\}$ of R^{n+1} . Let $|K| = S^n$, and for each $q \geq 0$ let the q-skeleton of K be the q-sphere

$$S^q = \begin{cases} S^n \cap R^{q+1} & q \leq n \\ S^n & q > n \end{cases}$$

Certainly S^n is a Hausdorff space, $S^0 \subset S^1 \subset \ldots \subset S^n$, each S^q is closed in S^n , and $S^n = \bigcup S^q$.

The 0-skeleton S^0 consists of a pair of points. The (q-1)-skeleton S^{q-1} is the equator in S^q , and divides S^q into two (open) hemispheres, σ_1^q and σ_2^q , whose union is $S^q - S^{q-1}$.



The map $f_{q,1}\colon E^q \longrightarrow \bar{\sigma}_1^q$ is the homeomorphism defined by vertical projection:

$$f_{q,1}(x_1,\ldots,x_q) = (x_1,\ldots,x_q, (1 - \sum_1^q x_i^2)^{1/2})$$

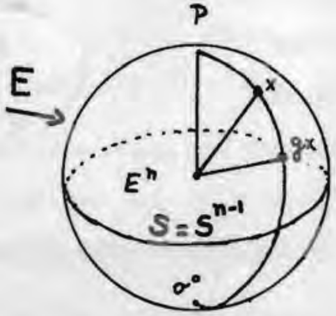
in $\bar{\sigma}_i^q$. A reversal of sign gives the homeomorphism for $\bar{\sigma}_2^q$. The verification that the σ_i^q are the cells of a regular complex on S^n is now trivial by Exercise 5.

2.2. <u>An irregular complex on</u> S^n .

Let $|K| = S^n$, let σ^0 denote the point $(0,\ldots,0,-1)$ in S^n , and let the q-skeleton

$$|K_q| = \begin{cases} \sigma^0 & \text{if } 0 \leq q \leq n-1 \\ S^n & \text{if } q \geq n \end{cases}$$

Conditions (1) through (3) are satisfied, and conditions (4) and (5) are satisfied vacuously for $q < n$. For the one n-cell, $(S^n - \sigma^o)$, we now define a relative homeomorphism $f\colon (E^n, S^{n-1}) \longrightarrow (S^n, \sigma^o)$ in two stages. First, let $P = (0,\ldots,0,1)$, the point diametrically opposite σ^o on S^n, and let E be the closed hemisphere of points of S^n with coordinate $x_{n+1} \geq 0$. Then E contains P, and has as boundary the set $S = S^{n-1}$ of points of S^n with coordinate $x_{n+1} = 0$. We define a relative homeomorphism $g\colon (E,S) \longrightarrow (S^n, \sigma^o)$ by doubling angles.



More precisely, for each point x in E, gx is that point of S^n which is located on the great circular arc that starts at P and passes through x to σ^o in such a way that

$$\measuredangle(P, \text{origin}, gx) = 2 \cdot \measuredangle(P, \text{origin}, x)\ .$$

The result of Exercise 2.1 can be used to show that g, which is clearly continuous, is a relative homeomorphism.

To complete the definition of f, compose g with the homeomorphism $h\colon (E^n, S^{n-1}) \longrightarrow (E,S)$ defined by

$$h(x_1,\ldots,x_n) = (x_1,\ldots,x_n, (1 - \sum_1^n x_i^2)^{1/2})\ .$$

The mapping $f = gh$ then satisfies condition (5). As in Example 2.1, (6) and (7) are satisfied because the complex is finite.

2.3. <u>A regular complex on</u> E^n.

We start with the regular complex for S^{n-1} described in Example 2.1. Take $|K_q| = S^q$ for $0 \leq q < n$, $|K_n| = E^n$, and $|K_q| = E^n$ for $q > n$. To show that these are the skeletons of a regular complex on E^n it remains only to check the properties required of the one n-cell $E^n - S^{n-1}$. It is certainly an open, connected subset of $|K_n| = E^n$, and the homeomorphism $f\colon (E^n, S^{n-1}) \longrightarrow (E^n, S^{n-1})$ may be taken to be the identity map.

2.4. <u>An irregular complex on</u> E^n.

We augment Example 2.2 by taking $|K_q| = |K_n| = E^n$ for $q \geq n$, and by letting $f\colon (E^n, S^{n-1}) \longrightarrow (E^n, S^{n-1})$ be the identity mapping.

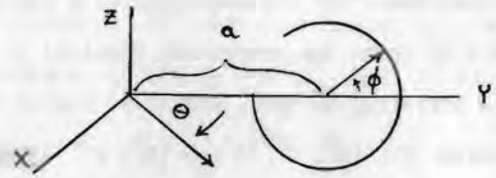
2.5. <u>An irregular complex on the torus</u> T.

Let θ and ϕ lie in the closed interval $[0, 2\pi]$, let a and b be positive real numbers with $a > b$, and let T denote the set of points of R^3 whose coordinates (x,y,z) satisfy the parametric equations

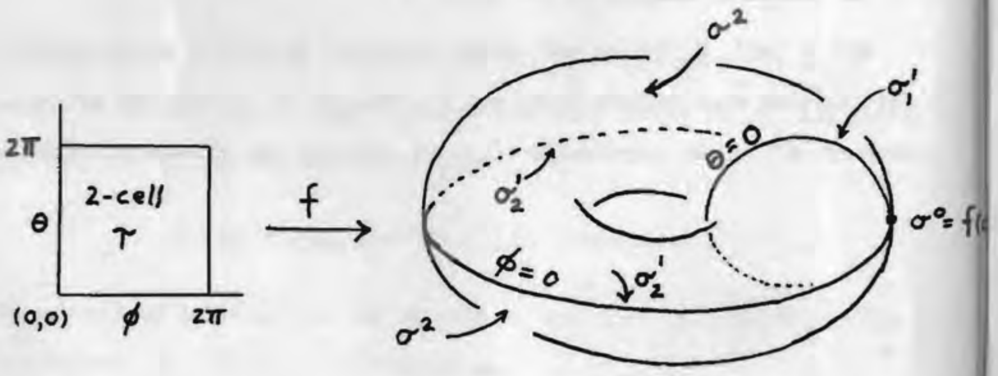
$$x = (a + b \cos \phi) \sin \theta$$

$$y = (a + b \cos \phi) \cos \theta$$

$$z = b \sin \phi$$



To obtain an irregular complex K on T, subdivide T into the cells shown in the following picture.



The cells can be described precisely by using the parametrization of T, which defines a mapping f of the 2-cell $\tau = [0, 2\pi] \times [0, 2\pi]$ of the $\theta\,\phi$-plane onto T.

Thus

$$\sigma^0 = f(0,0)$$

$$\sigma_2^1 = f((0, 2\pi) \times 0)$$

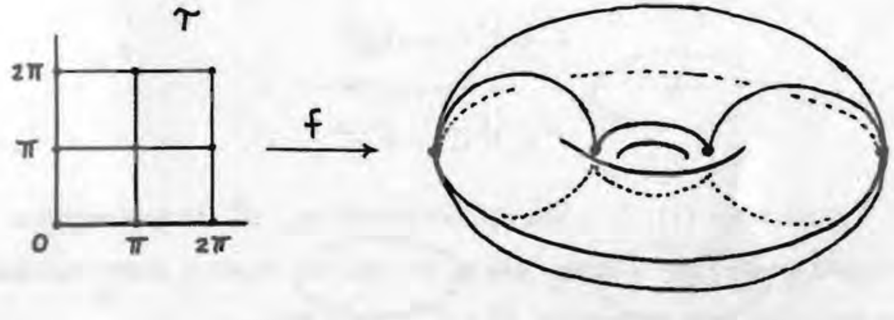
$$\sigma_1^1 = f(0 \times (0, 2\pi))$$

$$\sigma^2 = f((0, 2\pi) \times (0, 2\pi))$$

The mappings $f_{n,i}$ may be taken to be the appropriate restrictions of f.

2.6. A regular complex on T.

Subdivide as pictured below:



This subdivision of τ induces a subdivision of T into the cells of a regular complex: four vertices, eight edges (1-cells), and four 2-cells, for which f induces the required homeomorphisms.

2.7. An irregular complex on real projective n-space, P^n.

A point of P^n is an equivalence class of points of $\hat{R}^{n+1} = (R^{n+1}$ - the origin) under the relation

$$(x_1,\ldots,x_{n+1}) \approx (rx_1,\ldots,rx_{n+1}) ,$$

where r is a real number different from 0. The topology of P^n is in the quotient space topology obtained from $\hat{R}^{n+1}$. Recall that this topology is defined as follows: Let $p\colon \hat{R}^{n+1} \longrightarrow P^n$ be the transformation which sends each point onto its class in P^n. Then a subset A of P^n is closed if and only if $p^{-1}(A)$ is closed in $\hat{R}^{n+1}$. The map $g = p|S^n\colon S^n \longrightarrow P^n$ is a two-fold covering of P^n. The involution $T\colon S^n \longrightarrow S^n$ defined by

$$T(x_1,\ldots,x_{n+1}) = (-x_1,\ldots,-x_{n+1})$$

is the covering transformation of g.

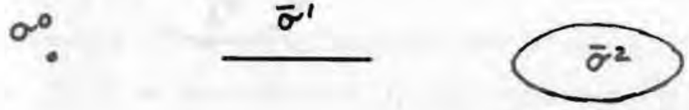
For each $k \geq 0$ define the k-skeleton of P^n to be $P^k = g(S^k)$.

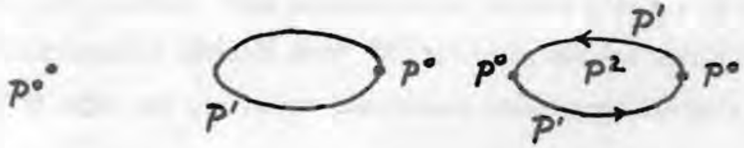
$$\begin{array}{ccccccc} S^o & \subset & S^1 & \subset & \cdots & \subset & S^n \\ \downarrow & & \downarrow & & & & \downarrow \\ P^o & \subset & P^1 & \subset & \cdots & \subset & P^n \end{array}$$

Conditions (1), (2), and (3) are satisfied, P^k is homeomorphic to real projective k-space, and we can exhibit relative homeomorphisms to show that each difference $P^k - P^{k-1}$ is a cell.

The set $g^{-1}(P^k - P^{k-1}) = S^k - S^{k-1}$ is the union of two open k-cells, each of which is mapped homeomorphically by g onto $P^k - P^{k-1}$. Let σ^k denote the upper cell, i.e., the one which has its last non-zero coordinate positive. Let $f_k\colon E^k \longrightarrow \bar{\sigma}^k$ denote the usual projection of E^k onto the closed hemisphere $\bar{\sigma}^k$.

Remark. Note that P^k has been obtained from P^{k-1} by attaching $\bar{\sigma}^k$ to P^{k-1} with the mapping g. This suggests an inductive construction for P^n. Start with a point $P^o = \sigma^o$. Attach a closed 1-cell $\bar{\sigma}^1$ to P^o by adjoining its end-points to P^o. The resulting space is P^1. Observe that $P^1 = S^1$. To obtain P^2, adjoin a 2-cell $\bar{\sigma}^2$ to P^1 by attaching its boundary S^1 to P^1 with the double covering map. This attaching operation amounts to wrapping S^1 around $P^1 (= S^1)$ twice.



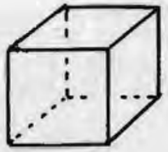
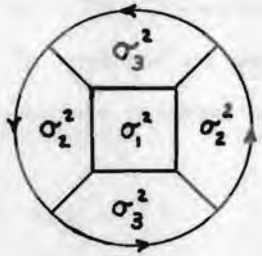


Given P^{n-1}, one constructs P^n by attaching the boundary S^{n-1} of $\bar{\sigma}^n$ to P^{n-1} by the double covering map g.

2.8. <u>A regular complex on</u> P^n.

We start with the regular complex on the boundary $\dot{I}^{n+1}$ of the unit (n+1)-cube in R^{n+1} . The cells of the complex on $\dot{I}^{n+1}$ are the open r-dimensional faces $(0 \leq r \leq n)$ of $\dot{I}^{n+1}$. On $\dot{I}^{n+1}$ we identify points that are diametrically opposite with respect to the center $(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$. We thus transform the complex on $\dot{I}^{n+1}$ into a regular complex on P^n by identifying diametrically opposite cells. To illustrate:

$\dot{I}^2$  **becomes**  P^1 **with two 1-cells**

$\dot{I}^3$  **becomes** 

P^2 **with three 2-cells**

2.9. A regular complex on R^n .

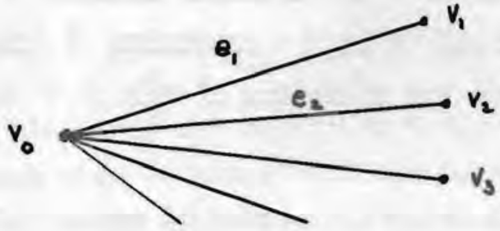
We obtain this complex by considering unit n-dimensional cubes whose vertices are the points of R^n with integral coordinates. The cubes, with all their lower dimensional faces, are the cells of R^n .

3. Locally-finite Complexes

A complex is locally-finite if each point has a neighborhood that is contained in the union of a finite number of cells of the complex. Such a complex is locally compact. The reticulation of R^n described in Example 2.9 is locally-finite, although the complex is infinite.

3.1. A complex that is not locally-finite.

Let $|K|$ be the union of countably many regular, closed 1-cells e_i joined to a common vertex v_0 . For each i let v_i be the other vertex of e_i . Give to $|K|$ the weak topology with respect to the e's and v's . The e's and v's determine a regular complex on $|K|$ that is not locally-finite.



Each neighborhood of v_0 meets infinitely many of the e's: it must meet each $\bar{e}_i$ in an open subset of $\bar{e}_i$, and this subset is never empty because it contains v_0 .

Exercise. Show that the space $|K|$ of 3.1 cannot be embedded in R^n for any n .

4. Subcomplexes

4.1. DEFINITION. A complex L is a subcomplex of a complex K (in symbols $L \subset K$) if (1) $|L|$ is a closed subspace of $|K|$, (2) $|L_q| = |L| \cap |K_q|$ for each q , and (3) each cell of L is a cell of K .

The proposition below gives a combinatorial characterization of subcomplexes. In particular, it shows that the collection of cells of dimension $\leq q$ of a complex K determines a subcomplex of K. This subcomplex is denoted by K_q; it is called the q-skeleton of K, and its underlying space is $|K_q|$.

4.2. PROPOSITION. (Characterization of subcomplexes.) The cells of a collection C of cells of K are those of a subcomplex L of K if and only if, for each cell σ in C, the boundary $\dot{\sigma}$ (the closure of σ in $|K|$, minus σ) lies in the union of cells of C of dimension less than $\dim \sigma$.

The proof of this proposition occupies the remainder of this section.

4.3. COROLLARY. The cells of K_q are those of a subcomplex of K.

This corollary is an immediate consequence of 4.2.

We now begin the proof of the proposition. Suppose that the cells of C are those of a subcomplex L of K, let σ be a q-cell of C, and let

$$\hat{\sigma} = (\text{the closure of } \sigma \text{ in } |L|) - \sigma .$$

We know from Exercise 2.3 that $\hat{\sigma} \subset |L_{q-1}|$, and that $|L_{q-1}|$ is the union of cells of L of dimension less than q. It remains to show that $\hat{\sigma} = \dot{\sigma}$. But this follows from the fact that $|L|$ is closed in $|K|$.

Now suppose that each cell σ of C satisfies the condition that $\dot{\sigma}$ lies in the union of cells of C of dimension less than $\dim \sigma$. Let $|L|$ denote the union of the cells of C with the subspace topology

from $|K|$, and let $|L_q| = |L| \cap |K_q|$. Let L consist of the space $|L|$ and the subspaces $|L_q|$. We will show that L is a complex whose cells are precisely the cells of C, and that $|L|$ is closed in $|K|$.

(1) $|L|$ is a Hausdorff space because it is a subspace of $|K|$.

(2) $|L_{q-1}| \subset |L_q|$, and each $|L_q|$ is closed in $|L|$ because $|L_q| = |L| \cap |K_q|$, the intersection of $|L|$ with a closed subspace of $|K|$.

(3) $|L| = |L| \cap (\cup|K_q|) = \cup(|L| \cap |K_q|) = \cup|L_q|$.

(4) $|L_q| - |L_{q-1}| = (|L| \cap |K_q|) - (|L| \cap |K_{q-1}|) = |L| \cap (|K_q| - |K_{q-1}|)$, which is the union of the q-cells of C. Each q-cell is a component of $|L_q| - |L_{q-1}|$ because each is connected and no two have a common point. Each q-cell is open in the relative topology of $|L_q|$. This follows from the fact that

$$|L_q| - \sigma = |L| \cap |K_q| - |L| \cap \sigma = |L| \cap (|K_q| - \sigma) ,$$

which is closed in $|L|$, and hence in $|L_q|$, because $|K_q| - \sigma$ is closed in $|K|$.

It follows from (4) that $|L_q|$ is the union of the cells of C of dimension $\leq q$.

(5) For each q-cell σ_i^q, the closure of σ_i^q in $|K|$ is $\bar{\sigma}_i^q = \sigma_i^q \cup \dot{\sigma}_i^q$. Now $\dot{\sigma}_i^q \subset |L_{q-1}|$ by hypothesis together with the fact that $|L_{q-1}|$ is the union of cells of C of dimension $< q$. Therefore, $\bar{\sigma}_i^q \subset |L|$, and $\bar{\sigma}_i^q$ is the closure of σ_i^q in $|L|$. Accordingly, the relative homeomorphism $f_{q,i}\colon (E^q, S^q) \longrightarrow (\bar{\sigma}_i^q, \dot{\sigma}_i^q)$ given for K may be taken as the one required for L.

So far we have been able to avoid the question of whether $|L_q|$ and $|L|$ are closed in $|K|$. It has been sufficient to know that $|L_q|$ is closed in $|L|$, and that if σ is a cell of C its closure in $|K|$ lies in $|L|$. The question must be settled before proceeding to properties (6) and (7). To settle it, and to show several related facts, we prove two lemmas.

4.4. LEMMA. _Let_ K _be a complex. Each compact subset of_ $|K|$ _meets only finitely many cells of_ K.

Let D be a compact subset of $|K|$. We show first that D meets cells of only finitely many different dimensions. Suppose that this is not so. Let $\{\sigma^{q_i}\}$ be an infinite sequence of cells of strictly ascending dimension, and let x_i be a point of $(\sigma^{q_i} \cap D)$ for each i. A given $|K_q|$ contains x_i only if $q_i \leq q$, so that $\{x_i\} \cap |K_q|$ is finite. (Here we use $\{x_i\}$ to denote the entire set $\{x_1, x_2, \ldots\}$ of the x_i.) If S is a subset of $\{x_i\}$, and if σ is a q-cell of K, then $(S \cap \bar{\sigma}) \subset (\{x_i\} \cap |K_q|)$ is finite, and therefore closed. Thus S is closed in $|K|$, and in D as well. It follows that $\{x_i\}$ can have no limit point in D, because $S = \{x_i\} - \{\text{a limit point}\}$ would still be closed. But this contradicts the compactness of D.

We show next that in each dimension D meets only finitely many cells. If D meets infinitely many q-cells $\{\sigma_i^q\}$, then there exists an infinite sequence of points $\{x_i\}$, with x_i in $(D \cap \sigma_i^q)$. If S is a subset of $\{x_i\}$, and if σ is a cell of $\dim \leq q$, then $S \cap \bar{\sigma}$ is either empty or consists of a single point. It follows that S is closed in the weak topology of $|K_q|$. By property (7) of the complex K, therefore, S is closed in $|K|$, and hence in D. It follows, as before, that $\{x_i\}$ can have no limit point in D, contradicting the compactness of D.

The lemma is now proved.

4.5. LEMMA. _Let_ C _be a collection of cells of a complex_ K _such that_ $\bar{\sigma}$ _lies in the union of cells of_ C, _for each cell_ σ _of_ C. _Let_ $|C|$ _be the union of the cells of_ C _with the relative topology inherited from_ $|K|$. _If_ A _is a subset of_ $|C|$, _such that_ $A \cap \bar{\sigma}$ _is closed in_ $\bar{\sigma}$ _for each_ σ _of_ C, _then_ A _is closed in_ $|K|$. _In particular,_ $|C|$ _is closed in_ $|K|$.

We prove the lemma by showing that if τ is a cell of K, then $A \cap \bar{\tau}$ is closed. We assume that $A \cap \bar{\tau} \neq \emptyset$, and we argue as follows.

By the preceding lemma, $\bar{\tau}$ meets only finitely many cells of K,

$$\rho_1, \rho_2, \ldots, \rho_n, \rho_{n+1}, \ldots, \rho_{n+h}$$

where the first n of these are the ones that lie in C. Since

$$\bar{\tau} \subset \bigcup_1^{n+h} \rho_i ,$$

we know that

$$\bar{\tau} = \bar{\tau} \cap \left(\bigcup_1^{n+h} \rho_i \right) ,$$

and therefore,

$$\bar{\tau} \cap |C| = \bar{\tau} \cap \left(\bigcup_1^{n+h} \rho_i \right) \cap |C| = \bar{\tau} \cap \left(\bigcup_1^{n} \rho_i \right) .$$

Since $\bar{\rho}_i \subset |C|$, for $1 \leq i \leq n$,

$$\bar{\tau} \cap \left(\bigcup_1^{n} \bar{\rho}_i \right) \subset \bar{\tau} \cap |C| = \bar{\tau} \cap \left(\bigcup_1^{n} \rho_i \right) \subset \bar{\tau} \cap \left(\bigcup_1^{n} \bar{\rho}_i \right) ,$$

so that

$$\overline{\tau} \cap (\bigcup_1^n \overline{\rho}_i) = \overline{\tau} \cap |C| .$$

Accordingly,

$$A \cap \overline{\tau} = A \cap \overline{\tau} \cap |C| = A \cap \overline{\tau} \cap (\bigcup_1^n \overline{\rho}_i) = \overline{\tau} \cap [\bigcup_1^n (A \cap \overline{\rho}_i)] .$$

Since each $A \cap \overline{\rho}_i$ is closed in $\overline{\rho}_i$, their union (finite) is closed in $|K|$, so that $A \cap \overline{\tau}$ is closed.

We now return to the proof of Proposition 4.2, and finish showing that L is a complex.

(6) The topology of $|L|$ is the weak topology with respect to the cells of $|L|$. First suppose that A is closed in $|L|$. Then $A \cap$ (any subset of $|L|$) is closed in that subset of $|L|$. Hence $A \cap \overline{\sigma}$ is closed in $\overline{\sigma}$ for any cell of C. Now suppose that $A \cap \overline{\sigma}$ is closed in $\overline{\sigma}$ for each cell σ of C. An application of 4.5 shows that A is closed in $|K|$ and hence in $|L|$.

(7) On $|L_q|$ the weak topology and the relative topology coincide. First suppose that $A \subset |L_q|$ is closed in the relative topology of $|L_q|$ Then $A \cap$ (any subset of $|L|$) is closed in that subset. Therefore, $A \cap \overline{\sigma}$ is closed in $\overline{\sigma}$ for any cell of L. Now suppose that $A \subset |L_q|$ and that $A \cap \overline{\sigma}$ is closed for each $\sigma \subset |L_q|$. By applying 4.5 with $|C| = |L_q|$, we see that A is closed in $|K|$. Hence A is closed in the relative topology of $|L_q|$.

A final application of Lemma 4.5 shows that $|L|$ is closed in $|K|$, and the proof of Proposition 4.2 is now complete.

4.6. Exercises.

1. Let σ be a cell of a complex. Show that $\dot{\sigma}$ lies in the union of finitely many cells of dimension less than $\dim \sigma$.

2. Let K be a complex and suppose that H is a compact subset of $|K|$. Show that H lies in the union of finitely many cells of K.

3. Show that each compact subset of a complex K lies in the union of the cells of a finite subcomplex of K.

4. Assume that K is a complex for which each cell boundary $\dot{\sigma}$ is actually equal to (not merely contained in) the union of a finite number of cells of dimension less than $\dim \sigma$. This will be seen later to be true whenever K is regular. Show that any union of closed cells of K is a closed subset of $|K|$.

An immediate consequence of Exercise 3 above is that if $|K|$ is locally compact then K is locally finite.

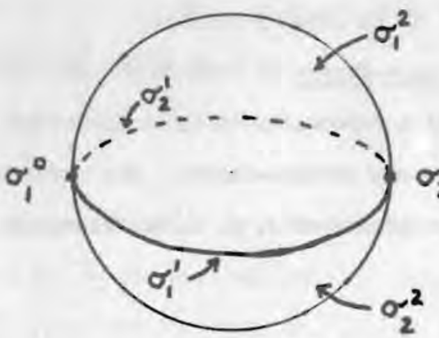
A complex is closure-finite if each closed cell lies in the union of the cells of a finite subcomplex of K. The result of Exercise 3 implies that complexes are closure-finite. The complexes that we have defined are the CW-complexes of J. H. C. Whitehead [1].

5. The Weak Topology for Skeletons

The proposition that we established in the preceding section shows that the q-skeleton of a complex K is a subcomplex of K. In examining the proof of the proposition to see how the proposition follows from known properties of K, one sees that property (7) of the definition of a complex is used in what may be an essential way in the proof of 4.4. It is used there to show that a set S is closed in $|K|$ because it is closed in the weak topology of $|K_q|$. In this section we consider an example which may throw some light on the extent to which property (7) is really needed.

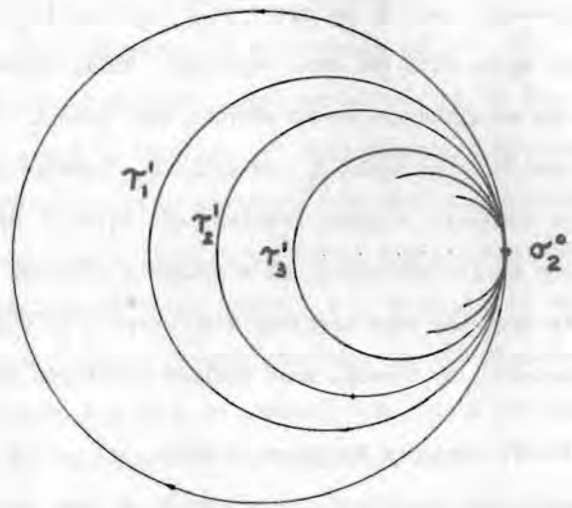
A quasi complex Q consists of a space $|Q|$ and a sequence of subspaces $|Q_n|$ of $|Q|$ satisfying conditions (1) through (6) of the definition of a complex. There are regular quasi complexes, irregular quasi complexes, finite ones, and so forth. Every complex is a quasi complex, but not every quasi complex is a complex. The quasi complex presented below does not satisfy condition (7). In Chapter VIII (Theorem 5.6) we prove that each regular quasi complex satisfies condition (7) (i.e., each regular quasi complex is a complex).

We start with the regular complex K on E^3 described in 2.1, and pictured below.



The quasi complex Q will have $|Q| = |K|$, but Q will have infinitely many cells. In dimension zero, there is no difference between K and Q: $|Q_0| = \sigma^0_1 \cup \sigma^0_2$. In dimensions one and two, new cells are defined by subdividing σ^2_1. The cell σ^2_2 will be included in $|Q_2|$ unchanged. The subdivision is carried out by constructing an infinite sequence of

circles $\overline{\tau}^1_i$, all of them tangent to the equator $\dot{\sigma}^2_1$ at σ^0_2. The diameters of the $\overline{\tau}^1_i$ decreases as i increases, so that as sets the $\overline{\tau}^1_i$ converge to σ^0_2 in the topology of $\overline{\sigma}^2_1$. See the diagram below.



<u>Subdivision of $\overline{\sigma}^2_1$</u>

Each pair $(\overline{\tau}^1_i, \sigma^0_2)$ is considered to be the image of (E^1, S^0) under a relative homeomorphism. The 1-cells of $|Q_1| - |Q_0|$ are σ^1_1, σ^1_2, and the sets $\tau_i = (\overline{\tau}_i - \sigma^0_2)$. The space $|Q_1|$ is the union of the closures of all these 1-cells, with the Euclidean subspace topology obtained from E^3. We then let $|Q_2| = |K_2|$. The 2-cells of Q, being the components of $|Q_2| - |Q_1|$, are

1) the crescent-shaped domains of $\sigma^2_1 - (\cup\, \tau_i)$, and

2) the single cell σ^2_2.

The construction of Q is completed by letting $|Q_n|$ equal $|K_3|$, for each $n \geq 3$. There is only one 3-cell in Q: the cell σ^3 . By definition, the topology on each $|Q_n|$ is the Euclidean subspace topology, inherited from $|Q| = |K| = E^3$.

In $|Q|$, the Euclidean topology and the weak topology agree. It is easily seen, however, that in neither $|Q_1|$ nor in $|Q_2|$ does the Euclidean topology agree with the weak topology. This, in fact, is why the example must be so elaborate as to contain the 3-cell σ^3 . We cannot stop our construction with Q_1 or with Q_2 because neither Q_1 or Q_2 is a quasi complex. A quasi complex that fails to be a complex does so because one of its skeletons, as a subspace with the relative topology, fails to have the weak topology with respect to its own closed cells. Such a skeleton, by itself, must violate condition (6).

Exercise. Verify that the Euclidean topology given for $|Q| = |K|$, in the quasi complex just described, agrees with the weak topology defined by the closed cells of Q .

6. Simplicial Complexes

An n-simplex s consists of a collection {A} of n+1 objects, called vertices, together with the set of all functions $\alpha\colon \{A\} \longrightarrow [0,1]$ such that $\sum_A \alpha(A) = 1$. The functions are called points of s , and the values of a function are called barycentric coordinates of the point. The point whose value is $\frac{1}{n+1}$ on each vertex is called the barycenter of s . We define

$$d(\alpha,\beta) = [\sum_A (\alpha(A) - \beta(A))^2]^{\frac{1}{2}}$$

as the distance between the points α and β . The points of s are

then topologized by this metric.

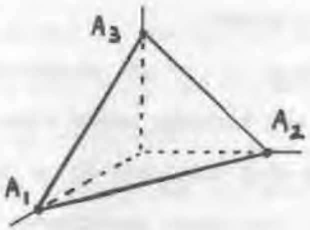
A q-dimensional face s' of an n-simplex s is a q-dimensional simplex whose vertices are included among those of s . Each point of s' is to be identified with the point of s having the same coordinates on the vertices of s' (the other coordinates of such a point of s are zero). This identification is an isometry of s' with the subset of s with which it is identified. Each vertex A of s determines a unique 0-face of s , and we identify A with the point of that 0-face.

The identification of vertices with functions allows us to regard a simplex as a single topological space, namely, the collection of functions α topologized by the metric d . We may also regard a face of a simplex as a subspace of that simplex. If two simplexes have vertices in common they have points in common: the points of the common face that is the simplex determined by the common vertices.

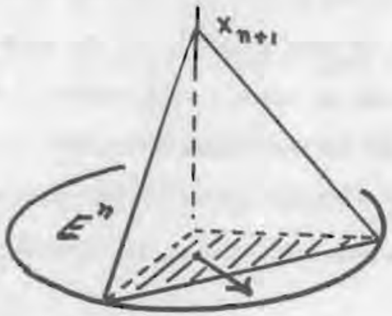
In R^{n+1} , let $A_i = (0,\ldots,0,1,0,\ldots,0)$, with the 1 in the i^{th} place, let s be the simplex defined by the objects A_i , and, using vector addition, let

$$\hat{\alpha} = \sum_1^{n+1} \alpha(A_i)\cdot A_i$$

for each α in s . Because of the restrictions $\sum_1^{n+1} \alpha(A_i) = 1$ and $0 \leq \alpha(A_i) \leq 1$, the collection $\hat{s} = \{\hat{\alpha}\}$ of points of R^{n+1} is the convex closure of the vertices A_i . The correspondence $\alpha \longleftrightarrow \hat{\alpha}$ defines an isometry of s and $\hat{s}$. The space $\hat{s}$ is called the standard simplex in R^{n+1} . Below is a picture for n = 2 .



Next we show that an n-simplex is homeomorphic to the n-ball E^n. Use the isometry to get s into R^{n+1}, project onto the hyperplane $x_{n+1} = 0$, and, with a similarity transformation, move s to the interior of E^n. Radial projection from an interior point of s onto E^n thus yields the desired homeomorphism.



If α and β are distinct points of an (n+1)-simplex s, the line segment containing α and β is defined to be the collection G of points $\{t\alpha + (1-t)\beta | 0 \leq t \leq 1\}$. It is easily seen that for each t the function $t\alpha + (1-t\beta)$ is a point of s. Note that under the isometry of s into R^{n+1} the image of G is an ordinary line segment in R^{n+1}.

6.1. DEFINITION. Suppose we are given (1) a countable collection of vertices $\{A\}$, and (2) a family of finite subsets of $\{A\}$ with the property that each subset of a listed subset is listed. Each listed subset determines a simplex, and the collection K of these simplices

is called a simplicial complex.

Since each subset of a listed set is listed, each face of a simplex of K is a simplex of K. The intersection of two simplices of K, if not empty, is a simplex of K which is a face of each of the intersecting simplices. The set K_n of simplices of K of dimension $\leq n$ is a simplicial complex. Take all simplices on the original list with $\leq n+1$ vertices. This new listing specifies K_n.

A weak topology with respect to the simplices of K can be defined for $|K|$, the union of the simplices of K, because each simplex has a topology.

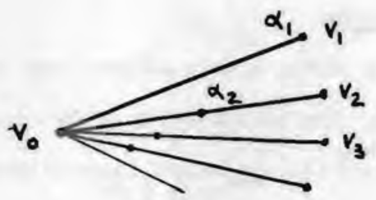
A metric topology for $|K|$ can also be defined in a natural way. Extend each point α of each simplex s of K to that mapping $\overline{\alpha}$ of the set of all vertices of K into [0,1] which 1) agrees with α on vertices of s and 2) is 0 on the other vertices of K. Let $\overline{K}$ be the collection of such extended functions, with the metric

$$\rho(\overline{\alpha},\overline{\beta}) = [\Sigma_A (\overline{\alpha}(A) - \overline{\beta}(A)^2]^{\frac{1}{2}} .$$

The bijection $\phi: \overline{K} \longrightarrow |K|$, defined by $\phi(\overline{\alpha}) = \alpha$, induces a metric d on $|K|$. For two points α and β of $|K|$, not necessarily from the same simplex, $d(\alpha,\beta) = \rho(\overline{\alpha},\overline{\beta})$. For each simplex s of K, the function $\phi^{-1}|s$ is an isometry.

If K is finite, that is if K has finitely many simplices, the weak topology and the metric topology agree. The proof requires showing that each simplex of K is closed in the metric topology of $|K|$. If K is infinite, the two topologies may differ, as the following example shows.

Start with the vertices v_i and the 1-simplices e_i of 2.12.



Let α_i be the point in $\overline{e}_i$ defined by

$$\alpha_i(v_o) = 1 - 1/i \quad \text{and} \quad \alpha_i(v_i) = 1/i .$$

Since $d(\alpha_i, v_o) = \sqrt{2}/i$ the α_i converge to v_o in the metric topology for $|K|$. However, in the weak topology for $|K|$, the set $\{\alpha_i\}$ is closed. Thus the two topologies for $|K|$ do not coincide.

An example of a finite simplicial complex is a simplex with all its faces. We will usually use a single symbol to denote both a simplex and the complex it determines. If s is a simplex, then $\dot{s}$ will denote the topological boundary of s as well as the simplicial complex consisting of the proper faces of s (those faces of dimension less than $\dim s$) .

6.2. An alternative definition of finite simplicial complex.

A finite simplicial complex K is a collection of faces of a single simplex, with the property that each face of a simplex of K is likewise in K .

Exercise. Show that 6.2 agrees with 6.1.

6.3. THEOREM. *Let K be a simplicial complex. The space $|K|$ with the weak topology, together with the subspaces $|K_n|$, determines a regular complex.*

Here, $|K_n|$ denotes the union of the simplices of K of dimension $\leq n$.

Conditions (1) and (3) present no difficulty. Also, $|K_n|$ is closed in $|K|$. If s is a simplex of K then $|K_n| \cap s$ is the union of faces of s , so that $|K_n| \cap s$ is the union of a finite number of closed sets of $|K|$.

The points of $|K_n| - |K_{n-1}|$ are precisely the interior points of the n-simplices of K: functions from some listed set $\{A_1,\ldots,A_{n+1}\}$ of K to the open interval $(0,1)$. The interior $\overset{\circ}{s} = (s - \dot{s})$ of an n-simplex s is connected, and is referred to as an open simplex. Each open n-simplex is open in $|K_n|$ because its complement is closed in $|K_n|$. Each point α of $\dot{s}$ lies in $|K_{n-1}|$. Thus $\dot{s} = s \cap |K_{n-1}|$.

It is easily seen that $\overset{\circ}{s}$ is an open n-cell of K and that s is a closed n-cell of K .

Mappings $f\colon (E^n, S^{n-1}) \longrightarrow (s,\dot{s})$ are provided by isometries with the standard n-simplex.

Since each simplex of K is a closed cell of K in the sense of condition (5) and since $|K|$ has the weak topology with respect to the simplices of K , condition (6) is satisfied. The proof of (7) is left to the reader.

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Chapter II.

HOMOLOGY GROUPS FOR REGULAR COMPLEXES

1. Redundant Restrictions on Regular Complexes. Homology Groups.

In this chapter we describe the process of associating with any given regular complex an infinite sequence of groups called the homology groups of the complex. Fundamental to this process are the concepts of chain complex and homology of a chain complex.

1.1. DEFINITION. A chain complex C over a ground ring R (R assumed to be commutative with unit) is a collection of unitary R-modules $\{C_q : q \in Z\}$ together with R-homomorphisms $\partial_q : C_q \to C_{q-1}$ such that for every q, $\partial_{q-1}\partial_q = 0$. For every q, C_q is called the R-module of q-chains of C, or the qth chain module, and ∂_q is called the qth boundary operator of C. We will often write ∂ for the collection $\{\partial_q\}$ and call ∂ the boundary operator of C. We then write C as $(\{C_q\},\partial)$.

1.2. DEFINITION. If $C = (\{C_q\},\partial)$ is a chain complex, then the complex of cycles of C, written $Z(C)$, is the chain complex $(\{Z_q\},0)$ whose qth chain module is $Z_q = \mathrm{Ker}\,\partial_q$ and whose boundary operators are zero. The complex of boundaries of C, written $B(C)$, is the chain complex $(\{B_q\},0)$, where $B_q = \mathrm{Im}\,\partial_{q+1}$.

For each q, we call Z_q and B_q the R-modules of q-cycles and q-boundaries, respectively, of C. Note that since $\partial_q\partial_{q+1} = 0$, we have $B_q \subseteq Z_q$.

1.3. DEFINITION. Given a chain complex $C = (\{C_q\},\partial)$, the qth homology module of C is the factor R-module Z_q/B_q and is denoted by $H_q(C)$. The homology chain complex of C is the chain complex written $H_*(C)$.

In this section the ring R will always be the ring Z of integers. We will sometimes write "chain complex" for "chain complex over Z". If C is a chain complex over Z, its chain modules are abelian groups.

Given a regular complex K, we will define homology groups for K by first giving K an orientation. Any such orientation gives rise to a chain complex, and then the homology groups for K will be the homology groups of this chain complex. It will be clear that there may be, for a given complex K, many different ways of defining an orientation of K. Thus it will appear that the homology groups of K were not well-defined. We will show, however, that the homology groups of K are independent of the orientation used to define them.

As we have just remarked, an orientation on a regular complex will induce a chain complex. We will see that only the boundary operators of the chain complexes so induced may vary as we vary orientations. The chain groups, however, are defined independent of orientation in the following way.

1.4. DEFINITION. Given a regular complex K, the group of q-chains of K, written $C_q(K)$, is the free abelian group on the q-cells of K.

Thus for $q \leq -1$, $C_q(K) = 0$. In order to define a boundary operator we will assume that the regular complex K satisfies certain restrictions. At this time we will state these restrictions and assume that they are satisfied by all regular complexes. We will prove later that this is the case by using the homology theory of simplicial complexes. Thus we will show now that the redundant restrictions for regular complexes are satisfied by the subclass of simplicial complexes.

Three Redundant Restrictions on a Regular Complex K :

R.R. 1: If $r < q$, and σ is a q-cell whose closure contains a point of the r-cell τ , then $\tau \subseteq \dot{\sigma}$.

To see that R.R. 1 is satisfied if K is simplicial, we note that a q-cell of K is then the interior of a simplex of K . If a point of the interior of an r-simplex lies in a closed q-cell, that is, in a q-simplex, then every vertex of the r-simplex is a vertex of the q-simplex, and the desired relation holds.

1.5. DEFINITION. In a regular complex K , if σ is a cell whose closure contains the cell τ , then τ is called a face of σ . If τ is a face of σ , we write $\tau < \sigma$. If $\tau < \sigma$ and $\tau \neq \sigma$ then τ is called a proper face of σ .

1.6. PROPOSITION. If K is a regular complex and σ a cell of K, then the collection of faces of σ is a subcomplex (necessarily finite by Lemma I.4.4) for $\bar{\sigma}$, and the collection of proper faces of σ is a finite subcomplex for $\dot{\sigma}$.

Proof: This proposition follows from Proposition I.4.2.

1.7. PROPOSITION. If σ is a cell of the regular complex K , then $\bar{\sigma}$ contains at least one vertex.

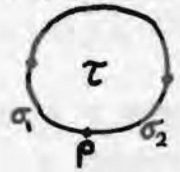
Proof: The proof is by induction on the dimension of σ . The proposition is obviously true if σ is a vertex. Suppose it to be true for cells of dimension less than q , and let σ be a q-cell. Then $\dot{\sigma} \subseteq K_{q-1}$ and $\dot{\sigma}$ is nonempty. If x is a point of $\dot{\sigma}$, then

x is a point of some r-cell τ with $r < q$. By the inductive hypothesis, $\bar{\tau}$ contains a vertex. By R.R.1, $\tau \subseteq \dot{\sigma}$. Since $\dot{\sigma}$ is closed, $\bar{\tau} \subseteq \dot{\sigma}$, and the conclusion follows.

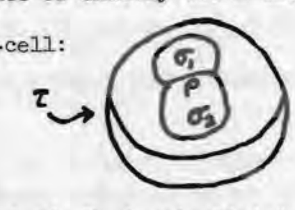
An example of a complex not satisfying R.R.1 is obtained by taking σ to be a 2-cell whose boundary is collapsed to a point of some 1-cell τ . The only vertices of the complex are the two vertices of τ . This complex is of course irregular.

R.R. 2: If ρ and τ are two cells of K such that $\rho < \tau$ and the dimensions of ρ and τ are q and $q+2$ respectively, then there are precisely two (q+1)-cells σ_1 and σ_2 such that $\rho < \sigma_1 < \tau$ and $\rho < \sigma_2 < \tau$.

If the dimension q is zero, this means that a vertex of a 2-cell is a vertex of exactly two 1-cells contained in the boundary of the 2-cell:



If $q = 1$, R.R.2 states that an edge ρ lying in the boundary of a 3-cell is the common face of exactly two 2-cells which also lie in the boundary of the 3-cell:



R.R.2 is satisfied if K is simplicial. Indeed, take ρ and τ to be open simplices such that $\rho < \tau$, with $\dim \rho = q$ and $\dim \tau = q+2$. Then ρ is a simplex on vertices $A_0, A_1, \ldots, A_q$, and τ is a simplex on the vertices of ρ and two other vertices

A_{q+1} and A_{q+2}. It is clear that we may take σ_1 to be the simplex on the vertices of ρ and A_{q+1}, and σ_2 the simplex on the vertices of ρ and A_{q+2}.

1.8. DEFINITION. Given the regular complex K, an incidence function α on K is a function assigning to each ordered pair of cells σ, τ of K an integer from the set $\{-1,0,+1\}$ denoted by $[\sigma:\tau]_\alpha$ (or $[\sigma:\tau]$ for short) such that:

(i) $[\sigma:\tau]$ is nonzero if and only if τ is a face of σ of one lower dimension.

(ii) If σ is an edge with vertices A and B, then

$$[\sigma:A] + [\sigma:B] = 0 .$$

(iii) If $\rho < \tau$ and $\dim \rho = \dim \tau - 2$, and σ_1 and σ_2 are the two cells with $\rho < \sigma_i < \tau$ for $i = 1,2$, given by R.R.2, then

$$[\tau:\sigma_1]\cdot[\sigma_1:\rho] + [\tau:\sigma_2][\sigma_2:\rho] = 0 .$$

$[\sigma:\tau]$ is called the incidence number of the pair σ,τ. We say that K is oriented by the incidence function α, or that α yields an orientation on K.

Remark on condition (ii) above. It follows easily from the definition of a regular complex that every 1-cell has precisely two vertices.

1.9. DEFINITION. If α is an incidence function on K, the associated boundary operator ∂^α, with $\partial^\alpha_q\colon C_q(K) \to C_{q-1}(K)$, is defined as follows. For σ a generator of $C_q(K)$,
$\partial^\alpha_q\sigma = \sum_{\tau\in K} [\sigma:\tau]_\alpha\tau$. Note that this sum has finitely many nonzero

terms by Lemma I.3.4. We extend ∂^α_q over all of $C_q(K)$ by linearity.

To justify this definition, we have the following lemma:

1.10. LEMMA. For all integers q, $\partial^\alpha_q\partial^\alpha_{q+1} = 0$.

Proof: Let τ be a generator of $C_{q+1}(K)$. Then, for $q \geq 1$, we have

$$\begin{aligned}
\partial^\alpha_q\partial^\alpha_{q+1}\tau &= \partial^\alpha_q(\Sigma[\tau:\sigma]\sigma) && \text{(by def.)}\\
&= \sum_\sigma [\tau:\sigma]\partial^\alpha_q\sigma && \text{(by linearity)}\\
&= \sum_\sigma [\tau:\sigma](\sum_\rho[\sigma:\rho]\rho) && \text{(by def.)}\\
&= \sum_\rho (\sum_\sigma[\tau:\sigma][\sigma:\rho])\rho &&\\
&= 0 && \text{(by Def. 2.18 (iii))}
\end{aligned}$$

Thus, by linearity, the result follows for all $(q+1)$-chains, $q \geq 1$. If $q < 1$, then the map ∂^α_q is the zero map. This completes the proof.

1.11. DEFINITION. If c is a 0-chain, the index of c, written $\mathrm{In}\, c$, is the sum of the coefficients of c.

Remark. Using methods similar to the proof of the above lemma, it is easy to see that for $\sigma \in C_1(K)$, $\mathrm{In}\,(\partial^\alpha_1\sigma) = 0$, as a consequence of Definition II.1.8 (ii).

1.12. DEFINITION. If K is a regular complex oriented by the incidence function α, then the chain complex induced by α, written $C^\alpha(K)$, is the chain complex $(\{C_q(K),\partial^\alpha)$. The groups of cycles and boundaries of K in dimension q are the groups of cycles and boundaries of the chain complex $C^\alpha(K)$, and are written $Z^\alpha_q(K)$ and $B^\alpha_q(K)$, respectively. The qth homology group of K is the group

$Z_q^\alpha(K)/B_q^\alpha(K)$, and is denoted by $H_q^\alpha(K)$. The collection of homology groups and the zero boundary operator form the homology chain complex of K , written $H_*^\alpha(K)$. Notation and minor definitions: A q-chain of a regular complex is sometimes referred to as a <u>chain on</u> K ; if it is zero off of the cells of some subcomplex L of K it is often said to <u>lie on</u> L , or to be a <u>chain on</u> L . Two q-cycles z_1 and z_2 are said to be <u>homologous</u> if they lie in the same coset of $B_q^\alpha(K)$ -- that is, if their difference is a boundary -- and we write $z_1 \sim z_2$. The collection of q-cycles homologous to a given cycle z is called a <u>homology class</u> and is denoted by $\{z\}$. We shall also say that two chains c_1, c_2 are <u>homologous</u>, and write $c_1 \sim c_2$, if their difference is a boundary.

Next we note that for a finite regular complex K , oriented by the incidence function α , the qth homology group $H_q^\alpha(K)$ is a factor group of two finitely generated abelian groups and so is finitely generated. By the fundamental theorem on finitely generated abelian groups, $H_q^\alpha(K)$ splits into a direct sum of a free abelian group and a finite group. The finite summand is called the <u>torsion subgroup</u> of $H_q^\alpha(K)$, and K is said to have <u>non-trivial torsion in dimension q</u> if the torsion subgroup of H_q^α is non-trivial. The rank of the free summand is called the <u>qth Betti number</u> of K and is written $R_q^\alpha(K)$.

<u>R.R. 3:</u> There is an incidence function for any regular complex K.

This restriction allows one to define homology groups for all regular complexes. It is satisfied if K is simplicial. To see this, order the vertices of K . If σ is a simplex spanned

by the (ordered) vertices $A_0, \ldots, A_q$, we define incidence numbers $[\sigma:\tau]$ for $\tau < \sigma$ as follows. If τ is of dimension $< q-1$, we set $[\sigma:\tau] = 0$. If τ is of dimension $q-1$, its vertices are those of σ with one vertex, A_i , say, omitted. We set $[\sigma:\tau] = (-1)^i$. We leave as an exercise to the reader the verification that the function so defined is an incidence function for K .

There is a practical procedure for defining an incidence function on a finite regular complex of dimension at most 3 :

(i) Associate an arrow with each 1-cell σ of K , and assign the value $+1$ to $[\sigma:A]$ for A the vertex at the head of the arrow, and the value -1 to $[\sigma:B]$ for B the vertex at the tail of the arrow.

$$B \xrightarrow[\sigma]{} A$$

(ii) Associate a circular indicatrix with each 2-cell σ . For each 1-cell $\tau < \sigma$, assign the value $+1$ to $[\sigma:\tau]$ if the directions of the indicatrix on σ and the arrow on τ agree, -1 if not.

$[\sigma:\tau_1] = -1$

$[\sigma:\tau_2] = [\sigma:\tau_3] = +1$

(iii) Associate a corkscrew with each 3-cell σ ; the corkscrew induces an orientation (indicatrix) on each 2-cell $\tau < \sigma$. We assign $+1$ to $[\sigma:\tau]$ if the orientation induced by the corkscrew on σ agrees with the circular indicatrix attached to τ in (ii), -1 if not.

$[\sigma:\tau] = +1$

Once we have an incidence function α on a regular complex K, we may compute the homology of K using the chain complex $C^{\alpha}(K)$. It appears at this point that the homology so obtained might depend upon the particular incidence function α on K. It will turn out that this is not the case: given two orientations α, β, on K, the homology groups of $C^{\alpha}(K)$ are isomorphic to those of $C^{\beta}(K)$. In fact, the chain complexes $C^{\alpha}(K)$ and $C^{\beta}(K)$ have the same structure. The precise sense in which this is true will be explained in section 5.

<u>Example:</u> Computation of the Homology Groups of the n-Sphere.

We have presented a procedure for constructing homology groups of a regular complex. Given a topological space X, such that X is the space of some regular complex K, we may define the homology groups of the space X to be the homology groups of the complex K. We must of course verify that if X can be realized as the space of another complex L, then K and L have isomorphic homology groups. In other words, we must demonstrate that the sequence of homology groups is a topological invariant for the category of spaces which realize regular complexes. This will be done in a later chapter. Assuming topological invariance for now, we compute the homology of the space X by giving X the structure of a regular complex K and by defining an orientation on K. We demonstrate this technique with the n-sphere S^n.

The 0-sphere is the space of a regular complex with two 0-cells; orientation is possible in only one way. $C_q(S^0) = 0$ except when $q = 0$, and $C_0(S^0) \approx Z \oplus Z$. Then $Z_0(S^0) = C_0(S^0)$ and

$B_0(S^0) = 0$, so $H_0(S^0) \approx Z \oplus Z$. $H_q(S^0) = 0$ for $q \neq 0$.

The 1-sphere is obtained from the 0-sphere by adjoining two 1-cells. Let T be the mapping $S^1 \to S^1$ interchanging antipodal points; then S^1 is the space of a regular complex with two 0-cells e_0 and Te_0 and two 1-cells e_1 and Te_1. (See diagram.) We construct an incidence function on S^1 as follows:

$$[e_1 : e_0] = -1 \qquad [Te_1 : e_0] = +1$$

$$[e_1 : Te_0] = +1 \qquad [Te_1 : Te_0] = -1$$

Then $\partial e_1 = Te_0 - e_0$, $\partial Te_1 = e_0 - Te_0$. $Z_0(S^1) = C_0(S^1) \approx Z \oplus Z$, generated by e_0, Te_0. $B_0(S^1) \approx Z$, generated by $e_0 - Te_0$, so $H_0(S^1) = C_0/B_0 \approx Z$. $Z_1(S^1) \approx Z$, generated by $e_1 + Te_1$, $B_1(S^1) = 0$, so $H_1(S^1) \approx Z$. $H_q(S^1) = 0$ for $q > 1$.

For $k \geq 0$, S^{2k+2} is obtained from S^{2k+1} by adjoining two $(2k+2)$ cells e_{2k+2}, Te_{2k+2}, with the following orientation:

$$[e_{2k+2} : e_{2k+1}] = [e_{2k+2} : Te_{2k+1}] = [Te_{2k+2} : e_{2k+1}] = [Te_{2k+2} : Te_{2k+1}] = +1 .$$

Other incidence numbers are as in S^{2k+1}.

For $k \geq 0$, S^{2k+3} is obtained from S^{2k+2} by adjoining two $(2k+3)$ cells e_{2k+3}, Te_{2k+3}, with the following orientation:

$$[e_{2k+3} : e_{2k+2}] = -1 \qquad [Te_{2k+s} : e_{2k+2}] = +1$$

$$[e_{2k+3} : Te_{2k+s}] = +1 \qquad [Te_{2k+3} : Te_{2k+2}] = -1$$

The reader may check that the functions given satisfy the requirements for incidence functions. Note that the regular complex for S^n so obtained is the same as in example 2.1, chapter I.

It is clear that $H_r(S^{n+1}) \approx H_r(S^n)$ for $n \geq 1$, $r \neq n$, $n+1$. In dimensions n, $n+1$:

$Z_{2k}(S^{2k+1}) = Z_{2k}(S^{2k}) \approx Z$ with generator $Te_{2k} - e_{2k}$

$Z_{2k+1}(S^{2k+2}) = Z_{2k+1}(S^{2k+1}) \approx Z$ with generator $e_{2k+1} + Te_{2k+1}$

$B_{2k}(S^{2k}) = 0$

$B_{2k+1}(S^{2k+1}) = 0$.

$\partial e_{2k+1} = Te_{2k} - e_{2k}$ $\qquad\qquad$ $\partial e_{2k+2} = e_{2k+1} + Te_{2k+1}$

Thus $H_k(S^{k+1}) = 0$, $H_{k+1}(S^{k+1}) \approx Z$.

Summary: For $n \geq 1$, $H_q(S^n) = \begin{cases} Z \text{ for } q = 0 \text{ or } n \\ 0 \text{ otherwise} \end{cases}$

$$H_q(S^0) = \begin{cases} Z \oplus Z \text{ for } q = 0 \\ 0 \text{ otherwise} \end{cases}$$

Exercise. a) Using a regular complex whose space is the torus T^2, compute $H_0(T^2) \approx Z$, $H_1(T^2) \approx Z \oplus Z$, $H_2(T^2) \approx Z$.
b) Using a regular complex whose space is the projective plane $\mathbb{P}^2$, compute $H_0(\mathbb{P}^2) \approx Z$, $H_1(\mathbb{P}^2) \approx Z_2$, $H_2(\mathbb{P}^2) = 0$.

2. The Euler Characteristic

2.1. DEFINITION. The <u>Euler characteristic</u> of a finite regular complex K is defined by: $\chi(K) = \Sigma_{q=0}^{\dim K}(-1)^q R_q(K)$.

2.2. THEOREM. <u>If</u> K <u>is a finite complex and</u> α_q <u>is the number of q-cells of</u> K, <u>then</u> $\chi(K) = \Sigma_{q=0}^{\dim K}(-1)^q \alpha_q$.

Proof: Let ρ_q = rank of Z_q, for each q. α_q is of course the rank of C_q. Consider the short exact sequence

$$0 \longrightarrow Z_q \longrightarrow C_q \xrightarrow{\partial_q} B_{q-1} \longrightarrow 0 .$$

B_{q-1} is free since it is a subgroup of the free abelian group C_{q-1}. Thus there is a homomorphism $r: B_{q-1} \longrightarrow C_q$ such that $\partial_q r = 1$. This implies that Z_q is a direct summand and $C_q = Z_q \oplus D_q$ where $D_q = rB_{q-1}$. Since $D_q \approx B_{q-1}$ we have that rank B_{q-1} = rank D_q = $\alpha_q - \rho_q$. $H_q(K)$ is the quotient of Z_q by B_q, so $R_q(K) = \rho_q - (\alpha_{q+1} - \rho_{q+1})$. The rest is arithmetic:

$$\Sigma(-1)^q R_q(K) = (\rho_0 + \rho_1 - \alpha_1) - (\rho_1 + \rho_2 - \alpha_2) + \dots$$
$$+ (-1)^{n-1}(\rho_{n-1} + \rho_n - \alpha_n) + (-1)^n \rho_n ,$$

where $n = \dim K$. Everything drops out except

$$\rho_0 - \alpha_1 + \alpha_2 - \alpha_3 + \dots + (-1)^n \alpha_n .$$

But $\rho_0 = \alpha_0$, and the proof is complete.

Once we have proved that the homology groups are topologically invariant, we will know that $\chi(K)$, as defined above, is also a topological invariant. By Theorem 2.2, so is $\Sigma(-1)^q \alpha_q$. We can compute $\chi(K)$ using the theorem and any regular decomposition of $|K|$.

For example, $\chi(S^n) = 1 + (-1)^n$

$\chi(E^n) = 1$

$\chi(\text{torus}) = 0$

$$\chi(\mathbb{P}^2) = 1$$

$$\chi(\mathbb{P}^n) = 1/2 + 1/2\,(-1)^n .$$

This last equality follows from the fact that the regular decomposition of S^n yields a regular decomposition of $\mathbb{P}^n$ with every two cells of the same dimension identified.

We will see later that the product of two finite regular complexes K and L can be given the structure of a regular complex $K \times L$. A q-cell of $K \times L$ is a pair (σ, τ) with σ a cell of K and τ a cell of L such that $\dim \sigma + \dim \tau = q$.

Exercise. Show that $\chi(K \times L) = \chi(K) \cdot (L)$ for any two finite regular complexes K and L. This shows, for example, that the n-dimensional torus, being the product of n circles, each of Euler characteristic zero, has Euler characteristic zero.

Given a simplicial decomposition of a space $|K|$, with α_0 vertices, α_1 edges, etc., we know that

$$\alpha_k \le \binom{\alpha_0}{k+1} \text{, the binomial coefficient.}$$

In particular, if K is a finite 2-dimensional simplicial complex, $\alpha_1 \le \frac{1}{2}\alpha_0(\alpha_0 - 1)$. If we triangulate a compact 2-manifold M (without boundary) every 1-simplex is the face of exactly two 2-simplices, so $2\alpha_1 = 3\alpha_2$. Then

$$\begin{aligned} 6\chi(M) &= 6\alpha_0 - 6\alpha_1 + 6\alpha_2 \\ &= 6\alpha_0 - 2\alpha_1 \\ &\ge 6\alpha_0 - \alpha_0(\alpha_0 - 1) . \end{aligned}$$

Thus $\alpha_0^2 - 7\alpha_0 + 6\chi(M) \ge 0$. Since $\alpha_0 \ge 3$, and the graph of

$y = x^2 - 7x + 6\chi(M)$ is symmetric about the line $x = 7/2$,

$$\alpha_0 \ge \frac{7 + \sqrt{49 - 24\chi(M)}}{2} .$$

The number on the right is called the Heawood number of the manifold M and is written $h(M)$. Any simplicial decomposition of M must have at least $h(M)$ vertices.

<u>Examples.</u> $h(S^2) = 4$. $h(\text{torus}) = 7$. $h(\mathbb{P}^2) = 6$.

<u>Exercise.</u> Exhibit simplicial triangulations of these manifolds with these least numbers of vertices.

3. Homology and Connectedness

Homology groups of a space are constructed in the hope that they will provide some information about the topological properties of the space. We find in this section that there is a characterization of the number of connected components of a space in terms of the 0th homology group. First we prove the following theorem.

3.1. THEOREM. <u>If</u> K <u>is a regular complex, and if the connected components of</u> $|K|$ <u>are the spaces of subcomplexes</u> $L_1, \ldots, L_n$, <u>then for every</u> q,

$$H_q(K) \approx \text{the direct sum of the } H_q(L_i) .$$

Proof: Clearly $C_q(K)$ is the direct sum of the $C_q(L_i)$. Since $[\sigma:\tau]$ is zero if σ and τ are not in the same component, we have that boundary maps $C_q(L_i) \to C_{q-1}(L_i)$ for each i, q. Thus the boundary preserves the direct sum and the conclusion follows.

3.2. DEFINITION. If K is a regular complex and A and B are vertices of K, then an edge path from A to B is a finite sequence of vertices $\{A_i\}$, $0 \leq i \leq n$, and edges $\{\sigma_i\}$, $0 \leq i \leq n-1$, such that $A = A_0$, $B = A_n$, and σ_i has A_i and A_{i+1} as vertices for each i.

3.3. THEOREM. The following three statements are equivalent for a complex K:

(i) $|K|$ is connected.

(ii) $|K_1|$ is connected.

(iii) There is an edge path between every two vertices of K.

Proof: We first prove by induction on dimension that (i) $\Longrightarrow$ (ii). This is clearly true for $\dim K = 1$. Suppose it is true for $\dim K \leq n$. Let K be a regular complex of dimension $n+1$ such that $|K|$ is connected. Suppose $|K_1| = P \cup Q$, P open, closed, and non-empty. We show that Q must be empty. Let P' be the union of all cells of K having a vertex in P, and define Q' similarly. If P contains a vertex of a 1-cell it contains that 1-cell, so if Q' is empty, so is Q. If σ is a cell of K, then $\dot{\sigma}$ is a subcomplex of K of dimension $\leq n-1$. By the induction hypothesis, $|\dot{\sigma}_1|$ (the 1-skeleton of $\dot{\sigma}$) is connected and must lie either in P or in Q. (It is non-empty by Prop. 1.7.) Thus σ and all its faces must lie either in P' or in Q'. It follows that P' and Q' are disjoint and exhaust K. Moreover, P' (and Q') meets each closed cell $\bar{\sigma}$ in the null set or in $\bar{\sigma}$ and so is closed by the weak topology property. By the connectedness of $|K|$, Q' is empty. Then so is Q, and so $|K_1|$ is connected.

If K contains cells of arbitrarily high dimension the same argument proves that $|K_1|$ is connected since we have now shown that $|\sigma_1|$ is connected for each cell σ of K.

Proof that (ii) $\Longrightarrow$ (iii). Let $|K_1|$ be connected, and let A be any vertex of K. Let U be the union of the cells of all edge paths connecting A with other vertices of K. A 1-cell τ lies in U if and only if one of its vertices does. So U either contains $\bar{\tau}$ or is disjoint from $\bar{\tau}$. Since K_1 is a subcomplex of K, U is open and closed in $|K_1|$ by the weak topology property. $|K_1|$ is connected, and U contains A and therefore all of $|K_1|$.

Proof that (iii) $\Longrightarrow$ (i). Since the union of the cells of a path of edges is a connected set, (iii) implies at once that $|K_1|$ is connected. Since every closed cell of K is connected and contains points of $|K_1|$, we use an elementary theorem on connected sets (finite chain theorem in Kelley's General Topology) to deduce that $|K|$ is connected.

In what follows we will say that K is a connected complex if $|K|$ is a connected space.

3.4. LEMMA. If K is a connected regular complex, A_0 a vertex in K, and c a 0-chain on K, then $c \sim (\mathrm{In}\ c)A_0$.

Proof: If c is a multiple of a vertex, say $c = mA_1$, then there exists, by the previous theorem, an edge path from A_0 to A_1. By R.R.3 we may choose an orientation on K. Let $\Sigma a_i\sigma_i$ be the 1-chain with the following properties: (i) the σ_i are the 1-cells of the edge path from A_0 to A_1, (ii) $a_i = \pm 1$, and (iii) $\partial(\Sigma a_i\sigma_i) = A_1 - A_0$. Then $mA_0 \sim mA_1$, so $c \sim (\mathrm{In}\ c)\,A_0$.

If $c = \Sigma_{i=0}^{k} m_i A_i$, join edge paths from each A_i to A_0 so that $A_i - A_0$ bounds. Then $m_i A_i - m_i A_0$ bounds, and $\Sigma(m_i A_i - m_i A_0) = c - (\text{In } c)A_0$. The latter bounds, so by definition $c \sim (\text{In } c)A_0$.

3.5. THEOREM. A regular complex K is connected if and only if $H_0(K) \approx Z$.

Proof: Let K be a connected regular complex. Let A be any vertex of K. Then, by the preceding lemma, any 0-cycle is homologous to a multiple of A. Thus, the coset of $B_0(K)$ to which A belongs generates $H_0(K)$. If mA bounds for some m, then $mA = \partial c$ for some 1-chain c. By the remark after Definition 1.11, $m = \text{In}(mA) = \text{In}(\partial c) = 0$. Thus $H_0(K) \approx Z$.

If K is not connected we may apply the first theorem of this section, after noting that by Prop. I.4.2 a connected component of a regular complex is a subcomplex.

3.6. COROLLARY. $H_0(K)$ is isomorphic to k copies of Z where k is the number of components of K. $R_0(K)$ = no. of components of K.

3.7. COROLLARY. $H_0(K)$ is invariant under change of orientation and under homeomorphism. That is, if K and L are complexes such that $|K|$ is homeomorphic to $|L|$, then $H_0(K) \approx H_0(L)$.

4. Computation of Homology Groups

4.1. DEFINITION. A tree is a connected regular 1-dimensional complex with no non-zero 1-cycles. A maximal tree in a regular complex K is a subcomplex which is a tree and contains all of the vertices of K.

4.2. PROPOSITION. Each connected regular complex K has a maximal tree.

Proof: The proof uses Zorn's Lemma. We must show that the set of trees in K, ordered by inclusion, has the property that any totally ordered subset has an upper bound. Let $\{L_\alpha, \alpha \in A\}$ be a totally ordered set of trees in K. The union L of the trees L_α is connected, because any two vertices of L must lie in some L_α, and we may apply 3.3. Any cycle on L, being a finite complex, must lie in some L_α. Thus L is a tree. By Zorn's Lemma, there is a tree T in K which is maximal with respect to inclusion. We claim that T is maximal in the sense of 4.1. Let A be a vertex of T, and let B be any vertex of K. Since K is connected, we may apply 3.3 to show that there exists a path of edges $A = A_1, e_1, A_2, e_2, \ldots, e_n, A_{n+1} = B$ from A to B. Let j be the largest integer $\leq n+1$ such that T contains A_j. If $j < n+1$, then $T \cup \overline{e}_j$ is a tree strictly containing T, which contradicts the maximality of T. Thus T contains $A_{n+1} = B$ and is a maximal tree in K.

Remark. The construction of a maximal tree can be carried out effectively whenever K has a finite or countable number of edges. We order them in a simple sequence $\{e_i\}$, and define a sequence of trees $\{L_n\}$ inductively by $L_1 = \overline{e}_1$, and L_n is the union of L_{n-1} and the closure of the first edge in the ordering with precisely one vertex in L_{n-1}. Then the union of the L_n is a maximal tree.

Given an oriented connected regular complex K we construct a basis for $Z_1(K)$ as follows. Let T be a maximal tree for K, and let $\{e_\alpha, \alpha \in A\}$ be the edges of K not in T. For each α, e_α connects two vertices in T. Thus $\partial e_\alpha = B-A$ for some A and B in T. Since T is a tree, there is precisely one 1-chain c_α in T such that $\partial c_\alpha = B-A$. If we set $z_\alpha = e_\alpha - c_\alpha$, then z_α is a cycle. For each α, z_α involves an edge, z_α, not in any z_β for $\beta \neq \alpha$, so the z_α's are independent. We show that the z_α's span $Z_1(K)$. Let z be a 1-cycle on K. Then z can be written $\Sigma_\alpha a_\alpha e_\alpha + d$, where d is a chain on T. Then $z - \Sigma a_\alpha z_\alpha = \Sigma a_\alpha c_\alpha + d$, the c_α as above. In this equation, the left side is a 1-cycle on K, while the right side is a 1-chain on T. Thus the right side is a cycle on T. But T is a tree, and has no non-zero cycles. Thus $z = \Sigma a_\alpha z_\alpha$, as desired, and the z_α's form a basis for $Z_1(K)$.

Suppose K is finite in the above computation. Since T is a tree, we have $\alpha_0(T) - \alpha_1(T) = 1$. But every vertex of K is in T, so $\alpha_0(T) = \alpha_0(K)$, and the number of z_α's is

$$\alpha_1(K) - \alpha_1(T) = \alpha_1(K) - \alpha_0(K) + 1 .$$

Thus $Z_1(K)$ is isomorphic to a sum of $\alpha_1(K) - \alpha_0(K) + 1$ copies of Z if K is finite. This result can also be obtained using purely algebraic methods.

We have shown, incidentally, that every 1-cycle of K is a sum of <u>simple 1-cycles.</u> A simple 1-cycle is a cycle carried by a simple closed curve -- a 1-cycle whose coefficients are all ± 1 such that no more than two of the 1-cells of the cycle have any vertex in common. We shall consider the analog of a simple 1-cycle, the fundamental n-cycle of an orientable n-circuit.

4.3. DEFINITION. An <u>n-circuit</u> is an n-dimensional regular complex K such that 1) each (n-1)-cell is a face of precisely two n-cells and 2) for any two n-cells σ and σ', there is a "path of n-cells" joining them; that is, a finite sequence of n-cells $\sigma_0, \sigma_1, \ldots, \sigma_S$ such that $\sigma = \sigma_0$, $\sigma' = \sigma_S$, and each pair (σ_i, σ_{i+1}) has an (n-1)-dimensional face in common. An example of an n-circuit is any regular complex K such that $|K|$ is an n-manifold.

If K is an oriented <u>finite</u> <u>n-circuit</u>, we compute $H_n(K)$ and $H_{n-1}(K)$ as follows. Order the n-cells of K in a sequence $\sigma_0, \sigma_1, \ldots, \sigma_S$, so that each σ_k for $k \geq 1$ has at least one (n-1)-dimensional face in common with a preceding σ_i; for each $k \geq 1$, choose such an (n-1)-cell, call it τ_k. Define integers a_k, $0 \leq k \leq S$, by induction as follows: Set $a_0 = 1$, and, given a_i for $i < k$, define a_k so that $\partial(a_0\sigma_0 + a_1\sigma_1 + \ldots + a_k\sigma_k)$ has coefficient 0 on τ_k. This can always be done since τ_k is always a face of precisely one n-cell among $\sigma_0, \sigma_1, \ldots, \sigma_{k-1}$. Note that a_k is always ± 1. The chain $\gamma = \Sigma_{k=0}^{S} a_k\sigma_k$ is crucial in the computation of the homology of K.

Define $|L|$ to be the subset of $|K|$ consisting of $|K_{n-2}|$ and all (n-1)-cells of K not among the τ_k's. By Prop. I.4.2., L is a subcomplex of K. $\partial\gamma$ is an (n-1)-chain on L, and since each (n-1)-cell of K is a face of exactly two n-cells, the coefficients of $\partial\gamma$ are members of the set $\{-2,0,+2\}$. Thus $\partial\gamma = 2\lambda$, where λ is an (n-1)-chain on L having coefficients ± 1.

4.4. LEMMA. <u>If an n-chain has its boundary on</u> L, <u>then it is a multiple of</u> γ.

Proof: Let $c = \Sigma_{j=0}^{S} b_j \sigma_j$, with ∂c lying on L. Then $c - b_0\gamma = \Sigma_{j=1}^{S} b_j' \sigma_j$ and $\partial(c-b_0\gamma)$ lies on L. We show by induction that all of the b_j' are zero. We know that τ_1 is a face of σ_0 and σ_1 only, so $\partial(c-b_0\gamma)$ has coefficient $\underline{+}b_1'$ on τ_1, the sign depending only on orientation. But $\partial(c-b_0\gamma)$ lies on L, so b_1' is zero. Given that $b_j' = 0$ for $j < k$, we have that τ_k is a face of σ_k and of no other cells σ_j for $j > k$. Consequently the coefficient of τ_k in $\partial(c-b_0\gamma)$ is $\underline{+}b_k'$, the sign depending on orientation. In any case this must be zero, so b_k' is zero. Thus $c = b_0\gamma$.

4.5. COROLLARY. If K has any non-zero n-cycles, they are all multiples of γ.

Thus γ is a cycle if and only if K has some non-zero n-cycle.

4.6. DEFINITION. If the chain γ is a cycle, then the n-circuit K is said to be orientable. In this case γ is called the fundamental cycle of K. If γ is not a cycle, then K is called non-orientable. (Note the difference between "oriented" and "orientable.")

4.7. COROLLARY. If K is orientable, $H_n(K) \approx Z$. If K is non-orientable, $H_n(K)$ is zero. (We assume here that homology is independent of orientation.)

4.8. LEMMA. Each (n-1)-cycle of K is homologous to an (n-1)-cycle on L.

Proof: Let z be an (n-1)-cycle of K. Then $z = \Sigma_{k=1}^{S} b_k \tau_k + d$, where d is an (n-1)-chain on L. We prove by induction that z

is homologous for every j to a cycle z_j which has coefficients zero on τ_k for $k < j$. This is vacuously true for $j = 1$. Suppose it is true for $j = m$, where $1 \leq m \leq S$. Then $z \sim z_m = \Sigma_{k=m}^{S} c_k \tau_k + d'$, for some coefficients c_k and some (n-1)-chain d' on L. We know that τ_m is a face of σ_m and of some σ_i with $i < m$. Also, $\partial\sigma_m = \epsilon\,\tau_m$ + terms involving no preceding τ_k's, where $\epsilon = \underline{+}1 = [\sigma_m : \tau_m]$. So $z_m - \epsilon\, c_m \partial\sigma_m = c_{m+1}'\tau_{m+1} \cdots + c_S'\tau_S + d''$, for suitable c_{m+1}', c_{m+2}', etc., and d" on L. We set $z_{m+1} = z_m - \epsilon\, c_m \partial\sigma_m$ and note that we have produced a cycle homologous to z_m which has coefficient zero on τ_m and all preceding τ_k's. Of course $z_{m+1} \sim z$. Thus z is eventually shown to be homologous to a cycle on L.

Note that if an (n-1)-cycle of L bounds in K, then it is a multiple of 2λ. For, if z is an (n-1)-cycle on L such that $z = \partial c$ for some n-chain c on K, then, by Lemma 4.4, $c = b\gamma$ for some integer b, so $z = \partial c = \partial b\gamma = 2b\lambda$.

4.9. THEOREM. If K is an orientable n-circuit, then $H_{n-1}(K) \approx Z_{n-1}(L)$. If K is non-orientable, then $H_{n-1}(K) \approx Z_{n-1}(L)/ < 2\lambda >$, where $< 2\lambda >$ is the subgroup of $Z_{n-1}(L)$ generated by 2λ. (Note that $Z_{n-1}(L) \approx H_{n-1}(L)$.)

Proof: Let K be orientable. γ is a cycle, so $\partial\gamma = 2\lambda = 0$. Thus λ is zero, since $C_{n-1}(K)$ is free. We define a homomorphism $f: H_{n-1}(K) \to Z_{n-1}(L)$ as follows. An (n-1)-cycle z on K is homologous to a cycle z' on L by Lemma 4.8. The remark preceding this theorem shows that z' is unique. For if $z' \sim z''$ and both are on L, then their difference bounds in K, and must be a multiple of λ . But λ is zero. Moreover, starting with any z_1 in the homology class of z , we get a cycle z_1' on L homologous to z_1 . But then $z' \sim z_1'$ and so they are equal. Therefore we may set $f(\{z\}) = z'$. Now f is obviously a homomorphism. It is a monomorphism, for if $f(\{z\}) = 0$, then $z \sim 0$. It is an epimorphism because any cycle on L is a member of some homology class on K . Thus f is an isomorphism, and $H_{n-1}(K) \approx Z_{n-1}(L)$, as desired.

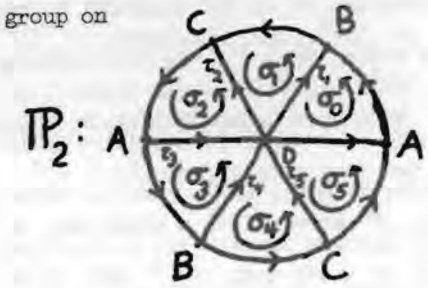
If K is non-orientable, we consider $< 2\lambda >$, the subgroup of multiples of 2λ . We define a map $f\colon H_{n-1}(K) \to Z_{n-1}(L)/ < 2\lambda >$ as follows. Given an (n-1)-cycle z on K , we use Lemma 4.8 to find z' on L homologous to z . Then z' is determined, up to an addition of a multiple of 2λ , so we may set $f(\{z\})$ equal to the coset of $< 2\lambda >$ in which z' lies. As before, f is a homomorphism. It is a monomorphism because if $f(\{z\})$ is zero, then z is homologous to a multiple of 2λ and thus bounds. Since f is clearly an epimorphism, it provides the desired isomorphism $H_{n-1}(K) \approx Z_{n-1}(L)/ < 2\lambda >$.

4.10. COROLLARY. <u>If</u> K <u>is a non-orientable n-circuit,</u> $H_{n-1}(K)$ <u>is isomorphic to a direct sum of</u> Z_2 <u>and a finite number of copies of</u> Z .

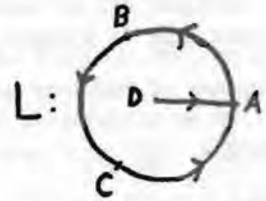
Proof: We may take a basis for $C_{n-1}(L)$ which contains λ since all of its coefficients are ± 1 . $Z_{n-1}(L)$ is a subgroup of $C_{n-1}(L)$. By a fundamental theorem on finitely generated abelian groups we can find a basis for the free group $Z_{n-1}(L)$ each of whose elements is a multiple of a basis element of $C_{n-1}(L)$.

But 2λ bounds in K and so is a cycle in L . Thus $2\partial\lambda = \partial 2\lambda = 0$, so λ itself is a cycle. Therefore λ can be taken to be a generator of $Z_{n-1}(L)$. The corollary now follows.

4.11. <u>Example.</u> The projective plane $\mathbb{P}^2$, a non-orientable 2-circuit. We triangulate $\mathbb{P}^2$ as in the diagram and label the n- and (n-1)-cells. Note that the chain $\gamma = \sigma_0 + \sigma_1 + \dots + \sigma_5$, so $\partial\gamma = 2AB + 2BC + 2CA \neq 0$. Thus $\mathbb{P}^2$ is non-orientable, and $H_2(\mathbb{P}^2) = 0$. $Z_1(L)$ is a free group on one generator, λ , so $Z_1(L)/ < 2\lambda > \approx Z_2$. Thus $H_1(\mathbb{P}^2) \approx Z_2$. Since $\mathbb{P}^2$ is connected, $H_0(\mathbb{P}^2) \approx Z$.



Exercises. Show that the homology of 1) the torus is $H_2 \approx Z$, $H_1 \approx Z \oplus Z$, $H_0 \approx Z$, and 2) the Klein bottle is $H_2 = 0$, $H_1 \approx Z \oplus Z_2$, $H_0 \approx Z$, by triangulating them as 2-circuits and using the methods of this section.



5. Change of Orientation in a Complex

For this section we will need the notions of chain mapping and chain isomorphism.

5.1. DEFINITION. Given two chain complexes $C = (\{C_q\},\partial)$ and $C' = (\{C_q'\},\partial')$, a <u>chain map</u> $\varphi\colon C \to C'$ is a collection $\{\varphi_q\}$ of homomorphisms $\varphi_q\colon C_q \to C_q'$ such that $\varphi_{q-1}\partial_q = \partial_q'\varphi_q\colon C_q \to C_{q-1}'$ for each q . A chain map is called a chain isomorphism if each of the φ_q's is an isomorphism.

The notion of chain map is important because a chain map is easily seen to carry cycles into cycles and boundaries into boundaries. Thus a chain map φ induces a chain map $\varphi_*: H_*(C) \to H_*(C')$ defined by $(\varphi_*)_q\{z\} = \{\varphi_q z\}$. As a matter of convention we shall usually omit the subscripts whenever referring to the homomorphisms of a chain map or to the boundary operators of a chain complex. Thus we write $\varphi\partial = \partial'\varphi$ as the defining relation of the chain map φ.

Note that isomorphic chain complexes have isomorphic homology groups and that a chain map $\varphi: C \to C'$ is a chain isomorphism if and only if there exists a chain map $\varphi^{-1}: C' \to C$ called the inverse of φ such that $\varphi\varphi^{-1}$ and $\varphi^{-1}\varphi$ are the identity on C and C' respectively. Here we use the fact that the composition of two chain maps is a chain map. The basic theorem on induced chain mappings of homology groups is:

5.2. THEOREM. If $\varphi: C \to C'$ and $\psi: C' \to C''$ are chain maps then $\psi_*\varphi_* = (\psi\varphi)_*: H_*(C) \to H_*(C'')$.

The proof is easy and will be omitted. This theorem, together with the fact that the identity chain map induces the identity homomorphisms of homology groups, shows that the assignment to C of the chain complex $H_*(C)$ and the assignment to a map $\varphi: C \to C'$ the map φ_* is a functor from the category of chain complexes and abelian groups into itself.* In the future we shall usually omit reference to the homology chain complex and talk of the homology groups or regard the collection of homology groups as a graded group.

*See Chapter IV.

If we are given a regular complex K with an incidence function α , we may reverse the orientation on a given cell σ of K in the following way. Let $C^\alpha(K)$ be the chain complex for K given by the incidence function α on K . We define a new incidence function β for K . β is defined to equal α except for incidence numbers involving σ . The β-incidence numbers involving σ are defined to be the negative of the α-incidence numbers involving σ (we assume that $\dim \sigma \geq 1$). Then β induces the chain complex $C^\beta(K)$. We say that $C^\beta(K)$ is obtained from $C^\alpha(K)$ by an orientation reversal. We define a chain map $\psi: C^\alpha(K) \to C^\beta(K)$ as follows. $\psi\tau = \tau$ for every cell τ different from σ ; $\psi\sigma = -\sigma$. This defines ψ on all the generators of the $C_q(K)$, and we extend ψ by linearity. It is easy to see that ψ is a chain map and has an inverse map which reverses the sign on σ once again. Thus ψ is a chain isomorphism and induces isomorphisms of the homology groups of $C^\alpha(K)$ and $C^\beta(K)$. And so $H_q^\alpha(K) \approx H_q^\beta(K)$ for every q .

For our general theorem on change of orientation we will need to assume the following lemma:

5.3. LEMMA. Any regular complex on the n-sphere $(n \geq 1)$ is an orientable n-circuit.

The proof of this lemma is given in Chapter VIII, section 4. At the present time we may regard the proposition as an additional redundant restriction imposed on a regular complex K : for each $q > 1$ and each q-cell σ of K , the boundary $\dot\sigma$ is an orientable (q-1)-circuit. Notice that R.R.2 asserts that each (q-2)-cell of $\dot\sigma$ is a face of exactly two (q-1)-cells of $\dot\sigma$, and that R.R.3 implies that $\dot\sigma$ contains the (q-1)-cycle $\partial\sigma$, and hence is orientable. Thus we are imposing additionally only condition 2 of definition 4.3 on the existence of a path of (q-1)-cells connecting any two (q-1)-cells. In case K is simplicial, the condition holds trivially because any two (q-1)-faces of $\dot{\sigma}$ have a common (q-2)-face.

5.4. THEOREM. Given a regular complex K and two incidence functions α and β on K, then the chain complexes $C^{\alpha}(K)$ and $C^{\beta}(K)$ are isomorphic. The isomorphism $\psi: C^{\alpha}(K) \approx C^{\beta}(K)$ may be chosen so that $\psi\sigma = \pm\sigma$ for each cell σ of K.

Proof: We define a chain isomorphism $\psi: C^{\alpha}(K) \to C^{\beta}(K)$ by starting at the chain group of dimension zero and working upwards. We let $\psi: C_0(K) \to C_0(K)$ be the identity isomorphism. To specify ψ on $C_1(K)$, let σ be a generator with vertices A and B. Now $\partial^{\alpha}\sigma = \epsilon_1(A-B)$ for $\epsilon_1 = \pm 1$. Similarly, $\partial^{\beta}\sigma = \epsilon_2(A-B)$ for $\epsilon_2 = \pm 1$. Set $\psi\sigma = \epsilon_1\epsilon_2\sigma$. Then

$$\psi\partial^{\alpha}\sigma = \epsilon_1(A-B) = \epsilon_1\epsilon_2\epsilon_2(A-B) = \partial^{\beta}\psi\sigma .$$

Extend ψ over all of $C_1(K)$ by linearity. It is clearly an isomorphism. Note that ψ is induced by a set of independent orientation reversals: namely, orientation is reversed on each cell σ such that $\epsilon_1\epsilon_2 = -1$.

Suppose now that ψ has been defined on $C_i(K)$ for $i < q$, where q is an integer ≥ 2. Suppose that ψ as defined is an isomorphism on each $C_i(K)$, commutes with the boundary operators ∂^{α} and ∂^{β}, and satisfies $\psi\sigma = \pm\sigma$ for each generator of each $C_i(K)$ for $i < q$. Then we extend ψ over $C_q(K)$ so as to have

the same properties: For σ a generator of $C_q(K)$ we define $\psi\sigma = \pm\sigma$, where the sign is determined by comparing $\partial^{\beta}\sigma$ with $\psi\partial^{\alpha}\sigma$ in the following way. First note that $\partial^{\beta}\psi\partial^{\alpha}\sigma = \psi\partial^{\alpha}\partial^{\alpha}\sigma = 0$, and of course $\partial^{\beta}\partial^{\beta}\sigma = 0$. Thus both $\partial^{\beta}\sigma$ and $\psi\partial^{\alpha}\sigma$ are cycles on the subcomplex $\dot{\sigma}$ with the orientation induced by β. Moreover, both have coefficients ± 1 on the (q-1)-cells of $\dot{\sigma}$. Now by the preceding lemma, $\dot{\sigma}$ is an orientable (q-1)-circuit, since $\dot{\sigma}$ is a regular complex on the (q-1)-sphere, $q \geq 2$. So $Z_{q-1}(\dot{\sigma}) \approx Z$, and $\partial^{\beta}\sigma = \epsilon\psi\partial^{\alpha}\sigma$ for some $\epsilon = \pm 1$. We then set $\sigma = \epsilon\sigma$. We do this for each generator of $C_q(K)$, and extend by linearity over the whole group. Then, if σ is a generator of $C_q(K)$, we have $\psi\partial^{\alpha}\sigma = \epsilon\partial^{\beta}\sigma = \partial^{\beta}\epsilon\sigma = \partial^{\beta}\psi\sigma$. This relation extends by linearity over the whole group, and so ψ commutes with the boundary operators. Clearly ψ is an isomorphism on $C_q(K)$, and $\psi\sigma = \pm\sigma$ for each σ a generator of $C_q(K)$. Note that ψ is obtained by a collection of orientation reversals. We extend ψ in this way over all the chain groups of K, and then the result is the desired chain isomorphism $C^{\alpha}(K) \approx C^{\beta}(K)$.

5.5. COROLLARY. If K is a regular complex with two orientations α and β, then $H_q^{\alpha}(K) \approx H_q^{\beta}(K)$ for each q. Homology groups are independent of orientation.

Proof: Isomorphic chain complexes have isomorphic homology groups.

From now on we will write $H_q(K)$ to mean $H_q^{\alpha}(K)$ for some α, etc.

The lemma stated before Theorem 5.4 has another important consequence which relates to the possibility of defining an incidence function on a regular complex K. In fact we have

5.6. THEOREM. Given a regular complex K, any incidence function α defined on a subcomplex L of K can be extended to an incidence function of K.

Proof: We extend α inductively over the subcomplexes $L \cup K_0$, $L \cup K_1, \ldots$. Clearly α can be regarded as an incidence function for $L \cup K_0$. We define α_1 over $L \cup K_1$ to be the following extension of α: If σ is a 1-cell of K not in L, then there are two vertices A and B lying in $\dot{\sigma}$, since $\bar{\sigma}$ is homeomorphic to a closed 1-ball. We define $[\sigma:A] = 1$, $[\sigma:B] = -1$ arbitrarily. In this way α is defined on all pairs of cells involving 1-cells of K not in L.

Suppose now that we are given an incidence function α_{q-1} defined on the subcomplex $L \cup K_q$, $q \geq 2$. We extend α_{q-1} to an incidence function α_q on $L \cup K_q$ as follows. For any q-cell σ of K not in L we must define incidence numbers for pairs of cells including σ. These will be zero unless the second cell of the pair is a (q-1)-face of σ. Since $\dot{\sigma}$ is a subcomplex of K oriented by α_{q-1}, by lemma 5.3, $Z_{q-1}^{\alpha_{q-1}}(\dot{\sigma}) \approx Z$. We choose a generator γ. This generator has coefficients ± 1 on the (q-1)-cells of $\dot{\sigma}$. If τ is a (q-1)-cell of $\dot{\sigma}$, we set $[\sigma:\tau]$ equal to the coefficient which γ has on τ. Then the only property of the incidence function α_q so defined which we need to verify is (iii) of Def. 2.1.8. Let ρ be a (q-2)-cell of $\dot{\sigma}$. By R.R.2 there are 2 (q-1)-cells of τ_1, τ_2 which are faces of σ and have ρ as a common face. Now $\partial^{\alpha_{q-1}}\gamma = 0$. So the sum $[\sigma:\tau_1][\tau_1:\rho] + [\sigma:\tau_2][\tau_2:\rho]$, which is the coefficient of $\partial^{\alpha_{q-1}}\gamma$ on ρ, must be zero. Thus (iii) is verified and α_q is an incidence function for $L \cup K_q$. We continue this process and extend α to all of K.

By taking L to be the empty subcomplex we derive the third redundant restriction R.R.3.

6. Homology with General Coefficients. Cohomology.

We have defined $C_q(K)$, for a regular complex K, as the free abelian group generated by the q-cells of K. An element of $C_q(K)$ is then a function mapping the set of q-cells of K into the integers which is zero on all but a finite number of q-cells. We generalize the definition of $C_q(K)$ by considering functions from the set of q-cells of K to an arbitrary R-module G. First we make the following definition.

6.1. DEFINITION. Given a regular complex K and a commutative ring R with unit, the qth dimensional R-module of chains of K with coefficients in R is the set of functions defined on the collection of q-cells of K mapping into R so as to be zero on all but a finite number of q-cells. The addition and scalar multiplication are defined as follows. If f and g are two such functions, $r \in R$, and σ a q-cell, then

$$(f+g)\sigma = f\sigma + g\sigma \quad \text{and} \quad (rf)\sigma = r(f\sigma),$$

where the addition and scalar multiplication on the right sides are in R. The qth dimensional R-module of chains of K with coefficients in R is denoted by $C_q(K;R)$. An incidence function α on

K induces a boundary operator $\partial^{\alpha}_q: C_q(K;R) \to C_{q-1}(K;R)$ because of the unit in R. More precisely, if σ is the chain which takes the value 1 on σ and 0 elsewhere (1 and 0 are the unit and zero of R) we set $\partial^{\alpha}_q \sigma = \Sigma_{\tau \in K}[\sigma:\tau]\tau$, where $[\sigma:\tau]$ lies in the set $\{-1, 0, +1\}$ of elements of R. With this boundary operator the chain modules $C^{\alpha}_q(K;R)$ form a chain complex denoted by $C^{\alpha}(K;R)$. Note that $C^{\alpha}(K;Z) = C^{\alpha}(K)$ as defined previously.

6.2. DEFINITION. Given a regular complex K and a unitary R-module G over a commutative ring R, the qth dimensional module of chains of K with coefficients in G is the R-module

$$C_q(K;R) \otimes_R G .$$

This R-module is denoted by $C_q(K;G)$. As in Definition 6.1, an orientation α induces boundary homomorphisms mapping $C_q(K;G)$ into $C_{q-1}(K;G)$ for each Q. The qth homomorphism is the mapping $\partial^{\alpha}_q \otimes 1$, where 1 is the identity mapping on G. The collection of modules $\{C_q(K;G)\}$ together with the boundary homomorphisms $\{\partial^{\alpha}_q \otimes 1\}$ form a chain complex which we will write as $C^{\alpha}(K;G)$.

6.3. DEFINITION. The qth dimensional modules of cycles and boundaries of the regular complex K with coefficients in the R-module G are the qth dimensional modules of cycles and boundaries, respectively, of the chain complex $C^{\alpha}(K;G)$, and are denoted by $Z^{\alpha}_q(K;G)$ and $B^{\alpha}_q(K;G)$, respectively. The qth dimensional homology module $H^{\alpha}_q(K;G)$ of K with coefficients in G is the factor module $Z^{\alpha}_q(K;G)/B^{\alpha}_q(K;G)$.

As in the previous section, one can show that all of the chain complexes $C^{\alpha}(K;G)$ are isomorphic, and that the homology groups $H^{\alpha}_q(K;G)$ are all isomorphic. The proof of these facts is easy and will be omitted.

Homology with general coefficients is useful in some cases. For example, if G is the group of rationals, there is no torsion. If G is the group Z_2, every n-circuit is orientable, because the boundary of the chain γ is $2\lambda = 0$. It turns out, however, that for the case where R is Z, the homology groups with coefficients in an arbitrary abelian group G can be calculated from the homology groups with coefficients in Z. Thus homology with coefficients in a group G does not give any new information about the complex K. This is the main result of this section. We need the following definition.

6.4. DEFINITION. If $C = (\{C_q\},\partial)$ and $C' = (\{C'_q\},\partial')$ are two chain complexes then the direct sum $C \oplus C'$ of C and C' is the chain complex $(\{C_q \oplus C'_q\},\partial \oplus \partial')$ whose qth boundary operator is the mapping defined by

$$(\partial \oplus \partial')(c,c') = (\partial c,\partial' c') .$$

This definition can be extended to any finite or infinite number of chain complexes. We remark that the homology of $C \oplus C'$ is the direct sum of the homology of C and of C'; that is, for each q, $H_q(C \oplus C') \approx H_q(C) \oplus H_q(C')$.

6.5. DEFINITION. An elementary chain complex is a chain complex over Z of one of the following three types:

i) (Free) All chain groups are zero except for an infinite cyclic group in a single dimension. (All boundary operators are of course zero.)

$$\cdots \to 0 \to Z \to 0 \cdots$$

ii) (Acyclic) All chain groups are zero except for infinite cyclic groups in adjacent dimensions, n and $n-1$, for some n, and all boundary operators are zero except ∂_n, which is an isomorphism onto.

$$\cdots \to 0 \to Z \xrightarrow{\approx} Z \to 0 \to \cdots$$

iii) (Torsion) The complex is the same as in the acyclic case except that ∂_n is multiplication by an integer other than 0 or ± 1.

$$\cdots \to 0 \to Z \xrightarrow{\theta_k} Z \to 0 \to \cdots$$

6.6. THEOREM. <u>If</u> C <u>is a chain complex over</u> Z <u>such that each chain group is free abelian and finitely generated, then</u> C <u>is the direct sum of elementary chain complexes.</u>

Proof: From the proof of Theorem II.2.2, we know that Z_q is a direct summand in C_q for each q. We set $C_q = Z_q \oplus D_q$. Since only the zero of D_q is a cycle, the boundary operator when restricted to D_q is a monomorphism. The image ∂D_q is a subgroup of the finitely generated free abelian group Z_{q-1}. Thus we may choose bases $\{d_1,\ldots,d_k\}$ and $\{z_1,\ldots,z_m\}$ for D_q and Z_{q-1} respectively, such that for $1 \le i \le k$ there are integers $p_i > 0$ such that $\partial d_i = p_i z_i$ and $p_i | p_{i+1}$. By our first assertion C is split into a direct sum over q of the chain complexes $\cdots \to 0 \to D_q \xrightarrow{\partial} Z_{q-1} \to 0 \to \cdots$. We now split each of these into a direct sum of elementary chain complexes. The elementary chain complexes we obtain are as follows: For every $p_i = 1$ we have an acyclic elementary chain complex with non-zero groups in dimensions q and $q-1$. For every $p_i \neq 1$ we have an elementary chain complex of the torsion type with the non-zero groups in dimension q, $q-1$, and the boundary operator the mapping θ_{p_i} which multiplies by p_i. For every z_i with $i > k$ we have free elementary chain complex with the non-zero group in dimension q. The direct sum over q of all these chain complexes is C.

We will use the theorem above to compute the homology groups of a regular complex K with coefficients in an arbitrary abelian group G. Let α be an orientation on K. The chain complex $C^\alpha(K)$ satisfies the hypotheses of Theorem 6.6 as long as K has finitely many cells in each dimension. We will use the result that $(\Sigma A_i) \otimes G$ is isomorphic to $\Sigma(A_i \otimes G)$. Now by Theorem 6.6, $C^\alpha(K)$ splits into a direct sum ΣN_i, where the N_i are elementary chain complexes. Then it is clear that $C^\alpha(K;G) = C^\alpha(K) \otimes G \approx \Sigma(N_i \otimes G)$, where the tensor product notation denotes tensoring in each dimension. For instance $C^\alpha(K) \otimes G$ is the chain complex $(\{C_q(K) \otimes G\}, \partial \otimes 1)$, where $(\partial \otimes 1)_q = \partial_q \otimes 1$. Therefore, by the remark following Definition 6.4,

$$H_*(K;G) \approx \Sigma[H_*(N_i \otimes G)] .$$

Computation of the homology of K is then reduced to the computation of the homology of the chain complexes $N_i \otimes G$.

If N_i is free, then $N_i \otimes G$ is the chain complex $\dots \to 0 \to G \to 0 \to \dots$ and so $H_q(N_i \otimes G) = G$, where q is the dimension of the single non-trivial chain group G of $N_i \otimes G$.

If N_i is acyclic, then $N_i \otimes G$ is the chain complex whose groups are all zero except for two adjacent dimensions q, $q-1$. The non-zero groups are copies of G, and the boundary operator between them is an isomorphism. In this case it is easy to see that $H_*(N_i \otimes G) = 0$.

Finally, if N_i is of the torsion type, then $N_i \otimes G$ is the chain complex $\dots \to 0 \to G \xrightarrow{\theta_k} G \to 0 \to \dots$, where θ_k is multiplication by k. Then $H_q(N_i \otimes G) \approx \operatorname{Ker} \theta_k$, the subgroup of elements of G whose order divides k. This group is written ${}_kG$. Also, $H_{q-1}(N_i \otimes G)$ is G/kG, which we write as G_k.

We show below that G_k $(= G/kG)$ is isomorphic to $Z_k \otimes G$. Thus in all three cases $H_*(N_i \otimes G)$ contains $H_*(N_i) \otimes G$ as a direct summand. The only non-trivial complementary summand occurs in the torsion case. We have therefore proved the following theorem.

6.7. THEOREM. For each q, $H_q^\alpha(K;G) \approx (H_q^\alpha(K) \oplus \Sigma\, {}_kG$ where the last sum is over all k for which there is a chain complex of the torsion type with qth boundary operator θ_k in the direct sum decomposition of $C^\alpha(K)$.

6.8. LEMMA. If G is an arbitrary abelian group then $G_k = G/kG$ is isomorphic to $G \otimes Z_k$. If G is finitely generated, then ${}_kG$, the subgroup of elements whose orders divide k, is isomorphic to $T \otimes Z_k$, where T is the torsion subgroup of G.

Proof: To prove the first assertion, we define $\varphi\colon G \to G \otimes Z_k$ by setting $\varphi(g) = g \otimes 1$ for all $g \in G$. The kernel of φ then contains kG, and so φ induces a homomorphism $\varphi_0\colon G/kG \to G \otimes Z_k$. Define $\psi\colon G \otimes Z_k \to G/kG$ by $\psi(g \otimes n) =$ coset containing ng, for all $g \in G$, $n \in Z_k$. It is easily verified that φ_0 and ψ are inverses of each other. Thus $G_k \approx G \otimes Z_k$.

By the fundamental theorem for finitely generated abelian groups, together with the relation ${}_k(G_1 \oplus G_2) \approx G_1 \oplus {}_kG_2$, the second assertion reduces to the case where G is a cyclic group. If G is infinite cyclic then ${}_kG$ and $T \otimes Z_k$ are both zero. Suppose that G is of order n, generated by g_0. Then $G \otimes Z_k$ is of order (n,k). Let r be an integer.

$$rg_0 \in {}_kG \iff n \text{ divides } kr$$
$$\iff \frac{n}{(n,k)} \text{ divides } r.$$

Thus ${}_kG$ is generated by $\frac{n}{(n,k)} g_0$ and so has order (n,k). So ${}_kG \approx G \otimes Z_k$ and the proof is complete.

6.9. COROLLARY. Let G be a finitely generated abelian group. Then, for each q,

$$H_q^\alpha(K;G) \approx (H_q^\alpha(K) \otimes G) \oplus (T_{q-1}^\alpha(K) \otimes T),$$

where $T_{q-1}^\alpha(K)$ is the torsion subgroup of $H_{q-1}^\alpha(K)$ and T is the torsion subgroup of G.

6.10. PROPOSITION. Let G be a torsion free abelian group. Then

$$H_q^\alpha(K;G) \approx H_q^\alpha(K) \otimes G.$$

Proof: For then ${}_kG = 0$ for all k.

Exercise. Compute the homology groups of the projective plane $\mathbb{P}^2$ with coefficients in Z_2. A tabulation of the homology groups with coefficients in Z and in Z_2 gives:

$$\begin{array}{ll} H_0(\mathbb{P}^2) \approx Z & H_0(\mathbb{P}^2 : Z_2) \approx Z_2 \\ H_1(\mathbb{P}^2) \approx Z_2 & H_1(\mathbb{P}^2; Z_2) \approx Z_2 \\ H_2(\mathbb{P}^2) \approx 0 & H_2(\mathbb{P}^2; Z_2) \approx Z_2 \end{array}$$

The 0th homology group on the left, Z, gives rise to the 0th group Z_2 on the right; and the 1st homology group on the left, Z_2, gives rise to both the 1st and 2nd groups on the right.

Given a regular complex K we define the cohomology groups of K by first associating with K a cochain complex.

6.11. DEFINITION. A cochain complex C over a ground ring R (commutative with unit) is a sequence $\{C^q\}$ of R-modules together with R-homomorphisms $\delta_q: C^q \to C^{q+1}$ such that for each q, $\delta_q \delta_{q-1} = 0$. The R-module C^q is called the module of q-cochains of C, and the homomorphism δ_q is called the qth coboundary operator of C. The modules of q-cocycles and q-coboundaries, denoted by Z^q, B^q are the R-modules $\operatorname{Ker} \delta_q$ and $\operatorname{Im} \delta_{q-1}$ respectively. The qth cohomology module of C is the factor R-module Z^q/B^q and is denoted $H^q(C)$. The collection of cohomology modules of C together with the zero boundary operators forms a cochain complex called the cohomology chain complex of C and denoted $H^*(C)$.

6.12 DEFINITION. The qth dimensional module of cochains of a regular complex K with coefficients in the R-module G, written $C^q(K;G)$, is the R-module $\operatorname{Hom}(C_q(K;R),G)$. The cochain complex associated with an oriented regular complex K is the cochain complex $C^\cdot(K) = (\{C^q(K\{G)\},\delta\})$, where $\delta_q = (-1)^q \operatorname{Hom}(\partial_q, 1)$. The modules of cocycles and coboundaries of this cochain complex form the modules of cocycles and coboundaries, respectively, of K with coefficients in G. These modules are written $Z^q(K;G)$, $B^q(K;G)$. The qth homology module of K with coefficients in G is the factor module $Z^q(K;G)/B^q(K;G)$, and is written $H^q(K;G)$.

If u is a cochain on K of dimension q, and c is a q-chain on K, then for the value of u on c we write (u,c). Note that $(\delta u, c) = (-1)^q (u, \partial c)$.

Exercise. 1. State and prove a theorem about decomposing cochain complexes over Z whose cochain groups are free and finitely generated into a direct sum of elementary cochain complexes. Deduce from the theorem that the cohomology of a finite regular complex with coefficients in an arbitrary abelian group G is determined by the cohomology with coefficients in Z.

2. Let $C = (\{C^q\},\delta)$ and $D = (\{D^q\},\delta')$ be two cochain complexes. Define a cochain map $\varphi: C \to D$ to be a collection of homomorphisms $\varphi_q: C^q \to D^q$ such that $\varphi_{q+1}\delta_q = \delta'_q \varphi_q$ for each q. Show that φ induces a cohomology homomorphism

$$\varphi^*: H^*(C) \to H^*(D).$$

Prove an analog of Theorem 5.2.

CHAPTER III

REGULAR COMPLEXES WITH IDENTIFICATIONS

1. The identifications

Let K be a regular complex, and let σ and τ be cells of K of equal dimension. A homeomorphism f of $\bar{\sigma}$ onto $\bar{\tau}$ is called an <u>identification on</u> K if whenever $\sigma_o < \sigma$ the restriction $f|\bar{\sigma}_o$ carries $\bar{\sigma}_o$ onto a (closed) face of τ of the same dimension as $\bar{\sigma}_o$.

1.1. DEFINITION. A collection F of identifications on K is a <u>family of identifications</u> on K if each of the following holds:

1. For each σ in K the identity homeomorphism $\bar{\sigma} \longrightarrow \bar{\sigma}$ lies in F.
2. For each f in F, f^{-1} is in F.
3. If $f: \bar{\rho} \longrightarrow \bar{\sigma}$ and $g: \bar{\sigma} \longrightarrow \bar{\tau}$ are in F, then $gf: \bar{\rho} \longrightarrow \bar{\tau}$ is in F.
4. If $f: \bar{\sigma} \longrightarrow \bar{\sigma}$ is in F, f is the identity homeomorphism.
5. If $f: \bar{\sigma} \longrightarrow \bar{\tau}$ is in F and $\sigma_o < \sigma$, then $f|\bar{\sigma}_o$ is in F.

A <u>complex with identifications</u> is a pair (K,F) with K a regular complex and F a family of identifications on K.

If (K,F) is a complex with identifications, the relation

$$\sigma \sim \tau \quad \text{if} \quad f: \bar{\sigma} \longrightarrow \bar{\tau} \text{ is in } F$$

is an equivalence relation on the cells of K. This follows from properties (1)-(3) for F. From properties (3) and (4) we conclude that if $f: \bar{\sigma} \longrightarrow \bar{\tau}$ and $g: \bar{\tau} \longrightarrow \bar{\sigma}$ are in F then $g = f^{-1}$. Accordingly, if $f: \bar{\sigma} \longrightarrow \bar{\tau}$ and $h: \bar{\sigma} \longrightarrow \bar{\tau}$ are in F then $h = f$. That is, if $\sigma \sim \tau$ then there exists one and only one map in F of $\bar{\sigma}$ onto $\bar{\tau}$. This fact will be used to construct an incidence function in the proof of 2.4.

The family F also determines an equivalence relation for points of K:

$$x \sim y \quad \text{if a map in } F \text{ carries } x \text{ to } y .$$

We let K/F denote the set of equivalence classes of points of $|K|$. The natural function $s: |K| \longrightarrow K/F$ (where sx is the class of x) defines a topology for K/F in the familiar way: a subset X of K/F is closed if and only if $s^{-1}X$ is closed in K. Throughout this chapter we will omit absolute value bars whenever we can to simplify notation.

We define the q-skeleton $(K/F)_q$ of K/F to be the subspace $s(K_q)$. The space K/F and the skeletons $(K/F)_q$ form a complex. The proof of this fact is routine, and we take time here to point out only a few of the more important considerations. Each open cell of K/F is the homeomorphic image, under s, of at least one open cell of K. If $s\sigma = u$, then the cells of K that s maps onto u are precisely the cells equivalent to σ. Property (4) listed for F shows that no two points of the same cell of K are ever identified by a map of F. Property (5) guarantees that if two cells are identified then so are their boundaries.

The complex K/F need not be regular, but relative homeomorphisms may be obtained for the cells of K/F from any set of homeomorphisms given for K by composition with s :

$$\begin{tikzcd}
(E^n,S^{n-1}) \arrow[r, "h"', "\text{homeomorphism}"] \arrow[dr, "(s|\bar{\sigma})h"'] & (\bar{\sigma},\dot{\sigma}) \arrow[d, "s|\bar{\sigma}"] \\
& (\bar{u},\dot{u})
\end{tikzcd}$$

The map $(s|\bar{\sigma})h$ is a relative homeomorphism because $s|\bar{\sigma}$ is a relative homeomorphism.

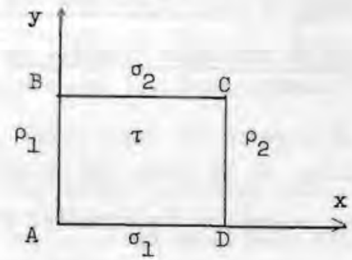
2. The homology of K/F

Let (K,F) be a complex with identifications. Although K/F is not necessarily regular it is possible to define a chain complex based on the cells of K/F whose homology is isomorphic to the homology of the space K/F. We mean that 1) the space K/F does carry a regular complex, so that its homology groups are well defined (see page 33 of Chapter II), and 2) these homology groups are isomorphic to the homology groups of the chain complex $C^{\alpha}(K/F)$ that we are about to define. Proofs of (1) and (2) appear in section 3 of Chapter IX. The justification for presenting the chain complex at this stage is the ease with which its homology (and hence that of K/F) may sometimes be computed.

2.1. DEFINITION. An incidence function α on K is invariant under F if whenever $f\colon \bar{\rho} \longrightarrow \bar{\sigma}$ is in F and $\tau < \rho$, then $[\rho:\tau]_{\alpha} = [f\rho: f\tau]_{\alpha}$.

2.2. EXAMPLE. The torus from a disk by identification.

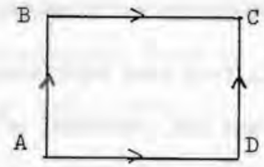
Let K be a regular complex on the unit square with four vertices, four 1-cells and one 2-cell:



Define $f\colon \bar{\rho}_1 \longrightarrow \bar{\rho}_2$ by $f(x,0) = (x,1)$, and define $g\colon \bar{\sigma}_1 \longrightarrow \bar{\sigma}_2$ by $g(0,y) = (1,y)$. Then take F to consist of

1. the identity map on $\bar{\tau}$
2. f, g, f^{-1} and g^{-1}
3. the restrictions of these mappings to bounding cells.

An incidence function invariant under F is given by the diagram below (following the arrow-notation introduced on page 35 of Chapter II):



2.3. EXAMPLE. Real projective n-space P^n from S^n by identifications.

Let K be the regular complex on S^n described in I.2.1. We take F to be the collection of maps obtained by restricting the involution T (see I.2.7) to the closed cells of K. The complex $P^n = K/F$ has one cell in each dimension. The incidence function given on pages 36 and 37 of Chapter II is invariant under F.

2.4. LEMMA. Let F be a family of identifications for a regular complex K. Then there exists an incidence function on K that is invariant under F.

We construct the function α by induction on successive skeletons of K. The argument resembles the proof of II.5.6.

For $n = 1$, choose a 1-cell from each equivalence class of 1-cells. Call the vertices of σ A_σ and B_σ. For each τ equivalent to σ, let f be the unique map in F that carries $\bar{\tau}$ to $\bar{\sigma}$. (Note that we allow $\tau = \sigma$, in which case f is the identity.) Define $A_\tau = f^{-1}A_\sigma$ and $B_\tau = f^{-1}B_\sigma$, and set $[\tau:A_\tau] = -1$ and $[\tau: B_\tau] = 1$. All other incidence numbers involving cells of K_1 we set equal to zero. It is easy to check that α as defined is an incidence function on K_1 which is invariant under F.

Suppose now that α has been defined as an invariant incidence function on K_{q-1}. We devote the next few paragraphs to showing that α can be extended to an invariant function on K_q.

Choose a preferred cell from each equivalence class of q-cells. The union of K_{q-1} and the preferred cells is a subcomplex L of K, and so by II.5.6 α can be extended over L. Let σ be a q-cell of K-L. Let τ be the unique preferred cell equivalent to σ, and let $f: \bar{\sigma} \longrightarrow \bar{\tau}$ be the unique identification. If ρ is a face of σ, define $[\sigma:\rho]_\alpha = [\tau:f\rho]_\alpha$. If ρ is not

a face of σ, set $[\sigma: \rho]_\alpha = 0$. We assert that the function α thus defined is an incidence function on K_q invariant under F. Properties (i) and (ii) of II.1.8 are clearly satisfied by α. Before verifying property (iii) we show that α is invariant under F. As α is invariant on cells of K_{q-1} by the inductive hypothesis, we need only check incidence numbers involving q-cells of K. Let σ_1 and σ_2 be q-cells of K with an identification $g: \bar{\sigma}_1 \longrightarrow \bar{\sigma}_2$. The unique preferred cell τ equivalent to σ_1 is also equivalent to σ_2, and so we have identifications $f_i: \bar{\sigma}_i \longrightarrow \bar{\tau}$, $i = 1,2$. By the remark on page 67, we have $f_1 = f_2 g$. Let ρ be a face of σ_1. Then

$$\begin{aligned} [\sigma_1:\rho]_\alpha &= [\tau:f_1\rho]_\alpha && \text{by construction} \\ &= [\tau:f_2 g\rho]_\alpha && \text{since } f_1 = gf_2 \\ &= [\sigma_2:g\rho]_\alpha && \text{by construction.} \end{aligned}$$

Thus α is invariant under F.

To show that α satisfies property (iii) of II.1.8, let σ be a q-cell of K-L, and let $f: \bar{\sigma} \longrightarrow \bar{\tau}$ identify σ with a preferred cell τ. Suppose ρ_1 and ρ_2 are (q-1)-dimensional faces of σ with a common (q-2)-dimensional face ρ_3. Then $f\rho_1$ and $f\rho_2$ are (q-1)-dimensional faces of $f\sigma = \tau$ with the common (q-2)-dimensional face $f\rho_3$. Thus

$$\begin{aligned} &[\sigma:\rho_1]_\alpha[\rho_1:\rho_3]_\alpha + [\sigma:\rho_2]_\alpha[\rho_2:\rho_3]_\alpha \\ &\qquad = [f\sigma:f\rho_1]_\alpha[f\rho_1:f\rho_3]_\alpha + [f\sigma:f\rho_2]_\alpha[f\rho_2:f\rho_3]_\alpha \end{aligned}$$

because α is invariant under F. The second expression is zero because α is an incidence function for L, and τ is a cell of L. The proof of 2.3 is complete.

Let (K,F) be a complex with identifications, and let α be an incidence function on K that is invariant under F. For each integer q we define $C_q^\alpha(K/F)$ to be the free abelian group on the q-cells of K/F.

For each q, the map $s:|K| \longrightarrow K/F$ induces a homomorphism

$$s_{\#q}: C_q^\alpha(K) \longrightarrow C_q^\alpha(K/F)$$

defined by

$$s_{\#q}(\Sigma\, a_i\sigma_i^q) = \Sigma\, a_i s(\sigma_i^q) .$$

A boundary operator $\partial_q': C_q^\alpha(K/F) \longrightarrow C_{q-1}^\alpha(K/F)$ is defined by

$$\partial_q' s\sigma = s_{\#q-1}\partial_q^\alpha \sigma .$$

To show that ∂_q' is well-defined, let $f: \bar{\sigma} \longrightarrow \bar{\tau}$ be in F. Then

$$\begin{aligned}\partial_q' s\sigma &= s_{\#q-1}\partial_q^\alpha\sigma \\ &= s_{\#q-1} \sum_{\rho<\sigma} [\sigma:\rho]_\alpha \rho \\ &= s_{\#q-1} \sum_{\rho<\sigma} [f\sigma:f\rho](f\rho) .\end{aligned}$$

The last equality holds because α is invariant under F, and because $s_\#\rho = s_\# f\rho$ for $\rho < \sigma$. As ρ varies over the faces of σ, $f\rho$ varies over the faces of τ, so the last expression equals $s_{\#q-1}\partial_q^\alpha \tau = \partial' s\tau$.

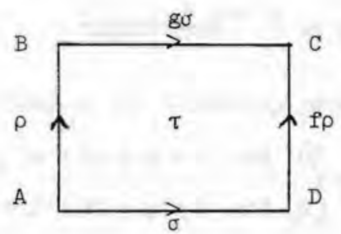
One easily verifies that $\partial'\partial' = 0$. Thus $\{C_q^\alpha(K/F), \partial_q'\}$ is

a chain complex, which we shall denote by $C^\alpha(K/F)$. Note that $s_\#: C^\alpha(K) \longrightarrow C^\alpha(K/F)$ is a chain map.

The q-th homology group of $C^\alpha(K/F)$ is denoted by $H_q^\alpha(K/F)$. In the examples which follow, we will anticipate the verifications in Chapter IX and refer to $H_q^\alpha(K/F)$ as the q-th homology group of the space K/F.

3. The homology of a torus

We begin where Example 2.1 left off, with the identifications and orientation exhibited in the diagram below:



The complex K/F has four cells: sA, $s\sigma$, $s\rho$ and $s\tau$. Each of these cells is a cycle in the chain complex $C^\alpha(K/F)$:

$$\begin{aligned}\partial_2'(s\tau) &= s_\#\partial\tau = s_\#(\sigma + f\rho - g\sigma - \rho) = 0 \\ \partial_1'(s\sigma) &= s_\#\partial\sigma = s_\#(D-A) = 0 \\ \partial_1'(s\rho) &= s_\#\partial\rho = s_\#(B-A) = 0 .\end{aligned}$$

Thus ∂' is trivial, so that $H(K/F) \approx C^\alpha(K/F)$ and

$$H_2(T^2) \approx Z , \quad H_1(T^2) \approx Z \oplus Z , \quad \text{and} \quad H_0(T^2) \approx Z .$$

4. The homology and cohomology of P^n

We start with Example 2.2. The complex $P^n = K/F$ has one cell se_q in each dimension, so that $C_q(P^n) \approx Z$ for $0 \leq q \leq n$. The boundary operator ∂' is determined by

$$\partial'(se_{2i}) = s_{\#}\partial e_{2i} = s_{\#}(e_{2i-1} + Te_{2i-1}) = 2se_{2i-1}$$

$$\partial'(se_{2i+1}) = s_{\#}\partial e_{2i+1} = s_{\#}(Te_{2i} - e_{2i}) = 0 .$$

Schematically the chain complex $C^{\alpha}(P^n)$ can be written

$$\begin{array}{ccccccccc} C_0 & & C_1 & & C_2 & & C_3 & & \cdots \\ Z & \xleftarrow{0} & Z & \xleftarrow{\theta_2} & Z & \xleftarrow{0} & Z & \xleftarrow{\theta_2} & \end{array}$$

where θ_2 is multiplication by 2. Accordingly

$$H_0(P^n) \approx Z$$

$$H_q(P^n) \approx \begin{cases} 0 & \text{for } 0 < q = 2i < n \\ Z_2 & \text{for } 0 < q = 2i+1 < n \end{cases}$$

$$H_n(P^n) \approx \begin{cases} 0 & n \text{ even} \\ Z & n \text{ odd.} \end{cases}$$

The homology sequence for P^n is

dimension	0	1	2	3	4	5	...	n
$H_q(P^n)$	Z	Z_2	0	Z_2	0	Z_2	...	$\{\begin{smallmatrix} 0 & n \text{ even} \\ Z & n \text{ odd}\end{smallmatrix}$

It follows from II.6.7 that the sequence of homology groups of P^n with coefficients in Z_2 is

dimension	0	1	2	3	4	...	n
$H_q(P^n:Z_2)$	Z_2	Z_2	Z_2	Z_2	Z_2		Z_2

The cochain complex obtained from $C^{\alpha}(P^n:Z)$ is

dimension	0	1	2	3	...	n
$C^q(P^n)$	Z	Z	Z	Z		Z

Here,

$$\delta'_q = (-1)^q \cdot \operatorname{Hom}(\partial'_q, 1) = \begin{cases} 0 & q \text{ odd} \\ \theta_2 & q \text{ even} \end{cases}$$

Thus,

$$H^0(P^n) \approx Z$$

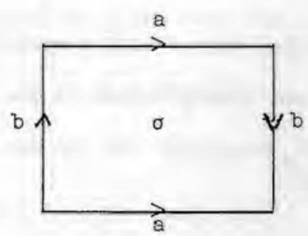
$$H^q(P^n) \approx \begin{cases} 0 & \text{for odd } q < n \\ Z_2 & \text{for even } q < n \end{cases}$$

$$H^n(P^n) \approx \begin{cases} Z_2 & \text{for even } n \\ Z & \text{for odd } n \end{cases}$$

Exercise. Compute the cohomology groups of P^n with coefficients in Z_2.

5. The homology of the Klein bottle

We construct the Klein bottle from a disk, with the identifications and orientation shown in the following diagram:



The non-trivial chain groups are

$$C_0 \approx Z , \qquad C_1 \approx Z \oplus Z \quad \text{and} \quad C_2 \approx Z .$$

The boundary operator is given by

$$\partial_1'(sa) = s_{\#}\partial a = 0$$
$$\partial_1'(sb) = s_{\#}\partial b = 0$$
$$\partial_2'(s\sigma) = s_{\#}\partial\sigma = s_{\#}(a-a-b-b) = -2sb$$

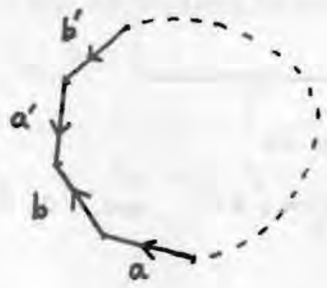
Therefore,

$$H_0 \approx Z\ , \quad H_1 \approx Z_2 \oplus Z \quad \text{and} \quad H_2 \approx 0\ .$$

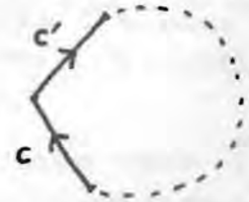
6. Compact 2-manifolds without boundary

We will assume that the reader is familiar with the fact that each compact 2-manifold M without boundary can be represented as a 2-sphere with $0 \leq h$ handles and $0 \leq k \leq 2$ crosscaps. If $M \neq S^2$, then M can, in fact, be obtained from a closed 2-cell $\bar{\sigma}^2$ by subdividing $\dot{\sigma}^2$ into an even number of edges and by identifying these edges in pairs in an appropriate fashion. Such a subdivision determines a regular complex on $\bar{\sigma}^2$, and the identifications are carried out in such a way that M is a complex with identifications. A good description of the identification process may be found in Chapter 6 of Seifert and Threlfall's Lehrbuch der Topologie.

In the process described there, each sequence of four consecutive edges, oriented and identified as in the following diagram,



produces a handle. Each pair of consecutive edges oriented and identified as in the following diagram



produces a crosscap. Each M is obtained by subdividing $\dot{\sigma}^2$ into h handle-producing sequences and k crosscap-producing sequences (for some h , and some $k \leq 2$).

We label the edges of the i-th handle-producing sequence with the letters a_i, b_i, a_i', b_i' , and we label the edges of the i-th crosscap-producing sequence with the letters c_i, c_i' . Let $s\colon \bar{\sigma}^2 \longrightarrow M$ denote the projection. Then M has one 2-cell, $2h+k$ 1-cells, and one vertex. Note that by attaching a circular indicatrix to σ^2 , and using the arrows already given, we obtain an incidence function on $\bar{\sigma}^2$ which is invariant under the identifications.

Computation of $H_*(M)$

Case 1): $k = 0$. We have

$$\partial' s\sigma^2 = s_{\#}\partial\sigma^2 = s_{\#}(\Sigma a_i - \Sigma a_i' + \Sigma b_i - \Sigma b_i') = 0\ .$$

Also, $\partial' sa_i = \partial sb_i = 0$. Thus $H_2(M) \approx Z$, and $H_1(M)$ is isomorphic to a direct sum of $2h$ copies of Z .

Case 2): $k = 1$. We have

$$\partial' s\sigma^2 = s_{\#}\partial\sigma^2 = s_{\#}(\Sigma a_i - \Sigma a_i' + \Sigma b_i - \Sigma b_i' + c_1 + c_1') = 2sc_1\ .$$

Thus $H_2(M) \approx 0$. As in Case 1, every 1-cell of M is a cycle. The boundaries are generated by $2s(c_1)$, and so $H_1(M)$ is isomorphic to the direct sum of $2h$ copies of Z and a single copy of Z_2 .

Case 3: $k = 2$. We have

$$\partial s\sigma^2 = s_{\#}\partial\sigma^2 = 2sc_1 + 2sc_2 .$$

Again $H_2(M) \approx 0$. As before, every one-cell of M is a cycle. Thus $Z_1(M)$ is free abelian on the $2h+2$ generators

$$sa_1, sb_1, \dots, sa_h, sb_h, sc_1, sc_2 .$$

Another system of generators for $Z_1(M)$ is

$$sa_1, sb_1, \dots, sa_h, sb_h, sc_1, sc_1 + sc_2 .$$

Thus $H_1(M)$ is generated by $2h+2$ elements and has a single relation, that twice $sc_1 + sc_2$ is zero. So $H_1(M)$ is isomorphic to the direct sum of $2h+1$ copies of Z and a single copy of Z_2 .

7. Lens spaces

In this section we define, for each pair (p,q) of relatively prime positive integers, a 3-manifold $L(p,q)$ called a <u>lens space.</u> We compute the homology of lens spaces, using a family of identifications on the 3-sphere.

Let X be a topological space and let G be a group of homeomorphisms of X onto itself. Then G determines an equivalence relation on the points of X , as follows: x and x' in

X are G-equivalent if there exists a g in G so that $g(x) = x'$. It is easy to verify that G-equivalence is an equivalence relation. The identification space whose points are G-equivalence classes is said to be obtained from X by <u>collapsing under the action of</u> G , and is denoted by X/G .

Let K be a regular complex. A homeomorphism f of $|K|$ onto itself is called a <u>(cellular) isomorphism</u> of K if f maps every cell of K onto a cell of K of the same dimension. It is clear that an isomorphism preserves the face relation: if $\sigma < \tau$, then $f\sigma < f\tau$. A group G of homeomorphisms of $|K|$ is called <u>cellular</u> if each g in G is an isomorphism.

Suppose that a cellular group G on $|K|$ satisfies the following property:

(*) If σ is a cell of K , and g in G is such that g maps σ onto σ , then g is the identity.

Then for each g in G , and each σ in K , $g|\overline{\sigma}$ is an identification on K , and the collection of these identifications forms a family F of identifications as is easily verified. The identification space K/F is the space $|K|/G$.

We specialize. Let S^3 be represented as the unit sphere in the complex plane C^2 . That is,

$$S^3 = \{(z_0, z_1) \in C^2 \mid z_0\overline{z}_0 + z_1\overline{z}_1 = 1\} .$$

Let p and q be relatively prime positive integers. Let $\lambda = e^{2\pi i/p}$, and define $T\colon S^3 \longrightarrow S^3$ by

$$T(z_0,z_1) = (\lambda z_0, \lambda^q z_1) .$$

Then T and its iterates $T^2, T^3, \dots, T^p = 1$ form a cyclic group G of order p. The collapsed space S^3/G is called the lens space $L(p,q)$. If $q \equiv q' \bmod p$ then $\lambda^q = \lambda^{q'}$ so $T = T'$ and $L(p,q) = L(p,q')$. In the special case $p = q = 1$, we have $\lambda = 1$ and $L(1,1) = S^3$. If $p = 2$ and $q = 1$, then $\lambda = -1$, and T is the antipodal map. Thus $L(2,1)$ is P^3.

We compute the homology of $L(p,q)$ by constructing a regular complex on S^3 for which G is cellular and satisfies condition (*).

Let p and q be fixed. Set

$\sigma^0 = (0,1)$

$\sigma^1 = \{(0,z_1) \in S^3 | 0 < \arg z_1 < 2\pi/p\}$

$\sigma^2 = \{(z_0,z_1) \in S^3 | z_0 \neq 0 \text{ and } \arg z_0 = 0\}$

$\sigma^3 = \{(z_0,z_1) \in S^3 | z_0 \neq 0 \text{ and } 0 < \arg z_0 < 2\pi/p\}$.

To show that the $T^n\sigma^k$ for $k = 0,1,2,3$ and $n = 0,1,\dots,p-1$, are the cells of a regular complex on S^3 with $\dim \sigma^k = k$, we shall appeal to Exercise 5 of I.1.2. It is clear that the $T^n\sigma^k$ are disjoint and that their union is S^3. Also, the $T^n\sigma^k$ with $k = 0$ and $k = 1$ clearly form a regular complex on the circle $z_0 = 0$.

The closed cell $\bar{\sigma}^2$ is the intersection of S^3 with the set

$$P = \{(z_0,z_1) \in C^2 | z_0 = 0 \text{ or } \arg z_0 = 0\} .$$

If we set $z_0 = x_0 + iy_0$ and regard C^2 as Euclidean 4-space, then P is the half-2-space $x_0 \geq 0$, $y_0 = 0$. The intersection of the whole 3-space $y_0 = 0$ with S^3 is a 2-sphere. Thus $P \cap S^3$ is a closed 2-disk, and $(\bar{\sigma}^2, \dot{\sigma}^2)$ is a regular 2-cell. A similar argument holds for $T^n\bar{\sigma}^2$ with $n = 1,\dots,p-1$.

The subset

$$E^3 = \{(z_0,z_1) \in S^3 | z_0 = 0 \text{ or } 0 \leq \arg z_0 \leq \pi\}$$

of S^3 is a hemisphere and thus a closed disk. There is an obvious map $f: E^3 \longrightarrow \bar{\sigma}^3$ defined as follows. If (z_0,z_1) lies in E^3 then $z_0 = \rho e^{i\theta}$ with $0 \leq \theta \leq \pi$. Define

$$f(z_0,z_1) = (\rho e^{i\theta/p}, z_1) .$$

Then f is a homeomorphism of E^3 onto $\bar{\sigma}^3$ which carries the boundary sphere onto $\dot{\sigma}^3$. Thus $(\bar{\sigma}^3, \dot{\sigma}^3)$, and similarly each $T^n(\bar{\sigma}^3, \dot{\sigma}^3)$ for $n = 1,\dots,p-1$, is a regular 3-cell.

Finally, it is an easy matter to show that each $T^n\dot{\sigma}^k$ lies in a union of cells of dimension less than k. By the conclusion of I.1, Exercise 5, then, the $T^n\sigma^k$ are the cells of a complex on S^3. The complex is regular because the cells are regular.

Since $T^m(T^n\sigma^k) = T^{m+n}\sigma^k$, it follows that G is a group of isomorphisms of the cellular structure. Moreover $T^m(T^n\sigma^k) = T^n\sigma^k$ if and only if $m \equiv 0 \bmod p$. Hence G satisfies condition * above.

An incidence function α that is invariant under G is given by:

$[T^n\sigma^3 : T^n\sigma^2] = 1$ and $[T^n\sigma^3 : T^{n+1}\sigma^2] = -1$ for $n = 0,1,\ldots,p-1$,
$[T^n\sigma^2 : T^m\sigma^1] = 1$ for all m,n ,
$[T^n\sigma^1 : T^n\sigma^0] = 1$ and $[T^n\sigma^1 : T^{n+1}\sigma^0] = -1$ for $n = 0,1,\ldots,p-1$.

(All other incidence numbers are set equal to zero.)

The collapsed space S^3/G has one cell in each dimension. If s denotes the projection of S^3 onto S^3/G then the boundaries in $C^\alpha(S^3/G) = C^\alpha(L(p,q))$ are given by

$$\partial' s\sigma^3 = s_\#\partial\sigma^3 = s_\#(\sigma^2 - T\sigma^2) = 0$$
$$\partial s\sigma^2 = s_\#\partial\sigma^2 = s_\#(\Sigma_{i=0}^{p-1} T^i\sigma^1) = ps\sigma^1$$
$$\partial s\sigma^1 = s_\#\partial\sigma^1 = s_\#(\sigma^0 - T\sigma^0) = 0$$

Thus the homology of $L(p,q)$ is given by

$$H_0 \approx Z , \quad H_1 \approx Z_p , \quad H_2 \approx 0 \quad \text{and} \quad H_3 \approx Z .$$

Note that $H_*(L(p,q))$ is independent of q .

For an alternative description of lens spaces, and for more information, the reader should consult Hilton and Wylie's <u>Homology Theory</u>, page 223.

8. Complex projective n-space

In this section we anticipate later chapters in order to state a result which allows us to compute the homology groups of certain spaces very quickly.

In Chapter VII, we define homology groups for arbitrary spaces in such a way that whenever a space X carries a regular

complex the homology groups $H_q(X)$ defined for X agree with the cellular homology groups of the complex. But the homology groups of X need not be computed from a regular complex on X . They may in certain cases be computed from an irregular complex on X .

5.1. THEOREM. <u>Let</u> X <u>be a space that carries a complex</u> K <u>with the property that for each</u> q <u>the topological boundary of each q-cell is contained in</u> K_{q-2} . <u>Then for each</u> q , $H_q(X)$ <u>is isomorphic to the free abelian group on the q-cells of</u> K .

For example, suppose that K is the irregular complex on S^n given in I.2.2. Then 5.1 gives the homology of S^n immediately.

The proof of 5.1 appears in Chapter IX.

Let C^n denote complex n-space, and let S^{2n+1} be represented as the unit sphere in C^{n+1} . Each λ in $S^1 \subset C$ defines an automorphism of S^{2n+1} given by scalar multiplication

$$\lambda(z_0,\ldots,z_n) = (\lambda z_0,\ldots,\lambda z_n) .$$

The space S^{2n+1}/S^1 is called <u>complex projective n-space</u>, and is denoted by CP^n . The space CP^0 consists of a single point; CP^1 is a 2-sphere. The projection map is denoted by $s\colon S^{2n+1} \longrightarrow CP^n$.

For $k < n$, we identify C^{k+1} with the subspace $\{(z_0,\ldots,z_k,0,\ldots,0)\}$ of C^{n+1} . Then $S^{2k+1} = C^{k+1} \cap S^{2n+1}$, and the inclusion $S^{2k+1} \subset S^{2n+1}$ is consistent with the action of S^1 . Thus we have a diagram of inclusions and projections:

$$\begin{array}{ccccccccc} S^1 & \subset & S^3 & \subset & S^5 & \subset \cdots \subset & S^{2n+1} \\ \downarrow s & & \downarrow s & & \downarrow s & & \downarrow s \\ CP^0 & \subset & CP^1 & \subset & CP^2 & \subset \cdots \subset & CP^n \end{array} .$$

For $k > 0$ the subspace

$$E^{2k} = \{(z_o,\ldots,z_k) \in S^{2k+1} | z_k \text{ is real } \geq 0\}$$

is a closed 2k-cell with boundary S^{2k-1}. We claim that $s|E^{2k}$ is a relative homeomorphism of (E^{2k}, S^{2k-1}) onto (CP^k, CP^{k-1}). It is sufficient to show that for each point k in $S^{2k+1} - S^{2k-1}$ the open cell $E^{2k} - S^{2k-1}$ contains exactly one point equivalent to x under the action of S^1, and this we show as follows: Let $x = (z_o,\ldots,z_k,0,\ldots,0)$. Since x lies in $S^{2k+1} - S^{2k-1}$, $z_k \neq 0$. For $\lambda = |z_k|/z_k$ we have

$$\lambda x = (\lambda z_o,\ldots,\lambda z_{k-1},|z_k|,0,\ldots,0) ,$$

which lies in $E^{2k} - S^{2k-1}$. Further, if $|\lambda'| = 1$, and if $\lambda' z_k$ is real and non-negative in

$$\lambda' x = (\lambda' z_o,\ldots,\lambda' z_k,0,\ldots,0) ,$$

then $\lambda'/\lambda = \lambda' z_k/|z_k|$ is real and non-negative and hence equal to 1. So $\lambda = \lambda'$, and x is the unique point in $S^{2k+1} - S^{2k-1}$ equivalent to x.

Now let $\sigma_o = CP^0$, and for each $k > 0$ let $\sigma_k = CP^k - CP^{k-1}$. For each $k \geq 0$, σ^k is a 2k-cell, and each

point CP^n lies in precisely one σ^k. Furthermore, since $s|E^{2k}: (E^{2k}, S^{2k-1}) \longrightarrow (CP^k, CP^{k-1})$ is a relative homeomorphism the cells σ_k satisfy the conditions of Exercise 5 of I.1.2. They are, therefore, the cells of an irregular complex K on CP^n with skeletons

$$|K_q| = \begin{cases} CP^{q/2} & q \text{ even} \\ CP^{(q-1)/2} & q \text{ odd.} \end{cases}$$

We may now apply 5.1 to obtain

$$H_q(CP^n) \approx \begin{cases} Z & q = 2k \leq 2n \\ 0 & \text{otherwise.} \end{cases}$$

CHAPTER IV

COMPACTLY GENERATED SPACES AND PRODUCT COMPLEXES

In this chapter we introduce the concepts of compactly generated spaces and of the product of complexes. Sections 1-6, 8 are based upon notes prepared by Martin Arkowitz for lectures he gave in Professor Steenrod's course in the fall of 1963.

1. Categories

1.1. DEFINITION. A <u>category</u> $\mathcal{C}$ is a non-empty class of objects, together with a set $M(A,B)$ for every two objects A and B in $\mathcal{C}$. For each triple A,B,C of objects there exists a function from the cartesian product $M(A,B) \times M(B,C)$ to the set $M(A,C)$. If $f \in M(A,B)$ and $g \in M(B,C)$, then the image in $M(A,C)$ of $f \times g$ is denoted by $g \circ f$, and is called the <u>composition</u> of f and g. Two conditions are imposed:

1. $f \circ (g \circ h) = (f \circ g) \circ h$.
2. For each object A in $\mathcal{C}$, there exists an element 1_A in $M(A,A)$ such that $f \circ 1_A = 1_B \circ f = f$ for each f in $M(A,B)$.

The set $M(A,B)$ is called <u>the set of morphisms</u> from A to B. We write $f\colon A \longrightarrow B$ to indicate that f is a morphism in $M(A,B)$. When no confusion is likely to result, we shall write gf for $g \circ f$.

1.2. <u>Examples of Categories</u>

The category $\mathcal{S}$ of sets and functions between them. The objects are the sets, the morphisms are the functions, and the composition of two morphisms is the usual composition of functions.

The category $\mathcal{T}$ of all topological spaces (objects) and continuous maps (morphisms) between them. The composition of morphisms is the usual composition of maps.

The category $\mathcal{A}$ of abelian groups (objects) and homomorphisms. The composition of homomorphisms is the usual one.

The category $\mathcal{G}$ of groups and homomorphisms, with the usual composition of homomorphisms.

The category $\mathcal{H}$, in which the objects (X,x_0) are the topological spaces with base point, and the morphisms are <u>homotopy classes</u> of maps $(X,x_0) \longrightarrow (Y,y_0)$.

2. Functors

2.1. DEFINITION. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A functor F from $\mathcal{C}$ to $\mathcal{D}$ is a function that assigns to each object A in $\mathcal{C}$ an object FA in $\mathcal{D}$ and to each morphism $f\colon A \longrightarrow B$ a morphism Ff, so as to satisfy one of the following two sets of conditions:

1. $F(1_A) = 1_{FA}$, $Ff\colon FA \longrightarrow FB$ and $F(f \circ g) = Ff \circ Fg$, or
2. $F(1_A) = 1_{FA}$, $Ff\colon FA \longrightarrow FB$ and $F(f \circ g) = Fg \circ Ff$.

If F satisfies (1) then F is a <u>covariant functor.</u> If F satisfies (2) then F is a <u>contravariant functor.</u>

2.2. <u>Examples of functors</u>

<u>The functor</u> Hom. Let A and G be abelian groups, and let $\mathrm{Hom}(A,G)$ denote the abelian group consisting of all homomorphisms $\mu\colon A \longrightarrow G$, with addition $\mu_1+\mu_2$ defined by

$$(\mu_1+\mu_2)a = \mu_1 a + \mu_2 a .$$

If $\alpha: A' \longrightarrow A$ and $\gamma: G \longrightarrow G'$ are homomorphisms, then the correspondence $\mu \longrightarrow \gamma\mu\alpha$ is a homomorphism

$$\mathrm{Hom}(\alpha,\gamma): \mathrm{Hom}(A,G) \longrightarrow \mathrm{Hom}(A',G') .$$

It is easy to see that

1. $\mathrm{Hom}(1_A,1_G)$ is the identity map of $\mathrm{Hom}(A,G)$.
2. If $\alpha: A' \longrightarrow A$, $\alpha': A'' \longrightarrow A'$, $\gamma: G \longrightarrow G'$ and $\gamma': G' \longrightarrow G''$ are all homomorphisms, then

$$\mathrm{Hom}(\alpha'\alpha,\gamma\gamma') = \mathrm{Hom}(\alpha,\gamma)\mathrm{Hom}(\alpha',\gamma') .$$

Therefore, for a fixed G the correspondence

$$A \longrightarrow \mathrm{Hom}(A,G)$$
$$(\alpha: A' \longrightarrow A) \longrightarrow (\mathrm{Hom}(\alpha,1_G): \mathrm{Hom}(A,G) \longrightarrow \mathrm{Hom}(A',G))$$

is a contravariant functor from $\mathcal{Q}$ to $\mathcal{Q}$. For fixed A, the correspondence

$$G \longrightarrow \mathrm{Hom}(A,G)$$
$$(\gamma: G \longrightarrow G') \longrightarrow (\mathrm{Hom}(1_A,\gamma): \mathrm{Hom}(A,G) \longrightarrow \mathrm{Hom}(A,G'))$$

is a covariant functor from $\mathcal{Q}$ to $\mathcal{Q}$.

The homology functor. For each space X in $\mathcal{J}$ let $H_n(X)$ be the nth singular homology group of X with respect to some fixed group G. For each map $f: X \longrightarrow Y$ let $H_n(f)$ be the induced homomorphism $f_*: H_n(X) \longrightarrow H_n(Y)$. The properties of homology groups (established later in these notes, and independently of this example)

then guarantee H_n to be a covariant functor from $\mathcal{J}$ to $\mathcal{Q}$.

The cohomology functor H^n . For each space X in $\mathcal{J}$ let $H^n(X)$ be the nth singular cohomology group of X with respect to some fixed group G . For each map $f: X \longrightarrow Y$, let $H^n(f)$ be the induced homomorphism $f^*: H^n(Y) \longrightarrow H^n(X)$. The properties to be established later about cohomology groups then guarantee H^n to be a contravariant functor from $\mathcal{J}$ to $\mathcal{Q}$.

The fundamental group functor. For each pair (X,x_o) in $\mathcal{H}$ define $\Pi_1(X,x_o)$ to be the fundamental group of X at x_o . For each homotopy class $\{f\}$ of maps from (X,x_o) to (Y,y_o) , define $\Pi_1 f$ to be the homomorphism $f_\#: \Pi_1(X,x_o) \longrightarrow \Pi_1(Y,y_o)$ induced by f . It is known that $f_\#$ depends only on the class of f , that $id_\#$ is the identity and that $(fg)_\# = f_\# g_\#$. Therefore Π_1 is a covariant functor from $\mathcal{H}$ to $\mathcal{G}$.

Exercises.

1. If G is a group, let $[G,G]$ denote the commutator subgroup. Show that the assignment to each group G of the abelian group $G/[G,G]$ induces a functor from $\mathcal{G}$ to $\mathcal{Q}$.
2. Make up a functor using $\otimes$.

3. Products in a category

3.1. DEFINITION. Let $\{A_\alpha\}$ be a collection of objects from a category $\mathcal{C}$. An object P of $\mathcal{C}$ is said to be the product of the A_α if there exists a collection $\{p_\alpha: P \longrightarrow A_\alpha\}$ of morphisms (called projections) with the following property: Given any object

X in $\mathcal{C}$ and any collection of morphisms $\{f_\alpha: X \longrightarrow A_\alpha\}$, there exists a unique morphism $f: X \longrightarrow P$ such that $p_\alpha f = f_\alpha$, for each α.

3.2. DEFINITION. A category $\mathcal{C}$ is a category with products if the product of any family $\{A_\alpha\}$ of objects of $\mathcal{C}$ exists in $\mathcal{C}$. If the product of any finite family of objects exists in $\mathcal{C}$, then $\mathcal{C}$ is called a category with finite products.

In a category $\mathcal{C}$ a morphism $f: A \longrightarrow B$ is called an equivalence if there exists $f': B \longrightarrow A$ in $\mathcal{C}$ such that $ff' = 1_B$ and $f'f = 1_A$. If $f: A \longrightarrow B$ is an equivalence we call A and B equivalent. In $\mathcal{J}$, for example, "equivalent" means "homeomorphic."

Exercise. Show that the product (if it exists) of a collection $\{A_\alpha\}$ of objects in a category is unique up to equivalence.

The uniqueness of products up to equivalence allows us to write ΠA_α for P.

Exercise. Give examples of products, using categories defined in 1.2. Show that $\mathcal{J}$ is a category with products, with P the cartesian product with the product topology.

4. Compactly generated spaces

4.1. DEFINITION. A space X is compactly generated if it has the following property: a set A is closed in X if and only if $A \cap H$ is closed in H for each compact subset H of X.

Clearly, "closed" may be replaced by "open" in 4.1.

Exercises.

1. Show that if K is a complex then K is compactly generated. (Hint: use I.4.4.)

2. Show that a function from one compactly generated space into another is continuous if and only if its restriction to each compact set is continuous.

4.2. PROPOSITION. Each locally compact topological space is compactly generated.

Proof. Let X be a topological space. If $F \subset X$ is closed and $H \subset X$ is compact, then $F \cap H$ is closed in H by definition of the relative topology of H.

To complete the proof, we show that if $F \subset X$ is not closed, then there exists a compact subset H of X with $H \cap F$ not closed.

Let x be a limit point of F that does not lie in F. Because X is locally compact there exists a compact neighborhood H of x. The set $F \cap H$ is not closed in H because it does not contain its limit point x.

Remarks. For Hausdorff spaces, 4.1 can be stated in a slightly simpler form because "closed in H" can be replaced by "closed" (meaning "closed in X"). With compactly generated space defined as it is in 4.1, Proposition 4.2 holds for any X in $\mathcal{J}$. To preserve this generality *we shall assume* in sections 5 and 6 which follow that X is any space in $\mathcal{J}$. The results of these sections are of course applicable to Hausdorff spaces. In later sections of this chapter, as well as in the other chapters of these notes, we shall revert to our assumption that spaces are Hausdorff spaces.

4.3. PROPOSITION. If X is any topological space satisfying the first axiom of countability, (for example, if X is a metric space) then X is compactly generated.

Proof. Let A be a subset of X such that for any compact subset H of X, $A \cap H$ is closed in H. Let x be a limit point of A. Since X satisfies the first axiom of countability, there exists a sequence $\{x_n\}$ of points of $A - \{x\}$ which converges to x. The set Y consisting of the points of the sequence $\{x_n\}$ together with x is a compact subset of X. Therefore $A \cap Y$ is closed in Y. But A already contains the sequence $\{x_n\}$, whose closure in Y includes x. Thus x is in A and A is closed.

5. The functor k

For each space X in $\mathcal{J}$, let $k(X)$ denote the space that is the underlying set X with the topology defined by

> A is closed (open) if and only if $A \cap H$ is closed (open) in H for each compact subspace H of X.

The topology of $k(X)$ is called the *weak topology with respect to compact subsets of* X. The identity map $k(X) \longrightarrow X$ is continuous because each set that is closed in X is closed in $k(X)$. If X is a Hausdorff space, then so is $k(X)$.

5.1. LEMMA. *X and $k(X)$ have the same compact sets.*

Proof. If H is compact in $k(X)$ then H is compact in X because the topology of $k(X)$ is finer than that of X.

Let H be a set that is compact in X, and let $\{U_\alpha\}$ be a covering of H by sets open in $k(X)$. By definition of the topology of $k(X)$, each $U_\alpha \cap H$ is open in H. Thus finitely many of the $U_\alpha \cap H$ cover H, so that H is compact in $k(X)$.

The following two corollaries are immediate.

5.2. COROLLARY. *For each X in $\mathcal{J}$, $k(X)$ is compactly generated.*

5.3. COROLLARY. *X is compactly generated if and only if X and $k(X)$ are homeomorphic.*

Let X and Y be spaces. If f is a function from X to Y then f is also a function from $k(X)$ to $k(Y)$. If $f: X \longrightarrow Y$ is continuous, then $f: k(X) \longrightarrow k(Y)$ is continuous, as follows: Since f is continuous on X, the restriction of f to each compact subset of X is continuous. Thus by 5.1 the restriction of f to each compact subset of $k(X)$ is continuous, and by Exercise 2 of the preceding section $f: k(X) \longrightarrow k(Y)$ is continuous.

We shall denote by $\mathcal{T}_k$ the category whose objects are compactly generated spaces and whose morphisms are continuous maps.

5.4. DEFINITION. The functor $k: \mathcal{T} \longrightarrow \mathcal{T}_k$ is the functor that assigns to each X the space $k(X)$ and to each map $f: X \longrightarrow Y$ the map $f: k(X) \longrightarrow k(Y)$.

6. Products in $\mathcal{T}_k$

6.1. PROPOSITION. <u>The category $\mathcal{T}_k$ is a category with products.</u>

<u>Proof.</u> If $\{X_i\}_{i\varepsilon I}$ (I an arbitrary index set) is a collection of objects of $\mathcal{T}_k$, then $\prod^{(k)}_{i\varepsilon I} X_i$, the product of the X_i in $\mathcal{T}_k$, is defined to be the space $k(\prod_{i\varepsilon I} X_i)$, where $\prod_{i\varepsilon I} X_i$ is the product of the X_i in $\mathcal{T}$. We show that $\prod^{(k)}_{i\varepsilon I} X_i$ is a product in $\mathcal{T}_k$. If $f_i: Z \longrightarrow X_i$ are morphisms in $\mathcal{T}_k$, then the f_i are also morphisms from Z to X_i in $\mathcal{T}$. Since $\mathcal{T}$ is a category with products, there exists a unique morphism $f: Z \longrightarrow \Pi X_i$ for which $p_i f = f_i$ for each i in I , where the p_i are the projections $\Pi X_i \longrightarrow X_i$. The functor k then furnishes us with morphisms $\{kp_i: k(\Pi X_i) \longrightarrow k(X_i)\}$ and with a unique morphism $kf: Z \longrightarrow k(\Pi X_i)$ such that $kp_i \circ kf = kf_i$.

$$\begin{tikzcd}
kZ \arrow[r, "kf"] & k(\Pi X_i) \arrow[r, "kp_i"] \arrow[d] & kX_i \arrow[d, equal] \\
Z \arrow[r, "f"] & \Pi X_i \arrow[r, "p_i"] & X_i
\end{tikzcd}$$

Thus, $k(\Pi X_i)$ is the product of the X_i in $\mathcal{T}_k$.

If X and Y are compactly generated spaces then their cartesian product $X \times Y$ in $\mathcal{T}$ is not necessarily compactly generated, as an example of C. H. Dowker [1] shows (see Section 8 of this chapter). In contrast, their product $X \times_k Y = k(X \times Y)$ in $\mathcal{T}_k$ is (by definition) always compactly generated. It will be important to distinguish between the two kinds of product when we consider products of complexes.

7. The product of two complexes

Let K and L be complexes. We define the <u>product</u> $K \times L$ of K and L to be the space $|K| \times_k |L|$ together with the subspaces

$$(K \times L)_n = \bigcup_{i=o}^{n} (K_i \times_k L_{n-i}), \qquad n = 0,1,2,\ldots$$

In this section we show that $K \times L$ is a complex. In Section 8 we discuss conditions on K and L which insure that the cartesian product $|K| \times |L|$ be a complex. We also given an example of two complexes K and L whose cartesian product is not compactly generated, and thus not a complex.

7.1. LEMMA. <u>The space $|K| \times_k |L|$ and the skeletons $(K \times L)_n$ satisfy conditions 1) through 5) of Definition I.1.1.</u>

<u>Proof.</u> The spaces $(K \times L)_n$ clearly form an ascending sequence of closed subsets of $|K| \times_k |L|$ whose union is $|K| \times_k |L|$. we have

$$(K \times L)_n - (K \times L)_{n-1} = \bigcup_{i=o}^{n} [(K_i - K_{i-1}) \times_k (L_{n-i} - L_{n-i-1})] .$$

Thus the components of $(K \times L)_n - (K \times L)_{n-1}$ are products of the form $\sigma^i \times \tau^{n-i}$, where σ^i is an i-cell of K and τ^{n-i} is an (n-i)-cell of L. To show that each cell $\sigma^i \times \tau^{n-i}$ is open in $(K \times L)_n$, we observe that

$$(K \times L)_n - \sigma^i \times \tau^{n-i} = (\bigcup_{j \neq i} (K_j \times_k L_{n-j})) \cup ((K_i - \sigma^i) \times_k L_{n-i})$$
$$\cup (K_i \times_k (L_{n-i} - \tau^{n-i})) .$$

The right hand side is the union of finitely many closed sets and so is closed.

Relative homeomorphisms for the cells of $K \times L$ are easily obtained. If $f: (E^i, S^{i-1}) \to (\bar{\sigma}, \dot{\sigma})$ and $g: (E^{n-i}, S^{n-i-1}) \to (\bar{\tau}, \dot{\tau})$ are given, then

$$f \times g: (E^i \times E^{n-i}, E^i \times S^{n-i-1} \cup S^{i-1} \times E^{n-i}) \to (\bar{\sigma} \times \bar{\tau}, \bar{\sigma} \times \dot{\tau} \cup \dot{\sigma} \times \bar{\tau}) .$$

Since the boundary $\sigma \times \tau$ is $\overline{\sigma \times \tau} - \sigma \times \tau = \bar{\sigma} \times \dot{\tau} \cup \dot{\sigma} \times \bar{\tau}$, $f \times g$ is the required relative homeomorphism (see the exercise below). This completes the proof of 7.1.

<u>Exercise.</u> Show that the product of two relative homeomorphisms is a relative homeomorphism.

7.2. LEMMA. If K and L are complexes, then each compact subset of $|K| \times_k |L|$ lies in a union of finitely many closed cells of $K \times L$.

<u>Proof.</u> Let H be a compact subset of $|K| \times_k |L|$. Let p and q be the projections of $|K| \times_k |L|$ onto $|K|$ and $|L|$ respectively. Then pH is compact in $|K|$. By I.4.4, pH is contained in a union of finitely many cells of K, and hence in the union of their closures: $pH \subseteq \bigcup_{i=1}^{m} \bar{\sigma}^i$. Similarly, qH lies in a finite union $\bigcup_{j=1}^{n} \bar{\tau}^j$ of closed cells of L. Thus H lies in the finite union $\bigcup_{i,j} \bar{\sigma}^i \times \bar{\tau}^j$ of closed cells of $K \times L$.

7.3. COROLLARY. Each compact subset of $(K \times L)_n$ lies in a union of finitely many closed cells of $(K \times L)_n$.

<u>Proof.</u> Let H be a compact subset of $(K \times L)_n$. Since $(K \times L)_n$ is a Hausdorff space, H is closed. Set $H_i = H \cap (K_i \times_k L_{n-i})$. Then H_i is closed in H and so is a compact subset of $K_i \times_k L_{n-i}$. By I.4.3, K_i and L_{n-i} are complexes. Thus 7.2 implies that H_i is contained in a union of finitely many closed cells of $K_i \times_k L_{n-i}$. Thus $H = \bigcup H_i$ is contained in a union of finitely many closed cells of $(K \times L)_n$.

Recall that a quasi complex Q consists of a Hausdorff space $|Q|$ and a sequence of subspaces Q_i satisfying the first six conditions of I.1.1.

7.4. LEMMA. <u>Suppose that the Hausdorff space $|Q|$ together with the subspaces Q_i, $i = 0,1,\ldots$, satisfies conditions 1) through 5) of I.1.1. Suppose that $|Q|$ is compactly generated and that, in addition, each compact subset of $|Q|$ is contained in a finite union of closed cells. Then Q is a quasi complex.</u>

<u>Proof.</u> Let A be a subset of $|Q|$ which meets each closed cell of Q in a closed set. We have to show that A is closed. That is, since $|Q|$ is compactly generated we have to show that A meets each compact set in a closed set. Let H be

a compact subset of $|Q|$. Then $H \subset \bigcup_{i=o}^{n} \bar{\sigma}^i$ where each σ^i is a cell of Q . Then

$$A \cap H = A \cap H \cap (\bigcup_{i=o}^{n} \bar{\sigma}^i) = H \cap (\bigcup_{i=o}^{n} (A \cap \bar{\sigma}^i)) .$$

Since $A \cap \bar{\sigma}^i$ is closed for each i , $A \cap H$ is closed. Thus A is closed in $|Q|$, which completes the proof.

7.5. LEMMA. <u>A closed subspace of a compactly generated space is compactly generated.</u>

<u>Proof.</u> Let X be compactly generated, and let A be a closed subset of X . Suppose that B is a subset of A such that $B \cap C$ is closed in C for every compact set $C \subset A$. We show that B is closed in X . Let H be a compact subset of X . Then $H \cap A$ is closed in H , and so is a compact subset of A . By hypothesis, $H \cap B = (H \cap A) \cap B$ is closed in $H \cap A$. Since A is closed, it follows that $H \cap B$ is closed in H . Thus B meets every compact subset of X in a relatively closed set. Since X is compactly generated, B is closed in X . Therefore B is closed in A and so A is compactly generated.

7.6. THEOREM. <u>If</u> K <u>and</u> L <u>are complexes, then</u> $K \times L$ <u>is a complex.</u>

<u>Proof.</u> By 7.1 we need only verify conditions 6) and 7). By 7.2 and 7.4, $K \times L$ is a quasi complex. That is, $K \times L$ satisfies condition 6). Note that condition 7) merely states that each skeleton of a complex has the weak topology with respect to closed cells. Since $(K \times L)_n$ is closed in $|K| \times_k |L|$, we know by 7.5 that $(K \times L)_n$ is compactly generated. Finally, by 7.3 and 7.4 $(K \times L)_n$ is a quasi complex. That is, $(K \times L)_n$ has the weak topology with respect to closed cells. Thus $K \times L$ satisfies 7) and the proof is complete.

8. The cartesian product $|K| \times |L|$

It follows from Exercise 1 of Section 4 and Theorem 7.6 that the cartesian product $|K| \times |L|$ of two complexes K and L is a complex if and only if $|K| \times |L|$ is a compactly generated space. We now give two results which show that $|K| \times |L|$ is compactly generated for a large class of pairs (K,L) .

8.1. THEOREM. (Milnor [2]) <u>If</u> K <u>and</u> L <u>are countable complexes, then</u> $|K| \times |L|$ <u>is compactly generated.</u>

8.2. THEOREM. (J. H. C. Whitehead [4]) <u>If</u> K <u>and</u> L <u>are complexes and</u> L <u>is locally finite, then</u> $|K| \times |L|$ <u>is compactly generated.</u>

For a definition of "locally finite" see I.3.

If X is a topological space, a collection $\mathcal{B}$ of compact sets of X is called a <u>base for compact subsets</u> if each compact subset of X is contained in some member of $\mathcal{B}$.

8.3. LEMMA. <u>If</u> K <u>is a complex, then</u> K <u>is countable if and only if</u> K <u>has a countable base for compact sets.</u>

<u>Proof.</u> Let K be countable. By I.4.4, a countable base for compact sets of K is given by the collection of finite unions of closed cells of K .

Conversely, suppose K has a countable base $\mathcal{B}$ for compact sets. Every point of K is contained in some member of $\mathcal{B}$. By I.4.4, each member of $\mathcal{B}$ is contained in a finite union of cells of K. Thus K is a countable union of finite unions of cells, and is therefore countable.

The following theorem thus implies 8.1.

8.4. THEOREM. (Weingram [3]) If X and Y are Hausdorff compactly generated spaces each having a countable base for compact subsets, then $X \times Y$ is compactly generated.

Proof. The proof is a restatement of Milnor's proof of 8.1:

We know that the identity function $f\colon X \times_k Y \longrightarrow X \times Y$ is continuous. To show that f is a homeomorphism, and hence that $X \times Y$ is compactly generated, we need only show that if U is open in $X \times_k Y$, then $f(U) = U$ is open in $X \times Y$.

Suppose that $x \times y$ is a point of U. We will find a neighborhood V of x, and a neighborhood W of y, such that $V \times W \subset U$.

We may assume that $X = \cup A_i$ and that $Y = \cup B_i$, with $A_0 \subset A_1 \subset \dots \subset A_n \subset \dots$ and $B_0 \subset B_1 \subset \dots \subset B_n \subset \dots$ all compact. We may also assume that $\{A_i\}$ and $\{B_i\}$ are bases, and that $x \in A_0$ and $y \in B_0$.

Now, $U \cap (A_0 \times B_0)$ is open in $A_0 \times B_0$, and contains $x \times y$. A_0 and B_0 are compact Hausdorff spaces and hence regular. Thus there exists a set V_0 open in A_0 and containing x, and a

set W_0 open in B_0 and containing y, such that $\overline{V}_0 \times \overline{W}_0 \subset U \cap (A_0 \times B_0)$.

Assume, by induction, that V_n and W_n, open in A_n and B_n, contain x and y, respectively, and that $\overline{V}_n \times \overline{W}_n \subset U \cap (A_n \times B_n)$. The set $\overline{V}_n$ is closed in A_{n+1} as well as in A_n, and is therefore compact in A_{n+1}. Similarly, $\overline{W}_n$ is compact in B_{n+1}. Also, $U \cap (A_{n+1} \times B_{n+1})$ is open in $A_{n+1} \times B_{n+1}$. There exist*, therefore, sets V_{n+1} and W_{n+1} containing $\overline{V}_n$ and $\overline{W}_n$, and open in A_{n+1} and B_{n+1}, such that

$$\overline{V}_{n+1} \times \overline{W}_{n+1} \subset U \cap (A_{n+1} \times B_{n+1}).$$

Let $V = \cup V_n$ and $W = \cup W_n$, Then

$$x \times y \in V \times W \subset U.$$

Furthermore, V is open in X and W is open in Y, since they meet each set of a base for compact sets in an open set. This concludes the proof of 8.4.

A complex K is called locally countable if each point of K has a neighborhood contained in a countable subcomplex of K.

8.5. COROLLARY. If K and L are locally countable complexes, then $|K| \times |L|$ is compactly generated.

*See the theorem of Wallace on p. 142 of J. L. Kelley's General Topology.

Proof. Let A be a subset of $|K| \times |L|$ meeting each compact subset of $|K| \times |L|$ in a (relatively) open set. Let (x,y) be a point of A. Then $x \in U \subset |K'|$, where U is open in $|K|$ and K' is a countable subcomplex of K. Similarly, $y \in V \subset |L'|$, with V open in $|L|$ and L' a countable subcomplex of L. Now $|K'| \times |L'|$ is compactly generated, and A meets every compact subset of $|K'| \times |L'|$ in a relatively open set. Therefore, $A \cap (|K'| \times |L'|)$ is open in $|K'| \times |L'|$. In particular, $A \cap (U \times V)$ is open, so that A is a neighborhood of (x,y) in $|K| \times |L|$. It follows that A, being a neighborhood of each of its points, is open in $|K| \times |L|$.

8.6. THEOREM. (Weingram [3]) If X is locally compact, and Y is compactly generated, then $X \times Y$ is compactly generated.

Proof. Let U be a subset of $X \times Y$ which meets each compact subset of $X \times Y$ in a (relatively) open set. We must show that U is open. Suppose that (x_0,y_0) is a point of U. Since X is locally compact, x_0 lies in an open set V whose closure is compact. The set $\overline{V} \times y_0$ is a compact subset of $X \times Y$, so $(\overline{V} \times y_0) \cap U$ is open in $\overline{V} \times y_0$. Therefore, there exists a neighborhood W_1 of x_0 in X such that $W_1 \times y_0 \subset U$ and $W_1 \subset V$. The compact Hausdorff space $\overline{V}$ is regular, so there exists a neighborhood W_2 of x_0 with $\overline{W}_2 \subset W_1$. Let Z be the collection of all points y in Y such that $\overline{W}_2 \times y \subset U$. We assert that Z is open in Y. Since Y is compactly generated, it will suffice to show that Z meets each compact subset of Y in a (relatively) open set. Let $H \subset Y$ be compact. Let y_1 be a point of $H \cap Z$. Then, since y_1 belongs to Z, $\overline{W}_2 \times y_1 \subset U$. The set U intersects the compact set $V \times H$ in an open set, by hypothesis. By Wallace's theorem there exists A open in V and B open in H with

$$\overline{W}_2 \times y_1 \subset A \times B \subset U \cap (\overline{V} \times H).$$

But this implies immediately that $B \subset Z$, and so $H \cap Z$ is open. Thus Z is open in Y. By construction, $W_2 \times Z \subset U$. Since x_0 lies in W_2 and y_0 lies in Z, U is a neighborhood of (x_0,y_0) and so is open. This completes the proof.

We conclude this section by describing an example due to C. H. Dowker [1] of two complexes M and N for which $|M| \times |N|$ is not compactly generated, and thus not a complex.

The complex M is a collection $\{A_i | i \in I\}$ of closed 1-cells, where I is an index set with the power of the continuum. The A_i have a common vertex, u_0. The complex N is a denumerable collection $\{B_j | j = 1,2,\ldots\}$ of closed 1-cells, all having a common vertex, v_0.

We suppose that each A_i is parametrized as a unit interval $0 \leq x_i \leq 1$, with $x_i = 0$ at u_0. Likewise, we suppose each B_j to be parametrized as a unit interval $0 \leq y_j \leq 1$, with $y_j = 0$ at v_0.

We now identify I with the set of all sequences $(i_1, i_2, \ldots)$ of positive integers. Thus, if (i,j) is a pair of indices, with j an integer and with $i = (i_1, i_2, \dots)$ a sequence of positive integers, we may define p_{ij} to be the point with coordinates $(1/i_j,\ 1/i_j)$ in $A_i \times B_j \subset |M| \times |N|$. Let $P = \{p_{ij}\}$. Then $P \cap (A_i \times B_j) = p_{ij}$ is closed.

If H is a compact subset of $|M| \times |N|$ then $P \cap H$ is finite because H lies in a union of finitely many closed cells of $M \times N$ (7.2). In particular, P meets each closed cell of $M \times N$ in a finite, and hence closed, set. Therefore, P is closed in $|M| \times_k |N|$.

We show now that P is not closed in $|M| \times |N|$.

For each i in I , and for each positive integer j , let a_i and b_j be positive real numbers. Let U be the neighborhood of u_0 in $|M|$ that is given by $x_i < a_i$, and let V be the neighborhood of v_0 in $|N|$ that is given by $y_j < b_j$. Then $U \times V$ is a neighborhood of $u_0 \times v_0$ in $|M| \times |N|$.

We shall now choose a pair $\bar{i}, \bar{j}$ of indices for which $p_{\bar{i}\bar{j}}$ lies in $U \times V$. This will imply that $u_0 \times v_0$ is a limit point of P in $|M| \times |N|$. Since $u_0 \times v_0$ is not a point of P , it will follow that P is not closed in $|M| \times |N|$. The pair $\bar{i}, \bar{j}$ is chosen so that $p_{\bar{i}\bar{j}}$ is a point of $(A_i \times B_j) \cap (U \times V)$. That is, $1/\bar{i}_{\bar{j}}$ is smaller than either $a_{\bar{i}}$ or $b_{\bar{j}}$. To accomplish this we pick a sequence $\bar{i} = (i_1, i_2, \dots, i_j)$ so that for each $j = 1, 2, \dots$ both $\bar{i}_j > j$ and $\bar{i}_j > 1/b_j$. We then choose $\bar{j}$ to be an integer larger than $1/a_{\bar{i}}$. With these choices, $1/\bar{i}_{\bar{j}} < 1/\bar{j} < a_{\bar{i}}$ and $1/\bar{i}_{\bar{j}} < b_{\bar{j}}$, so that $p_{\bar{i}\bar{j}}$ lies in $U \times V$.

Thus P is not closed in $|M| \times |N|$, and $|M| \times |N|$ is not compactly generated.

Exercise. If K and L are regular complexes, then $|K| \times |L|$ is compactly generated if and only if one of the following conditions holds:

a) One of K, L is locally finite.

b) Both K and L are locally countable.

(HINT: The sufficiency of these conditions is given by 8.5 and 8.6. To show necessity, suppose neither condition obtains. Then we may assume that K fails to be locally countable at a point x and that L fails to be locally finite at a point y . Embed Dowker's example $|M| \times |N|$ as a closed subset of $|K| \times |L|$, based on the point (x,y) . Suppose $|K| \times |L|$ compactly generated. The closed subset $|M| \times |N|$ would be compactly generated by 7.5, and this we know to be false. The contradiction establishes necessity.)

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Chapter 5

THE HOMOLOGY OF PRODUCTS AND JOINS

RELATIVE HOMOLOGY

1. The homology of $K \times L$

Let K and L be regular complexes. In section IV.7 we defined the product complex $K \times L$. The main result of this section (see 1.5) is the computation of $H_*(K \times L)$ in terms of $H_*(K)$ and $H_*(L)$ for regular complexes K and L that are finite in each dimension. For the moment, however, we allow K and L to be arbitrary regular complexes. Since K and L are regular, $K \times L$ is regular. The m-cells of $K \times L$ are the components of

$$(K \times L)_m - (K \times L)_{m-1} = \bigcup_{i+j=m} (K_i - K_{i-1}) \times (L_j - L_{j-1}).$$

The products on the right, as well as all other operations considered in this chapter, are carried out in the category of compactly generated spaces. (See Chapter IV.) Each m-cell has the form $\sigma \times \tau$, where σ is a cell of K, τ is a cell of L and $\dim \sigma + \dim \tau = m$.

If A is an arbitrary set, let $F(A)$ denote the free abelian group generated by the elements of A. It is easy to show that for any sets A and B

$$F(A \times B) \approx F(A) \otimes F(B).$$

From this it follows that

$$\psi_m\colon\ C_m(K \times L) \approx \sum_{i+j=m} C_i(K) \otimes C_j(L)$$

where the isomorphism ψ_m is defined by the correspondence

$$\sigma \times \tau \approx \sigma \otimes \tau.$$

If K and L are oriented by incidence functions $[\ \ ]_K$ and $[\ \ ]_L$, we may orient $K \times L$ in the following way. A face of a cell $\sigma \times \tau$ of $K \times L$ is a cell $\sigma' \times \tau'$ with $\sigma' < \sigma$ and $\tau' < \tau$. Accordingly, we set

$$[\sigma \times \tau : \sigma' \times \tau'] = \begin{cases} [\sigma:\sigma']_K & \text{if } \tau' = \tau \\ (-1)^{\dim \sigma} [\tau:\tau']_L & \text{if } \sigma = \sigma'. \end{cases}$$

All other incidence numbers are defined to be zero. We call the incidence function so defined the <u>product incidence function</u>.

To verify that the product incidence function satisfies property (iii) of II.1.8, let $\sigma \times \tau$ be an m-cell of $K \times L$ with $\dim \sigma = i$. A face $\sigma' \times \tau'$ of $\sigma \times \tau$ of dimension m-2 is of one of three forms:

a) $\sigma \times \tau'$ $\qquad \dim \tau' = m - i - 2$

b) $\sigma' \times \tau'$ $\qquad \dim \sigma' = i-1,\ \dim \tau' = m - i - 1$

c) $\sigma' \times \tau$ $\qquad \dim \sigma' = i-2.$

In case a), we have

$$\begin{aligned}
&\sum_{\tau'<\rho<\tau} [\sigma \times \tau : \sigma \times \rho][\sigma \times \rho : \sigma' \times \tau'] \\
&\quad = \sum_{\tau'<\rho<\tau} (-1)^{\dim \sigma} [\tau:\rho]_L \cdot (-1)^{\dim \sigma} [\rho:\tau']_L \\
&\quad = \sum [\tau:\rho]_L [\rho:\tau']_L = 0.
\end{aligned}$$

In case b) we have

$$[\sigma \times \tau : \sigma \times \tau'][\sigma \times \tau' : \sigma' \times \tau'] + [\sigma \times \tau : \sigma' \times \tau][\sigma' \times \tau : \sigma' \times \tau']$$

$$= (-1)^{\dim \sigma} [\tau : \tau']_L [\sigma : \sigma']_K + (-1)^{\dim \sigma'} [\sigma : \sigma']_K [\tau : \tau']_L$$

$$= 0, \quad \text{since} \quad \dim \sigma = \dim \sigma' + 1.$$

Case c) is similar to case a).

Properties (i) and (ii) are easily verified.

Thus we have associated with each pair of oriented complexes K and L the oriented complex $K \times L$. We denote the boundary operator associated with the product incidence function by $\partial_{K \times L}$, and we define $C(K \times L)$ to be the chain complex $(\{C_m(K \times L)\}, \partial_{K \times L})$.

1.1. DEFINITION. Given two chain complexes $C = (\{C_q\}, \partial)$ and $D = (\{D_q\}, \partial')$ over a ground ring R, the tensor product $C \otimes_R D$ is the chain complex whose mth chain group is

$$\sum_{i+j=m} C_i \otimes_R D_j$$

and whose boundary operator is defined by

$$\partial(c \otimes d) = \partial c \otimes d + (-1)^{\dim c}\, c \otimes \partial' d.$$

It is easy to see that $\partial\partial = 0$.

1.2. THEOREM. _The correspondence_ $\sigma \times \tau \longleftrightarrow \sigma \otimes \tau$ _induces an isomorphism of chain complexes:_ $C(K \times L) \approx C(K) \otimes C(L)$.

Proof: We know that the correspondence $\sigma \times \tau \longleftrightarrow \sigma \otimes \tau$ defines an isomorphism ψ_m of the mth chain groups of $C(K \times L)$ and $C(K) \otimes C(L)$. Thus we need only show that these isomorphisms commute with the boundary operators. Let $\sigma \times \tau$ be a generator of $C_m(K \times L)$. Then

$$\partial_{K \times L}(\sigma \times \tau) = \sum_{\substack{\sigma' < \sigma \\ \tau' < \tau}} [\sigma \times \tau : \sigma' \times \tau'](\sigma' \times \tau')$$

$$= \sum_{\sigma' < \sigma} [\sigma : \sigma']_K (\sigma' \times \tau) + \sum_{\tau' < \tau} (-1)^{\dim \sigma} [\tau : \tau']_L (\sigma \times \tau').$$

Under the isomorphism ψ_{m-1}, $\partial_{K \times L}(\sigma \times \tau)$ maps to

$$\sum_{\sigma' < \sigma} [\sigma : \sigma']_K \sigma' \otimes \tau + \sum_{\tau' < \tau} (-1)^{\dim \sigma} [\tau : \tau']_L \sigma \otimes \tau'$$

$$= \Big(\sum_{\sigma' < \sigma} [\sigma : \sigma']_K \sigma'\Big) \otimes \tau + (-1)^{\dim \sigma}\, \sigma \otimes \Big(\sum_{\tau' < \tau} [\tau : \tau']_L \tau'\Big)$$

$$= \partial\sigma \otimes \tau + (-1)^{\dim \sigma}\, \sigma \otimes \partial'\tau$$

$$= \partial(\sigma \otimes \tau).$$

Thus

$$\partial(\psi_m(\sigma \times \tau)) = \psi_{m-1}(\partial_{K \times L}(\sigma \times \tau))$$

and the proof is complete.

We now turn to the problem of computing $H_*(K \times L)$ in terms of $H_*(K)$ and $H_*(L)$. Because of 1.2 we may consider the problem in an algebraic setting: given chain complexes C and D, compute $H_*(C \otimes D)$ in terms of $H_*(C)$ and $H_*(D)$.

We wish to define a homomorphism

$$\alpha: \ H_*(C) \otimes H_*(D) \longrightarrow H_*(C \otimes D).$$

Let z_1 and z_2 be cycles in C and D respectively. Then in $C \otimes D$ we set

$$\alpha(\{z_1\} \otimes \{z_2\}) = \{z_1 \otimes z_2\}.$$

First we verify that this definition is independent of the choice of cycles:

$$\alpha(\{z_1 + \partial c_1\} \otimes \{z_2 + \partial' c_2\}) - \alpha(\{z_1\} \otimes \{z_2\})$$
$$= \{z_1 \otimes \partial' c_2 + \partial c_1 \otimes z_2 + \partial c_1 \otimes \partial' c_2\}.$$

But this latter cycle is the boundary of

$$(-1)^{\dim z_1} z_1 \otimes c_2 + c_1 \otimes z_2 + c_1 \otimes \partial' c_2,$$

so that

$$\alpha(\{z_1 + \partial c_1\} \otimes \{z_2 + \partial' c_2\}) = \alpha(\{z_1\} \otimes \{z_2\}).$$

In order to complete the verification that α is well-defined, we need only note that

$$\alpha(\{z_1\} \otimes \{z_2\}) + \alpha(\{z_3\} \otimes \{z_2\}) = \alpha(\{z_1 + z_3\} \otimes \{z_2\})$$

and similarly that α is linear in the second component. Thus α is a well-defined homomorphism

$$\alpha: \ H_r(C) \otimes H_s(D) \longrightarrow H_{r+s}(C \otimes D).$$

We extend α to $H_*(C) \otimes H_*(D)$ by linearity, and obtain a degree-preserving homomorphism

$$\alpha: \ H_*(C) \otimes H_*(D) \longrightarrow H_*(C \otimes D).$$

<u>We now assume</u> that C and D are chain complexes over Z that are free and finitely generated in each dimension. We apply Theorem II.6.6 and write C and D each as a sum of elementary chain complexes of types i (free), ii (acyclic) and iii (torsion):

$$\phi_1: \ C \approx \Sigma M_i \quad \text{and} \quad \phi_2: \ D \approx \Sigma N_j \ .$$

Using the fact that the tensor product of direct sums of chain complexes is the direct sum of the tensor product of chain complexes, we have

$$\phi_1 \otimes \phi_2: \ C \otimes D \approx \sum_{i,j} M_i \otimes N_j.$$

By the remark following II.6.4,

$$H_*(C \otimes D) \approx \sum_{i,j} H_*(M_i \otimes N_j).$$

Also,

$$H_*(C) \otimes H_*(D) \approx H_*(\Sigma M_i) \otimes H_*(\Sigma N_j)$$
$$\approx (\Sigma H_*(M_i)) \otimes (\Sigma H_*(N_j))$$
$$\approx \sum_{i,j} H_*(M_i) \otimes H_*(N_j).$$

For each pair (i,j) we have the homomorphism

$$\alpha_{ij}: \ H_*(M_i) \otimes H_*(N_j) \longrightarrow H_*(M_i \otimes N_j),$$

which extends by linearity to give the homomorphism

$$\Sigma\alpha_{ij}: \ \sum_{i,j} H_*(M_i) \otimes H_*(N_j) \longrightarrow \sum_{i,j} H_*(M_i \otimes N_j).$$

1.3. THEOREM. The following diagram is commutative:

$$\begin{tikzcd} H_*(C \otimes D) \arrow[r, "\approx", "(\phi_1 \otimes \phi_2)_*"'] & \sum_{i,j} H_*(M_i \otimes N_j) \\ H_*(C) \otimes H_*(D) \arrow[u, "\alpha"] \arrow[r, "\approx", "\phi_{1*} \otimes \phi_{2*}"'] & \sum_{i,j} H_*(M_i) \otimes H_*(N_j) \arrow[u, "\Sigma\alpha_{ij}"'] \end{tikzcd} .$$

This theorem will allow us to replace α by the homomorphism $\Sigma\alpha_{ij}$.

Proof of 1.3. Let z_1 and z_2 be cycles of C and D respectively. We have

$$\phi_1 z_1 = \sum_i x_i \quad \text{and} \quad \phi_2 z_2 = \sum_j y_j$$

where each x_i lies in $Z(M_i)$ and each y_j lies in $Z(N_j)$. Then

(1) $$(\phi_1 \otimes \phi_2)(z_1 \otimes z_2) = \sum_{i,j} x_i \otimes y_j.$$

Thus

$$\begin{aligned} &(\phi_1 \otimes \phi_2)_*\alpha(\{z_1\} \otimes \{z_2\}) \\ &= (\phi_1 \otimes \phi_2)_*\{z_1 \otimes z_2\} \quad \text{(by def. of } \alpha) \\ &= \sum_{i,j} \{x_i \otimes y_j\} \quad \text{(by (1) above)} \\ &= (\sum_{i,j} \alpha_{ij}) \sum_{i,j} \{x_i\} \otimes \{y_j\} \quad \text{(by def. of } \alpha_{ij}) \\ &= (\Sigma\alpha_{ij})((\sum_i \{x_i\}) \otimes (\sum_j \{y_j\})) \\ &= (\Sigma\alpha_{ij})(\phi_{1*}\{z_1\} \otimes \phi_{2*}\{z_2\}) \\ &= (\Sigma\alpha_{ij})(\phi_{1*} \otimes \phi_{2*})(\{z_1\} \otimes \{z_2\}), \end{aligned}$$

and the proof is complete.

We remark in passing that α is a natural transformation of functors. That is, given chain complexes C, C', D, D' and maps $f\colon C \longrightarrow C'$ and $g\colon D \longrightarrow D'$, then the diagram below is commutative

$$\begin{tikzcd} H_*(C) \otimes H_*(D) \arrow[r, "f_* \otimes g_*"] \arrow[d, "\alpha"] & H_*(C') \otimes H_*(D') \arrow[d, "\alpha"] \\ H_*(C \otimes D) \arrow[r, "(f \otimes g)_*"] & H_*(C' \otimes D') \end{tikzcd}$$

The proof is easy and is left to the reader.

We now investigate the homomorphism $\Sigma\alpha_{ij}$. This homomorphism is completely determined by the individual

$$\alpha_{ij}\colon H_*(M_i) \otimes H_*(N_j) \longrightarrow H_*(M_i \otimes N_j),$$

and each α_{ij} in turn depends only on the type of M_i and N_j. We shall see that α_{ij} is an isomorphism except in the case that M_i and N_j are both of the torsion type with torsion numbers which are not relatively prime. In this case we shall prove that α_{ij} is a monomorphism. To reduce nine cases to four, we say that an elementary chain complex is of type (ii)' if it is of type (ii) or (iii).

Case (i) $\otimes$ (i). Let M and N be free elementary chain complexes. That is, for some p, q we have

$$C_i(M) \approx \begin{cases} Z, & \text{generated by } a, \text{ if } i = p \\ 0 & \text{if } i \neq p \end{cases}$$

$$C_j(N) \approx \begin{cases} Z, & \text{generated by } b, \text{ if } i = q \\ 0 & \text{if } i \neq q. \end{cases}$$

Then $M \otimes N$ is also a free elementary chain complex, with

$$C_k(M \otimes N) \approx \begin{cases} Z, & \text{generated by } a \otimes b, \text{ if } k = p+q \\ 0 & k \neq p+q \end{cases}$$

The boundary operators in M, N and $M \otimes N$ are all zero, and so

$$H_i(M) \approx \begin{cases} Z, & \text{generated by } \{a\} \text{ if } i = p \\ 0 & i \neq p \end{cases}$$

$$H_j(N) \approx \begin{cases} Z, & \text{generated by } \{b\} \text{ if } j = q \\ 0 & j \neq q \end{cases}$$

$$H_k(M \otimes N) \approx \begin{cases} Z, & \text{generated by } \{a \otimes b\} \text{ if } k = p+q \\ 0 & k \neq p+j \end{cases}$$

Thus $\alpha\colon H_*(M) \otimes H_*(N) \longrightarrow H_*(M \otimes N)$, which maps $\{a\} \otimes \{b\}$ to $\{a \otimes b\}$, is an isomorphism.

Case (i) $\otimes$ (ii)'. Let M be a free elementary chain complex with one generator a in dimension p. Let N be a chain complex with two generators, c in dimension $q+1$, and d in dimension q, with $\partial c = nd$ for some $n \neq 0$. We show that $\alpha\colon H_*(M) \otimes H_*(N) \longrightarrow H_*(M \otimes N)$ is an isomorphism.

We have

$$H_i(M) \approx \begin{cases} Z, & \text{generated by } a \text{ if } i = p \\ 0 & i \neq p \end{cases}$$

$$H_j(M) \approx \begin{cases} Z_n, & \text{generated by } \{d\} \text{ if } j = q \\ 0 & j \neq q \end{cases}$$

Thus $H_*(M) \otimes H_*(N)$ has a single generator, $\{a\} \otimes \{d\}$, of dimension $p + q$, and of order n.

The chain complex $M \otimes N$ is elementary and of type (ii)', generated by $a \otimes c$ in dimension $p+q+1$ and $a \otimes d$ in dimension $p+q$, with $\partial(a \otimes c) = a \otimes nd = n(a \otimes d)$. Thus $H_*(M \otimes N)$ has a single generator, $\{a \otimes d\}$, of dimension $p+q$ and of order n. So α, which maps $\{a\} \otimes \{d\}$ to $\{a \otimes d\}$, is an isomorphism.

Case (ii)' $\otimes$ (i) is like case (i) $\otimes$ (ii)'.

Case (ii)' $\otimes$ (ii)'. Let M be a chain complex with two generators, a in dimension $p+1$ and b in dimension p, with $\partial a = mb$, $m \neq 0$. Let N be a chain complex with two generators, c in dimension $q+1$ and d in dimension q, with $\partial c = nd$, $n \neq 0$. Then

$$H_i(M) \approx \begin{cases} Z_m, & \text{generated by } \{b\}, \text{ if } i = p \\ 0 & i \neq p \end{cases}$$

$$H_j(N) \approx \begin{cases} Z_n, & \text{generated by } \{d\}, \text{ if } j = q \\ 0 & j \neq q. \end{cases}$$

The tensor product $Z_m \otimes Z_n$ is cyclic and of order $\theta = (m,n)$, the greatest common divisor of m and n. Thus $H_*(M) \otimes H_*(N)$ has a single generator $\{b\} \otimes \{d\}$, of dimension $p+q$ and of order θ.

The chain complex $M \otimes N$ has four generators, $a \otimes c$ of dimension $p+q+2$, $a \otimes d$ and $b \otimes c$ of dimension $p+q+1$, and $b \otimes d$ of dimension $p+q$. The boundary operator is defined by

$$\begin{aligned} \partial(a \otimes c) &= \partial a \otimes c + (-1)^{p+1}\, a \otimes \partial c = m(b \otimes c) + (-1)^{p+1}\, n(a \otimes d) \\ \partial(a \otimes d) &= m(b \otimes d) \\ \partial(b \otimes c) &= (-1)^p\, n(b \otimes d) \\ \partial(b \otimes d) &= 0. \end{aligned}$$

We first compute $H_{p+q}(M \otimes N)$. The boundaries in dimension $p+q$ are generated by $m(b \otimes d)$ and $(-1)^p\, n(b \otimes d)$. As x and y vary over the integers, $xm + (-1)^p yn$ describes the subgroup generated by $\theta = (m,n)$. Thus the $(p+q)$-boundaries are generated by $\theta(b \otimes d)$. Since $b \otimes d$ generates the cycles, we have

$$H_{p+q}(M \otimes N) \approx Z_\theta, \quad \text{generated by } \{b \otimes d\}.$$

Next we compute $H_{p+q+1}(M \otimes N)$. In order that

$$\partial(x(a \otimes d) + y(b \otimes c)) = 0,$$

it is necessary and sufficient that

$$xm + (-1)^p yn = 0.$$

Dividing by θ,

$$x\,\frac{m}{\theta} = (-1)^{p+1}\, y\,\frac{n}{\theta}.$$

Since m/θ and n/θ are relatively prime, there is an integer k such that

$$x = (-1)^{p+1}\, k\,\frac{n}{\theta} \quad \text{and} \quad y = k\,\frac{m}{\theta}.$$

Thus the $(p+q+1)$-cycles are generated by

$$u = (-1)^{p+1}\left(\frac{n}{\theta}\right)(a \otimes d) + \left(\frac{m}{\theta}\right)(b \otimes c).$$

The $(p+q+1)$-boundaries are of course generated by $\partial(a \otimes c) = \theta u$. Thus

$$H_{p+q+1}(M \otimes N) \approx Z_\theta, \quad \text{generated by } \{u\}.$$

The homomorphism

$$\alpha\colon\ H_*(M) \otimes H_*(N) \longrightarrow H_*(M \otimes N)$$

maps $\{b\} \otimes \{d\}$ to $\{b \otimes d\}$ in dimension $p+q$. Thus $\alpha\colon Z_\theta \longrightarrow Z_\theta$ is an isomorphism in dimension $p+q$. In dimension $p+q+1$, $\alpha\colon 0 \longrightarrow Z_\theta$ is a monomorphism. In dimension $p+q+2$, α is an isomorphism because both the domain and range of α are 0.

We are now ready to prove

1.4. THEOREM. If C and D are chain complexes of abelian groups that are free and finitely generated in each dimension, then α is a monomorphism of $H_*(C) \otimes H_*(D)$ onto a direct summand of $H_*(C \otimes D)$. The complementary summand, in dimension m, is isomorphic to $\sum_{q+r=m-1} T_q(C) \otimes T_r(D)$, where $T_q(C)$ denotes the torsion subgroup of the qth homology group of C.

Proof. By 1.3 it is sufficient to show that the theorem is true if C and D are elementary chain complexes. In cases (i) $\otimes$ (i), (i) $\otimes$ (ii)' and (ii)' $\otimes$ (i) we have shown that α is an isomorphism. The statement about the complementary summand is true because in each case at least one of the chain complexes has torsion-free homology groups. In case (ii)' $\otimes$ (ii)' we know that α is a monomorphism onto a direct summand whose complementary summand is

$$H_{p+q+1}(M \otimes N) \approx Z_\theta.$$

The only torsion in $H_*(M)$ is Z_m in dimension p. The only torsion in $H_*(N)$ is Z_n in dimension q. Thus

$$H_{p+q+1}(M \otimes N) \approx Z_\theta \approx Z_m \otimes Z_n \approx \sum_{r+s=p+q} H_r(M) \otimes H_s(N).$$

The proof of the theorem is complete.

1.5. COROLLARY (the Künneth relations). If K and L are regular complexes with finitely many cells in each dimension, then

$$H_m(K \times L) \approx \sum_{p+q=m} H_p(K) \otimes H_q(L) \oplus \sum_{p+q=m-1} T_p(K) \otimes T_q(L),$$

where $T_p(K)$ denotes the torsion subgroup of $H_p(K)$.

Proof. This follows immediately from 1.1 and 1.4.

1.6. COROLLARY. If K and L are as above, and G is either Z_p for a prime p or the group of rationals, then

$$H_*(K \times L; G) \approx H_*(K;G) \otimes H_*(L;G).$$

Proof. Apply 1.5, II.6.9, and II.6.10.

Exercises.

1. Compute the homology of $P^2 \times P^2$.

2. Künneth relations for cohomology. Using the results of the exercise at the end of chapter II, and ideas similar to those used in the proofs of 1.3 and 1.4 obtain a formula relating $H^*(C^\cdot \otimes D^\cdot)$ to $H^*(C^\cdot)$ and $H^*(D^\cdot)$.

2. Joins of Complexes.

Let X and Y be compactly generated spaces*. The <u>join</u> $X \circ Y$ of X and Y is the quotient space of $X \times I \times Y$ under the following identifications: for all $x, x' \in X$ and $y, y' \in Y$,

$$(x,0,y) \sim (x,0,y') \quad \text{and} \quad (x,1,y) \sim (x',1,y).$$

The projection map $X \times I \times Y \longrightarrow X \circ Y$ is denoted by p. The function $i\colon X \longrightarrow X \circ Y$ defined by $i(x) = p(x,0,y)$ embeds X as a subspace of $X \circ Y$. Similarly, we can regard Y as embedded in $X \circ Y$.

If $x \in X$ and $y \in Y$ then the <u>line segment</u> from x to y in $X \circ Y$ is the subset

$$[x,y] = \{(x,t,y) \mid 0 \leq t \leq 1\}.$$

Each point of $X \circ Y$ with $t \neq 0, 1$ lies on a unique $[x,y]$.

The <u>mapping cylinder</u> of a map $f\colon X \longrightarrow Y$ is the subspace of $X \circ Y$ that includes all line segments $[x, fx]$, $x \in X$, together with the points of Y. The join $X \circ Y$ is homeomorphic to the space obtained by identifying the two copies of $X \times Y$ in the disjoint union of the mapping cylinders of the projections $X \times Y \longrightarrow X$ and $X \times Y \longrightarrow Y$:

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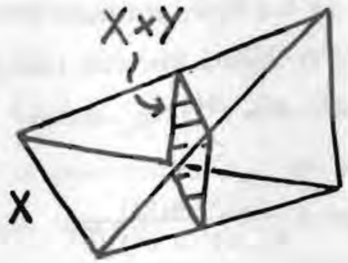
*Recall that throughout this chapter we work in the category of compactly generated spaces.



The join of a point and a space X is called the <u>cone</u> on X, and is written CX. The natural inclusion $X \subseteq CX$ embeds X as the <u>base</u> of the cone. A space is <u>contractible</u> if there exists a map $F\colon I \times X \longrightarrow X$ with F_0 the identity map and F_1 constant. The cone on any space X is contractible, and provides the simplest way of embedding X in a contractible space.

2.1. LEMMA. <u>The join of a closed</u> p-<u>cell and a closed</u> q-<u>cell is a closed</u> $(p+q+1)$-<u>cell. The join of a</u> $(p-1)$-<u>sphere and a closed</u> q-<u>cell is a closed</u> $(p+q)$-<u>cell. The join of a</u> $(p-1)$-<u>sphere and a</u> $(q-1)$-<u>sphere is a</u> $(p+q-1)$-<u>sphere</u>.

<u>Proof</u>: Let $\sigma_p = (A_0, \ldots, A_p)$ and $\sigma_q = (B_0, \ldots, B_q)$ be simplexes. Let σ_{p+q+1} be the simplex on the vertices $(A_0, \ldots, A_p, B_0, \ldots, B_q)$. We define a map $\phi\colon \sigma_p \times I \times \sigma_q \longrightarrow \sigma_{p+q+1}$ by setting

$$[\phi(f,t,g)]A_i = (1-t)\, f(A_i)$$
$$[\phi(f,t,g)]B_j = tg(B_j),$$

for $f \in \sigma_p$, $g \in \sigma_q$, and $t \in I$. If $f' \in \sigma_p$ and $g' \in \sigma_q$, then $\phi(f,0,g) = \phi(f,0,g')$ and $\phi(f,1,g) = \phi(f',1,g)$. Thus ϕ induces a continuous map $\bar{\phi}\colon \sigma_p \circ \sigma_q \longrightarrow \sigma_{p+q+1}$. Furthermore, $\bar{\phi}$ is one-one and onto. Since $\sigma_p \circ \sigma_q$ is compact and σ_{p+q+1} Hausdorff, $\bar{\phi}$

is a homeomorphism. This proves the first statement of the lemma.

Let $E^p = \{x \in R^p \mid \|x\| \leq 1\}$ denote the unit ball in R^p. We define a map ψ: $S^{p-1} \times I \times E^q \longrightarrow R^{p+q}$ by setting

$$\psi(x,t,y) = \frac{((1-t)x,ty)}{((1-t)^2 + t^2)^{1/2}}$$

for $x \in S^{p-1}$, $y \in E^q$, $t \in I$. If $x' \in S^{p-1}$, $y' \in E^q$, then $\psi(x,0,y) = \psi(x,0,y')$ and $\psi(x,1,y) = \psi(x',1,y)$. Thus ψ induces a continuous map $\overline{\psi}$: $S^{p-1} \circ E^q \longrightarrow R^{p+q}$. A straightforward calculation shows that $\overline{\psi}$ is actually a homeomorphism of $S^{p-1} \circ E^q$ onto E^{p+q}. The restriction of $\overline{\psi}$ to $S^{p-1} \circ S^{q-1}$ is a homeomorphism onto S^{p+q-1}. This completes the proof of the lemma.

The reader should note that the homeomorphism $\overline{\phi}$: $\sigma_p \circ \sigma_q \longrightarrow \sigma_{p+q+1}$ carries $\dot{\sigma}_p \circ \sigma_q \cup \sigma_p \circ \dot{\sigma}_q$ onto $\dot{\sigma}_{p+q+1}$.

Exercise. Show that the join of two spaces is arcwise connected.

We now introduce a gimmick that will simplify the computation of the homology of the join of two complexes. Let K be a regular complex. We adjoin to the collection of cells of K an ideal cell e_K of dimension -1. We stipulate that e_K be a face of every cell of K. Since every 1-cell has precisely two vertices the redundant restrictions are all satisfied. Given an incidence function on K, we define additional incidence numbers involving e_K by

$$[\sigma:e_K] = \begin{cases} 0 & \text{if } \dim \sigma > 0 \\ 1 & \text{if } \dim \sigma = 0. \end{cases}$$

We define the augmented chain complex $\tilde{C}(K)$ of the oriented complex K by setting $\tilde{C}_q(K)$ = the free abelian group on the q-cells of K and by defining $\tilde{\partial}_q(\sigma) = \sum_{\tau \in K} [\sigma:\tau]\tau$ for any q-cell σ. Thus $\tilde{C}_{-1}(K) = Z$, and is generated by e_K. To show that $\tilde{C}(K)$ is a chain complex, we observe that $\tilde{C}(K) = C(K)$ in dimensions ≥ 0, and if c is a zero-chain, then $\partial c = (\text{In } c)e_K$, where $\text{In } c$ is the index of c, defined in II.1.11. We remarked following II.1.11 that $\text{In } \partial c = 0$ for any 1-chain c. The homology groups of $\tilde{C}(K)$ are denoted by $\tilde{H}_*(K)$ and are called the <u>reduced homology groups</u> of K. The obvious chain map φ: $\tilde{C}(K) \longrightarrow C(K)$ induces an isomorphism $\tilde{H}_q(K) \approx H_q(K)$ for $q > 0$. φ_* maps $\tilde{H}_0(K)$ isomorphically onto a direct summand of $H_0(K)$. A generator for the complementary summand is given by the homology class of v, where v is any vertex in K. Thus $H_0(K) \approx Z \oplus \tilde{H}_0(K)$. For a categorical definition of reduced homology, see the exercise at the end of VI.4.

Let K and L be regular complexes, with ideal cells e_K and e_L of dimension -1. We have inclusions i: $|K| \subseteq |K| \circ |L|$ and j: $|L| \subseteq |K| \circ |L|$. For notational convenience we write $i(x) = x \circ e_L = x \circ L_{-1}$ and $j(y) = e_K \circ y = K_{-1} \circ y$. Thus we have an inclusion $K_m \circ L_n \subseteq |K| \circ |L|$. We define the join complex $K \circ L$ of K and L to be the space $|K| \circ |L|$ together with the subspaces

(1) $$(K \circ L)_m = \bigcup_{i=-1}^{m} (K_i \circ L_{m-1-i}).$$

We leave to the reader the job of verifying that $K \circ L$ thus defined is a complex. We write $e_K \circ e_L$ for the ideal cell of $K \circ L$. The cells of $K \circ L$ are then given as follows. Let σ and τ be (possibly ideal) cells of K and L, respectively, of dimensions m and

n respectively. Then by 2.1, $\bar{\sigma} \circ \bar{\tau}$ is a closed cell of $K \circ L$ of dimension $m+n+1$, with interior $\sigma \circ \tau - (\sigma \cup \tau)$. Thus the open cells of $K \circ L$ are not joins of open cells. We will, however, write $\sigma \circ \tau$ to denote the (open) cell $\mathrm{Int}(\bar{\sigma} \circ \bar{\tau})$. By the remark following 2.1, the boundary of $\sigma \circ \tau$ is the union $\dot{\sigma} \circ \bar{\tau} \cup \bar{\sigma} \circ \dot{\tau}$, so $\sigma \circ \tau$ is a face of $\sigma' \circ \tau'$ only when either $\sigma = \sigma'$ and $\tau < \tau'$ or $\sigma < \sigma'$ and $\tau = \tau'$ or $\sigma < \sigma'$ and $\tau < \tau'$.

Given incidence functions α on K and β on L, we define an incidence function $\alpha \circ \beta$ on $K \circ L$ by setting

$$(2) \qquad [\sigma\circ\tau:\rho]_{\alpha\circ\beta} = \begin{cases} [\sigma:\sigma']_\alpha & \text{if } \rho = \sigma' \circ \tau \\ (-1)^{1+\dim \sigma}[\tau:\tau']_\beta & \text{if } \rho = \sigma \circ \tau' \\ 0 & \text{otherwise} \end{cases}$$

for all cells σ, σ' of K and τ, τ' of L. Note that if we set $\sigma = e_K$, and $\rho = e_K \circ \tau'$, then

$$[e_K \circ \tau : e_K \circ \tau'] = (-1)^{1+\dim e_K}[\tau:\tau']_\beta = [\tau:\tau']_\beta .$$

Thus $\alpha \circ \beta$ extends the incidence functions α and β on the subcomplexes K and L of $K \circ L$. The verification that $\alpha \circ \beta$ is indeed an incidence function is routine and is left to the reader.

It follows from (1) that $\tilde{C}_q(K \circ L) \approx \sum_{i=-1}^{q} \tilde{C}_i(K) \otimes \tilde{C}_{q-1-i}(L)$.

The boundary operator in the chain complex $\tilde{C}(K \circ L)$ is given by (2). If σ is a generator of $\tilde{C}_i(K)$ and τ is a generator of $\tilde{C}_j(L)$, then $\partial^{\alpha\circ\beta}(\sigma \circ \tau) = \partial\sigma \circ \tau + (-1)^{1+i}\sigma \circ \partial\tau$. Here we write $\sigma \circ \tau$

for the chain $\sigma \otimes \tau$ in the summand $\tilde{C}_i(K) \otimes \tilde{C}_j(L)$ of $\tilde{C}_{i+j+1}(K \circ L)$. Thus the computation of $\tilde{H}_*(K \circ L)$ is reduced to the following algebraic problem: Given two chain complexes $C = (\{C_i\},\partial)$ and $D = (\{D_j\},\partial')$, define a new chain complex $C \circ D$, with boundary operator $\partial \circ \partial'$, by

$$(C \circ D)_q = \sum_{i+j=q-1} C_i \otimes D_j$$

$$(\partial \circ \partial')(c \otimes d) = \partial c \otimes d + (-1)^{1+\dim c}\, c \otimes \partial d.$$

Describe the homology of $C \circ D$ in terms of the homology of C and of D.

If $C = (\{C_i\},\partial)$ is a chain complex, define the <u>suspension</u> of C, written sC, with boundary operator ∂^s, by

$$(sC)_i = C_{i-1}$$

$$\partial^s c = \partial c \quad \text{for } c \in (sC)_i = C_{i-1}.$$

2.2. LEMMA. <u>The function</u> $\varphi\colon C \circ D \longrightarrow s(C \otimes D)$ <u>defined by</u> $\varphi(c \otimes d) = (-1)^{\dim d}(c \otimes d)$ <u>is a chain isomorphism</u>.

<u>Proof</u>: It is clearly sufficient to show that φ is a chain map. Let $c \in C_i$, $d \in D_j$.

$$\begin{aligned}(s(\partial \otimes \partial'))\varphi(c \otimes d) &= (s(\partial \otimes \partial'))[(-1)^{\dim d}(c \otimes d)] \\ &= (-1)^{\dim d}(\partial \otimes \partial')(c \otimes d) \\ &= (-1)^{\dim d}(\partial c \otimes d + (-1)^{\dim c}\, c \otimes \partial d).\end{aligned}$$

$$\varphi(\partial \circ \partial')(c \otimes d) = \varphi(\partial c \otimes d + (-1)^{1+\dim c} c \otimes \partial d)$$
$$= (-1)^{\dim d}(\partial c \otimes d) + (-1)^{1+\dim c}(-1)^{\dim \partial d} c \otimes \partial d$$
$$= (-1)^{\dim d}(\partial c \otimes d + (-1)^{\dim c} c \otimes \partial d).$$

2.3. COROLLARY. Let K and L be regular complexes with finitely many cells in each dimension. Then

$$\tilde{H}_q(K \circ L) = \sum_{i+j=q-1} \tilde{H}_i(K) \otimes \tilde{H}_j(L) + \sum_{i+j=q-2} T_i(K) \otimes T_j(L)$$

Proof: This follows from 1.4 and 2.2.

2.4. DEFINITION. A regular complex K is called acyclic if $\tilde{H}_*(K) = 0$. Equivalently, any cycle in $\tilde{C}(K)$ bounds in K.

2.5. PROPOSITION. Let K be a regular complex, and let L be a point. Then $K \circ L$, the cone on K, is acyclic.

Proof: If K is finite, 2.5 is a corollary of 2.3. Let K be arbitrary. Let z be a cycle on $K \circ L$. Since z is a finite linear combination of cells of $K \circ L$, there exists a finite subcomplex K' of K such that z lies on $K' \circ L$. Applying 2.3, z must bound in $K' \circ L$, and therefore in $K \circ L$.

Alternate proof of 2.5 which does not use 2.3: Suppose $z \in \tilde{Z}_q(K \circ L)$. Then $z = c \circ e_L + d \circ L$, where $c \in \tilde{C}_q(K)$, $d \in \tilde{C}_{q-1}(K)$. We have

$$0 = \partial z = \partial(c \circ e_L + d \circ L)$$
$$= \partial c \circ e_L + (-1)^{1+q} c \circ \partial e_L + \partial d \circ L + (-1)^q d \circ \partial L$$
$$= \partial c \circ e_L + \partial d \circ L + (-1)^q d \circ e_L .$$

Thus $\partial d = 0$ and $\partial c = (-1)^{q+1} d$. It then follows that z is the boundary of the $(q+1)$-chain $(-1)^{q+1} c \circ L$, and so $K \circ L$ is acyclic.

3. The homology of SK

The join of a complex K with a 0-sphere is called the suspension of K, and is denoted by SK. In this section we prove

3.1. THEOREM. If K is a regular complex then

$$\tilde{H}_*(SK) \approx s\tilde{H}_*(K).$$

If K is finite in each dimension this result follows from Corollary 2.3.

To give a proof for general K we define a chain map $\phi\colon \tilde{C}(K) \longrightarrow \tilde{C}(SK)$ that raises dimension by 1, and that induces an isomorphism of homology groups. For each chain c on K we define

$$\phi(c) = (-1)^{\dim c}(A \circ c - B \circ c)$$

where A and B are the vertices of the 0-sphere with which K is joined. Thus if $c = e_K$ we have $\partial\phi(e_K) = 0$. If $\dim c \geq 0$ then

$$\partial\phi(c) = (-1)^{\dim c}(c - A \circ \partial c - c + B \circ \partial c)$$
$$= (-1)^{-1+\dim c}(A \circ \partial c - B \circ \partial c)$$
$$= \phi(\partial c).$$

Thus ϕ induces a homomorphism

$$\phi_*: \tilde{H}_m(K) \longrightarrow \tilde{H}_{m+1}(SK), \quad m \geq 0.$$

To show that ϕ_* is a monomorphism, let c be a q-chain on K with $\phi(c) = \partial d$. Each chain d in $C_{q+1}(SK)$ is a sum $d = A \circ d_1 + B \circ d_2 + e \circ d_3$, with d_1, d_2 in $C_q(K)$, e the ideal element of the 0-sphere, and d_3 in $C_{q+1}(K)$. Accordingly

$$\partial d = e \circ d_1 + e \circ d_2 + e \circ \partial d_3 - A \circ \partial d_1 - B \circ \partial d_2 = \phi(c) = (-1)^q (A \circ c - B \circ c).$$

This implies that $-A \circ \partial d_1 = (-1)^q A \circ c$, so that c is a boundary.

To show that ϕ_* is onto, suppose that $d = A \circ d_1 + B \circ d_2 + e \circ d_3$ is a q-cycle on SK. Then $\partial d = 0$, so that $d_1 + d_2 + \partial d_3 = 0$. Also $\partial d_1 = 0$, so that d_1 is a cycle. Therefore

$$\begin{aligned}(-1)^q \phi(d_1) &= A \circ d_1 - B \circ d_1 \\ (-1)^q \phi(d_1) - d &= -B \circ d_1 - B \circ d_2 - e \circ d_3 \\ &= -B \circ (d_1 + d_2) - e \circ d_3 \\ &= B \circ \partial d_3 - e \circ d_3 \\ &= \partial(B \circ d_3).\end{aligned}$$

This says that $\phi((-1)^q d_1) \sim d$, so that ϕ_* is onto. Thus ϕ_* is an isomorphism, as desired.

As an application we show that it is possible to construct a complex with prescribed finitely generated homology groups, provided that the given H_o is free abelian of rank ≥ 1.

If K and L are complexes, we shall use $K \vee L$, read "K wedge L", to denote a complex obtained from identifying a vertex

of K with a vertex of L. In general $K \vee L$ depends on the choice of vertices, but it is easy to show that if K and L are both regular then $K \vee L$ is regular and

$$H_q(K \vee L) \approx H_q(K) \oplus H_q(L), \quad q > 0$$

$$H_o(K \vee L) \oplus Z \approx H_o(K) \oplus H_o(L).$$

Let X_p be a regular complex obtained from wrapping the boundary of a 2-cell p times around S^1, where p is an integer greater than one. For example, $|X_2| = P^2$. Then the homology groups of X_p are given as follows:

$$H_i(X_p) \approx \begin{cases} Z_p & i = 1 \\ Z & i = 0 \\ 0 & \text{otherwise.} \end{cases}$$

If $S^k X_p$ denotes the k-th suspension of X_p (defined inductively by $S^1 X_p = SX_p$, $S^{k+1} X_p = S(S^k X_p)$), then

$$H_i(S^k X_p) \approx \begin{cases} Z_p & i = k+1 \\ Z & i = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Let G be a finitely generated abelian group. Then $G \approx F \oplus \sum_1^n Z_{p_i}$, where F is free abelian. Let Y denote the space $S^{k+1} \vee S^{k+1} \vee \dots \vee S^{k+1} \vee S^k X_{p_1} \vee S^k X_{p_2} \vee \dots \vee S^k X_{p_n}$, where the number of spheres equals the rank of F. It follows that

$$H_i(Y) \approx \begin{cases} G & \text{if } i = k+1 \\ Z & \text{if } i = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Given a sequence $G_1, G_2, \ldots$ of finitely generated abelian groups, set $W = Y_1 \vee Y_2 \vee \cdots \vee Y_j \vee \cdots$, where for each j, Y_j has been constructed so that

$$H_i(Y_j) \approx \begin{cases} G_j & \text{if } i = j \\ Z & \text{if } i = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Then $H_i(W) \approx G_i$ if $i > 0$, and $H_0(W) \approx Z$. If G_0 is a free abelian group of rank $r \geq 1$ we can construct a complex with homology $H_i \approx G_i$, $i \geq 0$ by forming the disjoint union of W with $r-1$ points.

4. Relative Homology.

A <u>pair</u> (K,L) of regular complexes is a regular complex K and a subcomplex L of K. The pair is <u>oriented</u> if K is oriented by an incidence function, γ say, and L is oriented by the restriction of γ.

For each pair (K,L), we have an inclusion homomorphism $i\colon C_q(L) \longrightarrow C_q(K)$. We define the group of relative q-chains of the pair (K,L), written $C_q(K,L)$, to be the quotient group

$C_q(K)/C_q(L)$. If c is a chain on K, we denote by $[c]$ the coset of $C_q(L)$ in $C_q(K)$ which contains c.

Let (K,L) be an oriented pair. Let ∂ denote the boundary operator both in L and in K. We define a homomorphism $\hat{\partial}\colon C_q(K,L) \longrightarrow C_{q-1}(K,L)$ by setting $\hat{\partial}[c] = [\partial c]$ for any $c \in C_q(K)$. The map $\hat{\partial}$ is well defined because ∂ maps $C_q(L)$ to $C_{q-1}(L)$. Also, $\hat{\partial}\hat{\partial}[c] = \hat{\partial}[\partial c] = [\partial\partial c] = 0$. Thus the collection $(\{C_q(K,L)\}, \hat{\partial})$ forms a chain complex, written $C(K,L)$.

4.1. DEFINITION. The <u>qth (relative) homology group</u> of the oriented pair (K,L), written $H_q(K,L)$, is the qth homology group of the chain complex $C(K,L)$. If we wish to stress the orientation γ of the pair (K,L), we shall write $C^\gamma(K,L)$, $H_q^\gamma(K,L)$.

4.2. LEMMA. <u>The inclusion map</u> $i\colon C(L) \longrightarrow C(K)$ <u>is a chain map.</u> <u>The projection map</u> $j\colon C(K) \longrightarrow C(K,L)$ <u>is a chain map.</u>

<u>Proof</u>: The first assertion is precisely the statement that L is a subcomplex oriented by restriction of the incidence function on K. To prove the second assertion, let $c \in C_q(K)$. Then $\hat{\partial} jc = \hat{\partial}[c] = [\partial c] = j(\partial c)$.

We thus have the following commutative diagram:

$$\begin{tikzcd} C_q(L) \arrow[r, "i"] \arrow[d, "\partial"] & C_q(K) \arrow[r, "j"] \arrow[d, "\partial"] & C_q(K,L) \arrow[d, "\hat{\partial}"] \arrow[dll, dashed, "\psi"'] \\ C_{q-1}(L) \arrow[r, "i"'] & C_{q-1}(K) \arrow[r, "j"'] & C_{q-1}(K,L) \end{tikzcd}$$

By 4.2, i and j induce homomorphisms $i_*: H_q(L) \longrightarrow H_q(K)$ and $j_*: H_q(K) \longrightarrow H_q(K,L)$. We wish to define a homomorphism $\partial_*: H_q(K,L) \longrightarrow H_{q-1}(L)$. Let $z \in Z_q(K,L)$. Then $z = [c]$ for some $c \in C_q(K)$. We have $j\partial c = \hat{\partial} jc = \hat{\partial} z = 0$, so ∂c is a chain, in fact a cycle, on L. We set $\partial_*\{z\} = \{\partial c\}$. If we vary c by a chain on L, we vary ∂c by a boundary, and so ∂_* is well-defined. The map ∂_* is called the *boundary* or *connecting homomorphism* for the oriented pair (K,L).

4.3. THEOREM. *The sequence of groups and homomorphisms*

$$(1) \quad \cdots \xrightarrow{\partial_*} H_q(L) \xrightarrow{i_*} H_q(K) \xrightarrow{j_*} H_q(K,L) \xrightarrow{\partial_*} H_{q-1}(L) \xrightarrow{i_*} \cdots$$

is exact.

Proof: (i) Exactness at $H_q(L)$. Let $z \in Z_{q+1}(K,L)$, $z = [c]$, $c \in C_{q+1}(K)$. Then $i_*\partial_*\{z\} = i_*\{\partial c\} = \{i\partial c\} = \{\partial ic\} = 0$. Thus $\ker i_* \supseteq \operatorname{im} \partial_*$. Suppose $i_*\{z\} = 0$ for $z \in Z_q(L)$. Then $z = \partial c$ for some $c \in C_{q+1}(K)$. Also, $\hat{\partial}[c] = [\partial c] = [iz] = 0$, so $[c] \in Z_{q+1}(K,L)$. It is then obvious that $\partial_*\{[c]\} = \{z\}$. Thus $\ker i_* = \operatorname{im} \partial_*$.

(ii) Exactness at $H_q(K)$. Let $z \in Z_q(L)$. Then $j_*i_*\{z\} = \{jiz\} = 0$. So $\ker j_* \supseteq \operatorname{im} i_*$. Let $z \in Z_q(K)$ such that $j_*\{z\} = 0$. Then $jz = \hat{\partial}[c]$, $c \in C_{q+1}(K)$. We have $j(z-\partial c) = jz - j\partial c = jz - \hat{\partial}[c] = 0$, so z is homologous to a cycle on L. Thus $z \in \operatorname{im} i_*$ and $\ker j_* = \operatorname{im} i_*$.

(iii) Exactness at $H_q(K,L)$. Let $z \in Z_q(K)$. Then $\partial_* j_*\{z\} = \partial_*\{jz\} = \{\partial z\} = 0$. Thus $\ker \partial_* \supseteq \operatorname{im} j_*$. Let $z \in Z_q(K,L)$ be such that $\partial_*\{z\} = 0$. If $z = [c]$, $c \in C_q(K)$, then ∂c bounds a

chain d on L. Thus $\partial(c-d) = 0$ and $\{z\} = j_*\{c-d\}$. We have shown that $\ker \partial_* = \operatorname{im} j_*$. The proof of 4.3 is complete.

The sequence (1) is called the *relative homology sequence* of the oriented pair (K,L).

Suppose that γ, γ' are orientations of the pair (K,L). Let $\phi: C^{\gamma}(K) \longrightarrow C^{\gamma'}(K)$ be the chain isomorphism of II.5.4. Then ϕ carries chains on L to chains on L and so induces $\hat{\phi}: C^{\gamma}(K,L) \longrightarrow C^{\gamma'}(K,L)$. If $c \in C_q^{\gamma}(K)$, then $\hat{\partial}\hat{\phi}[c] = \hat{\partial}[\phi c] = [\partial\phi c] = [\phi\partial c] = \hat{\phi}[\partial c] = \hat{\phi}\hat{\partial}[c]$. Thus $\hat{\phi}$ is a chain isomorphism, and induces an isomorphism $\hat{\phi}_*: H_q^{\gamma}(K,L) \approx H_q^{\gamma'}(K,L)$. Relative homology groups are independent of orientation. Furthermore, the triple $(\phi, \phi|L, \hat{\phi})$ induces an isomorphism of the exact homology sequences associated with the orientations γ and γ'. That is to say, the diagram below is commutative:

$$\begin{tikzcd}
\arrow[r] & H_q^{\gamma}(L) \arrow[r, "i_*"] \arrow[d, "(\phi|L)_*"', "\approx"] & H_q^{\gamma}(K) \arrow[r, "j_*"] \arrow[d, "\phi_*"', "\approx"] & H_q^{\gamma}(K,L) \arrow[r, "\partial_*"] \arrow[d, "\hat{\phi}_*"', "\approx"] & H_{q-1}^{\gamma}(L) \arrow[r] \arrow[d, "(\phi|L)_*"', "\approx"] & {} \\
\arrow[r] & H_q^{\gamma'}(L) \arrow[r, "i_*"'] & H_q^{\gamma'}(K) \arrow[r, "j_*"'] & H_q^{\gamma'}(K,L) \arrow[r, "\partial_*"'] & H_{q-1}^{\gamma'}(L) \arrow[r] & {}
\end{tikzcd}$$

The proof that this diagram is commutative is routine and is left to the reader.

Remark: The exact sequence

$$0 \longrightarrow C_q(L) \xrightarrow{i} C_q(K) \xrightarrow{j} C_q(K,L) \longrightarrow 0$$

is of course split exact. Thus we have maps $h\colon C_q(K) \longrightarrow C_q(L)$ and $k\colon C_q(K,L) \longrightarrow C_q(K)$ such that $hi = 1$, $jk = 1$. The map h restricts chains on K to the cells of L. We may take k to be defined as follows. If c is a chain on K, then $c = d+c'$, where d is a chain on L and c' is a chain involving only cells of K not in L. Then $k([c]) = c'$. Using the splittings h and k, we could define a map $\psi\colon C_q(K,L) \longrightarrow C_{q-1}(L)$ by $\psi = h\partial k$. (See diagram p. 131) A tedious calculation shows that $\psi\hat{\partial} = -\partial\psi$. Thus ψ carries cycles to cycles and boundaries to boundaries, and induces a map $\psi_*\colon H_q(K,L) \longrightarrow H_{q-1}(L)$. It is easy to prove that $\psi_* = \partial_*$. Note that the splitting maps h and k are <u>not</u> chain maps.

Exercise: Let K and L be complexes, v a vertex of K, v' a vertex of L. The complex $K \vee L$, obtained by identifying the vertices v and v' of the disjoint union $K \cup L$, is called the <u>wedge</u> of K and L. The wedge can be regarded as the subcomplex $K \times v' \cup v \times L$ of $K \times L$. Using the Künneth relations for the homology of a product, compute $H_*(K \times L, K \vee L)$. Show that

$$sH_*(K \times L, K \vee L) \approx \tilde{H}_*(K \circ L),$$

using the results in section 2 concerning $\tilde{H}_*(K \circ L)$.

In defining relative homology groups we had occasion to refer to the exact sequence

$$0 \longrightarrow C_q(L) \longrightarrow C_q(K) \longrightarrow C_q(K,L) \longrightarrow 0$$

associated with any pair (K,L). More generally, we may define a

<u>short exact sequence</u> of chain complexes to be a sequence

(1) $$0 \longrightarrow L \xrightarrow{i} K \xrightarrow{j} M \longrightarrow 0$$

of chain complexes and chain maps such that for each q the sequence

(2) $$0 \longrightarrow L_q \xrightarrow{i} K_q \xrightarrow{j} M_q \longrightarrow 0$$

is exact. We may proceed as above to define a connecting homomorphism $\partial_*\colon H_q(M) \longrightarrow H_{q-1}(L)$ for each q. If $z \in Z_q(M)$, since j is onto, we have $z = jc$ for some $c \in K_q$. Since j is a chain map, $j\partial c = \partial jc = \partial z = 0$. By the exactness of (2), $\partial c = id$ for some $d \in L_{q-1}$. Then $i\partial d = \partial id = \partial\partial c = 0$, and since i is a monomorphism, d is a cycle. We set $\partial_*\{z\} = \{d\}$. To show that ∂_* is well-defined, we vary c by id' for some $d' \in L_q$. Then $\partial(c+id') = \partial c + \partial id' = i(d+\partial d')$. Thus d varies by a boundary. We receive a sequence

(3) $$\cdots \longrightarrow H_q(L) \xrightarrow{i_*} H_q(K) \xrightarrow{j_*} H_q(M) \xrightarrow{\partial_*} H_{q-1}(L) \longrightarrow \cdots$$

which is called the homology sequence of the exact sequence (1). One proves just as in 4.3 that the homology seuqence (3) is exact.

If $0 \longrightarrow L' \xrightarrow{i'} K' \xrightarrow{j'} M' \longrightarrow 0$ is another short exact sequence of chain complexes, then a <u>homomorphism</u> of short exact sequences of chain complexes is a triple $(\bar{\varphi}, \varphi, \hat{\varphi})$ of chain maps such that the following diagram is commutative:

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r, "i"] \arrow[d, "\bar{\varphi}"] & K \arrow[r, "j"] \arrow[d, "\varphi"] & M \arrow[r] \arrow[d, "\hat{\varphi}"] & 0 \\
0 \arrow[r] & L' \arrow[r, "i'"] & K' \arrow[r, "j'"] & M' \arrow[r] & 0
\end{tikzcd}$$

4.4. PROPOSITION. <u>A homomorphism of short exact sequences of chain complexes induces a homomorphism of the associated homology sequences. In other words, the following diagram is commutative.</u>

$$\begin{tikzcd}
{} \arrow[r, "\partial_*"] & H_q(L) \arrow[r, "i_*"] \arrow[d, "\bar{\varphi}_*"] & H_q(K) \arrow[r, "j_*"] \arrow[d, "\varphi_*"] & H_q(M) \arrow[r, "\partial_*"] \arrow[d, "\hat{\varphi}_*"] & H_{q-1}(L) \arrow[r, "i_*"] \arrow[d, "\bar{\varphi}_*"] & {} \\
{} \arrow[r, "\partial_*'"] & H_q(L') \arrow[r, "i_*'"] & H_q(K') \arrow[r, "j_*'"] & H_q(M') \arrow[r, "\partial_*'"] & H_{q-1}(L') \arrow[r, "i_*'"] & {}
\end{tikzcd}$$

<u>Proof</u>: $\varphi_* i_* = (\varphi i)_* = (i\varphi)_* = i_*\varphi_*$.

$$\hat{\varphi}_* j_* = (\hat{\varphi} j)_* = (j\hat{\varphi})_* = j_*\hat{\varphi}_* .$$

The third square is more complicated. Let $z \in Z_q(M)$. Then $z = jc$ for some $c \in C_q(K)$, $\partial c = id$ for some $d \in Z_{q-1}(L)$. Then $\partial_*\{z\} = \{d\}$, by definition. We have $\hat{\varphi}z = j'\varphi c$, $\partial'\varphi c = i'\varphi d$, and so $\partial_*'\{\hat{\varphi}z\} = \{\varphi d\}$. Thus $\partial_*'\hat{\varphi}_*\{z\} = \partial_*'\{\hat{\varphi}z\} = \{\varphi d\} = \varphi_*\{d\} = \varphi_*\partial_*\{z\}$.

<u>Exercise</u>: (Relative homology with coefficients in an arbitrary abelian group G.) Verify that the following constructions are valid: Let G be an abelian group, and let $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ be an exact sequence of free chain complexes. Then

$$0 \longrightarrow A \otimes G \longrightarrow B \otimes G \longrightarrow C \otimes G \longrightarrow 0$$

is exact, and, as in the case $G = Z$, we have a relative homology sequence

$$\longrightarrow H_q(A;G) \longrightarrow H_q(B;G) \longrightarrow H_q(C;G) \longrightarrow H_{q-1}(A;G) \longrightarrow$$

which is exact. For an arbitrary oriented complex pair (K,L) set

$A = C^{\alpha}(L)$, $B = C^{\alpha}(K)$, $C = C^{\alpha}(K,L)$. Then we may define $H_q(K,L;G)$, the relative homology group with coefficients in G, to be the group $H_q(C;G)$.

<u>Exercise</u>: (Relative cohomology with coefficients in G.) Let G be an abelian group, and let $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ be an exact sequence of free chain complexes. Prove that the associated sequence

$$0 \longleftarrow \mathrm{Hom}(A,G) \xleftarrow{\varphi} \mathrm{Hom}\,(B,G) \xleftarrow{\psi} \mathrm{Hom}(C,G) \longleftarrow 0$$

is exact. Define coboundaries in these cochain complexes as in Chapter II and verify that φ and ψ are chain maps. Finally, define a connecting homomorphism δ^*: $H^q(A;G) \longrightarrow H^{q+1}(C;G)$. Obtain a sequence

$$\xrightarrow{\xi^*} H^q(C;G) \xrightarrow{\psi^*} H^q(B;G) \xrightarrow{\varphi^*} H^q(A;G) \xrightarrow{\xi^*} H^{q+1}(C;G) \xrightarrow{\psi^*} .$$

Prove that this sequence is exact.

If (K,L) is a pair of complexes, set $A = C^{\alpha}(L)$, $B = C^{\alpha}(K)$, $C = C^{\alpha}(K,L)$. The group $H^q(C;G)$ is called the qth relative cohomology group of (K,L) with coefficients in G and is written $H^q(K,L;G)$. It follows from the exactness of the sequence above that we get an exact sequence:

$$\longrightarrow H^q(K,L;G) \longrightarrow H^q(K;G) \longrightarrow H^q(L;G) \longrightarrow H^{q+1}(K,L;G) \longrightarrow$$

This sequence is called the exact cohomology sequence with coefficients in G of the pair (K,L).

Chapter VI. THE INVARIANCE THEOREM

1. Remarks on the Proof of Topological Invariance.

Let K and K' be regular complexes. Suppose that $f\colon K \longrightarrow K'$ is a continuous mapping with the following property: for each cell σ of K, there is a cell τ of K' of the same dimension such that $f\sigma = \tau$. (We say that f maps cells onto cells.) Let β be an incidence function for K'. Define an incidence function α for K by

$$[\sigma:\tau]_\alpha = [f\sigma:f\tau]_\beta$$

where σ and τ are arbitrary cells of K. Then it is easy to verify that α is an incidence function for K. We define a chain map $\varphi\colon C^\alpha(K) \longrightarrow C^\beta(K')$ as follows: if $\Sigma a_i\sigma_i$ is a q-chain of K, then $\varphi(\Sigma a_i\sigma_i) = \Sigma a_i f\sigma_i$. Then φ is a homomorphism. Moreover, if ∂^α and ∂^β are the boundary operators associated with α and β, respectively, and if σ is a generator of $C_q(K)$, then

$$\begin{aligned} \varphi\partial^\alpha\sigma &= \varphi \sum_{\tau\in K} [\sigma:\tau]_\alpha \tau \\ &= \sum_{\tau\in K} [f\sigma:f\tau]_\beta f\tau \\ &= \partial^\beta f\sigma \\ &= \partial^\beta \varphi\sigma. \end{aligned}$$

By linearity, this relation holds over all of $C_q(K)$. Thus φ is a chain map. Consequently, φ induces homomorphisms $\varphi_*\colon H_q(K) \longrightarrow H_q(K')$ for every q. We then say that the homomorphisms φ_* are induced by the mapping f and write them as f_*.

Examples of mappings satisfying the condition given above are a) the inclusion of a subcomplex in a complex and b) the projection of

a complex K to the complex K/F obtained from K by identification (as long as K/F is regular).

In this chapter we extend this definition of induced homomorphism. First, in section 2, we extend it to mappings f which have the property that the minimal subcomplex containing the f-image of any cell is acyclic*. Such mappings we call proper mappings. Then, in sections 3 and 4, using the notion of subdivision, we show that an arbitrary mapping $f\colon K \longrightarrow K'$ of finite complexes can be factored into proper mappings. We define the homology homomorphisms induced by f to be the composition of the homomorphisms induced by the factors of f.

Returning to our first topic, we note that if $f\colon K \longrightarrow K$ is the identity mapping then f maps cells onto cells, and $f_*\colon H_q(K) \longrightarrow H_q(K)$ is the identity isomorphism for each q. Also, if $f\colon K \longrightarrow K'$ and $g\colon K' \longrightarrow K''$ map cells onto cells, then so does gf, and $(gf)_* = g_*f_*$. These two properties are fundamental in homology theory. In section 4 we show that they hold for the induced homomorphisms of arbitrary continuous mappings of finite regular complexes. The invariance theorem (4.9) is a corollary of this result. We emphasize that in this chapter we prove the invariance theorem only for finite regular complexes.

2. Chain Homotopy, Carriers, and Proper Mappings.

Suppose we are given two topological spaces X and Y, and two continuous mappings f_0 and f_1 from X to Y. We say that f_0 is

*See V.2.4. for a definition.

homotopic to f_1, written $f_0 \simeq f_1$, if there is a continuous mapping $F\colon I \times X \longrightarrow Y$ such that $F(0,x) = f_0 x$ and $F(1,x) = f_1 x$. We can reformulate this condition as follows. We define mappings p_0 and p_1 from X into $I \times X$ by $p_0 x = (0,x)$ and $p_1 x = (1,x)$. Then two mappings f_0 and f_1 from X to Y are homotopic if and only if there exists a mapping $F\colon I \times X \longrightarrow Y$ such that $Fp_0 = f_0$ and $Fp_1 = f_1$.

We define in a similar way the notion of chain homotopy. Suppose that we are given two chain complexes C and C', and two chain maps f_0 and f_1 from C to C'. Let I be the cell complex for the closed unit interval consisting of a 1-cell I and two vertices $\bar{0}$ and $\bar{1}$. Let I also represent the chain complex for I with the boundary operator ∂ defined by $\partial I = \bar{1} - \bar{0}$. Then in the chain complex $I \otimes C$ we have

$$\partial(\bar{0} \otimes c) = \bar{0} \otimes \partial c, \quad \partial(\bar{1} \otimes c) = \bar{1} \otimes \partial c,$$

and

$$\partial(I \otimes c) = \bar{1} \otimes c - \bar{0} \otimes c - I \otimes \partial c,$$

for c a chain of C. We define chain maps φ_0 and φ_1 from C to $I \otimes C$ by

$$\varphi_0(c) = \bar{0} \otimes c \quad \text{and} \quad \varphi_1(c) = \bar{1} \otimes c.$$

2.1. DEFINITION. Two chain maps $f_0, f_1\colon C \longrightarrow C'$ are chain homotopic if there exists a chain map $F\colon I \otimes C \longrightarrow C'$ such that $F\varphi_0 = f_0$ and $F\varphi_1 = f_1$. F is called a chain homotopy of f_0 and f_1, and we write $f_0 \simeq f_1$ to indicate that f_0 and f_1 are chain homotopic.

2.2. LEMMA. Let f_0 and f_1 be chain maps from C to C'. Then f_0 and f_1 are chain homotopic if and only if there are homomorphisms $\mathcal{D}_q\colon C_q \longrightarrow C'_{q+1}$ such that $\partial'\mathcal{D}_q + \mathcal{D}_{q-1}\partial = f_1 - f_0$ for each q, where ∂ and ∂' are the boundary operators of C and C' respectively.

Proof: If $f_0 \simeq f_1$ and F is a chain homotopy of f_0 and f_1, set $\mathcal{D}_q c = F(I \otimes c)$ for $c \in C_q$. Then

$$\begin{aligned}
\partial'\mathcal{D}_q c &= \partial' F(I \otimes c) \\
&= F\partial(I \otimes c) \\
&= F(\bar{1} \otimes c - \bar{0} \otimes c - I \otimes \partial c) \\
&= F\varphi_1 c - F\varphi_0 c - \mathcal{D}_{q-1}\partial c \\
\partial'\mathcal{D}_q c + \mathcal{D}_{q-1}\partial c &= f_1 c - f_0 c.
\end{aligned}$$

If $\{\mathcal{D}_q\}$ is given satisfying the conditions of the Lemma, define a chain map $F\colon I \otimes C \longrightarrow C'$ by

$$F(\bar{0} \otimes c) = f_0 c, \quad F(\bar{1} \otimes c) = f_1 c, \quad \text{and} \quad F(I \otimes c) = \mathcal{D}_q c,$$

for c a q-chain of C. Then F clearly maps q-chains of $I \otimes C$ into q-chains of C', and

$$\begin{aligned}
\partial' F(I \otimes c) = \partial'\mathcal{D} c &= f_1 c - f_0 c - \mathcal{D}\partial c \\
&= F(\bar{1} \otimes c) - F(\bar{0} \otimes c) - F(I \otimes \partial c) \\
&= F(\partial(I \otimes c)).
\end{aligned}$$

Thus F is a chain map and provides the desired chain homotopy.

The collection $\mathcal{D}$ of homomorphisms $\mathcal{D}_q$ is called a chain deformation for f_0 and f_1, and is said to exhibit a chain homotopy of f_o and f_1.

2.3. THEOREM. Let f_o and f_1 be chain maps from C to C'. If f_o and f_1 are chain homotopic, then

$$(f_o)_* = (f_1)_*:\ H_*(C) \longrightarrow H_*(C').$$

Proof: Let $\mathcal{D} = \{\mathcal{D}_q\}$ be a chain deformation exhibiting a chain homotopy of f_o and f_1. Let z be a q-cycle of C. Then

$$\begin{aligned}(f_1)_*\{z\} - (f_o)_*\{z\} &= \{f_1 z - f_o z\}\\ &= \{\partial' \mathcal{D}_q z + \mathcal{D}_{q-1}\partial z\}\\ &= \{\partial' \mathcal{D}_q z\} \quad \text{since } z \text{ is a cycle}\\ &= 0.\end{aligned}$$

Analogously, let $C^\cdot$ and $D^\cdot$ be cochain complexes, and let $f_o, f_1:\ C^\cdot \longrightarrow D^\cdot$ be cochain maps. A collection $\mathcal{D} = \{\mathcal{D}_q\}$ of homomorphisms $\mathcal{D}_q:\ C^q \longrightarrow D^{q-1}$ satisfying

$$f_1 - f_o = \delta\mathcal{D} + \mathcal{D}\delta$$

is called a cochain deformation. One proves as in 2.3 that if $\mathcal{D}$ is a cochain deformation for the cochain maps $f_o, f_1:\ C^\cdot \longrightarrow D^\cdot$, then $(f_o)^* = (f_1)^*:\ H^*(D^\cdot) \longrightarrow H^*(C^\cdot)$.

Now let K and K' be oriented regular complexes, and suppose that $f_o, f_1:\ C(K) \longrightarrow C(K')$ are chain maps. Let G be an abelian group. Then f_o and f_1 induce chain maps $f_o \otimes 1,\ f_1 \otimes 1:\ C(K;G) \longrightarrow C(K';G)$ and cochain maps $\mathrm{Hom}(f_o,1),\ \mathrm{Hom}(f_1,1):\ C^\cdot(K';G) \longrightarrow C^\cdot(K;G)$. Suppose that f_o and f_1 are chain homotopic. If $\mathcal{D}$ is a chain deformation exhibiting a

chain homotopy of f_o and f_1, then $\mathcal{D} \otimes 1$ is a chain deformation for $f_o \otimes 1$ and $f_1 \otimes 1$. Furthermore, $\{(-1)^{q+1}\mathrm{Hom}(\mathcal{D}_q,1)\}$ is a cochain deformation for $\mathrm{Hom}(f_o,1)$ and $\mathrm{Hom}(f_1,1)$. It follows that

$$(f_o)_* = (f_1)_*:\ H_*(K;G) \longrightarrow H_*(K';G)$$

and

$$(f_o)^* = (f_1)^*:\ H^*(K';G) \longrightarrow H^*(K;G).$$

A <u>carrier</u> from a regular complex K to a regular complex K' is a function X which assigns to each cell σ of K a non-empty subcomplex $X(\sigma)$ of K' in such a way that if $\sigma < \tau$ then $X(\sigma)$ is a subcomplex of $X(\tau)$. A carrier X from a pair of complexes (K,L) to a pair (K',L') is a carrier from K to K' which when regarded as a function on the cells of L is a carrier from L to L'.

If $f:\ K \longrightarrow K'$ is a continuous function then a <u>carrier for</u> f is a carrier X such that for each cell σ of K, $f\sigma \subseteq X(\sigma)$. X is said to <u>carry</u> f. Carriers for a map always exist -- set $X(\sigma) = K'$ for each σ. The <u>minimal carrier for</u> f is the carrier assigning to each σ in K the smallest subcomplex in K' containing $f\sigma$.

A carrier X from K to K' is <u>acyclic</u> if for each σ in K the subcomplex $X(\sigma)$ of K' is acyclic. A map $f:\ K \longrightarrow K'$ is called <u>proper</u> if the minimal carrier for f is acyclic. We may relativize these notions in the obvious way, noting that if $f:\ (K,L) \longrightarrow (K',L')$ is given, then the minimal carrier for $f|K$ is automatically a carrier from (K,L) to (K',L'). The concept of a proper mapping is fundamental in cellular homology theory because a proper mapping from one regular complex to another can be shown to induce homomorphisms of homology groups.

Finally, we call a chain map $\varphi\colon\ C(K) \longrightarrow C(K')$ proper if for each 0-chain c on K, $\operatorname{In} \varphi c = \operatorname{In} c$.

2.4. LEMMA. *Let K and K' be oriented regular complexes. If X is an acyclic carrier from K to K' then there exists a proper chain map $\varphi\colon\ C(K) \longrightarrow C(K')$ such that for each cell σ of K $\varphi\sigma$ is a chain on $X(\sigma)$. (We say X carries φ.) Furthermore, if φ_0 and φ_1 are any two proper chain maps carried by X, then $\varphi_0 \simeq \varphi_1$ and there exists a chain deformation $\mathcal{D}$ exhibiting this chain homotopy which is also carried by X.*

Proof: We construct φ by induction on dimension. For each 0-cell A of K, we select a vertex B of the non-empty subcomplex $X(A)$ of K'. We then set $\varphi(A) = B$, and extend over all of $C_0(K)$ by linearity. Then for c a 0-chain of K we have $\operatorname{In} \varphi c = \operatorname{In} c$. Given a 1-cell σ of K, $\varphi\partial\sigma$ is a 0-chain on $X(\sigma)$, and $\operatorname{In} \varphi\partial\sigma = \operatorname{In} \partial\sigma = 0$. Since $X(\sigma)$ is acyclic, there exists a 1-chain c on $X(\sigma)$ such that $\partial' c = \varphi\partial\sigma$. We set $\varphi\sigma = c$. Then $\varphi\sigma$ is a 1-chain on $X(\sigma)$ and $\partial'\varphi\sigma = \partial' c = \varphi\partial\sigma$ so that φ commutes with the boundary operators ∂ and ∂'. We extend φ over $C_1(K)$ by linearity.

Suppose that φ has been defined on $C_q(K)$ for all $q \leq n-1$ so that it commutes with the boundary operators and is carried by X. Let σ be an n-cell of K. Then $\varphi\partial\sigma$ is an $(n-1)$-chain on $X(\sigma)$

since $\partial\sigma$ is a chain on σ. Also, $\varphi\partial\sigma$ is a cycle on $X(\sigma)$ because $\partial'\varphi\partial\sigma = \varphi\partial\partial\sigma = 0$. Thus by the acyclicity of $X(\sigma)$ we may choose an n-chain c on $X(\sigma)$ such that $\partial' c = \varphi\partial\sigma$. We set $\varphi\sigma = c$. We extend by linearity over $C_n(K)$ and then φ commutes with the boundary operators and is carried by X. In this manner we define φ for all n.

To prove the second assertion of the Lemma, let φ_0 and φ_1 be two proper chain maps of $C(K)$ into $C(K')$, both carried by the acyclic carrier X. We construct a chain deformation $\mathcal{D}$ exhibiting a chain homotopy of φ_0 and φ_1 by induction as follows. If A is a vertex of K, $\varphi_0 A - \varphi_1 A$ has index zero, and thus bounds in $X(A)$. We set $\mathcal{D}A$ equal to a 1-chain on $X(A)$ having $\varphi_0 A - \varphi_1 A$ as boundary and extend by linearity over $C_0(K)$. Then if c is a 0-chain on K, $\partial'\mathcal{D}c = \varphi_0 c - \varphi_1 c - \mathcal{D}\partial c$ since the last term is zero.

Suppose that σ has been defined for all dimensions less than n so that it is carried by X and satisfies $\partial'\mathcal{D} = \varphi_0 - \varphi_1 - \mathcal{D}\partial$. Let σ be an n-cell of K. Then $\varphi_0\sigma - \varphi_1\sigma - \mathcal{D}\partial\sigma$ is a chain on $X(\sigma)$. We have

$$\begin{aligned}\partial'(\varphi_0\sigma - \varphi_1\sigma - \mathcal{D}\partial\sigma) &= \partial'\varphi_0\sigma - \partial'\varphi_1\sigma - \partial'\mathcal{D}\partial\sigma\\ &= \varphi_0\partial\sigma - \varphi_1\partial\sigma - (\varphi_0 - \varphi_1 - \mathcal{D}\partial)\partial\sigma\\ &= 0.\end{aligned}$$

Then since $X(\sigma)$ is acyclic we can choose an $(n+1)$-chain c on $X(\sigma)$ whose boundary is $\varphi_0\sigma = \varphi_1\sigma - \mathcal{D}\partial\sigma$. We set $\mathcal{D}\sigma = c$, and extend by linearity over $C_n(K)$. We continue in this manner for all n. $\mathcal{D}$ so defined is carried by X and exhibits the desired chain homotopy of φ_0 and φ_1. The proof of the Lemma is complete.

Now suppose that X is an acyclic carrier from (K,L) to (K',L'). By Lemma 2.4, X, as an acyclic carrier from K to K', carries a proper chain map $\varphi\colon C(K) \longrightarrow C(K')$. The restriction of φ to $C(L)$ is a proper chain map $\bar{\varphi}\colon C(L) \longrightarrow C(L')$ which is carried by X as an acyclic carrier from L to L'. Thus if we set

$$\hat{\varphi}[c] = [\varphi c], \quad c \in C_q(K),$$

we obtain a well-defined homomorphism $\hat{\varphi}\colon C_q(K,L) \longrightarrow C_q(K',L')$. Since $\hat{\partial}\hat{\varphi}[c] = \hat{\partial}[\varphi c] = [\partial\varphi c] = [\varphi\partial c] = \hat{\varphi}[\partial c] = \hat{\varphi}\hat{\partial}[c]$, $\hat{\varphi}$ is a chain map. Furthermore, let φ_o, φ_1 be proper chain maps carried by X. Then by 2.4, there exists a chain deformation $\mathcal{D}$ exhibiting a chain homotopy of φ_o and φ_1, and we may suppose $\mathcal{D}$ is carried by X. Thus if $c \in C_q(L)$, $\mathcal{D}c \in C_{q+1}(L')$. It follows that if we set $\hat{\mathcal{D}}[c] = [\mathcal{D}c]$ for $c \in C_q(K)$, then $\hat{\mathcal{D}}\colon C_q(K,L) \longrightarrow C_{q+1}(K',L')$ is well-defined.

$$\begin{aligned}(\hat{\mathcal{D}}\hat{\partial} + \hat{\partial}'\hat{\mathcal{D}})([c]) &= [\mathcal{D}\partial c + \partial'\mathcal{D}c] \\ &= [\varphi_o c - \varphi_1 c] \\ &= \hat{\varphi}_o[c] - \hat{\varphi}_1[c].\end{aligned}$$

Thus $\hat{\varphi}_o$ and $\hat{\varphi}_1$ are chain homotopic. In other words, X induces the triple of chain maps $(\varphi, \bar{\varphi}, \hat{\varphi})$, defined uniquely up to a chain homotopy.

2.5. COROLLARY. *An acyclic carrier* X *from* (K,L) *to* (K',L') *induces a unique set of homology and cohomology homomorphisms*

$$\begin{aligned}
\varphi_*&\colon H_q(K;G) \longrightarrow H_q(K';G) & \varphi^*&\colon H^q(K';G) \longrightarrow H^q(K;G)\\
\bar{\varphi}_*&\colon H_q(L;G) \longrightarrow H_q(L';G) & \bar{\varphi}^*&\colon H^q(L';G) \longrightarrow H^q(L;G)\\
\hat{\varphi}_*&\colon H_q(K,L;G) \longrightarrow H_q(K',L';G) & \hat{\varphi}^*&\colon H^q(K',L';G) \longrightarrow H^q(K,L;G)
\end{aligned}$$

for any coefficient group G. *The triple of homomorphisms* φ_*, $\bar{\varphi}_*$, $\hat{\varphi}_*$ *defines a homomorphism of the relative homology sequences of the pairs* (K,L) *and* (K',L'). *In other words, the diagram below is commutative:*

$$\begin{tikzcd}
\cdots \arrow[r, "\partial_*"] & H_q(L;G) \arrow[r, "i_*"] \arrow[d, "\bar{\varphi}_*"] & H_q(K;G) \arrow[r, "j_*"] \arrow[d, "\varphi_*"] & H_q(K,L;G) \arrow[r, "\partial_*"] \arrow[d, "\hat{\varphi}_*"] & H_{q-1}(L;G) \arrow[r, "i_*"] \arrow[d, "\bar{\varphi}_*"] & \cdots \\
\cdots \arrow[r, "\partial'_*"] & H_q(L';G) \arrow[r, "i'_*"] & H_q(K';G) \arrow[r, "j'_*"] & H_q(K',L';G) \arrow[r, "\partial'_*"] & H_{q-1}(L';G) \arrow[r, "i'_*"] & \cdots
\end{tikzcd}$$

A similar result holds for cohomology.

Proof: Everything but the commutativity of the diagram has been proved, and that follows from V.4.4.

2.6. COROLLARY. *If* $f\colon (K,L) \longrightarrow (K',L')$ *is proper, and if* X_o *and* X_1 *are any two acyclic carriers for* f, *then the associated homology and cohomology homomorphisms for* X_o *and* X_1 *coincide. We say that* f *induces these homomorphisms and write them as*

$$\begin{aligned}
f_*&\colon H_q(K,L;G) \longrightarrow H_q(K',L';G)\\
(f|K)_*&\colon H_q(K;G) \longrightarrow H_q(K';G)\\
(f|L)_*&\colon H_q(L;G) \longrightarrow H_q(L';G)
\end{aligned}$$

and similarly for cohomology.

Proof: Let χ be the minimal carrier for f. Then for each σ of K, $\chi(\sigma) \subseteq \chi_0(\sigma) \cap \chi_1(\sigma)$. Thus, if φ is a proper chain map carried by χ, then φ is also carried by χ_0 and χ_1; here we are regarding χ, χ_0, and χ_1 as carriers from K to K'. The unique homomorphisms induced by χ_0 and χ_1 are φ_*, φ^*, $\bar{\varphi}_*$, etc.

2.7. COROLLARY. If χ is an acyclic carrier from K to K' such that for each cell σ of K, the dimension of $\chi(\sigma)$ is at most that of σ, then there is exactly one proper chain map carried by χ.

Proof: By the Lemma, there is at least one proper chain map φ carried by χ. Suppose that ψ is also carried by χ. Then by the Lemma there is a collection of homomorphisms $\mathcal{D}_q$ carried by χ which gives a chain homotopy $\varphi \simeq \psi$. If σ is a q-cell of K, then $\chi(\sigma)$ is of dimension at most q. So $\mathcal{D}_q\sigma$, which is a (q+1)-chain on $\chi(\sigma)$, must be zero. Thus each $\mathcal{D}_q$ is zero and $\varphi = \psi$.

Examples of proper mappings:

We assume throughout that a regular complex on a closed cell is acyclic. This will be proved in section 4 of this chapter.

(a) Inclusion of a subcomplex: Let (K,L) be a pair of regular complexes, and let i: $L \longrightarrow K$ denote the inclusion map. The minimal carrier for i assigns to each cell σ of L the subcomplex $\bar{\sigma}$ of K, and so i is proper. The induced homomorphisms i_* and i^* are those of the homology and cohomology sequences of the pair (K,L).

b) Automorphism of a complex:

Exercise. Let T: $K \times L \longrightarrow L \times K$ be defined by $T(x,y) = (y,x)$ for $x \in K$, $y \in L$. Then T is proper, and the minimal carrier preserves dimension, so, by 2.7, there is a unique chain map φ induced by T. Show that $\varphi(c \otimes d) = (-1)^{pq}(d \otimes c)$ for $c \in C_p(K)$, $d \in C_q(L)$. Note that the mapping ψ: $C(K) \otimes C(L) \longrightarrow C(L) \otimes C(K)$ defined by $\psi(c \otimes d) = d \otimes c$ is not a chain map.

Exercise. Let T: $R^{n+1} \longrightarrow R^{n+1}$ be given by a diagonal matrix, with k minus ones starting from the upper left, and then $n-k+1$ plus ones. Compute T_* in $H_n(S^n)$.

c) Projection of a product onto its factors: Let f: $K \times L \longrightarrow K$ be the projection mapping. The minimal carrier χ for f satisfies $\chi(\sigma \times \tau) = \bar{\sigma}$. Thus f is proper and induces a unique chain map φ, by 2.7. Note that

$$\varphi(\sigma \otimes \tau) = \begin{cases} 0 & \text{if } \dim \tau > 0 \\ \sigma & \text{if } \tau \text{ is a vertex.} \end{cases}$$

d) Simplicial mappings: Let K and L be simplicial complexes. Then f: $K \longrightarrow L$ is a simplicial mapping if f carries the vertices of each simplex of K into vertices of a simplex of L and is linear in terms of barycentric coordinates. A mapping of the vertices of K into the vertices of L can be extended to a simplicial mapping if and only if it maps vertices spanning a simplex of K onto vertices spanning a simplex of L. If σ is a simplex, and τ is the face opposite a vertex A, then $\bar{\sigma}$ is the cone on $\bar{\tau}$. Thus $\bar{\sigma}$ is acyclic by V.2.5. It follows that a simplicial mapping induces a unique chain map φ: $C(K) \longrightarrow C(L)$.

3. Subdivision of a Regular Complex

3.1. DEFINITION. Let K be a regular complex. The <u>(first iterated) subdivision of</u> K, written $Sd\,K$, is the simplicial complex whose vertices are the cells of K and whose simplexes are defined as follows: A finite collection of cells of K form the vertices of a simplex of $Sd\,K$ if and only if the cells of the collection can be arranged in order so that each is a proper face of the next.

It is obvious from the definition that $Sd\,K$ is a simplicial complex. We topologize $|Sd\,K|$ by giving it the weak topology with respect to closed simplices. Note that if L is a subcomplex of K, then $Sd\,L$ is a simplicial subcomplex of $Sd\,K$. Note also that in general $Sd(K_q)$ is not equal to $(Sd\,K)_q$.

3.2. THEOREM. <u>If</u> K <u>is a regular complex then</u> $|K|$ <u>is homeomorphic to</u> $|Sd\,K|$.

In the proof of this theorem we will use the following lemma.

3.3. LEMMA. <u>If</u> σ <u>is a cell of a regular complex, then</u> $Sd\,\bar{\sigma}$, <u>as a complex is the join of the vertex</u> σ <u>with the subcomplex</u> $Sd\,\dot{\sigma}$.

Proof: A cell τ of $Sd\,\bar{\sigma}$ is a collection of faces of σ which can be arranged in an increasing order: $\sigma_0 < \sigma_1 < \ldots < \sigma_k$. Either $\sigma_k = \sigma$ or τ is a cell of $Sd\,\dot{\sigma}$. If $\sigma_k = \sigma$, then $\bar{\tau}$ is the join of the vertex σ with the cell $\sigma_0 < \ldots < \sigma_{k-1}$, unless $k = 0$. If $k = 0$ then τ is the vertex σ. Thus the cells of $Sd\,\bar{\sigma}$ are those of the join of the vertex σ with the subcomplex $Sd\,\dot{\sigma}$.

Proof of Theorem 3.2: We define a homeomorphism $h\colon |Sd\,K| \longrightarrow |K|$ by step wise extension over the subcomplexes $Sd(K_q)$. In dimension zero, every vertex of K is also a vertex of $Sd\,K$, so we have the obvious homeomorphism $h_0\colon Sd(K_0) \longrightarrow K_0$.

Suppose that we have extended h_0 to a homeomorphism h_{q-1} of $Sd(K_{q-1})$ onto K_{q-1}. Let σ be a q-cell of K. Then, by Lemma 3.3, $Sd\,\bar{\sigma} = \sigma \circ Sd\,\dot{\sigma}$. We choose a homeomorphism $f\colon E^q \longrightarrow \bar{\sigma}$. E^q is homeomorphic to the join of the origin with S^{q-1}. By hypothesis, $h_{q-1}\colon Sd\,\dot{\sigma} \longrightarrow \dot{\sigma}$ is a homeomorphism and so the mapping $f^{-1}h_{q-1}\colon Sd\,\dot{\sigma} \longrightarrow S^{q-1}$ is a homeomorphism. We extend $f^{-1}h_{q-1}$ to a homeomorphism g mapping $Sd\,\bar{\sigma}$ onto E^q which sends the vertex σ into the origin. We define h_q on $Sd\,\bar{\sigma}$ to be fg. We note that on $Sd\,\dot{\sigma}$ we have $h_q = fg = ff^{-1}h_{q-1} = h_{q-1}$ so that h_q extends h_{q-1}. We proceed in this manner for every q-cell of K. The resulting mapping h_q extends h_{q-1} and is a homeomorphism since a function on a regular comples is continuous if and only if it is continuous on each closed cell of the complex. The collection of mappings $\{h_q\}$ defines a function $h\colon |Sd\,K| \longrightarrow |K|$. The inverse of h is continuous because it is continuous on each skeleton of K. The function h is also continuous. This follows from the fact that if a regular complex L is expressed as a union of subcomplexes L_α, then the topology on $|L|$ is the weak topology with respect to the closed subsets $|L_\alpha|$. Thus h is a homeomorphism and the proof is complete.

Note that the homeomorphism h of $|Sd\,K|$ onto $|K|$ given in the proof of 3.2 preserves the cellular structure, in the sense that

h carries $\mathrm{Sd}\,\bar{\sigma}$ onto $\bar{\sigma}$ for each cell σ of K. The construction of the homeomorphism h involves choosing, for each cell σ, a homeomorphism with the Euclidean ball of the same dimension. We shall call a function $h\colon |\mathrm{Sd}\,K| \longrightarrow |K|$ and its inverse h^{-1} <u>subdivision homeomorphisms</u> if h satisfies the following two properties:

a) h is a homeomorphism which carries $\mathrm{Sd}\,\bar{\sigma}$ onto $\bar{\sigma}$ for each cell σ of K.

b) If K is itself a simplicial complex, then for each simplex σ of K, h carries the vertex σ of $\mathrm{Sd}\,K$ onto the barycenter of σ, and for each simplex τ of $\mathrm{Sd}\,\bar{\sigma}$, h maps $\bar{\tau}$ linearly, in terms of barycentric coordinates, into $\bar{\sigma}$.

We leave as an exercise to the reader the verification that if K is a simplicial complex, there is one and only one homeomorphism $|\mathrm{Sd}\,K| \longrightarrow |K|$ satisfying a) and b) above.

3.4. COROLLARY. <u>Given a finite regular complex</u> K, <u>there exists</u> n <u>such that</u> $|K|$ <u>can be embedded in</u> R^n.

Proof: $|K|$ is homeomorphic to $|\mathrm{Sd}\,K|$ by 3.2. $\mathrm{Sd}\,K$ is a finite simplicial complex, a subcomplex of the simplex $\bar{\sigma}$ on all of the vertices of $\mathrm{Sd}\,K$. If n is the number of vertices of $\mathrm{Sd}\,K$, then $\bar{\sigma}$ can be embedded in R^{n-1}. Thus K can be embedded in Euclidean space of dimension one less than the number of cells of K.

<u>Exercise</u>: Prove that a countable regular complex of dimension n can be embedded in R^{2n+1}. [Hint: Suppose that K_q is embedded in R^{2q+1}. For each $(q+1)$-cell σ of K, choose a hyperplane P_σ in R^{2q+3} which contains R^{2q+1}, in such a way that no P_σ contains a limit point of the set consisting of all the other P_τ's, except in R^{2q+1}. One easily embeds each $\bar{\sigma}$ in P_σ so that $\bar{\sigma} \cap R^{2q+1} = \dot{\sigma}$, and then one must verify that the result is indeed an embedding of K_{q+1} in R^{2q+3}.]

3.5. DEFINITION. The <u>0th iterated subdivision</u> of a regular complex K is K itself. The <u>(n+1)st iterated subdivision</u> of K is the first subdivision of the nth iterated subdivision of K. We write $\mathrm{Sd}^n K$ for the nth iterated subdivision of K.

We now describe a way of metrizing a finite regular complex and all of its iterated subdivisions. Let K be a finite regular complex. Choose a subdivision homeomorphism $g_0\colon \mathrm{Sd}\,K \longrightarrow K$ and an embedding $h_1\colon \mathrm{Sd}\,K \longrightarrow R^n$ which is linear, in terms of barycentric coordinates, on each closed simplex of $\mathrm{Sd}\,K$. (It suffices to embed $\mathrm{Sd}\,K$ as a subcomplex of the standard simplex with $n+1$ vertices, where $n+1$ is the number of cells of K.) The embeddings $h_1 g_0^{-1}$ and h_1 define a metric on $|K|$ and $|\mathrm{Sd}\,K|$, using the metric on R^n, in the obvious way. By the remarks preceding 3.4, we have uniquely defined homeomorphisms $g_k\colon \mathrm{Sd}^{k+1}K \longrightarrow \mathrm{Sd}^k K$. We metrize $|\mathrm{Sd}^k K|$, $k \geq 2$, in terms of the metric on $\mathrm{Sd}\,K$ by demanding that each g_k be an isometry. In the future we shall always regard a finite complex K and its subdivisions as metrized in this manner. We note that the metrization depends upon the choices of the subdivision homeomorphism $g_0\colon \mathrm{Sd}\,K \longrightarrow K$ and of the embedding $h_1\colon \mathrm{Sd}\,K \longrightarrow R^n$.

3.6. DEFINITION. Given a metrized finite regular complex K, the <u>mesh</u> of K is the maximum of the diameters of the cells of K.

It is obvious that the diameter of a triangle linearly embedded in Euclidean space is the length of its longest edge. From this it follows that if $\bar{\sigma}$ is an n-simplex embedded in some Euclidean space, and x and y are two points of $\bar{\sigma}$ such that the distance between them is the diameter of $\bar{\sigma}$, then neither x nor y lies in the interior of a straight line segment contained in $\bar{\sigma}$. Thus x and y must be vertices of $\bar{\sigma}$, and so the diameter of $\bar{\sigma}$ is the length of its longest edge. Therefore the mesh of a finite simplicial complex linearly embedded in some Euclidean space is the length of its longest 1-simplex.

3.7. THEOREM. <u>Let</u> K <u>be a finite regular complex. Then</u> $\lim_{r\to\infty} \mathrm{Mesh}(Sd^r K) = 0.$

Proof: It is clearly sufficient to show that if L is a finite simplicial complex of dimension q, linearly embedded in R^n, then

$$\mathrm{Mesh}(Sd\ L) \le \frac{q}{q+1}\ \mathrm{Mesh}\ L.$$

The metric on $Sd\ L$ is defined by requiring that the subdivision homeomorphism $h\colon |Sd\ L| \longrightarrow |L|$ be an isometry.

Let e be the longest edge of $Sd\ L$. Suppose that the vertices of e are the simplices σ_1 and σ_2 of L, with $\sigma_1 < \sigma_2$. We may assume that the vertices of σ_1 are $A_0, A_1, \ldots, A_j$, and that the vertices of σ_2 are those of σ_1 together with $A_{j+1}, A_{j+2}, \ldots, A_k$, where $k \le q$. Under the subdivision homeomorphism $h\colon |Sd\ L| \longrightarrow |L|$, the

vertices σ_1 and σ_2 are mapped into the barycenters of the simplices σ_1 and σ_2 respectively. To compute the length of the edge e we compute the distance between the barycenters of σ_1 and σ_2. The linear embedding $g\colon L \longrightarrow R^n$ carries each vertex A_i into a point B_i, and the barycenters of σ_1 and σ_2 into the points $\frac{1}{j+1}\sum_{i=0}^{j} B_i$ and $\frac{1}{k+1}\sum_{i=0}^{k} B_i$ respectively. (Here we add the B_i as vectors in R^n). The distance between these two points is the length of the vector joining them. This vector is:

$$\frac{1}{j+1}\sum_{i=0}^{j} B_i - \frac{1}{k+1}\sum_{i=0}^{k} B_i$$

$$= \left(\frac{1}{j+1} - \frac{1}{k+1}\right)\sum_{i=0}^{j} B_i - \frac{1}{k+1}\sum_{i=j+1}^{k} B_i$$

$$= \frac{k-j}{(j+1)(k+1)}\sum_{i=0}^{j} B_i - \frac{1}{k+1}\sum_{i=j+1}^{k} B_i$$

$$= \left(\frac{k-j}{k+1}\right)\left(\frac{1}{j+1}\sum_{i=0}^{j} B_i - \frac{1}{k-j}\sum_{i=j+1}^{k} B_i\right).$$

The inverse g-image of $\frac{1}{j+1}\sum_{i=0}^{j} B_i$ is the barycenter of σ_1; the inverse g-image of $\frac{1}{k-j}\sum_{i=j+1}^{k} B_i$ is the barycenter of the simplex of L spanned by $A_{j+1}, A_{j+2}, \ldots, A_k$. These barycenters both belong to the closed simplex $\bar{\sigma}_2$ and so the distance between them is at most Mesh L. Thus the length of the edge e is at most $(k-j)/(k+1) \le k/(k+1) \le q/(q+1)$ times the mesh of L. Thus, by the comments after Definition 3.6, the proof of 3.7 is complete.

We are about to prove the main theorem of this section. Given a mapping $f\colon K \longrightarrow K'$ of finite regular complexes, we have, for each r and s, the following diagram:

$$\begin{tikzcd} K \arrow[r, "f"] \arrow[d, "g"'] & K' \\ Sd^s K \arrow[r, "f'"] & Sd^r K' \arrow[u, "g'"'] \end{tikzcd}$$

In this diagram the mappings g' and g are the iterations of subdivision homeomorphisms. (See comments before 3.4.) The mapping f' is defined by the relation $f = g'f'g$. We say that f induces f' or that f' is the mapping f as a mapping of $Sd^s K$ into $Sd^r K'$. We shall sometimes write f for f'. Before stating our theorem we prove the following lemma from general topology.

3.8. LEMMA. (Lebesgue Covering Lemma). *Let* X *be a compact metric space and let* $\{U_i\}$ *be an arbitrarily indexed open covering of* X. *Then there exists a number* $\delta > 0$, *called a Lebesgue number of the covering* $\{U_i\}$, *such that every set of* X *of diameter less than* δ *is contained in some* U_i.

Proof: For each $x \in X$ and every i, let $d(x,i)$ be the radius of the largest open ball around x contained in U_i. Define
$$d(x) = \underset{i}{\ell.u.b.}\ d(x,i).$$
Then $d(x)$ is finite for each x since X is bounded. We show that $d\colon X \longrightarrow R$ is continuous. Letting ρ be the metric on X, we have $|d(x) - d(y)| < \epsilon$ whenever $\rho(x,y) < \epsilon$. Next we note that $d(x)$ is positive for each x, since the U_i cover X. Thus d maps X

into a compact subset of the positive real numbers. So $d(X)$ is bounded away from zero by some $\epsilon > 0$. Choose any δ such that $0 < \delta < \epsilon$. Let W be a set of X of diameter $< \delta$. Choose $x \in W$. W is contained in the open δ-ball V about x. But $\delta < \epsilon \leq d(x)$. Thus for some i, $W \subseteq V \subseteq U_i$. So δ is a Lebesgue number of the covering $\{U_i\}$.

3.9. THEOREM. *Let* K *and* K' *be finite regular complexes,* f *a continuous function mapping* K *into* K', *and* r *a positive integer. Then there exists an integer* N *such that for each* $s > N$, *the map* f *is proper as a map of* $Sd^s K$ *into* $Sd^r K'$.

Proof: The complex K is covered by the collection $\mathcal{C}$ of open sets $f^{-1}(\mathrm{St}\ A_j)$*, where A_j varies over the vertices of $Sd^r K'$. Since K is a compact metric space, we may by 3.8 choose a Lebesgue number δ for the covering $\mathcal{C}$. Let N be such that the mesh of $Sd^N K$ is less than δ. Given $s > N$, we show that the minimal carrier for $f\colon Sd^s K \longrightarrow Sd^r K'$ is acyclic. We note first that the smallest subcomplex of $Sd^r K'$ containing a given point set X is the union of the closures of those cells of $Sd^r K'$ which have a non-trivial intersection with X.

Let σ be a cell (open simplex) of $Sd^s K$. Then the diameter of σ is less than δ so that $\sigma \subseteq f^{-1}\mathrm{St}A$ for some vertex A of $Sd^r K'$. Thus $f\sigma \subseteq \mathrm{St}A$. If $f\sigma$ meets a cell τ of $Sd^r K'$, then A is a vertex of τ. We can thus write $\bar{\tau}$ as the join of A with the face $\bar{\tau}_A$ of $\bar{\tau}$ opposite A: $\bar{\tau} = A \circ \bar{\tau}_A$. Thus the smallest subcomplex of $Sd^r K'$ containing $f\sigma$ is $\bigcup_{f\sigma \cap \tau \neq \emptyset} \bar{\tau} = \bigcup_{\tau} A \circ \bar{\tau}_A = A \circ \bigcup_{\tau} \bar{\tau}_A$. Since the

*If A is a vertex of a regular complex K, the (open) star of A, written $\mathrm{St}\ A$, is the union of all (open) cells of K having A as a vertex.

join of a vertex and a complex is acyclic, by V.2.5, the smallest subcomplex containing $f\sigma$ is acyclic for each σ of Sd^sK. Thus the minimal carrier for f is acyclic, and f is proper.

4. The Induced Homomorphisms and the Invariance Theorem.

In this section we describe the process of associating with each continuous mapping of one finite regular complex into another a sequence of homomorphisms of the homology groups of the complexes, and similarly for cohomology. We have already shown that a proper mapping induces such homomorphisms. Thus Theorem 3.9, which associates with an arbitrary mapping of a finite regular complex a proper mapping, will play an essential role in all that follows.

4.1. THEOREM. <u>Each continuous mapping</u> $f\colon (K,L) \longrightarrow (K',L')$ <u>of pairs of oriented finite regular complexes induces homomorphisms</u> $f_*\colon H_q(K,L;G) \longrightarrow H_q(K',L';G)$ <u>and</u> $f^*\colon H^q(K',L';G) \longrightarrow H^q(K,L;G)$ <u>for each</u> q, <u>such that the following conditions hold</u>:

a) <u>If</u> $f\colon (K,L) \longrightarrow (K,L)$ <u>is the identity mapping, then, for each</u> q, $f_*\colon H_q(K,L;G) \longrightarrow H_q(K,L;G)$ <u>and</u> $f^*\colon H^q(K,L;G) \longrightarrow H^q(K,L;G)$ <u>are the identity isomorphisms</u>.

b) <u>If</u> $f\colon (K,L) \longrightarrow (K',L')$ <u>and</u> $g\colon (K',L') \longrightarrow (K'',L'')$ <u>are arbitrary continuous mappings, then, for each</u> q,

$$(gf)_* = g_*f_*\colon H_q(K,L;G) \longrightarrow H_q(K'',L'';G)$$

and

$$(gf)^* = f^*g^*\colon H^q(K'',L'';G) \longrightarrow H^q(K,L;G) .$$

Proof: Let A_n be the statement that the theorem holds for all finite complexes of dimension at most n. Let B_n be the following statement: If σ is a cell of an oriented regular complex and the dimension of σ is at most n, then $\bar{\sigma}$ is an acyclic complex.

Then B_0, A_0, and B_1 are all obvious.

$A_{n-1} \Longrightarrow B_n$: Let σ be a cell of a regular complex and let $\dim \sigma = m \leq n$. Then the dimension of $\dot{\sigma}$ is $\leq n-1$. We have a homeomorphism $f\colon \dot{\sigma} \longrightarrow S^{m-1}$, where S^{m-1} is the regular complex on the $(m-1)$ sphere given in Section I.2. Then by A_{n-1}, there are induced homomorphisms $f_*\colon H_q(\dot{\sigma}) \longrightarrow H_q(S^{m-1})$ and $(f^{-1})_*\colon H_q(S^{m-1}) \longrightarrow H_q(\dot{\sigma})$ for each q. From b) and a) we derive that

$$f_*(f^{-1})_* = (ff^{-1})_* = 1_* = 1 .$$

Similarly, $(f^{-1})_* f_* = 1$. Thus f_* is an isomorphism so that

$$H_q(\dot{\sigma}) \approx \begin{cases} Z & \text{if } q \text{ is } 0 \text{ or } m-1 \\ 0 & \text{otherwise.} \end{cases}$$

Consider the following diagram:

$$\begin{array}{ccccccc} Z \approx C_m(\bar{\sigma}) & \longrightarrow & C_{m-1}(\bar{\sigma}) & \longrightarrow & \cdots & \longrightarrow & C_o(\bar{\sigma}) \\ & & \uparrow \text{equality} & & & & \uparrow \text{equality} \\ & & C_{m-1}(\dot{\sigma}) & \longrightarrow & \cdots & \longrightarrow & C_o(\dot{\sigma}) \end{array}$$

Here $C_m(\bar{\sigma})$ is generated by σ. We have $Z_{m-1}(\bar{\sigma}) = Z_{m-1}(\dot{\sigma}) = H_{m-1}(\dot{\sigma}) \approx Z$. The cycle $\partial\sigma$ has coefficient ± 1 on each $(m-1)$ cell of $\bar{\sigma}$. This implies first that $\partial\sigma \neq 0$, so that $H_m(\bar{\sigma}) = 0$, and secondly that $\partial\sigma$ actually generates $Z_{m-1}(\bar{\sigma})$, so that $H_{m-1}(\bar{\sigma}) = 0$. $H_q(\bar{\sigma})$ is zero for $1 \leq q \leq m-2$ by the above diagram. We have therefore shown that $\bar{\sigma}$ is acyclic.

$B_n \Longrightarrow A_n$. We first derive some preliminary results, which follow

from B_n.

4.2. LEMMA. If the dimension of the oriented regular complex K is at most n and if $g\colon (\mathrm{Sd}K, \mathrm{Sd}L) \longrightarrow (K,L)$ is a subdivision homeomorphism, then

a) g and g^{-1} are proper maps.

b) g_* and g^*, the homology and cohomology homomorphisms induced by g given by 2.5, are independent of the choice of g.

c) g_* and g^* are isomorphisms.

Proof: By 3.3 and V.2.5, if σ is a cell of a regular complex, then $\mathrm{Sd}\bar{\sigma}$ is acyclic. The minimal carrier X_1 for g^{-1} maps a cell σ of K into the subcomplex $\mathrm{Sd}\bar{\sigma}$ of $\mathrm{Sd}K$. Thus g^{-1} is proper. Next we show that g is also proper. If σ is a cell of $\mathrm{Sd}K$, then σ is a collection of cells of K which can be arranged in an increasing sequence: $\sigma_0 < \sigma_1 < \ldots < \sigma_q$. The vertex σ_q of σ is mapped by g into an interior point of σ_q. All of σ is mapped into $\bar{\sigma}_q$, so the smallest subcomplex of K containing $g\sigma$ is $\bar{\sigma}_q$. Since σ_q is a cell of dimension at most n, we have from B_n that $\bar{\sigma}_q$ is acyclic. Thus the minimal carrier X_2 for g is acyclic and so g is proper. We have proved a).

Since the minimal carriers X_1 and X_2 for g^{-1} and g are independent of the choice of g, it follows that the homology and cohomology homomorphisms induced by g and by g^{-1} are independent of the choice of g. This proves b).

Now let φ_1 and φ_2 be proper chain maps carried by X_1 and X_2 respectively. Let σ be a cell of K. Then $\varphi_1\sigma$ is a chain

$\Sigma\, a_\tau \tau$ on $\chi_1(\sigma) = Sd\bar{\sigma}$. For each cell τ of $Sd\bar{\sigma}$, $\chi_2(\tau)$ is a closed cell contained in $\bar{\sigma}$. Thus $\varphi_2\varphi_1(\sigma)$ is a chain on $\bar{\sigma}$. The carrier X from K to K sending each cell σ to $\bar{\sigma}$ is acyclic by B_n. X carries $\varphi_2\varphi_1$ and also the identity chain map 1. Since X does not increase dimension, we have from Corollary 2.7 the fact that $\varphi_2\varphi_1 = 1$. This of course implies that $g_*(g^{-1})_* = 1$ and $(g^{-1})^* g^* = 1$.

Let σ be a cell of SdK. If σ is the collection $\sigma_0 < \sigma_1 < \ldots < \sigma_q$ of cells of K then we have $\chi_2(\sigma) = \bar{\sigma}_q$. Thus $\varphi_2(\sigma)$ is a chain on $\bar{\sigma}_q$. For each cell τ of $\bar{\sigma}_q$, we have $\chi_1(\tau) = Sd\bar{\tau} \subseteq Sd\bar{\sigma}_q$. Consequently, $\varphi_1\varphi_2(\sigma)$ is a chain on $Sd\bar{\sigma}_q$. $\varphi_1\varphi_2$ is then carried by the acyclic carrier which sends each cell $\sigma_0 < \ldots < \sigma_q$ of SdK into $Sd\bar{\sigma}_q$. But so is the identity chain map, and so $\varphi_1\varphi_2 \simeq 1$ by 2.4. It follows that $(g^{-1})_* g_* = 1$ and $g^*(g^{-1})^* = 1$. Thus g_* and g^* are isomorphisms and the lemma is proved.

4.3. COROLLARY. *If* K *is a regular complex and* σ *is a cell of* K *of dimension at most* n, *then* $Sd^r\bar{\sigma}$ *is acyclic for every* $r \geq 0$.

Proof: $\bar{\sigma}$ is acyclic by B_n. For $r \geq 0$, and q arbitrary, $H_q(Sd^r\bar{\sigma}) \approx H_q(Sd^{r+1}\bar{\sigma})$, by Lemma 4.2. Thus $Sd^r\bar{\sigma}$ is acyclic for $r \geq 0$.

4.4. NOTATION. For each pair of nonnegative integers r and s we have a homeomorphism $g_{r,s}\colon (Sd^rK, Sd^rL) \longrightarrow (Sd^sK, Sd^sL)$ obtained by composing subdivision homeomorphisms.

4.5. LEMMA. *If* K *is of dimension at most* n, *then* $g_{r,s}$ *is proper for all* r, s. *If* $r < s$, *then*

$$(g_{r,s})_* = (g_{s-1,s})_*(g_{s-2,s-1})_* \cdots (g_{r,r+1})_*$$

$$(g_{r,s})^* = (g_{r,r+1})^*(g_{r+1,r+2})^* \cdots (g_{s-1,s})^* .$$

If $r > s$, *then*

$$(g_{r,s})_* = (g_{s+1,s})_*(g_{s+2,s+1})_* \cdots (g_{r,r-1})_*$$

$$(g_{r,s})^* = (g_{r,r-1})^*(g_{r-1,r-2})^* \cdots (g_{s+1,s})^* .$$

Proof: Suppose that $r < s$, and let σ be a cell of Sd^rK. The minimal carrier X for $g_{r,s}$ sends σ to $Sd^{s-r}\bar{\sigma}$, and so is acyclic by 4.3. Thus $g_{r,s}$ is proper. If φ_i is a proper chain map carried by the minimal carrier X_i for $g_{i,i+1}$, $r \leq i \leq s-1$, then $\varphi_{s-1}\varphi_{s-2} \cdots \varphi_r(\sigma)$ is a chain on $Sd^{s-r}\bar{\sigma}$. By Corollary 2.5, and by Theorem II.5.2,

$$\begin{aligned}(g_{r,s})_* &= (\varphi_{s-1}\varphi_{s-2} \cdots \varphi_r)_* \\ &= (\varphi_{s-1})_* \cdots (\varphi_r)_* \\ &= (g_{s-1,s})_* \cdots (g_{r,r+1})_* .\end{aligned}$$

Also, by Corollary 2.5 and the exercise at the end of Chapter II, we may derive analogously that

$$(g_{r,s})^* = (g_{r,r+1})^* \cdots (g_{s-1,s})^* .$$

Now suppose that $r > s$. We have

$$g_{r,s} = g_{s+1,s}\, g_{s+2,s+1} \cdots g_{r,r-1} .$$

Each $g_{i,i-1}$, for $s+1 \leq i \leq r$, maps cells into (open) cells since it is a subdivision homeomorphism. Thus $g_{r,s}$ maps cells into cells and so is proper by B_n. More explicitly, let σ be a cell of $Sd^r K$. There are uniquely determined cells σ_i in $Sd^i K$, $s \leq i \leq r$, with $\sigma_r = \sigma$ and

$$(g_{i,i-1})\sigma_i \subsetneq \sigma_{i-1} \quad \text{for } s+1 \leq i \leq r .$$

The minimal carrier X_i for $g_{i,i-1}$ sends σ_i to $\bar{\sigma}_{i-1}$. Also, X_i sends each face of σ_i to a subcomplex of $\bar{\sigma}_{i-1}$. Thus if φ_i is a proper chain map carried by X_i, and c is a chain on $\bar{\sigma}_i$, then $\varphi_i(c)$ is a chain on $\bar{\sigma}_{i-1}$. From this it follows immediately that $\varphi_{s+1}\varphi_{s+2} \cdots \varphi_r\sigma$ is a chain on σ_s. Thus the minimal carrier for $g_{r,s}$ carries the proper chain map $\varphi_{s+1} \cdots \varphi_r$. The desired relations involving $(g_{r,s})_*$ and $(g_{r,s})^*$ follow immediately.

4.6. COROLLARY. <u>Given nonnegative integers</u> r,s, <u>and</u> t,

$$(g_{r,t})_* = (g_{s,t})_*(g_{r,s})_*$$

<u>and</u>

$$(g_{r,t})^* = (g_{r,s})^*(g_{s,t})^* .$$

Proof: This corollary follows immediately from Lemma 4.5 and the fact that $(g_{i,i+1})_* = [(g_{i+1,i})_*]^{-1}$, proved in Lemma 4.2.

We proceed with the proof that $B_n \Longrightarrow A_n$. Let $f\colon (K,L) \longrightarrow (K',L')$ be an arbitrary mapping of pairs of finite regular complexes of dimension $\leq n$. Let r be a positive integer. Then by Theorem 3.9, there exists an integer N such that for $s > N$, f is proper as a mapping of $(Sd^s K, Sd^s L)$ into $(Sd^r K', Sd^r L')$. Thus we have the following diagram:

$$\begin{array}{ccc} (K,L) & \xrightarrow{f} & (K',L') \\ g_{o,s}\downarrow & & \uparrow g'_{r,o} \\ (Sd^s K,\ Sd^s L) & \xrightarrow{f'} & (Sd^r K',\ Sd^r L') \end{array}$$

In this diagram $g_{o,s}$ and $g'_{r,o}$ are iterated subdivision homeomorphisms and $f = g'_{r,o} f' g_{o,s}$. The mappings $g_{o,s}, f', g'_{r,o}$ are all proper, and we make the following definition.

4.7. DEFINITION. If $f\colon (K,L) \longrightarrow (K',L')$ is a mapping of finite regular complexes of dimension at most n, then the <u>homology and cohomology homomorphisms induced by</u> f, written f_* and f^*, respectively, are the compositions

$$f_* = (g'_{r,o})_*(f')_*(g_{o,s})_*$$

and

$$f^* = (g_{o,s})^*(f')^*(g'_{r,o})^* .$$

In order to justify this definition, we prove:

4.8. LEMMA. <u>The induced homomorphisms</u> f_* <u>and</u> f^* <u>are independent of</u> r <u>and</u> s.

Proof: Let r be given and suppose s and t, with $s < t$, are such that f, as a mapping of both $(Sd^s K, Sd^s L)$ and $(Sd^t K, Sd^t L)$ into $(Sd^r K', Sd^r L')$, is proper. In the diagram below, f_0 and f_1 are induced by f:

$$\begin{tikzcd}
(K,L) \arrow[rr, "f"] \arrow[d, "g_{o,s}"] \arrow[dd, bend right, "g_{o,t}"'] & & (K',L') \\
(Sd^sK,\ Sd^sL) \arrow[rr, "f_0"] \arrow[d, "g_{s,t}"] & & (Sd^rK',\ Sd^rL') \arrow[u, "g'_{r,o}"'] \\
(Sd^tK,\ Sd^tL) \arrow[urr, "f_1"'] & &
\end{tikzcd}$$

By 4.6, $(g_{s,t})_*(g_{o,s})_* = (g_{o,t})_*$, and similarly for cohomology. Thus we need prove only that

$$(f_0)_* = (f_1)_*(g_{s,t})_*$$

and that

$$(f_0)^* = (g_{s,t})^*(f_1)^* .$$

Let X, X_0, X_1 <u>be the minimal carriers for</u> $g_{s,t}, f_0$, <u>and</u> f_1 , and let $\varphi, \varphi_0, \varphi_1$ be proper chain maps carried by X, X_0 , and X_1 . If σ is a cell of Sd^sK , then $X(\sigma) = Sd^{t-s}\bar{\sigma}$. So $\varphi\sigma$ is a chain on $Sd^{t-s}\bar{\sigma}$, and $\varphi_1\varphi\sigma$ is a chain on $\bigcup_{\tau \in Sd^{t-s}\bar{\sigma}} X_1(\tau)$.

If $\tau \in Sd^{t-s}\bar{\sigma}$, then $f_1\,\tau \subseteq f_0\,\bar{\sigma} \subseteq X_0(\sigma)$. Thus $\varphi_1\varphi\sigma$ is a chain on $X_0(\sigma)$ and so $\varphi_1\varphi \simeq \varphi_0$. The desired equalities follow.

Let r and r' be given, with $r < r'$, and suppose s is chosen so that f is proper as a mapping of (Sd^sK, Sd^sL) into (Sd^rK', Sd^rL') and into $(Sd^{r'}K', Sd^{r'}L')$. In the diagram below, f_0 and f_1 are induced by f :

$$\begin{tikzcd}
(K,L) \arrow[r, "f"] \arrow[d, "g_{o,s}"] & (K',L') \\
(Sd^sK,\ Sd^sL) \arrow[r, "f_0"] \arrow[dr, "f_1"'] & (Sd^rK',\ Sd^rL') \arrow[u, "g'_{r,o}"'] \\
 & (Sd^{r'}K',\ Sd^{r'}L') \arrow[u, "g'_{r',r}"'] \arrow[uu, bend right, "g_{r',o}"']
\end{tikzcd}$$

By 4.6, we need only prove that

$$(f_0)_* = (g'_{r',r})_*(f_1)_*$$

and

$$(f_0)^* = (f_1)^*(g'_{r',r})^* .$$

Let σ be a cell of Sd^sK . Let X_0, X_1 be the minimal carriers for f_0, f_1 and φ_0, φ_1 proper chain maps carried by X_0, X_1 . Then $\varphi_1\sigma$ is a chain on $X_1(\sigma)$. $X_0(\sigma)$ is a subcomplex containing $f_0\sigma$, so $Sd^{r'-r}\, X_0(\sigma)$ is a subcomplex of $Sd^{r'}K'$ containing $f_1\sigma$. Thus

$$X_1(\sigma) \subseteq Sd^{r'-r}\, X_0(\sigma) \text{ , and } g'_{r',r}(X_1(\sigma)) \subseteq X_0(\sigma) .$$

Thus if φ is a chain map carried by the minimal carrier for $g'_{r',r}$, $\varphi\varphi_1\sigma$ is a chain on $X_0(\sigma)$. Consequently $\varphi_0 \simeq \varphi\varphi_1$, and the desired equalities follow. The proof of the lemma is complete.

We show now that the induced homomorphism satisfies properties a) and b) of the theorem. To prove a) we note that the identity map is proper by B_n . If $f\colon (K,L) \longrightarrow (K',L')$ is an arbitrary proper mapping of finite regular complexes of dimension $\leq n$, then by an argument similar to the proof of Lemma 4.8, the induced homomorphisms

f_* and f^* given by Definition 4.7 coincide with those previously defined (Corollary 2.6) for f as a proper mapping. Property a) follows as a special case.

To prove b), let $f\colon (K,L) \longrightarrow (K',L')$ and $g\colon (K',L') \longrightarrow (K'',L'')$ be mappings of pairs of finite regular complexes of dimension $\leq n$. Let r be a positive integer. We subdivide (K',L') so finely that if A is a vertex of Sd^sK', then there is a vertex B of Sd^rK'' such that $g(\overline{St\ A}) \subseteq St\ B$. For, if δ is a Lebesgue number of the covering

$$\{g^{-1}(St\ B) \mid B \text{ a vertex of } Sd^rK''\} ,$$

then it suffices to take S so large that $\mathrm{Mesh}(Sd^sK') < \frac{\delta}{2}$. In order to ensure that stars of vertices of Sd^sK' are acyclic, we also require that $s \geq 1$. Then we take t so large that if σ is a cell of Sd^tK, there is a vertex A of Sd^sK' such that $f\sigma \subseteq St\ A$, as in the proof of Theorem 3.9. In the diagram below, f' and g' are the mappings induced by f and g respectively:

$$\begin{tikzcd}
(K,L) \arrow[d, "g_{o,t}"'] & (K',L') \arrow[d, shift left, "g'_{o,s}"] & (K'',L'') \\
(Sd^tk,\ Sd^tL) \arrow[r, "f'"] & (Sd^sK',\ Sd^sL') \arrow[u, shift left, "g'_{s,o}"] \arrow[r, "g'"] & (Sd^rK'',\ Sd^rL'') \arrow[u, "g''_{r,o}"']
\end{tikzcd}$$

Arguing as in Theorem 3.9, we show easily that f', g', and $g'f'$ are all proper. Let their minimal carriers be X_1, X_2, and X_3, respectively. Then each X_i carries a proper chain map φ_i. Given a cell σ of Sd^tK, $f'\sigma \subseteq St\ A$ for some vertex A of Sd^sK'. Thus $X_1(\sigma) \subseteq St\ A$, and $\varphi_1\sigma$ is a chain on $X_1(\sigma)$. But $g'(\overline{St\ A}) \subseteq St\ B$ for some vertex B of Sd^rK''. This implies that for each cell τ

of $X_1(\sigma)$, $g'(\tau) \subseteq St\ B$ and so $X_2(\tau) \subseteq St\ B$. As in the proof of Theorem 3.9, $X_2(\tau)$, which is the minimal subcomplex containing $g'(\tau)$, can be written in the form $B \circ M_\tau$, where M_τ is a subcomplex of Sd^rK'' depending on τ. Thus $\bigcup_{\tau \in X_1(\sigma)} X_2(\tau) =$ $\bigcup_{\tau \in X_1(\sigma)} B \circ M_\tau = B \circ \bigcup_{\tau \in X_1(\sigma)} M_\tau$ is an acyclic subcomplex of $\overline{St\ B}$. We define

$$X_4(\sigma) = \bigcup_{\tau \in X_1(\sigma)} X_2(\tau) .$$

If $\sigma' < \sigma$ then $X_1(\sigma') \subseteq X_1(\sigma)$ so $X_4(\sigma') \subseteq X_4(\sigma)$. Thus X_4 is an acyclic carrier. Since $\varphi_1\sigma$ is a chain on $X_1(\sigma)$, $\varphi_2\varphi_1(\sigma)$ is a chain on $X_4(\sigma)$. So X_4 is an acyclic carrier for $\varphi_2\varphi_1$. But

$$g'f'(\sigma) \subseteq g'X_1(\sigma) = \bigcup_{\tau \in X_1(\sigma)} g'(\tau) \subseteq \bigcup_{\tau \in X_1(\sigma)} X_2(\tau) = X_4(\sigma)$$

so X_4 carries $g'f'$. Thus $\varphi_2\varphi_1$ and φ_3 are both carried by X_4 and so we have $\varphi_2\varphi_1 \simeq \varphi_3$. Consequently,

$$(g'f')_* = (g')_*(f')_* \quad\text{and}\quad (g'f')^* = (f')^*(g')^* .$$

From Lemma 4.5 and the proof of Lemma 4.2,

$$(g'_{o,s})_*(g'_{s,o})_* = 1 .$$

Thus

$$\begin{aligned}
g_*f_* &= (g''_{r,o})_*(g')_*(g'_{o,s})_*(g'_{s,o})_*(f')_*(g_{o,t})_* \\
&= (g''_{r,o})_*(g')_*(f')_*(g_{o,t})_* \\
&= (g''_{r,o})_*(g'f')_*(g_{o,t})_* \\
&= (gf)_* .
\end{aligned}$$

A similar argument shows that $(gf)^{\pi} = f^{\pi}g^{\pi}$. Thus $B_n \Longrightarrow A_n$.

We have thus shown that A_n is true for all n . We note that the processes used for different n in defining the induced homomorphisms lead to the same result. More precisely, if $f\colon (K,L) \longrightarrow (K',L')$ is a mapping of finite regular complexes, then for n and n' both greater than the dimensions of the complexes, the induced homomorphisms given by A_n and $A_{n'}$ coincide. Also, the properties a) and b) follow since in any one instance we may take n large enough and apply A_n . Theorem 4.1 is proved.

It is important for later considerations to note that in our proof we do not assume Lemma II.5.3. We only assume, in effect, that K and K' satisfy the redundant restrictions. Stated another way, no matter how we orient K and K' our proof yields the fact (see below) that their homology groups are isomorphic. This is one way to prove, incidentally, that homology groups are independent of orientation! (See Chapter VIII, Section 4.)

4.9. COROLLARY. (The Invariance Theorem). <u>If</u> $f\colon (K,L) \longrightarrow (K',L')$ <u>is a homeomorphism of oriented finite regular complexes, then for each</u> q <u>we have</u>

$$f_*\colon\ H_q(K,L;\, G) \approx H_q(K',L';\, G)$$

$$f^*\colon\ H^q(K',L';\, G) \approx H^q(K,L;\, G)\ .$$

<u>Thus the homology and cohomology groups are topological invariants.</u>

Proof: $f_*(f^{-1})_* = (ff^{-1})_* = 1_* = 1$. Similarly, $(f^{-1})_*f_* = 1$. Thus f_* is an isomorphism. By similar reasoning, f^* is also.

If we are given a map $f\colon (K,L) \longrightarrow (K',L')$ of oriented finite regular complexes, then f as a mapping of K into K' induces homomorphisms of the homology groups of K into those of K' which we shall write $(f|K)_*$. Similarly, we have homomorphisms $(f|L)_*$. It is easy to see that the triple $f_*, (f|K)_*, (f|L)_*$ defines a homomorphism of the exact homology sequences associated with (K,L) and (K',L') , using Corollary 2.5 and the definition of the induced homomorphism. Thus we have

4.10. COROLLARY. <u>The exact homology and cohomology sequences of a pair are topological invariants.</u>

Exercise. (Reduced homology groups.) For each complex K , the unique map $f\colon K \longrightarrow$ point induces homomorphisms $H_*(K) \xrightarrow{f_*} H_*(\text{point})$. Set $\tilde{H}_*(K) = \ker f_*$. Show that if $g\colon K_1 \longrightarrow K_2$ is a continuous map, then g induces a map $\tilde{g}_*\colon \tilde{H}_*(K_1) \longrightarrow \tilde{H}_*(K_2)$. Show that the sequence below is exact, if L is a subcomplex of K :

$$\cdots \longrightarrow \tilde{H}_q(L) \xrightarrow{\tilde{i}_*} \tilde{H}_q(K) \xrightarrow{\tilde{j}_*} H_q(K,L) \xrightarrow{\tilde{\partial}_*} \tilde{H}_{q-1}(L) \longrightarrow \cdots$$

Here $\tilde{i}_*$ is induced by $i\colon L \subseteq K$, $\tilde{j}_*$ is given by the composition $\tilde{H}_q(K) \subseteq H_q(K) \xrightarrow{j_*} H_q(K,L)$, where $j\colon K \subseteq (K,L)$ is the inclusion, and $\tilde{\partial}_*$ is induced by the boundary homomorphism of the ordinary homology sequence. One must show that the composition $H_q(K,L) \xrightarrow{\partial_*} H_{q-1}(L) \longrightarrow H_{q-1}(\text{point})$ is zero.

More generally, let $\mathcal{C}$ be a category with a maximal object P , satisfying

(i) For each object $A \in \mathcal{C}$, there is one and only one morphism $f_A \in M(A,P)$.

(ii) If A and B are objects of $\mathcal{C}$, and $f \in M(A,B)$, then $f_B \circ f = f_A$.

Let $F: \mathcal{C} \longrightarrow \mathcal{A}$ be a covariant functor from $\mathcal{C}$ to the category of abelian groups and homomorphisms. Define the reduced functor $\tilde{F}$ by setting

$$\tilde{F}(A) = \ker F(f_A) \quad \text{for } A \in \mathcal{C}.$$

If $f \in M(A,B)$, since $f_B \circ f = f_A$, it follows that $F(f)$ maps $\tilde{F}(A)$ into $\tilde{F}(B)$. Thus we may define

$$\tilde{F}(f) = F(f) | \ker F(f_A)$$

for $f \in M(A,B)$. Verify that $\tilde{F}$ is a covariant functor from $\mathcal{C}$ to $\mathcal{A}$. Carry out an analogous definition for a contravariant functor $F: \mathcal{C} \longrightarrow \mathcal{A}$, using cokernels.

5. The Properties of Cellular Homology Theory.

Cellular homology theory assigns to each pair of regular complexes a sequence of homology groups and to each continuous mapping of pairs of regular complexes induced homomorphisms of the homology groups.*

In this section we state the seven basic properties of cellular homology theory. It turns out that these properties characterize cellular homology theory. A description of the precise sense in which this is true and a proof of the characterization may be found in <u>Foundations of Algebraic Topology</u>, by Eilenberg and Steenrod.

We keep the coefficient group G fixed throughout this section.

I. <u>If</u> $f: (K,L) \longrightarrow (K,L)$ <u>is the identity, then</u> f_* <u>is the identity isomorphism in each dimension</u>.

II. <u>If</u> $f: (K,L) \longrightarrow (K',L')$ <u>and</u> $g: (K',L') \longrightarrow (K'',L'')$ <u>are arbitrary continuous mappings, then</u> $(gf)_* = g_*f_*$.

III. <u>Let</u> $f: (K,L) \longrightarrow (K',L')$ <u>be an arbitrary continuous mapping</u>. <u>If</u> ∂_* <u>and</u> ∂'_* <u>are the boundary homomorphisms of the exact homology sequences associated with</u> (K,L) <u>and</u> (K',L'), <u>respectively, then</u> $\partial'_* f_* = (f|L)_* \partial_*$. Property III follows from the remarks after Corollary 4.9.

IV. <u>The homology sequence of a pair is exact</u>. Property IV was proved in Chapter V. We note that properties II and III imply that

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* We restrict ourselves in this section to finite regular complexes. We will use singular homology theory in the next chapter to extend our results to infinite complexes.

the induced homomorphism is actually a homomorphism of the exact homology sequences.

V. <u>If</u> f <u>and</u> g <u>are homotopic mappings of</u> (K,L) <u>into</u> (K',L'), <u>then</u> $f_* = g_*$.

Proof of Property V: Let p_0 and p_1 mapping K, into $I \times K$ be the embeddings defined in the first paragraph of Section 2. Then p_0 and p_1 are proper and their minimal carriers preserve dimension. Thus p_0 and p_1 induce unique chain maps φ_0 and φ_1 respectively. Note that φ_0 and φ_1 are the chain maps defined at the start of Section 2. Now, since f and g are homotopic, there is a mapping $F\colon (I \times K, I \times L) \longrightarrow (K',L')$ such that $f = Fp_0$ and $g = Fp_1$. Then $f_* = F_*(p_0)_*$ and $g_* = F_*(p_1)_*$, and we must show that $(p_0)_* = (p_1)_*$. We define $\mathcal{D} : C_q(K) \longrightarrow C_{q+1}(I \times K)$ by $\mathcal{D}(c) = I \otimes c$ for $c \in C_q(K)$. If ∂ and ∂' are the boundary operators for $C(K)$ and $C(I \times K)$ respectively, we have

$$\begin{aligned} \partial' \mathcal{D}(c) &= \overline{1} \otimes c - \overline{0} \otimes c - I \otimes \partial c \\ &= \varphi_1 c - \varphi_0 c - \mathcal{D}(\partial c) . \end{aligned}$$

Consequently, $\mathcal{D}$ is a chain deformation exhibiting a chain homotopy of φ_0 and φ_1. Thus $(p_0)_* = (p_1)_*$ and our proof is complete.

5.1. DEFINITION. Two pairs of topological spaces (X,A) and (Y,B) are said to be <u>homotopically equivalent</u> if there exist mappings $f\colon (X,A) \longrightarrow (Y,B)$ and $g\colon (Y,B) \longrightarrow (X,A)$ such that $gf \simeq 1$ on (X,A) and $fg \simeq 1$ on (Y,B). This means we can find a homotopy $F\colon (I \times X, I \times A) \longrightarrow (X,A)$ of fg and the identity, and similarly for (Y,B). The map f is called a <u>homotopy equivalence with homotopy inverse</u> g.

It is easy to show that homotopy equivalence is an equivalence relation. A property which, when possessed by an arbitrary space X, is possessed by all spaces homotopically equivalent to X, is called a <u>homotopy invariant</u>. Properties I, II, and V imply that homology groups are homotopy invariants.

5.2. THEOREM. <u>If</u> $f\colon (K,L) \longrightarrow (K',L')$ <u>is a homotopy equivalence then</u> $f_*\colon H_q(K,L;G) \approx H_q(K',L';G)$ <u>for each</u> q. <u>Homotopically equivalent complexes have isomorphic homology groups</u>.

Proof: Let g be a homotopy inverse for f. By properties II, V, and I, in that order, we have

$$g_*f_* = (gf)_* = 1_* = 1 .$$

Similarly $f_*g_* = 1$. Thus f_* is an isomorphism.

The sixth property of cellular homology theory is called <u>invariance under excision</u>. An inclusion mapping $i\colon (K,L) \longrightarrow (K',L')$ is called an <u>excision</u> if $K - L = K' - L'$.

VI. <u>If</u> $i\colon (K,L) \longrightarrow (K',L')$ <u>is an excision, then</u> $i_*\colon H_q(K,L;G) \approx H_q(K',L';G)$ for each q.

Proof of Property VI: It is only necessary to observe that if an orientation is chosen for K, inducing orientations on L, K', and L', then $C(K,L) = C(K',L')$.

VII. If K is a complex whose only cell is a single vertex, then

$$H_q(K) \approx \begin{cases} G & \text{if } q = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Thus, a point is an acyclic complex.

Exercise: State and prove the corresponding seven basic properties of cellular cohomology theory.

Chapter VII. SINGULAR HOMOLOGY THEORY

1. Direct Spectra.

Up until now, we have defined homology groups only for regular complexes and induced homomorphisms only for mappings of finite regular complexes. The process of direct limit, introduced in this section, will enable us to extend our domain of definitions over the entire category of topological spaces and continuous mappings. We will define the singular homology groups of a pair of spaces (X,A) in terms of mappings of pairs of oriented finite regular complexes (K,L) into (X,A). More specifically, we will associate with each mapping $f\colon (K,L) \longrightarrow (X,A)$ the qth homology group of (K,L), and the collection of these groups will form a direct spectrum of groups. The limit group of this direct spectrum will be by definition the qth singular homology group of the pair (X,A).

1.1. DEFINITION. A directed set (class) is a set (class) A together with a binary relation $\leq$ on A. The relation $\leq$ is required to be reflexive and transitive, and to satisfy the following property: For any m and n in A there is an element $p \in A$ such that $m \leq p$ and $n \leq p$. The ordering $\leq$ is said to direct A.

1.2. DEFINITION. A direct spectrum of abelian groups is a triple $\mathcal{G} = (\{G_\alpha\}, A, \leq)$, where $\{G_\alpha\}$ is a collection of abelian groups indexed by the directed set (or class) $(A,\leq)$, satisfying the following conditions: (i) For each pair (α,β) of elements of A such that $\alpha \leq \beta$ there is given a nonempty collection $\{\sigma^i_{\alpha\beta}\}$ of homomorphisms mapping G_α into G_β; these homomorphisms are called admissible; (ii) Whenever $\sigma^i_{\alpha\beta}\colon G_\alpha \longrightarrow G_\beta$ and $\sigma^j_{\beta\gamma}\colon G_\beta \longrightarrow G_\gamma$ are admissible, then so is the composition $\sigma^j_{\beta\gamma}\sigma^i_{\alpha\beta}: G_\alpha \longrightarrow G_\gamma$. (iii) If $\sigma^i_{\alpha\beta}$ and $\sigma^j_{\alpha\beta}$ are admissible, and $g \in G_\alpha$, then there exists a γ and an admissible homomorphism $\sigma^k_{\beta\gamma}$ such that $\sigma^k_{\beta\gamma}\sigma^i_{\alpha\beta}g = \sigma^k_{\beta\gamma}\sigma^j_{\alpha\beta}g$.

A direct spectrum which has at most one admissible homomorphism $\sigma_{\alpha\beta}$ for any pair (α,β) is called a direct system of abelian groups. Note: We do not require in this case that $\sigma_{\alpha,\alpha}$ be the identity.

Let $\mathcal{G} = (\{G_\alpha\},A,\leq)$ be a direct spectrum. Given an element g of some G_α, and an admissible homomorphism $\sigma^i_{\alpha\beta}$, we say that $\sigma^i_{\alpha\beta}g$ is a $\mathcal{G}$-successor (or successor, for short) of g, and that g is a $\mathcal{G}$-ancestor (or ancestor, for short) of $\sigma^i_{\alpha\beta}g$.

We show that any two elements having a common ancestor have a common successor. We note first that condition (iii) of 1.2 states that two elements of the same group which have a common ancestor must have a common successor. Let $g_1 \in G_\alpha$ and $g_2 \in G_\beta$ have the common ancestor $g_3 \in G_\gamma$. Thus $g_1 = \sigma^i_{\gamma\alpha}g_3$, $g_2 = \sigma^j_{\alpha\beta}g_3$ for some admissible homomorphisms $\sigma^i_{\gamma\alpha}$, $\sigma^j_{\gamma\beta}$. Now, since the indexing set of the G_α's is directed, there are admissible homomorphisms $\sigma^k_{\alpha\delta}$ and $\sigma^{k'}_{\beta\delta}$ for some δ. The elements $\sigma^k_{\alpha\delta}\sigma^i_{\gamma\alpha}g_3$ and $\sigma^{k'}_{\beta\delta}\sigma^j_{\gamma\beta}g_3$ are both elements of G_δ. By conditions (ii) and (iii), these elements have a common successor g_4. But then g_4 is a common successor of g_1 and g_2.

We define a relation R on the set theoretic disjoint union of the G_α as follows. Given $g_1 \in G_\alpha$ and $g_2 \in G_\beta$, then g_1Rg_2 if and only if g_1 and g_2 have a common successor. Then R is clearly reflexive and symmetric. To show that R is transitive, suppose elements g_1, g_2, and g_3 are given such that g_1Rg_2 and

g_2Rg_3. Then g_1 and g_2 have a common successor g_4, and g_2 and g_3 have a common successor g_5. But then g_4 and g_5 have the common ancestor g_2, so by what we proved above, they must have a common successor g_6. The element g_6 is a common successor of g_1 and g_3. Thus R is an equivalence relation. Given an element $g_1 \in G_\alpha$, we denote the R-equivalence class to which g_1 belongs by $\{g_1\}$.

1.3. DEFINITION. Given a direct spectrum of abelian groups $\mathcal{G} = (\{G_\alpha\},A,\leq)$, the limit group of $\mathcal{G}$ is the abelian group whose elements are the equivalence classes of the relation R defined above. Addition is defined as follows. Given two equivalence classes $\{g_1\}$ and $\{g_2\}$ represented by $g_1 \in G_\alpha$ and $g_2 \in G_\beta$, there exist admissible homomorphisms $\sigma^i_{\alpha\gamma}$ and $\sigma^j_{\beta\gamma}$ for some γ. We define $\{g_1\} + \{g_2\} = \{\sigma^i_{\alpha\gamma}g_1 + \sigma^j_{\beta\gamma}g_2\}$, where the addition in the brackets on the right is the addition in G_γ. The limit group is written $\operatorname{Lim}_\alpha G_\alpha$.

To justify this definition, one must show that addition as defined is independent of the various choices made and gives a group structure on the R-equivalence classes. This we leave as an exercise to the reader. We note that the zero of $\operatorname{Lim}_\alpha G_\alpha$ is the R-equivalence class containing all the zeroes of the G_α's.

We prove now a theorem which gives an interesting application of the direct limit process. Let K be an arbitrary oriented regular complex. Let $\mathcal{C}$ be the collection of finite subcomplexes of K, ordered by inclusion. Then $\mathcal{L}$ is a directed set. Let q be a fixed integer. We define a direct spectrum of groups by setting

$$G(L) = H_q(L) \quad \text{for each} \quad L \in \mathcal{L}.$$

Whenever $L \subseteq L'$, the inclusion induces a homomorphism $H_q(L) \xrightarrow{i(L,L')} H_q(L')$. The collection $\{H_q(L)\}$ together with the inclusion homomorphisms $\{i(L,L')\}$ forms a direct system of abelian groups.

1.4. THEOREM. <u>For each</u> q, $\operatorname{Lim}_{L \in \mathcal{L}} H_q(L) \approx H_q(K)$.

Proof: We define a homomorphism

$$\varphi\colon \operatorname{Lim}_{L \in \mathcal{L}} H_q(L) \longrightarrow H_q(K).$$

An element $\{u\}$ of $\operatorname{Lim}_{L \in \mathcal{L}} H_q(L)$ is represented by $u \in H_q(L)$ for some $L \subseteq K$. Denoting the inclusion of $L \subseteq K$ by i, we set $\varphi\{u\} = i_* u$. To show that φ is well-defined, suppose that $u_0 \in H_q(L')$ is another representative of $\{u\}$. Then there is a finite subcomplex L'' such that $i(L,L'')(u) = i(L',L'')(u_0) \in H_q(L'')$. It follows that if $i'\colon L' \subseteq K$ and $i''\colon L'' \subseteq K$, then

$$\begin{aligned} i_* u = i''_*\, i(L,L'')(u) &= i''_*\, i(L',L'')(u_0) \\ &= i'_*\, u_0 \end{aligned}$$

and so φ is well defined. It is clear that φ is a homomorphism. Suppose z is a cycle of a homology class $u \in H_q(L)$ such that $\varphi(\{u\}) = 0$. Then z bounds in K. Suppose $z = \partial c$. The chain c

lies on some finite subcomplex L' containing L. We then have

$$i(L,L')u = 0$$

and so $\{u\} = 0$. Thus φ is one-one. To see that φ is onto, let $u \in H_q(K)$. If z is a cycle representing u, then z lies on some finite subcomplex L. If $u' \in H_q(L)$ is the homology class of z as a cycle on L, then $\varphi\{u'\} = u$. Thus φ is an isomorphism and the theorem is proved.

1.5. DEFINITION. Let $\mathcal{G} = (\{G_\alpha\}, A, \leq)$ and $\mathcal{H} = (\{H_\beta\}, B, \leq')$ be direct spectra. Let $f\colon A \longrightarrow B$ be an order preserving function. Suppose that for each $\alpha \in A$ we have a homomorphism $\varphi_\alpha\colon G_\alpha \longrightarrow H_{f(\alpha)}$. The collection $\{\varphi_\alpha\}$ is called a <u>homomorphism</u> of direct spectra if, whenever $x \in G_\alpha$ has a $\mathcal{G}$-successor $y \in G_{\alpha'}$, then $\varphi_{\alpha'}(y)$ is an $\mathcal{H}$-successor of $\varphi_\alpha(y)$.

We shall sometimes write $\varphi\colon \mathcal{G} \longrightarrow \mathcal{H}$ to indicate that the collection $\{\varphi_\alpha\}$ is a homomorphism. It is clear that a homomorphism of direct spectra induces a homomorphism of their limits.

We say that the direct spectrum $\mathcal{G} = (\{G_\alpha\}, A, \leq)$ is <u>contained</u> in the direct spectrum $\mathcal{H} = (\{H_\beta\}, B, \leq')$ if (1) $(A, \leq)$ is a subset (or subclass) of $(B, \leq')$ (2) for all $\alpha \in A$, $G_\alpha = H_\alpha$, and (3) every $\mathcal{G}$-admissible homomorphism is an $\mathcal{H}$-admissible homomorphism. Thus if $\mathcal{G}$ is contained in $\mathcal{H}$, the inclusion $\mathcal{G} \subseteq \mathcal{H}$ is a homomorphism of direct spectra.

If $\mathcal{G}$ is contained in $\mathcal{H}$, we say that $\mathcal{G}$ is <u>cofinal</u> in $\mathcal{H}$ if the following two conditions are satisfied:

(a) For every $\beta \in B$, there is an $\alpha \in A$ and an $\mathcal{H}$-admissible homomorphism mapping H_β into G_α.

(b) If $g \in G_\alpha$, and $\sigma_{\alpha\beta}$: $G_\alpha \longrightarrow G_\beta$ is an $\mathcal{H}$-admissible homomorphism, then g and $\sigma_{\alpha\beta}g$ have a common $\mathcal{G}$-successor.

1.6. THEOREM. If $\mathcal{G}$ is cofinal in $\mathcal{H}$ then the inclusion $\mathcal{G} \subseteq \mathcal{H}$ induces an isomorphism $\operatorname{Lim}_\alpha G_\alpha \approx \operatorname{Lim}_\beta H_\beta$.

Proof: Condition (a) of cofinality implies that the homomorphism φ: $\operatorname{Lim}_\alpha G_\alpha \longrightarrow \operatorname{Lim}_\beta H_\beta$ induced by the inclusion $\mathcal{G} \subseteq \mathcal{H}$ is onto. For if $h \in H_\beta$, h has an $\mathcal{H}$-successor $g \in G_\alpha$ for some α by (i). Since $\alpha \le \alpha$ in A, we have a $\mathcal{G}$-admissible (and thus $\mathcal{H}$-admissible) homomorphism $\sigma_{\alpha\alpha}$: $G_\alpha \longrightarrow G_\alpha$. Thus h and g have the common $\mathcal{H}$-successor $\sigma_{\alpha\alpha}g$. (g might not be a successor of itself.) It follows that applying φ to the class $\{g\} \in \operatorname{Lim}_\alpha G_\alpha$ yields the class $\{h\} \in \operatorname{Lim}_\beta H_\beta$. Next we use both conditions of cofinality to show that φ is one-one. Let $g_1 \in G_{\alpha_1}$ and $g_2 \in G_{\alpha_2}$ be such that $\varphi\{g_1\} = \varphi\{g_2\}$. Then g_1 and g_2 have a common $\mathcal{H}$-successor $h \in H_\beta$ for some β. Now h has an $\mathcal{H}$-successor $g_3 \in G_{\alpha_3}$ for some α_3 by (a). Thus g_1 and g_2 have the common $\mathcal{H}$-successor $g_3 \in G_{\alpha_3}$. By (b), g_1 and g_3 have a common $\mathcal{G}$-successor, and the same is true of g_2 and g_3. Thus in $\operatorname{Lim}_\alpha G_\alpha$ we have

$$\{g_1\} = \{g_3\} = \{g_2\}.$$

It follows that φ is one-one. The theorem is proved.

2. Singular Homology Groups

For convenience, in this and the next two sections of this chapter, the word "complex" will mean "oriented finite regular complex". When we make reference to a pair of complexes (K,L), we will assume that the orientation of L is induced by restriction of the orientation of K.

Let (X,A) be a pair of topological spaces. Let $S(X,A)$ denote the class of all continuous mappings of pairs of complexes into (X,A). We define an ordering in $S(X,A)$ as follows. If f: $(K,L) \longrightarrow (X,A)$ and g: $(K',L') \longrightarrow (X,A)$ are given, then $f \le g$ if and only if there is an isomorphic embedding* h: $(K,L) \longrightarrow (K',L')$ such that $f = gh$. The ordering $\le$ is obviously reflexive and transitive. To see that $\le$ directs $S(X,A)$, let f and g, mapping (K,L) and (K',L') respectively, into (X,A), be given. Let (K'',L'') be the disjoint union of (K,L) and (K',L'). Then f and g induce a mapping $f \cup g$: $(K'',L'') \longrightarrow (X,A)$, and using the obvious embeddings we see that $f \le f \cup g$ and $g \le f \cup g$. Then $(S(X,A),\le)$ is a directed set.

For each mapping f: $(K,L) \longrightarrow (X,A)$ (i.e., for each $f \in S(X,A)$), let $H_q^f(K,L;G)$ be a labeled copy of $H_q(K,L;G)$. Technically, $H_q^f(K,L;G)$ is a group isomorphic to $H_q(K,L;G)$ together with a definite isomorphism $H_q^f(K,L;G) \longleftrightarrow H_q(K,L;G)$. If $u \in H_q(K,L;G)$, we write $[u,f]$ for the element corresponding to u in $H_q^f(K,L;G)$.

*An isomorphic embedding is an embedding which maps cells onto cells. We shall often use "embedding" to mean "isomorphic embedding".

We wish to show that the triple $(\{H_q^f(K,L;G)\}, S(X,A), \leq)$, together with appropriate admissible homomorphisms, is a direct spectrum. To define the admissible homomorphisms, we suppose that $f,g \in S(X,A)$ and that $f \leq g$. Thus $f\colon (K,L) \longrightarrow (X,A)$ and $g\colon (K',L') \longrightarrow (X,A)$, and there is at least one embedding $h\colon (K,L) \longrightarrow (K',L')$ satisfying $f = gh$. The admissible homomorphism mapping $H_q^f(K,L;G)$ to $H_q^g(K',L';G)$ are then the homology homomorphisms induced by embeddings h such that $f = gh$. To be precise, if h is such an embedding, then the mapping $[u,f] \longmapsto [h_*u,g]$ is an admissible homomorphism. Note that there may be more than one admissible homomorphism between a given pair of groups.

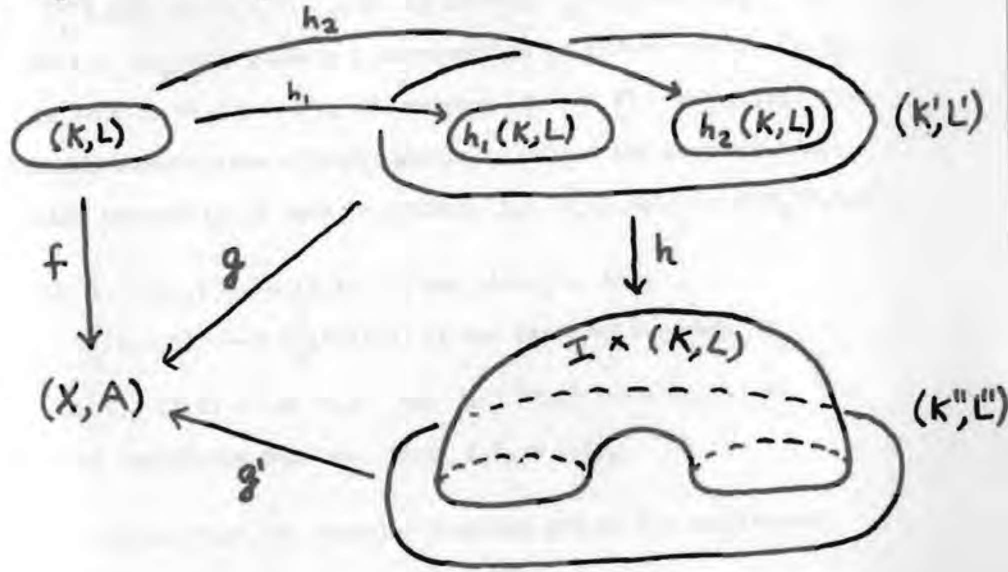
2.1. PROPOSITION. For each q, the triple $(\{H_q^f(K,L;G)\}, S(X,A), \leq)$, with the admissible homomorphisms defined above, is a direct spectrum of abelian groups, written $\mathcal{G}_q(X,A;G)$.

Proof: We verify the properties of a direct spectrum, referring to Definition 1.2. Property (i) is clear. Property (ii) follows from Property II of cellular homology theory. To prove (iii), let $f\colon (K,L) \longrightarrow (X,A)$ and $g\colon (K',L') \longrightarrow (X,A)$ be given. Suppose that h_1 and h_2 are embeddings of (K,L) in (K',L') satisfying $f = gh_1 = gh_2$, and let $[u,f] \in H_q^f(K,L;G)$. We find an admissible homomorphism carrying $[(h_1)_*u,g]$ and $[(h_2)_*u,g]$ into the same element. We define a complex (K'',L'') as follows. Let I be a complex on the unit interval with vertices at 0, $1/2$, and 1. (K'',L'') is then the complex obtained from the disjoint union $I \times (K,L) \cup (K',L')$

by the identifications $(0,x) \sim h_1x$, $(1,x) \sim h_2x$, for all $x \in (K,L)$. Let $h\colon (K',L') \subseteq (K'',L'')$ be the obvious inclusion. Define $g'\colon (K'',L'') \longrightarrow (X,A)$ by setting

$$g'x = \begin{cases} g(h^{-1}x) & \text{for } x \in h(K') \\ fy & \text{for } x = (t,y) \in I \times K. \end{cases}$$

Then g' is well defined because $f = gh_1 = gh_2$. It is easy to see that g' is continuous. Also, $g'h = g$ by construction, so the map $h_*\colon H_q^g(K',L') \longrightarrow H_q^{g'}(K'',L'')$ is an admissible homomorphism.



Since hh_1 and hh_2 map (K,L) onto opposite ends of $I \times (K,L)$ in (K'',L''), we have $hh_1 \simeq hh_2$. Thus $(hh_1)_* = (hh_2)_*$ by Property V of cellular homology theory. Thus

$$[h_*(h_1)_*u,g'] = [(hh_1)_*u,g'] = [(hh_2)_*u,g'] = [h_*(h_2)_*u,g']$$

and the proof of the Lemma is complete.

2.2. DEFINITION. The qth singular homology group of (X,A) with coefficients in G, written $H^S(X,A;G)$, is $\operatorname{Lim}_f H_q^f(K,L;G)$, the limit group of the direct spectrum $\mathcal{S}_q(X,A;G)$.

Notation: If $f\colon (K,L) \longrightarrow (X,A)$ is given and $[u,f] \in H_q^f(K,L;G)$, then we write $\{[u,f]\}$ for the equivalence class of $H_q^S(X,A;G)$ containing $[u,f]$.

3. The Properties of Singular Homology Theory

In cellular theory we found it relatively easy to compute homology groups and difficult to prove that they are topological invariants. The reverse is true of singular theory.

Given a continuous mapping $f\colon (X,A) \longrightarrow (Y,B)$ of pairs of topological spaces, we define the induced homomorphism $f_*\colon H_q^S(X,A;G) \longrightarrow H_q^S(Y,B;G)$ as follows. If we associate with each mapping g of a pair of complexes (K,L) into (X,A) the mapping $fg\colon (K,L) \longrightarrow (Y,B)$, we obtain an order-preserving function from $S(X,A)$ to $S(Y,B)$. For each $g \in S(X,A)$, define $\varphi_g\colon H_q^g(K,L;G) \longrightarrow H_q^{fg}(K,L;G)$ by setting $\varphi_g[u,g] = [u,fg]$. The collection $\{\varphi_g\}$ is clearly a homomorphism of direct spectra (1.5), and is said to be induced by the mapping $f\colon (X,A) \longrightarrow (Y,B)$. The resulting homomorphism of limit groups is called the homology homomorphism induced by f and is written $f_*\colon H_q^S(X,A;G) \longrightarrow H_q^S(Y,B;G)$.

I. If $1\colon (X,A) \longrightarrow (X,A)$ is the identity, then $(1)_*\colon H_q^S(X,A;G) \longrightarrow H_q^S(X,A;G)$ is the identity for each q.

II. If $f\colon (X,A) \longrightarrow (Y,B)$ and $g\colon (Y,B) \longrightarrow (Z,C)$ are arbitrary continuous mappings, then $g_*f_* = (gf)_*$.

It follows that the singular homology groups are topological invariants.

If (X,A) is given, we define a boundary homomorphism $\partial_*^S\colon H_q^S(X,A;G) \longrightarrow H_{q-1}^S(A;G)$ for each q as follows. If we associate to each mapping $f\colon (K,L) \longrightarrow (X,A)$ the mapping $(f|L)\colon L \longrightarrow A$, we obtain an order-preserving function from

$S(X,A)$ to $S(A)$. For each $f \in S(X,A)$, define $\psi_f\colon H^f_q(K,L;G) \longrightarrow H^{(f|L)}_{q-1}(L;G)$ by setting $\psi_f[u,f] = [\partial_* u,(f|L)]$, where ∂_* is the boundary homomorphism of the relative homology sequence of the pair (K,L). To show that the collection $\{\psi_f\}$ preserves the successor relation and thus is a homomorphism of direct spectra, let $g\colon (K',L') \longrightarrow (X,A)$ be a mapping such that there is an embedding $h\colon (K,L) \longrightarrow (K',L')$ with $f = gh$. Let $[u,f] \in H^f_q(K,L;G)$; then $[h_* u,g]$ is a successor of $[u,f]$. By property III of cellular homology (VI.5), $[\partial'_* h_* u,(g|L')] = [(h|L)_*\partial_* u, g|L']$ where ∂'_* is the boundary homomorphism for (K',L'). Thus $[\partial'_* h_* u,(g|L')]$ is a successor of $[\partial_* u,(f|L)]$ in the direct spectrum $\mathscr{G}_{q-1}(A;G)$. Therefore $\{\psi_f\}$ is a homomorphism of direct spectrum. We define $\partial^S_*\colon H^S_q(X,A;G) \longrightarrow H^S_{q-1}(A;G)$ to be the homomorphism of the limit groups of $\mathscr{G}_q(X,A;G)$ and $\mathscr{G}_{q-1}(A;G)$ induced by $\{\psi_f\}$.

III. If $f\colon (X,A) \longrightarrow (Y,B)$ is a continuous mapping of pairs of spaces, then $\partial^S_* f_* = (f|A)_*\partial^S_*$.

Proof of III: Consider the following diagram

$$\begin{array}{ccc}
\mathscr{G}_q(X,A;G) & \xrightarrow{\ \varphi\ } & \mathscr{G}_q(Y,B;G) \\
\Big\downarrow{\scriptstyle \psi} & & \Big\downarrow{\scriptstyle \psi'} \\
\mathscr{G}_{q-1}(A;G) & \xrightarrow{\ \bar{\varphi}\ } & \mathscr{G}_{q-1}(B;G)
\end{array}$$

Here φ is the homomorphism of direct spectra induced by f, $\bar{\varphi}$ is induced by $(f|A)$, and ψ and ψ' are defined as in the discussion

before III. The composition $\psi'\varphi$ is a homomorphism of direct spectra; the homomorphism of limits induced by $\psi'\varphi$ is the composition of the homomorphisms of limits induced by ψ' and by φ. Similarly for $\bar{\varphi}\psi$; thus, to show that $\partial^S_* f_* = (f|A)_*\partial^S_*$, it suffices to show that the above diagram is commutative.

Let $g\colon (K,L) \longrightarrow (X,A)$ be given. Then if $[u,g] \in H^g_q(K,L;G)$, we have

$$\begin{aligned}
\psi'_{fg}\varphi_g[u,g] &= \psi'_{fg}[u,fg] \\
&= [\partial_* u,(fg|L)].
\end{aligned}$$

On the other hand

$$\begin{aligned}
\bar{\varphi}_{(g|L)}\psi_g[u,g] &= \bar{\varphi}_{(g|L)}[\partial_* u,(g|L)] \\
&= [\partial_* u,(f|A)(g|L)] \\
&= [\partial_* u,(fg|L)].
\end{aligned}$$

Thus $\psi'\varphi = \bar{\varphi}\psi$ and the proof is complete.

IV. If (X,A) is a pair of spaces, let $i\colon A \subseteq X$ and $j\colon X \subseteq (X,A)$ denote the inclusion mappings. Then the sequence below, called the homology sequence of the pair (X,A), is exact:

$$\cdots \xrightarrow{j_*} H^S_{q+1}(X,A;G) \xrightarrow{\partial^S_*} H^S_q(A;G) \xrightarrow{i_*} H^S_q(X;G) \xrightarrow{j_*} H^S_q(X,A;G) \xrightarrow{\partial^S_*} \cdots$$

Proof of IV: We prove exactness at $H^S_q(X;G)$ and leave the rest as an exercise to the reader.

a) $\operatorname{Im} i_* \subseteq \ker j_*$. Suppose that $f\colon L \longrightarrow A$ is given. Let $\lambda\colon L \longrightarrow (L,L)$ denote the inclusion. Define $g\colon (L,L) \longrightarrow (X,A)$

so that the diagram below is commutative:

$$\begin{tikzcd} L \arrow[r, "f"] \arrow[d, "\lambda"'] & A \arrow[d, "ji"] \\ (L,L) \arrow[r, "g"] & (X,A) \end{tikzcd}$$

If $[u,f] \in H_q^f(L;G)$, then $j_*i_*\{[u,f]\} = \{[u,jif]\}$. Since λ is an embedding, $\{[u,jif]\} = \{[\lambda_*u,g]\}$; but $\lambda_*u = 0$, so $j_*i_*\{[u,f]\} = 0$.

b) Ker $j_* \subseteq$ im i_*. Suppose that $f\colon K \longrightarrow X$ is given and that $[u,f] \in H_q^f(K;G)$ satisfies $j_*\{[u,f]\} = 0$. Then there exists a map $g\colon (K',L') \longrightarrow (X,A)$ and an embedding $h\colon K \subseteq (K',L')$ such that $gh = jf$ and $h_*u = 0$. We denote the embedding $K \subseteq K'$ by h', and we have inclusions $L' \xrightarrow{k_1} K' \xrightarrow{k_2} (K',L')$. Then $(k_2)_*h'_*u = h_*u = 0$. By the exactness of the relative homology sequence of the pair (K',L'), $h'_*u = (k_1)_*v$ for some $v \in H_q(L')$. We show that $i_*\{[v,(g|L')]\} = \{[u,f]\}$.

$$\begin{tikzcd} L' \arrow[rr, "k_1"] \arrow[dd, "(g|L')"'] & & K' \arrow[rr, "k_2"] \arrow[dd, "(g|K')"'] & & (K',L') \arrow[dd, "g"] \\ & & & K \arrow[ul, "h'"'] \arrow[ur, "h"'] \arrow[dl, "f"] & \\ A \arrow[rr, "i"'] & & X \arrow[rr, "j"'] & & (X,A) \end{tikzcd}$$

Now $i_*\{[v,(g|L')]\}$ is represented by $[v,i(g|L')]$. Since $i(g|L') = (g|K')k_1$ and $(k_1)_*v = h'_*u$, it follows that $[v,i(g|L')]$ and $[u,f]$ have the common successor $[h'_*u,(g|K')]$ in the direct spectrum $\mathcal{S}_q(X;G)$. Thus $i_*\{[v,(g|L')]\} = \{[u,f]\}$ and $\ker j_* = \text{im } i_*$.

Note that II and III imply that the homomorphisms of singular

homology groups induced by a map $f\colon (X,A) \longrightarrow (Y,B)$ commute with the homomorphisms of the exact homology sequences of the pairs (X,A) and (Y,B). Thus f induces a homomorphism of exact sequences.

V. Invariance under homotopy: If f_0 and f_1 are maps of (X,A) into (Y,B) such that $f_0 \simeq f_1$, then $(f_0)_* = (f_1)_*\colon H_q^S(X,A;G) \longrightarrow H_q^S(Y,B;G)$.

Proof of V: Let $g\colon (K,L) \longrightarrow (X,A)$ be given, and suppose that $[u,g] \in H_q^g(K,L;G)$. Let $F\colon I \times (X,A) \longrightarrow (Y,B)$ be a homotopy connecting f_0 and f_1. Define $\rho_i\colon (K,L) \longrightarrow I \times (K,L)$ for $i = 0, 1$ by $\rho_0(x) = (0,x)$ and $\rho_1(x) = (1,x)$, for $x \in K$. Define $G\colon I \times (K,L) \longrightarrow (Y,B)$ by $G(t,x) = F(t,g(x))$. Then $G\rho_0 = f_0g$ and $G\rho_1 = f_1g$. Since $\rho_0 \simeq \rho_1$ property V of cellular homology theory implies that $(\rho_0)_* = (\rho_1)_*$. Thus $[u,f_0g]$ and $[u,f_1g]$ have the common successor $[(\rho_0)_*u,G]$ in the direct spectrum $\mathcal{S}_q(Y,B;G)$. (See diagram below.) It follows that $(f_0)_*\{[u,g]\} = (f_1)_*\{[u,g]\}$.

$$\begin{tikzcd} (K,L) \arrow[r, shift left, "\rho_0"] \arrow[r, shift right, "\rho_1"'] \arrow[rr, bend left, "f_0g"] \arrow[rr, bend right, "f_1g"'] & I \times (K,L) \arrow[r, "G"] & (Y,B) \end{tikzcd}$$

3.1. COROLLARY. <u>A homotopy equivalence between two pairs of spaces induces isomorphisms of their singular homology groups.</u>

3.2. DEFINITION. An inclusion mapping $(X,A) \subseteq (Y,B)$ is called an <u>excision</u> if $Y - B = X - A$. The excision is called <u>proper</u> if the closure of $Y - X$ is contained in the interior of B.

VI. If $(X,A) \subseteq (Y,B)$ is a proper excision, then the induced homomorphism $H_q^S(X,A;G) \longrightarrow H_q^S(Y,B;G)$ is an isomorphism for each q.

We give the proof of VI in section 4.

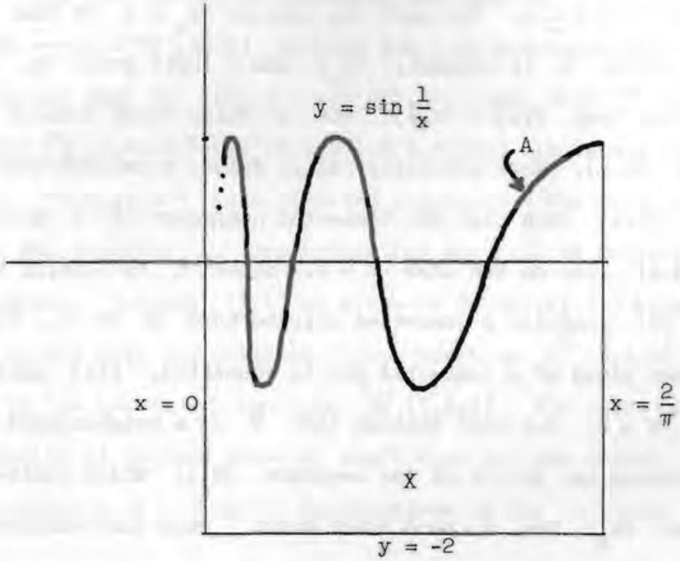
VII. If X is a point then

$$H_q^S(X;G) \approx \begin{cases} G & \text{if } q = 0 \\ 0 & \text{otherwise.} \end{cases}$$

VII follows from the fact that a map $f\colon K \longrightarrow X$ can be factored through an acyclic complex.

4. The excision property.

4.1. A counterexample for improper excision: Let X be the closed region of the xy-plane that lies below the curve $y = \sin\frac{1}{x}$, above the line $y = -2$, and between the lines $x = 0$ and $x = 2/\pi$. X is homotopically equivalent to a point, so $H_2^S(X) = H_1^S(X) = 0$. Let A be the boundary of X. By the exactness of the homology sequence for (X,A), $\partial_*^S\colon H_2^S(X,A) \approx H_1^S(A)$.



We show first that $H_1^S(A) = 0$. We claim that it is enough to show that

(1) If $f\colon L \longrightarrow A$ is a mapping of the finite regular complex L into A, then f is homotopic to a constant.

Suppose that we have proved (1). Let $f\colon L \longrightarrow A$ be given, and suppose that $[u,f] \in H_1^f(L)$. We have the inclusion $h\colon L \subseteq CL$ of L as the base of the cone on L. Since $f \simeq$ constant, there exists a map $g\colon CL \longrightarrow A$ such that $gh = f$. Thus $[u,f]$ has the class $[h_*u,g]$ as a successor. But CL is acyclic (V.2.5), so $h_*u = 0$. That is, $\{[u,f]\} = 0$ and so (1) implies that $H_1^S(A) = 0$.

Proof of (1): Suppose that $f\colon L \longrightarrow A$ is given, and let S be that part of $f(L)$ which lies on $y = \sin \frac{1}{x}$. We show that S is bounded away from the line $x = 0$. Suppose not: Then there exists a sequence $\{v_n\}$ of points in S which converges to a point of the form $(0,a)$, $-1 \leq a \leq 1$. For each n, choose $w_n \in L$ so that $f(w_n) = v_n$. Since L is compact, $\{w_n\}$ has a limit point w, and then it follows that $f(w) = (0,a)$. Now A fails to be locally connected at $(0,a)$. More precisely, there exists a neighborhood U (in A) of $(0,a)$ such that the connected component of U which contains $(0,a)$ lies on the line $x = 0$. Since L is locally connected, $f^{-1}(U)$ contains a connected neighborhood W of w. Since the continuous image of a connected set is connected, $f(W)$ must lie on the line $x = 0$. But that implies that W is a neighborhood of w not containing any points of the sequence $\{w_n\}$, which contradicts the fact that $\{w_n\}$ has w as a limit point. This contradiction establishes (1).

Let Y be the closed region bounded by $x = 0$, $x = \frac{2}{\pi}$, $y = -2$, $y = +1$, and let $B = Y - (X-A)$. We have as before that $H_2^S(Y,B) \approx H_1^S(B)$. But B is homotopically equivalent to a circle, and so $H_1^S(B) \approx Z$. Consequently $H_2^S(X,A) = 0$, $H_2^S(Y,B) \approx Z$, and the homomorphism induced by the excision $(X,A) \subseteq (Y,B)$ is *not* an isomorphism.

To prove that a proper excision induces isomorphisms of singular

homology groups we find it convenient to define a new direct spectrum $\mathcal{H}_q(X,A;G)$ associated with a pair (X,A). We define a new ordering on $S(X,A)$ by saying that if $f\colon (K,L) \longrightarrow (X,A)$ and $g\colon (K',L') \longrightarrow (X,A)$, then $f \ll g$ if there exists a *continuous* mapping $h\colon (K,L) \longrightarrow (K',L')$ such that $f = gh$. The groups of the direct spectrum $\mathcal{H}_q(X,A;G)$ are then the same as before: $\{H_q^f(K,L;G) \mid f\colon (K,L) \longrightarrow (X,A)\}$. If $f,g \in S(X,A)$ as above and $f \ll g$, then the admissible homomorphisms mapping $H_q^f(K,L;G) \longrightarrow H_q^g(K',L';G)$ are the homology homomorphisms induced by *continuous* maps $h\colon (K,L) \longrightarrow (K',L')$ such that $f = gh$. We claim that $\mathcal{H}_q(X,A;G) = (\{H_q^f(K,L;G)\}, S(X,A), \ll)$ is then a direct spectrum. $(S(X,A),\ll)$ is a directed class since the relation $\ll$ contains the relation $\leq$. Properties (i) and (ii) of Definition 1.2 are immediate. Property (iii) is given by Corollary 4.3 below.

We assert that our original direct spectrum $\mathcal{S}_q(X,A;G)$ is *cofinal* in the new direct spectrum $\mathcal{H}_q(X,A;G)$. The first condition of cofinality is obvious since we don't have any new groups. The second condition of cofinality is contained in the following lemma:

4.2. LEMMA. *If* $f\colon (K,L) \longrightarrow (X,A)$, $g\colon (K',L') \longrightarrow (X,A)$ *and* $h\colon (K,L) \longrightarrow (K',L')$ *are continuous mappings satisfying* $f = gh$, *then there exist isomorphic embeddings* h_1 *and* h_2 *of* (K,L) *and* (K',L'), *respectively, in a pair* (K'',L''), *and a mapping* $f'\colon (K'',L'') \longrightarrow (X,A)$, *such that* $f'h_1 = f$, $f'h_2 = g$, *and* $(h_1)_* = (h_2)_* h_*$.

4.3. COROLLARY. $\mathcal{H}_q(X,A;G)$ *satisfies condition (iii) of Definition 1.2 and is thus a direct spectrum.*

Proof of Corollary: Suppose we are given the commutative diagram below:

$$\begin{tikzcd}(K,L) \arrow[r, shift left, "f_1"] \arrow[r, shift right, "f_2"'] \arrow[rr, bend left, "f"] & (K',L') \arrow[r, "g"] & (X,A)\end{tikzcd}$$

Here f_1 and f_2 are arbitrary continuous mappings. Suppose that $[u,f] \in H_q^f(K,L;G)$. Then by the Lemma, $[u,f]$ and $[(f_1)_* u,g]$ have a common $\mathcal{G}$-successor v_1, and $[u,f]$ and $[(f_2)_* u,g]$ have a common $\mathcal{G}$-successor v_2. Since v_1 and v_2 have the common $\mathcal{G}$-ancessor $[u,f]$, they have a common $\mathcal{G}$-successor v_3 (see page 178). Consequently $[(f_1)_* u,g]$ and $[(f_2)_* u,g]$ have the common $\mathcal{G}$-successor (and thus $\mathcal{H}$-successor) v_3.

We prepare for the proof of Lemma 4.2 with a few definitions and results.

If $f\colon K \longrightarrow L$ is a map of regular complexes, then for each $s > 0$, f induces a map $f'\colon Sd^s K \longrightarrow Sd\, L$ such that the following diagram is commutative

$$\begin{tikzcd}K \arrow[r, "f"] \arrow[d, "k"] & L \arrow[d, "k'"] \\ Sd^s K \arrow[r, "f'"] & Sd\, L\end{tikzcd}$$

Here k and k' are subdivision homeomorphisms. (See VI.3 for details.)

4.4 DEFINITION. A simplicial approximation to f is a simplicial map $g\colon Sd^s K \longrightarrow Sd\, L$ such that, for each $x \in Sd^s K$, $g(x)$ lies

in the closure of the unique open simplex containing $f'(x)$.

Since $g(x)$ and $f'(x)$ lie in a closed simplex, there is a line segment joining $g(x)$ and $f'(x)$ in $Sd\, L$ for each $x \in Sd^s K$. Thus f' and g are homotopic.

4.5. THEOREM. Let K be a finite regular complex, $f\colon K \longrightarrow L$ a map of K to an arbitrary regular complex L. Then for some integer $s > 0$ there exists a simplicial approximation $g\colon Sd^s K \longrightarrow Sd\, L$ to f.

Proof: We proceed as in the proof of Theorem VI, 3.9, except that we choose N so that the mesh of $Sd^N K$ is less than $\delta/2$. Let s be any integer larger than N. For each vertex B of $Sd^s K$, there exists a vertex A of $Sd\, L$ such that $f'(\overline{St\, B}) \subseteq St\, A$, since the diameter of $\overline{St\, B}$ is less than δ. Define a function g from the vertices of $Sd^s K$ to those of $Sd\, L$ by choosing, for each B, a vertex A such that $f'(\overline{St\, B}) \subseteq St\, A$, and setting $g(B) = A$. To show that g extends to a simplicial map of $Sd^s K$, it is sufficient to show that g maps the vertices of any simplex of $Sd^s K$ onto the vertices of some simplex of $Sd\, L$. Let σ be a simplex of $Sd^s K$, with vertices $B_0, B_1, \ldots, B_q$. Then $\sigma \subseteq \bigcap_j St\, B_j$, and so

$$f'(\sigma) \subseteq \bigcap_j St(gB_j).$$

This implies that $\bigcap_j St\, g(B_j)$ is non-empty. Let τ be any simplex whose interior meets $\bigcap_j St\, g(B_j)$ in a non-empty set. For each j, $St\, g(B_j) \cap \tau \neq 0$, and so $g(B_j)$ is a vertex of τ. Thus the vertices $\{g(B_j)\}$ span some face τ' of τ. It follows that g

extends to a simplicial map of $\mathrm{Sd}^s K$ into $\mathrm{Sd}\, L$. Let $x \in \sigma$. Then $f'(x) \in \bigcap_j \mathrm{St}\, g(B_j)$. Thus the unique simplex of $\mathrm{Sd}\, L$ containing $f'(x)$ in its interior contains each $g(B_j)$ as a vertex, and so has τ' as a face. Therefore $g(x)$, which of course lies in τ', is contained in the closed simplex containing $f'(x)$, and so g is a simplicial approximation to f'.

4.6. DEFINITION. Let K and L be simplicial complexes, and let $f\colon K \longrightarrow L$ be a simplicial mapping. Then the <u>simplicial mapping cylinder</u> L_f of the mapping f is the simplicial complex given as follows. The vertices of L_f are the vertices of K together with those of L. (K and L are assumed to be disjoint.) A collection $\{A_1,\dots,A_s, B_1,\dots,B_t\colon A_i \in K, B_j \in L\}$ of vertices spans a simplex of L_f if the A_i's span a simplex σ in K and the B_j's span the simplex $f\sigma$. We include all faces of such simplexes together with all simplexes of L.

Note that K and L are embedded as subcomplexes in L_f. Note also that $|L_f|$ is not homeomorphic to the mapping cylinder of f, defined in V.2.

4.7. LEMMA. <u>Let</u> $i\colon K \subseteq L_f$, $j\colon L \subseteq L_f$ <u>denote the inclusions.</u> <u>Then</u> $i \simeq jf$.

Proof: Suppose that $x \in K$. Then x lies in some simplex $(A_0,\dots,A_s)$, and fx lies in the simplex $(fA_0,\dots,fA_s)$, since f is a simplicial map. Thus the simplex $(A_0,\dots,A_s, fA_0,\dots,fA_s)$ of L_f contains the line segment from ix to jfx. It follows that $i \simeq jf$.

Proof of Lemma 4.2: Let $h'\colon (\mathrm{Sd}^s K, \mathrm{Sd}^s L) \longrightarrow (\mathrm{Sd} K', \mathrm{Sd}\, L')$ be a simplicial approximation to h. Consider the diagram below, in which k and k' are subdivision homeomorphisms, and $h'' = k'h'k$. Thus $h'' \simeq h$.

(1)
$$\begin{tikzcd}
(K,L) \arrow[r, "h''"] \arrow[d, "k"] & (K',L') \arrow[r, "g"] & (X,A) \\
(\mathrm{Sd}^s K, \mathrm{Sd}^s L) \arrow[r, "h'"] & (\mathrm{Sd}\, K', \mathrm{Sd}\, L') \arrow[u, "k'"] &
\end{tikzcd}$$

We define the complex (K'',L'') as follows. Let $(\mathrm{Sd}\, K', \mathrm{Sd}\, L')_{h'}$ be the simplicial mapping cylinder of the simplicial map h'. (K'',L'') is obtained from the disjoint union

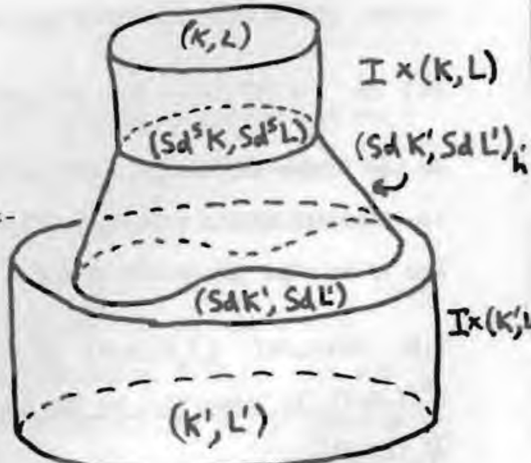
$$[I \times (K,L)] \cup (\mathrm{Sd}\, K', \mathrm{Sd}\, L')_{h'} \cup [I \times (K',L')]$$

by making the following identifications:

$$(1,x) \sim ikx \qquad x \in K,\ i\colon (\mathrm{Sd}^s K, \mathrm{Sd}^s L) \subseteq (\mathrm{Sd}\, K', \mathrm{Sd}\, L')_{h'}$$

$$(0,x') \sim j(k')^{-1}x' \qquad x' \in K',\ j\colon (\mathrm{Sd}\, K', \mathrm{Sd}\, L') \subseteq (\mathrm{Sd}\, K', \mathrm{Sd}\, L')_{h'}$$

Then (K'',L'') is a regular complex. Furthermore, we have embeddings h_1, i, j, h_2 of (K,L), $(\mathrm{Sd}^s K, \mathrm{Sd}^s L)$, $(\mathrm{Sd}\, K', \mathrm{Sd}\, L')$, and (K',L') respectively, in (K'',L''). (See picture.) We want to show that $(h_1)_* = (h_2)_* h_*$. It is clear that $h_1 \simeq ik$, and that $h_2 k' \simeq j$. In addition, Lemma 4.7 implies that



$i \simeq jh'$. It follows that

$$
\begin{aligned}
(h_1)_* &= i_*k_* && (\text{since } h_1 \simeq ik)\\
&= j_*h'_*k_* && (\text{since } i \simeq jh')\\
&= (h_2)_*k'_*h'_*k_* && (\text{since } h_2k' \simeq j)\\
&= (h_2)_*h''_* && (\text{since } h'' = k'h'k)\\
&= (h_2)_*h_* && (\text{since } h'' \simeq h).
\end{aligned}
$$

We must now define the mapping $f'\colon (K'',L'') \longrightarrow (X,A)$.

(a) On $I \times (K,L)$: Since $f = gh \simeq gh'' = gk'h'k$, we may use the homotopy between f and $gk'h'k$ to define f' on $I \times (K,L)$ so that

$$
\begin{aligned}
f'(0,x) &= fx && (x \in K)\\
f'(1,x) &= gk'h'kx && (x \in K).
\end{aligned}
$$

(b) On $(\mathrm{Sd}\,K', \mathrm{Sd}\,L')_{h'}$: Define a simplicial map $m\colon (\mathrm{Sd}\,K', \mathrm{Sd}\,L')_{h'} \longrightarrow (\mathrm{Sd}\,K', \mathrm{Sd}\,L')$ by setting $m(iA) = h'A$ for A a vertex of $\mathrm{Sd}^s K$, and $m(jB) = B$ for B a vertex of $\mathrm{Sd}\,K'$. Define f' on $(\mathrm{Sd}\,K', \mathrm{Sd}\,L')_{h'}$ to be the composition $gk'm$.

(c) On $I \times (K',L')$: Set $f'(t,x') = g(x')$ for $x \in K'$.

As the reader may verify, f' as defined above is consistent with the identifications yielding (K'',L''). By construction, $f'h_1 = f$ and $f'h_2 = g$. The proof of Lemma 4.2 is complete.

4.8. COROLLARY. $\mathcal{G}_q(X,A;G)$ <u>is cofinal in</u> $\mathcal{H}_q(X,A;G)$, <u>and so</u> $H^S_q(X,A;G)$ <u>is isomorphic to the limit group of the direct spectrum</u> $\mathcal{H}_q(X,A;G)$.

In the proof of Theorem 4.9, we will work with the direct spectrum $\mathcal{G}_q(X,A;G)$. Now that we have proved Lemma 4.2, we can use results concerning the successor relation in $\mathcal{H}_q(X,A;G)$ to deduce results about $\mathcal{G}_q(X,A;G)$.

4.9. THEOREM. <u>If</u> $i\colon (X,A) \subseteq (Y,B)$ <u>is a proper excision, then</u> $i_*\colon H^S_q(X,A;G) \approx H^S_q(Y,B;G)$.

Proof: We prove first that i_* is a monomorphism. Suppose that we are given a map $f\colon (K,L) \longrightarrow (X,A)$ and $[u,f] \in H^f_q(K,L;G)$ such that $i_*\{[u,f]\} = 0$. This means there exists a map $g\colon (K',L') \longrightarrow (Y,B)$ and an embedding $h\colon (K,L) \longrightarrow (K',L')$ such that $if = gh$ and $h_*u = 0$.

$$
\begin{tikzcd}
(K,L) \arrow[r, "f"] \arrow[d, "h"'] & (X,A) \arrow[d, "i"] \\
(K',L') \arrow[r, "g"] & (Y,B)
\end{tikzcd}
$$

The sets $\mathrm{Int}\,B$ and $Y - \overline{Y-X}$ form an open covering of Y. Thus $g^{-1}\,\mathrm{Int}\,B$ and $g^{-1}(Y - \overline{Y-X})$ form an open covering of K'. We choose an integer r so that:

(1) Every closed simplex of $\mathrm{Sd}^r K'$ lies in an open set of the covering $\{k_1g^{-1}\,\mathrm{Int}\,B,\ k_1g^{-1}(Y - \overline{Y-X})\}$, where $k_1\colon K' \longrightarrow \mathrm{Sd}^r K'$ is a subdivision homeomorphism. Let $L_1 = \mathrm{Sd}^r L' \cup \{$closed simplexes which are mapped by gk_1^{-1} into $\mathrm{Int}\,B\}$. Let K'' be the subcomplex of $\mathrm{Sd}^r K'$ of all closed simplexes mapped into X by gk_1^{-1}. Set $L'' = K'' \cap L_1$. Finally, since h is an embedding, $h(K,L)$ is a subcomplex of (K',L'). We set $(K''',L''') = (\mathrm{Sd}^r h(K), \mathrm{Sd}^r h(L))$. Then we have the diagram below, where k_2, k_3, and k_4 are the obvious

inclusions and $k_5 = k_1 h$.

$$\begin{tikzcd}
(K,L) \arrow[d, "k_5"] \arrow[drr, "f"] \arrow[ddddd, bend right, "h"'] & & \\
(K''',L''') \arrow[d, "k_4"] & & (X,A) \arrow[ddd, "i"] \\
(K'',L'') \arrow[d, "k_3"] \arrow[urr, "(gk_1^{-1}|K'')"'] & & \\
(Sd^r K',L_1) \arrow[drr, "gk_1^{-1}"] & & \\
(Sd^r K',Sd^r L') \arrow[u, "k_2"] & & (Y,B) \\
(K',L') \arrow[u, "k_1"] \arrow[urr, "g"'] & &
\end{tikzcd}$$

We show that $Sd^r K' - L_1 = K'' - L''$. In other words, k_3 is an excision of complexes. Let σ be an open simplex of $Sd^r K' - L_1$. Then $gk_1^{-1}\bar{\sigma}$ intersects $Y - \text{Int } B$ nontrivially. By (1), $\bar{\sigma} \subseteq k_1 g^{-1}(Y - \overline{Y-X})$, and so $gk_1^{-1}\bar{\sigma} \subseteq X$. Thus $\bar{\sigma} \subseteq K''$, and since $L'' = K'' \cap L_1$, $\sigma \subseteq K'' - L''$. We have shown that $Sd^r K' - L_1 \subseteq K'' - L''$; the reverse inclusion is obvious, and k_3 is an excision of complexes. By Property VI of cellular homology theory, $(k_3)_*\colon H_q(K'',L'';G) \longrightarrow H_q(Sd^r K',L_1;G)$ is an isomorphism. We have $k_3 k_4 k_5 = k_2 k_1 h\colon (K,L) \longrightarrow (Sd^r K',L_1)$, and so $(k_3 k_4 k_5)_* u = (k_2 k_1 h)_* u = (k_2 k_1)_* h_* u = 0$. Since $(k_3)_*$ is an isomorphism, $(k_4 k_5)_* u = 0$. Note that $k_4 k_5$ is not an embedding. Let g' denote the mapping $(gk_1^{-1}|K'')\colon (K'',L'') \longrightarrow (X,A)$. In the direct spectrum $\mathcal{H}_q(X,A;G)$, $[u,f] \in H_q^f(K,L;G)$ has the successor

$[(k_4 k_5)_* u, g'] \in H_q^{g'}(K'',L'';G)$. But $(k_4 k_5)_* u = 0$, and so by Lemma 4.2, $[u,f]$ represents the zero of $H_q(X,A;G)$. It follows that $i_*\colon H_q^S(X,A;G) \longrightarrow H_q^S(Y,B;G)$ is a monomorphism.

We prove in a similar manner that i_* is onto. Suppose that $g\colon (K',L') \longrightarrow (Y,B)$ is given, and that $[u,g] \in H_q^g(K',L';G)$. Proceeding as in the proof that i_* is a monomorphism, construct the complexes and maps in the diagram below:

$$\begin{tikzcd}
(K'',L'') \arrow[r, "g'"] \arrow[d, "k_3"] & (X,A) \arrow[dd, "i"] \\
(Sd^r K',L_1) \arrow[dr, "g''"] & \\
(Sd^r K',Sd^r L') \arrow[u, "k_2"] & (Y,B) \\
(K',L') \arrow[u, "k_1"] \arrow[ur, "g"'] &
\end{tikzcd}$$

Here k_1 is a subdivision homeomorphism, k_2 is an inclusion, and k_3 is an excision of complexes. Thus $(k_3)_*\colon H_q(K'',L'';G) \longrightarrow H_q(Sd^r K',L_1)$ is an isomorphism. Let $v = (k_3)_*^{-1}(k_2)_*(k_1)_* u \in H_q(K'',L'';G)$. Let g'' denote the map $gk_1^{-1}\colon (Sd^r K',L_1) \longrightarrow (Y,B)$. Then $[u,f] \in H_q^f(K',L';G)$ and $[v,ig'] \in H_q^{ig'}(K'',L'';G)$ have the common successor $[(k_3)_* v, g''] \in H_q^{g''}(Sd^r K',L_1;G)$ in the direct spectrum $\mathcal{H}_q(Y,B;G)$. It follows from Lemma 4.2 that $i_*\{[v,g']\} = \{[u,g]\}$. This completes the proof of 4.9.

5. Equivalence of Cellular and Singular Homology Theories. The Invariance Theorem for Infinite Complexes.

Let (K,L) be an oriented pair of (not necessarily finite) regular complexes. Let $\mathscr{C}$ denote the collection of finite subcomplexes of K, ordered by inclusion. We define a direct spectrum of abelian groups by setting

$$G(M) = H_q(M, M \cap L) \quad \text{for each} \quad M \in \mathscr{C}.$$

The collection $\{G(M)\}$, together with the homomorphisms induced by inclusions, forms a direct system of abelian groups. We claim that $\{G(M)\}$ is <u>cofinal</u> in the direct spectrum $\mathscr{H}_q(|K|,|L|)$. First, if $f\colon (K',L') \longrightarrow (K,L)$ is a map of the pair of finite regular complexes (K',L') into (K,L), then, since $|K'|$ is compact, f can be factored through some finite subcomplex of K. This proves condition (a) of cofinality. Condition (b) is trivial in this case, because if M and M' are finite subcomplexes of K such that $M \subseteq M'$, then the only admissible homomorphism mapping $H_q(M, M \cap L)$ to $H_q(M', M' \cap L)$ in the direct spectrum $\mathscr{H}_q(|K|,|L|)$ is the homomorphism induced by inclusion. The latter is of course admissible in the direct system $\{G(M)\}$. Thus the following theorem follows from 1.4 relativized, 1.6, and 4.2:

5.1. THEOREM. $H^S_q(|K|,|L|;G) \approx H_q(K,L;G)$.

5.2. COROLLARY. <u>Cellular homology groups of infinite regular complexes are topological invariants.</u>

We can prove a little more. We restrict ourselves to the category of finite regular complexes. Let $f\colon (K,L) \longrightarrow (K',L')$ be a continuous mapping of complexes. Then we have induced homology homomorphisms f_* and f^S_* for cellular and singular homology, respectively. Since (K,L) is finite the single group $H^1_q(K,L;G)$ with $1\colon (K,L) \longrightarrow (|K|,|L|)$ the identity, is cofinal in the direct spectrum $\mathscr{H}_q(|K|,|L|;G)$. Likewise, $H^1_q(K',L';G)$ is cofinal in $\mathscr{H}_q(|K'|,|L'|;G)$. So we have the following diagram, where φ denotes the isomorphism defined by taking equivalence classes under the relation of having a common successor:

(1)
$$\begin{tikzcd}
H_q(K,L;G) = H^1_q(K,L;G) \arrow[r, "\varphi", "\approx"'] \arrow[d, "f_*"] & H^S_q(|K|,|L|;G) \arrow[d, "f^S_*"] \\
H_q(K',L';G) = H^1_q(K',L';G) \arrow[r, "\varphi", "\approx"'] & H^S_q(|K'|,|L'|;G)
\end{tikzcd}$$

5.3. DEFINITION. Let $\mathscr{C}$ and $\mathscr{D}$ be categories, and let F and G be two covariant functors from $\mathscr{C}$ to $\mathscr{D}$. A <u>natural transformation</u> φ from F to G is a collection $\{\varphi_C | C \in \mathscr{C}\}$, where each $\varphi_C \in M(F(C),G(C))$, such that if C and D are objects of $\mathscr{C}$ and $f \in M(C,D)$, then

$$\varphi_D \circ F(f) = G(f) \circ \varphi_C.$$

One defines a natural transformation of contravariant functors analogously.

5.4 THEOREM. <u>The diagram (1) above is commutative. Thus</u> φ <u>yields a natural transformation of functors.</u>

Proof: Let $[u,1] \in H^1_q(K,L;G)$. Then $f^S_*([u,1])$ is represented by $[u,f] \in H^f_q(K,L;G)$. But $[f_*u,1] \in H^1_q(K',L';G)$ is a successor of

$[u,f]$ in the direct spectrum $\mathcal{H}_q(|K'|,|L'|;G)$, and so $f_*^S([u,1]) = \{[f_*u,1]\}$.

Exercise. Let (K,L) be a pair of finite regular complexes. Prove that the diagram below is commutative for every q:

$$\begin{array}{ccc} H_q(K,L;G) & \xrightarrow[\approx]{\varphi} & H_q^S(|K|,|L|;G) \\ \Big\downarrow \partial_* & & \Big\downarrow \partial_*^S \\ H_{q-1}(L;G) & \xrightarrow[\approx]{\varphi} & H_{q-1}^S(|L|;G) \end{array}$$

This fact, together with Theorem 5.3 and the result that φ is an isomorphism of cellular and singular homology groups for finite regular complexes, constitute the statement that φ defines an equivalence of cellular and singular homology theories on the category of finite regular complexes. (See Eilenberg and Steenrod, Foundations of Algebraic Topology.)

Chapter VIII

INTRODUCTORY HOMOTOPY THEORY AND THE PROOFS OF THE REDUNDANT RESTRICTIONS

In this chapter we will be touching upon a main problem in topology — the extension problem as defined in the Introduction, page ii. Our main goals are the homotopy extension theorem and the theorem on invariance of domain. We will complete the chapter by proving the redundance of the restrictions imposed on regular complexes in Chapter II, and by proving that each regular quasi complex is a complex. (See I.5.)

1. Solid spaces. Retracts.

1.1. DEFINITION. A space Y is solid if whenever there is given a normal space X, a closed subspace A of X, and a continuous function $f:A \longrightarrow Y$, then f extends over X.

The property of being solid is topological. The space consisting of a single point is solid. The Tietze extension theorem asserts that the closed unit interval is solid. The next proposition thus implies that the closed n-ball is solid.

1.2. PROPOSITION. A product of solid spaces is solid.

Proof: Let Y_α, $\alpha\in\Lambda$, be an arbitrary family of solid spaces. Let X be normal, A closed in X, and $f:A \longrightarrow \prod_{\alpha\in\Lambda} Y_\alpha$ a continuous mapping. If $p_\beta:\prod_{\alpha\in\Lambda} Y_\alpha \longrightarrow Y_\beta$ is the projection then for each α, $p_\alpha f:A \longrightarrow Y_\alpha$ can be extended to a mapping $g_\alpha:X \longrightarrow Y_\alpha$, since Y_α is solid. A point of $\prod_{\alpha\in\Lambda} Y_\alpha$ is a function τ on Λ mapping each α to a point $\tau(\alpha)\in Y_\alpha$. Define $g:X \longrightarrow \pi Y_\alpha$ by $[g(x)](\alpha) = g_\alpha x$. Then $p_\alpha g = g_\alpha$ for each α, and so g is continuous and extends f.

1.3. DEFINITION. Let A be a subset of a space X. Then A is called a retract of X if there exists a map $r:X \longrightarrow A$ whose restriction to A is the identity. The map r is called a retraction of X onto A.

Note that if X is a Hausdorff space, each retract of X is closed. It is easy to see that A is a retract of X if and only if for any space Y, every map of A to Y extends to a map of X to Y. Examples of retracts: S^{n-1} is a retract of R^n minus the origin and of E^n minus the origin. (But not of R^n or of E^n, as we shall prove later in this chapter.) In the product $X \times Y$ of arbitrary spaces a cross section $x_0 \times Y$ is a retract. Also, if $f:X \longrightarrow Y$ is continuous, then the function f, considered as a set of ordered pairs, is a retract of $X \times Y$. Thus the diagonal $X \subseteq X \times X$ is a retract of $X \times X$. Finally, suppose X is normal. If A is solid and closed in X, then A is a retract of X, for the identity map $1:A \longrightarrow A$ extends over X. A space is called an absolute retract if it is a retract of any normal space containing it as a closed subspace. Thus a solid space is an obsolute retract.

Given any diagram of spaces and maps we obtain a corresponding diagram of homology groups and induced homomorphisms. If we let $i:A \subseteq X$ denote inclusion, then the extension problem for a mapping $f:A \longrightarrow Y$ corresponds to the extension problem diagrammed below:

$$\begin{tikzcd} & H_n(X) \arrow[dr, dashed, "\phi"] & \\ H_n(A) \arrow[ur, "i_*"] \arrow[rr, "f_*"'] & & H_n(Y) \end{tikzcd}$$

For each n, is there a homomorphism ϕ such that $\phi\, i_* = f_*$? If there is a map $g:X \longrightarrow Y$ extending f then we can set $\phi = f_*$ and obtain a solution to the algebraic extension problem. Thus the existence, for each n, of a homomorphism ϕ such that $\phi\, i_* = f_*$ is a necessary condition for the existence of a mapping g such that $gi = f$. (It is not in general a sufficient condition.)

The above reasoning was used in the introduction to prove that S^{n-1} is not a retract of E^n. Note that if A is a retract of X, so that we have $i:A \subseteq X$ and $r:X \longrightarrow A$ satisfying $ri = 1$, then $r_* i_* = 1$: $H_n(A) \longrightarrow H_n(A)$, for each n. It follows that i_* is a monomorphism and $H_n(X) = \mathrm{Im}\, i_* \otimes \mathrm{Ker}\, r_*$ for each n. Suppose, for example, that $X = S^1 \times D^2$, a solid torus. Let A be a simple closed curve in X which, as a cycle, represents twice the generator of $H_1(X) \approx Z$. Although $i_*:H_1(A) \longrightarrow H_1(X)$ is a monomorphism the image is not a direct summand and so A is not a retract of X.

The problem of determining whether a given subset A of a space X is a retract — which we call a retraction problem — is of course an extension problem. The proposition below shows that every extension problem is actually equivalent to a retraction problem.

1.4. DEFINITION. Let A be a subspace of X and suppose that $f:A \longrightarrow Y$ is a continuous mapping. The adjunction space of the mapping f, written $X \cup_f Y$, is the space obtained from the disjoint union $X \cup Y$ by identifying each $x \in A$ with its image $f(x) \in Y$.

Note that $X \cup_f Y$ contains Y as a subspace.

1.5. PROPOSITION. Let $f:A \longrightarrow Y$ be given. Then there exists an extension of f to X if and only if Y is a retract of the adjunction space $X \cup_f Y$.

Proof: Let $j: X \longrightarrow X \cup_f Y$ map $x \in X$ to its equivalence class in $X \cup_f Y$. If $r: X \cup_f Y \longrightarrow Y$ is a retraction, then rj is an extension of f. On the other hand, suppose $g: X \longrightarrow Y$ extends f. Then we define $r: X \cup_f Y \longrightarrow Y$ by setting

$$r(z) = \begin{cases} g(z) & \text{if } z \in X \\ z & \text{if } z \in X \end{cases}$$

for all $z \in X \cup_f Y$. Then r is well-defined and continuous, and retracts $X \cup_f Y$ onto Y.

The following porposition is called a Polish addition theorem.

1.6. PROPOSITION. Suppose Y is a metric space. If Y' and Y'' are closed subspaces of Y satisfying $Y = Y' \cup Y''$, and if Y', Y'', and $Y' \cap Y''$ are all solid, then Y is solid.

Proof: Let X be normal, A closed in X, and let $f: A \longrightarrow Y$. Set $A' = f^{-1}Y'$, $A'' = f^{-1}Y''$. Suppose we have found closed subspaces X' and X'' of X such that $X = X' \cup X''$, $A' \subseteq X'$, $A'' \subseteq X''$, and $X' \cap X'' \cap A = A' \cap A''$. Then we can extend $f|A' \cap A''$ to a map $f: X' \cap X'' \longrightarrow Y' \cap Y''$ since $Y' \cap Y''$ is solid. Similarly, we can extend $f|(X' \cap X'') \cup A'$ to a map $g_1: X' \longrightarrow Y'$ and we can extend $f|(X' \cap X'') \cup A''$ to a map $g_2: X'' \longrightarrow Y''$. Then since g_1 and g_2 agree on $X' \cap X''$, they define a mapping $g: X \longrightarrow Y$ which extends f. Thus we only have to find X' and X'' satisfying the conditions given above.

We define $Y'_n = \{y \in Y' \mid d(Y' \cap Y'', y) \geq \frac{1}{n}\}$

and $Y''_n = \{y \in Y'' \mid d(Y' \cap Y'', y) \geq \frac{1}{n}\}$

Then Y'_n and Y''_n are closed for each n. We set $A'_n = f^{-1}Y'_n$ and $A''_n = f^{-1}Y''_n$. Since $\cup_n Y'_n = Y' - Y' \cap Y''$, we have $\bigcup_n A'_n = A' - A' \cap A''$. Similarly, $\bigcup_n A''_n = A'' - A' \cap A''$. For each n, $A'_n \cap A'' = A''_n \cap A' = \emptyset$. Since X is normal, we may choose open sets U'_n and U''_n such that $A'_n \subseteq U'_n$, $A''_n \subseteq U''_n$, and $\overline{U'_n} \cap A'' = \overline{U''_n} \cap A' = \emptyset$. For each n, we set $V'_n = U'_n - \bigcup_{m \leq n} \overline{U''_n}$ and $V''_n = U''_n - \bigcup_{m \leq n} \overline{U'_n}$. Then, for each n, V'_n contains A'_n and V''_n contains A''_n. Suppose $m \leq n$. Then $V'_n \subseteq U'_n$ and so $V''_m \subseteq U''_m \subseteq X - V'_n$. Thus $V'_n \cap V''_m = \emptyset$. Similarly, $V'_m \cap V''_n = \emptyset$. Thus $V' = \bigcup_n V'_n$ and $V'' = \bigcup_n V''_n$ are disjoint open sets such that $A' - A' \cap A'' = \bigcup_n A'_n \subseteq V'$, $A'' - A' \cap A'' = \bigcup_n A''_n \subseteq V''$, and $V' \cap A'' = V'' \cap A' = \emptyset$. Thus we may take $X'' = X - V'$ and $X' = X - V''$.

2. The Homotopy Extension Theorem.

Let X and Y be topological spaces. The relation of homotopy is an equivalence relation on the collection of mappings from X to Y. Thus the mappings from X to Y are partitioned into equivalence classes of homotopic mappings. We denote by $\pi(X;Y)$ the collection of these equivalence classes. Suppose X' is another space, and $f: X' \longrightarrow X$ is a map. To each map $g: X \longrightarrow Y$ we can assign the composition $gf: X' \longrightarrow Y$. This assignment preserves the relation of homotopy and so f induces a function $f^*: \pi(X;Y) \longrightarrow \pi(X;Y)$. Similarly, if Y' is another space, and $f: Y \longrightarrow Y'$ is a map, then f induces, by composition, a function $f_*: \pi(X;Y) \longrightarrow \pi(X; Y')$. Thus the assignment to every two spaces X and Y of the set $\pi(X;Y)$, with the induced functions described above, is a function of two variable, contravariant in X and covariant in Y.

If Y is arc-wise connected, then any two constant maps from X to Y are homotopic. Thus the constant maps are all contained in a single class of $\pi(X;Y)$. A map from X to an arc-wise connected space Y is called *homotopically trivial* or *inessential* if it is homotopic to a constant map. All other maps are called *essential*.

The *homotopy classification problem* for the spaces X and Y is to enumerate the homotopy classes in $\pi(X;Y)$ and to give an algorithm for deciding to which homotopy class any given map from X to Y belongs.

The problem of deciding whether two maps f and g from X to Y are homotopic is an extension problem. That is, we have a map F defined on $0 \times X \cup 1 \times X$ in $I \times X$ by

$$F(0,X) = fx$$
$$F(1,X) = gx$$

and $f \simeq g$ if and only if F extends to a map of $I \times X$ to Y. It follows that if Y is solid and $I \times X$ is normal, then $\pi(X;Y)$ consists of one class only, that of the inessential maps.

Less trivially, suppose X is a solid metric space, $I \times X$ is normal, and Y is arc wise connected. By the Tietze extension theorem, I is solid. By 1.6, the set $Z = 0 \times X \cup I \times X_0 \cup 1 \times X$ is solid for any $x_0 \in X$. It follows that Z is a retract of $I \times X$. Since Y is arc wise connected, we can extend the map F defined above over Z, and then we can use the retraction $I \times X \longrightarrow Z$ to extend F over all of $I \times X$.

Finally, suppose K and K' are finite simplicial complexes. Then by the simplicial approximation theorem (VII.4.5) every continuous map $f:K \longrightarrow K'$ is homotopic to a simplicial map of Sd^nK to K' for some n. Since for each n there are only finitely many simplicial maps from Sd^nK to K', it follows that $\pi(K;K')$ is countable.

2.1. DEFINITION. Let (X,A) be a pair of spaces. The space A has the *homotopy extension property in* X *with respect to a space* Y if, given any map $f:X \longrightarrow Y$, and a homotopy $G:I \times A \longrightarrow Y$ of $f|A$ (i.e. a map such that $G(0,x) = fx$ for $x \in A$), then there exists a homotopy $F: I \times X \longrightarrow Y$ of f which extends G. The space A has the *absolute homotopy extension property* in X if A has the homotopy extension property in X with respect to every space Y.

2.2. PROPOSITION. *Let (X,A) be a pair of spaces, and assume that A is closed in X. If A has the homotopy extension property in X with respect to an arc wise connected space Y, then the following sequence of sets and functions is exact at $\pi(X;Y)$*

$$\pi(X/A;Y) \xrightarrow{p^*} \pi(X;Y) \xrightarrow{i^*} \pi(A;Y)$$

Here $X \xrightarrow{p} X/A$ is projection onto the space obtained from X by identifying A to a point.

Proof: Since the composition $A \longrightarrow X \longrightarrow X/A$ maps A to a point, the composition i^*p^* sends every class in $\pi(X/A;Y)$ to the inessential class. On the other hand, suppose $\alpha \in \pi(X;Y)$ satisfies $i^*\alpha = 0$. Let $f:X \longrightarrow Y$ represent α. Since $i^*\alpha = 0$, $f|A$ is homotopic to a constant map of A to some point $y_0 \in Y$. Let $G:I \times A \longrightarrow Y$ be a homotopy of $f|A$ and a constant map. Since A has the homotopy extension property in X with respect to Y, G extends to a map $F:I \times X \longrightarrow Y$ satisfying $F(0,x)=fx$ and $F(1,x) = y_0$ for all $x \in A$. The map g defined by $g(x) = F(1,x)$ is homotopic to f and maps A to y_0. Thus g defines a map $h:X/A \longrightarrow Y$

such that $hp = g$. It then follows that if h represents $\beta \epsilon \pi(X/A;Y)$, $p^*\beta = \alpha$.

Exercise. Let v be a point of S^1. Prove that $S^1 v S^1 = S^1 \times v \cup v \times S^1 \subset S^1 \times S^1$ has the absolute homotopy extension property in $S^1 \times S^1$.

2.3. PROPOSITION. Let (X,A) be a pair of spaces with A closed in X. Then A has the absolute homotopy extension property in X if and only if $I \times A \cup 0 \times X$ is a retract of $I \times X$.

Proof: Suppose A has the absolute homotopy extension property in X. Let $f: X = 0 \times X \subset I \times A \cup 0 \times X$ be the obvious embedding. Let $G: I \times A \longrightarrow I \times A \cup 0 \times X$ be the inclusion. G and f together define the identity map $I \times A \cup 0 \times X \longrightarrow I \times A \cup 0 \times X$. Using the homotopy extension property for A in X with respect to the space $I \times A \cup 0 \times X$, this map extends to give a retraction of $I \times X$ onto $I \times A \cup 0 \times X$.

On the other hand, suppose $r: I \times X \longrightarrow I \times A \cup 0 \times X$ is a retraction. If $f: X \longrightarrow Y$ and a homotopy $G: I \times A \longrightarrow Y$ such that $G(0,x) = fx$ for $x \in A$ are given, then f and g define a map $H: I \times A \cup 0 \times X \longrightarrow Y$. Extend H over $I \times X$ by composing with the retraction r.

2.4. THEOREM: (The Homotopy Extension Theorem for Regular Complexes). If (K,L) is a pair of regular compleses, then L has the absolute homotopy extension property in K.

Proof: We show that $I \times L \cup 0 \times K$ is a retract of $I \times K$. Proposition 2.3 then applies to give the theorem.

2.5. LEMMA. Let E^n be the closed unit n-ball. Then $I \times S^{n-1} \cup 0 \times E^n$ is a retract of $I \times E^n$.

Proof: Embed $I \times E^n$ as the subset of R^{n+1} consisting of all points $(x_0, x_1, \ldots, x_n)$ such that $0 \leq x_0 \leq 1$ and $\sum_{i=1}^{n} x_i^2 = 1$. Then project $I \times E^n$ to $0 \times E^n \cup I \times S^{n-1}$ from the point $(2,0,0,\ldots,0)$.

We obtain a retraction of $I \times K$ onto $I \times L \cup 0 \times K$ by applying the retraction of 2.5 to the individual cells of $K - L$. For each n, let $\overline{K_n} = K_n \cup L$ and $M_n = I \times \overline{K_n} \cup 0 \times K$, with $M_{-1} = I \times L \cup 0 \times K$. Then for each $n > 0$ we define as retraction $r_n: M_n \longrightarrow M_{n-1}$ as follows. If e is an n-cell of K-L, we have the retraction r_n defined on $I \times \overline{e}$ using 2.5. These retractions together with the identity map on M_{n-1} define the function r_n on all of M_n. Since the topology on M_n is given by the weak topology with respect to the cells of M_n, r_n is continuous. For $n = 0$ there is an obvious retraction r_0 of M_0 onto M_{-1}.

For each $n \geq 0$, let $r_n': M_n \longrightarrow M_{-1}$ be the composition $r_0 r_1 \cdots r_n$. Then r_n' retracts M_n onto M_{-1}. Define $r: I \times K \longrightarrow M_{-1}$ by taking the union of all of the mappings r_n'. Since r_m' and r_n' agree on M_m for $m \leq n$, r is well-defined. The map r is continuous because $I \times K$ has the weak topology with respect to closed cells. This completes the proof of 2.4.

It follows from the homotopy extension theorem that extendibility of a map depends only on its homotopy class. Thus, in the light of the remarks before 2.1, if K and K' are finite simplicial complexes there are only countably many extension problems involving maps of a subcomplex L of K into K'. Also, if a map $f: L \longrightarrow Y$ is homotopic to a constant map, then f extends to all of K.

2.6. COROLLARY. If the dimension of L is less than n, any map $f: L \longrightarrow S^n$ is extendible over K.

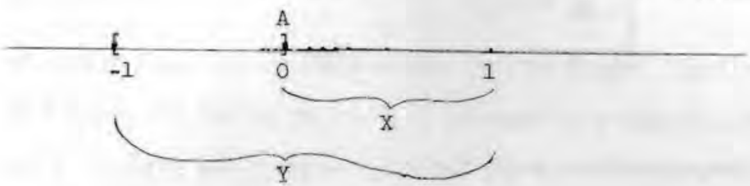
Proof: Let $h: Sd^m L \longrightarrow S^n$ be a simplicial approximation to f, using some simplicial decomposition of S^n. If e is an n-simplex of S^n, and p is a point in the interior of e, then h maps $Sd^m L$ into $S^n - p$. But $S^n - p$ is contractible in S^n and so h is homotopic to a constant.

__Exercise__. Show that in the group of linear fractional transformations of the Riemann Shpere, the identity is homotopic to the map $\frac{1}{z}$. This means there is a homotopy $F: I \times S^2 \longrightarrow S^2$ of the identity and $\frac{1}{z}$ such that for each t, $F(t,z)$ is a linear fractional transformation.

__Exercise__. Define the __degree__ of a map $f: S^2 \longrightarrow S^2$ to be the unique integer n such that $f_*(u) = nu$, where u generates $H_2(S^2)$. If $f(z) = P(z)/Q(z)$ with P and Q polynomials, show that the degree of f is the maximum of t algebraic degrees of the polynomials P and Q, assuming that P and Q have no common roots. If the degree of f is n, exhibit a homotopy $f \simeq g$ in the space of rational functions, where $g(z) = z^n$.

__Example__. Let $P(z)$ be a polynomial in z of degree n with leading coefficient 1. Let $P(z) = z^n + \sum_{i=0}^{n-1} a_i z^i$. Define $F(t,z) = z^n + (1-t)\sum_{i=0}^{n-1} a_i$ If we define $P(\infty) = \infty$ and $F(t,\infty) = \infty$ then F is a homotopy of P as a mapping of S^2 to itself. $F(0,z) = P(z)$ and $F(1,z) = z^n$. It was shown in the second exercise above that the degree of the map $z \longrightarrow z^n$ is n. Therefore if $n > 0$, P is not homotopic to a constant, by property $\overline{V}$ of cellular homology theory, section VI.5. Thus P is onto and so maps some point to zero. In other words, P must have a root. We have sketched a proof of the fundamental theorem of algebra using methods of algebraic topolog

We give an example to show that the conclusion of the homotopy extension theorem is false in general. Let X be the subspace of R^1 consisting of the origin together with all points of the form $\frac{1}{n}$ with n a positive integer. Let A be the origin, and set $Y = X \cup [-1,0]$.



Suppose $f: X \longrightarrow Y$ is the inclusion. Then $f|A$ is homotopic in Y to the map sending 0 to -1. We claim that there is no homotopy of f to a map which sends 0 to -1. This is so because, except at the origin, X is discrete, and so any homotopy $F(t,x)$ of f must satisfy $F(t,x) = fx$ for all t.

2.7. THEOREM. __If Y is a finite simplicial complex, A is closed in X, and X and $I \times X$ are normal, then A has the homotopy extension property in X with respect to Y.__

2.8. LEMMA. __If K is a finite simplicial complex and L is a subcomplex, then there exists an open neighborhood U of L such that L is a retract of U.__

Proof of 2.8: Let S be the set of vertices of K which lie in L, and let T be the set of vertices of K not in L. We define

$$U = \left\{ \alpha \in K \;\middle|\; \sum_{v \in S} \alpha(v) > \frac{1}{2} \right\}$$

If $\alpha \in L$, then $\sum_{v \in S} \alpha(v) = 1$, so $L \subseteq U$. The set U is open because it meets every simplex of K in an open set. We define a retraction of U onto L by setting

$$(r(\alpha))(v) = \begin{cases} \dfrac{\alpha(v)}{\sum_{v' \in S} \alpha(v')} & \text{if } v \in S \\ 0 & \text{if } v \in T \end{cases}$$

We may describe the retraction as follows.

If σ is a simplex of K, then σ may be exmessed as the join of two of its faces, $\sigma = \tau \circ \tau'$, where $\tau = \sigma \cap L$. Each point $\alpha \in U \cap \operatorname{Int} \sigma$ lies on a unique line segment joining a point of τ' to a point β of τ. We set $r(\alpha) = \beta$.

Remark: We note that the retraction r is homotopic to the identity mapping on U. For example, we may define a homotopy F by $F(t,x)=(1-t)\alpha + t\cdot r(\alpha)$. This works because the line segment joining a point α with its image $r(\alpha)$ lies in U.

In general, a retraction $r:X \longrightarrow A$ is a deformation retraction if it is homotopic to the identity on X; that is, if there exists a mapping $F:I \times X \longrightarrow X$ with $F(0,x) = x$ and $F(1,x) = rx$. In this case A is called a deformation retract of x. In the proof of 2.8, L is a deformation retract of U.

2.9. LEMMA. Let Y be a finite simplicial complex. If X is a normal space and A a closed subspace of X, then for each map $f:A \longrightarrow Y$ there is an open nieghborhood V of A and an extension of f to a map $g:V \longrightarrow Y$.

Proof of 2.9: Regard Y as a subcomplex of the simplex σ on the vertices of Y. Since σ is solid, we may extend f to a mapping $h:X \longrightarrow \sigma$. By 2.8, there exists a neighborhood U of Y in σ and a retraction $r:U \longrightarrow Y$. Then set $V = h^{-1}U$ and define $g:V \longrightarrow Y$ to be the composition rh.

Proof of 2.7: Let $G:I \times A \cup 0 \times X \longrightarrow Y$ be given. By 2.9, since $I \times A \cup 0 \times X$ is closed in $I \times X$, G extends to a mapping $H:V \longrightarrow Y$ where V is open and contains $I \times A \cup 0 \times X$. It is easy to see, using the compactness of I, that for each $x \in A$ there exists a set $N(x)$ containing x which is open X and such that $I \times N(x) \subseteq V$. Let $W = \bigcup_{x \in A} N(x)$. Then W is an open set containing A and $I \times A \subseteq I \times W \subseteq V$. Since X is normal there exists a Urysohn function $h:X \longrightarrow [0,1]$ such that $h(A) = 1$ and $h(X - W) = 0$. Let $E = \{(t,x) | 0 \leq t \leq h(x)\}$. Then $I \times A \cup 0 \times X \subseteq E \subseteq I \times W \subseteq V$, and we define a retraction $r:I \times X \longrightarrow E$ by

$$r(t,x) = \begin{cases} (h(x),x) & \text{if } t \geq h(x) \\ (t,x) & \text{if } t \leq h(x) \end{cases}$$

Then $Hr:I \times X \longrightarrow Y$ extends G and the proof of 2.7 is complete.

The hypotheses of 2.7 can be relaxed. The exercises in Chapter I of Hu's book, Homotopy Theory, provide references. In particular, we note that O. Hanner proved (Archiv För Mathematik, 1951) that a locally finite simplicial complex is an absolute neighborhood retract (abbreviated ANR). A metric space X is an ANR if given an embedding of X as a closed subspace of a metric space Z, then X is a retract of some open neighborhood in Z. Lemma 2.9 implies that a finite complex is an ANR. From Hanner's result it follows that 2.7 holds with Y a locally finite simplicial complex.

2.10. DEFINITION. A space X is contractible if the identity map of X is homotopic to a constant map.

Exercise. Prove that if X is contractible then $H_0^S(X) \approx Z$ and $H_q^S(X) \approx 0$ for $q > 0$. (Hint: Show that a contractible space has the homotopy type of a point.)

Note that if X is solid and $I \times X$ is normal, then X is contractible. Also, if X is any space, then the cone CX (See V.2 for a definition) is contractible.

2.11. PROPOSITION. Suppose that (X,A) is a pair of spaces such that A is closed in X and has the absolute homotopy extension property in X. Suppose also that A is contractible. If (Y,y_0) is the pair obtained from (X,A) by identifying A to the point y_0, then the identification map $f:(X,A) \longrightarrow (Y,y_0)$ is a homotopy equivalence of pairs.

Proof: Since A is contractible there exists a homotopy $F:I \times A \longrightarrow A$ of the identity to the constant map sending A to a point $a_0 \in A$. Since A has the absolute homotopy extension property in X, there exists a homotopy $G:I \times X \longrightarrow X$ which extends F and satisfies $G(0,x) = x$ for all $x \in X$. Define $h:X \longrightarrow X$ by $h(x) = G(1,x)$. Then h maps A to the point a_0. Noting that the identification map $f:(X,A) \longrightarrow (Y,y_0)$ is one-one on $X - A$, we define g mapping Y to X by

$$g(y) = \begin{cases} hf^{-1}y & \text{for } y \in Y - y_0 \\ a_0 & \text{if } y = y_0 \end{cases}$$

Then for $a \in A$, $gfa = gy_0 = a_0 = ha$, and so $h = gf$. The map g is continuous since (Y,y_0) has the quotient topology induced by f. By composing g with the inclusion $(X,a_0) \subset (X,A)$ we may regard g as mapping (Y,y_0) into (X,A). We show that g is a homotopy inverse for f.

First, $gfx = hx = G(1,x)$, so G is a homotopy of gf and the identity. (Note that G maps $I \times A$ into A.) Next, define $H(t,y) = fG(t,f^{-1}y)$. Since f is one-one on $X - A$, $H(t,y)$ is well defined for $y \in Y - y_0$. if $y = y_0$, then $f^{-1}y = A$, but since $G(t,A) \subset A$ we have $fG(t,f^{-1}y) = y_0$. Thus H is single valued and leaves y_0 fixed. To show that H is continuous, we use the fact that the topology on $I \times Y$ is the quotient topology induced by the map $1 \times f:I \times X \longrightarrow I \times Y$. (For a proof, see Hilton, An Introduction to Homotopy Theory, p. 109.) We have the following diagram:

$$\begin{array}{ccc} (I \times X, I \times A) & \xrightarrow{G} & (X,A) \\ \downarrow{\scriptstyle 1 \times f} & & \downarrow{\scriptstyle f} \\ (I \times Y, I \times y_0) & \xrightarrow{H} & (Y,y_0) \end{array}$$

Since fG is continuous, the fact that $I \times Y$ has the quotient topology with respect to $1 \times f$ implies that H is continuous. Since $H(0,y) = fG(0,f^{-1}y) = ff^{-1}y = y$ and $H(1,y) = fG(1,f^{-1}y) = fhf^{-1}y = fgy$, H is a homotopy of fg and the identity on (Y,y_0). This completes the proof of 2.11.

Examples: 1. Let K be a finite connected 1-dimensional complex. Letting A be a maximal tree, 2.11 shows that K is homotopically equivalent to a wedge of circles.

2. Let K be the 2-torus with two discs adjoined along the generators of the fundamental group. Then if A is the union of these discs, pinching A to a point shows that K is homotopically equivalent to the 2-sphere.

2.12. PROPOSITION. Let (X,A) be a pair of spaces such that A is closed in X and has the absolute homotopy extension property in X. Then the identification map $f:(X,A) \longrightarrow (Y,y_0)$ defined in 2.11 induces isomorphisms of singular homology groups.

Proof: Let $j:I \times A \subset CA$ denote the inclusion of $I \times A$, as the bottom half of the cone CA. Consider the following commutative diagram, where $X \cup_k CA$ is the adjunction space of the inclusion k of A as the base of the cone CA:

$$\begin{tikzcd}
(X,A) \arrow[r, "i"] \arrow[rrd, "f"'] & (X \cup I \times A, I \times A) \arrow[r, "J"] & (X \cup_k CA, CA) \arrow[d, "g"] \\
& & (Y,y_0)
\end{tikzcd}$$

The maps are given as follows: i is the inclusion, J is induced by $j:I \times A \subset CA$, and g is the projection defined by identifying CA to the point y_0. The inclusion i is a homotopy equivalence of pairs, and J is a proper excision. Using 2.3, it is easy to see that CA has the absolute homotopy extension property in $X \cup_k CA$, and so, by 2.11, g is a homotopy equivalence. Thus the composition $f = gJi$ induces isomorphisms of singular homology groups.

3. Invariance of Domain

In this section we prove some classical theorems about the topology of the n-sphere. Our principal result is the theorem on invariance of domain which states that if $U \subset R^m$ and $V \subset R^n$ are homeomorphic open sets then $m = n$. The proofs of the following classical theorem and corollary were given in the Introduction.

3.1. THEOREM. If $n \geq 1$, then the unit $(n-1)$ sphere S^{n-1} is not a retract of the closed unit n-ball E^n.

3.2. COROLLARY. (The Brouwer Fixed-Point Theorem.) Any continuous map of the closed unit n-ball E^n into itself has a fixed point.

Note that the proof of 3.1 utilizes the concept of induced homology homomorphism and thus depends upon Theorem VI 4.1. Since we are going to use 3.1 to prove results which will lead to proofs of the redundant restrictions, it is essential that we remark that the proof of 3.1 may be obtained using simplicial homology theory alone. We verified at the start that simplicial complexes satisfy the redundant restrictions. Thus we may use Theorem VI.4.1, stated for simplicial complexes, to derive Theorem 3.1 above.

If a space X is homeomorphic to $|K|$ for some regular complex K, thus X is said to be triangulable and K is called a triangulation of X.

3.3. THEOREM. (Characterization of dimension of a finite regular complex) Let K be a finite regular complex. Then the following two statements are equivalent for each n:

(i). The dimension of K is less than or equal to n.

(ii). For each closed set $A \subset |K|$, every mapping $f:A \longrightarrow S^n$ is extendible over $|K|$.

Statement (ii) expresses a topological property. Thus the dimensions of two triangulations of $|K|$ are equal and dimension is a topological property of triangulable spaces.

Proof that (i) => (ii): Suppose $\dim K \leq n$, and that $f:A \longrightarrow S^n$ is given with A closed in K. Then by 2.9, there exists an open neighborhood U of A and a map $g:U \longrightarrow S^n$ which extends f. Using the compactness of A, we may subdivide K so finely that each simplex of Sd^sK meeting A lies in U. Set L equal to the subcomplex of Sd^sK consisting of all closed simplices meeting A. Then $L \subset U$ and $(g|L):L \longrightarrow S^n$ extends f. Let $M = L_{n-1} \cup \overline{K-L}$. Then M is a subcomplex of Sd^sK. By 2.6, we may extend $g|L_{n-1}$ to a map $g':M \longrightarrow S^n$. Then $g|L$ and g' agree on L_{n-1} since $M \cap L = L_{n-1}$. Thus we may define $h:K \longrightarrow S^n$ by

$$h(x) = \begin{cases} g(x) & x \in L \\ g'(x) & x \in M \end{cases}$$

Proof that (ii) => (i): Suppose $\dim K > n$. Let σ be an $(n+1)$-cell of K, and let $f:E^{n+1} \longrightarrow \bar{\sigma}$ be a homeomorphism sending S^n onto $\dot{\sigma}$. Define $g:\dot{\sigma} \longrightarrow S^n$ to be the inverse of $(f|S^n)$. If g were extendible to a map $h:K \longrightarrow S^n$, then $hf:E^{n+1} \longrightarrow S^n$ would be a retraction. This is impossible, and the proof is complete.

3.4. THEOREM. (Borsuk) Let X be a closed proper subset of S^{n+1}, where $n \geq 0$. If X does not separate S^{n+1}, then every mapping $f:X \longrightarrow S^n$ is homotopically trivial.

Theorem 3.4 follows from the following theorem.

3.5. THEOREM. Let X be a closed proper subset of S^{n+1}. If F is a set consisting of exactly one point from each component of $S^{n+1} - X$, and if $f:X \longrightarrow S^n$ is an arbitrary continuous mapping, then there exists a finite subset F' of F and an extension $g:(S^{n+1} - F') \longrightarrow S^n$ of the mapping f.

Proof that 3.5 => 3.4: Let X be a closed proper subset of S^{n+1} which does not separate S^{n+1} and let $f:X \longrightarrow S^n$. Since $S^{n+1} - X$ is connect[ed,] F has just one point, and so $F' = F$ or $\emptyset$. In either case 3.5 asserts the existence of an extension $g:(S^{n+1} - x_0) \longrightarrow S^n$ of the mapping f for some point $x_0 \in S^{n+1} - X$. Since $(S^{n+1} - x_0)$ is contractible, g, and hence f, is homotopic to a constant map.

Proof of 3.5: Using an argument similar to the one used in the proof of Theorem 3.3, we take (K,L) to be a pair of finite regular complexes such that $|K| = S^{n+1}$, $X \subset L$, and such that there exists an extension $g:L \longrightarrow S^n$ of the mapping f. Since the dimension of $L_n \cup K_n$ is n, we may, by 3.3, extend $(g|L_n)$ to a map $g':(L_n \cup K_n) \longrightarrow S^n$. Then g and g' define a map $h:L \cup K_n \longrightarrow S^n$, and h of course extends f. For each $(n+1)$-cell σ of K not in L, choose a point $x_\sigma \in \sigma$ and a retraction $r_\sigma:(\bar{\sigma} - x_\sigma) \longrightarrow \dot{\sigma}$. The composition of these retractions with h gives a mapping $h':(K - \{x_\sigma\}) \longrightarrow S^n$.

Now let $\{V_i : 1 \leq i \leq k\}$ be the components of $S^{n+1} - X$ which contain points of $\{x_\sigma\}$. Each $x_\sigma \in V_i$ is an interior point. Let y_i be the point of F which lies in V_i. Since V_i is open and connected, there exists a homeomorph B_i^{n+1} of the closed unit $(n+1)$-ball containing simultaneously each $x_\sigma \in V_i$ and y_i, and such that $B_i^{n+1} \subset V_i$. Let r_i be a retraction of

$B_i^{n+1} - y_i$ onto the boundary $\dot{B}_i^{n+1}$. Let $Y = S^{n+1} - \bigcup_i \operatorname{Int} \dot{B}_i^{n+1}$. Then the retractions r_i together with the mapping $(h'|Y)$ give a mapping $h'':(S^{n+1} - \{y_i\}) \longrightarrow S^n$ which extends f. To be precise, h'' is defined by

$$h''x = \begin{array}{ll} h'x & \text{for } x \in Y \\ h'r_i x & \text{for } x \in B_i^{n+1} \end{array}$$

The Theorem is now proved.

3.6. THEOREM. Let X be a compact subset of R^{n+1}. A point $x_0 \in R^{n+1} - X$ lies in the unbounded component of $R^{n+1} - X$ if and only if the mapping $g_{x_0}:X \longrightarrow S^n$ defined by

$$g_{x_0}(x) = \frac{x - x_0}{|x - x_0|}$$

is homotopically trivial.

Proof: Suppose x_0 lies in the unbounded component U of $R^{n+1} - X$. Since X is compact there exists a solid ball B about the origin containing X. Choose a point $x_1 \in U$ lying outside B. Since U is open and connected, there exists a path $h:[0,1] \longrightarrow R^{n+1}$ such that $h(0) = x_0$, $h(1) = x_1$, and $h(t) \in U$ for every $t \in [0,1]$. Define a homotopy $H(t,x)$ by

$$H(t,x) = \frac{x - h(t)}{|x - h(t)|} \quad x \in X,\ t \in [0,1]$$

$H(0,x) = g_{x_0}(x)$ and $H(1,x) = g_{x_1}(x)$. Thus g_{x_0} and g_{x_1} are homotopic. Since x_1 lies outside of the ball B containing X, the image of $g_{x_1}:X \longrightarrow S^n$ is contained in a hemisphere. Thus g_{x_1}, and so also g_{x_0}, is homotopic to a constant map.

Now suppose x_0 lies in a bounded component V of $R^{n+1} - X$. We may assume that x_0 is the origin of R^{n+1} and that X is contained in the unit ball B^{n+1}. Suppose that the map g_{x_0}, which is defined by $g_{x_0}(x) = \frac{x}{|x|}$, is inessential. Then we shall apply the homotopy extension theorem (2.7) to obtain an extension of g_{x_0} to $X \cup V$. We must verify that the hypotheses of 2.7 are satisfied. Note that the boundary of V is contained in X. Thus $X \cup V = X \cup \overline{V}$ is compact. Consequently $X \cup V$ and $I \times (X \cup V)$ are normal. Clearly X is closed in $X \cup V$, and so we may apply 2.7 to obtain a mapping $g':X \cup V \longrightarrow S^n$ which extends g. We define $g'':B^{n+1} \longrightarrow S^n$ by

$$g''x = \begin{cases} g'x & \text{if } x \in X \cup V \\ \frac{x}{|x|} & \text{if } x \in B^{n+1} - V \end{cases}$$

Then g'' is continuous since V is a neighborhood of the origin x_0. But g'' is a retraction of B^{n+1} onto S^n, which is impossible. Consequently g_{x_0} is essential and the proof is complete.

3.7. THEOREM. (Borsuk) If X is a closed proper subset of S^{n+1}, then X separates S^{n+1} if and only if there exists a mapping $f:X \longrightarrow S^n$ which is essential.

Proof: If X separates S^{n+1}, let x_0 and x_1 be points in different components of $S^{n+1} - X$. Under stereographic projection $p:S^{n+1} \longrightarrow R^{n+1}$ from x_0, x_1 projects into a bounded component of the projection of $S^{n+1} - X$. By theorem 3.6, the map $g_{p(x_1)}p:X \longrightarrow S^n$ is essential.

If X does not separate S^{n+1}, then apply Theorem 3.4.

3.8. COROLLARY. Let $h:B^n \longrightarrow S^n$ be an embedding of the closed unit n-ball in S^n. Then $S^n - h(S^{n-1})$ has exactly two components, $h(B^n - S^{n-1})$ and $S^n - h(B^n)$.

Proof: Since B^n is contractible, so is $h(B^n)$. Thus any map of $h(B^n)$ into S^{n-1} is homotopic to a constant. By 3.7, $h(B^n)$ does not separate S^n. Therefore $S^n - h(B^n)$ is connected. Since $B^n - S^{n-1}$ is connected, so is $h(B^n - S^{n-1})$. But $h(S^{n-1})$ separates S^n since h is an embedding, by 3.7. It follows that $S^n - h(B^n), h(B^n - S^{n-1})$ are the components of $S^n - h(B^n)$.

3.9. COROLLARY. (Invariance of Domain)(Brouwer) If U is an open set of R^n (or S^n), and h embeds U in R^n (or S^n), then the h-image of U is open in R^n (or S^n).

Proof: Let x be a point of U. There exists a closed ball B containing x such that $B \subseteq U$. By 3.8, $h(B - \dot{B})$ is open in R^n (or S^n). Thus $h(B)$ is a neighborhood of hx, and so $h(U)$ is open in R^n (or S^n).

3.10. COROLLARY. A nonempty open set of R^n can not be embedded in R^k for any $k < n$.

3.11. COROLLARY. If $U \subseteq R^m$ and $V \subseteq R^n$ are homeomorphic open sets, then $m = n$.

4. Proofs of the Redundant Restrictions.

Let $P(q,r)$ be the following statement: for all regular complexes K, if σ is a q-cell of K and τ an r-cell which meets $\dot{\sigma}$, then $\tau \subseteq \dot{\sigma}$. We prove

R.R.1. $P(q,r)$ is true for all q and r.

Proof: The proof is most conveniently given in three steps.

Step 1. $P(q,r)$ is vacuously true for all q and r with $q \leq r$.

Step 2. $P(q+1,q)$ is true for all q.

Proof: Let σ be a $(q+1)$-cell of K, and let τ be a q-cell such that $\tau \cap \dot{\sigma} \neq \emptyset$. Let $f:S^q \xrightarrow{=} \dot{\sigma}$ be a homeomorphism. Since $\dot{\sigma} \subseteq K_q$ and τ is open in K_q, $f^{-1}\tau$ is open in S^q. The restriction of f to $f^{-1}\tau$ is a homeomorphism of $f^{-1}\tau$ onto $\tau \cap \dot{\sigma}$. Since $\tau \cap \dot{\sigma} \subseteq \tau$, the restriction of f to $f^{-1}\tau$ is an embedding of $f^{-1}\tau$ in τ. But τ is homeomorphic to R^q. By the theorem on invariance of domain (3.9), $\tau \cap \dot{\sigma}$ is open in τ. Since $\dot{\sigma}$ is closed, $\tau \cap \dot{\sigma}$ is closed in τ. Since τ is connected, and $\tau \cap \dot{\sigma} \neq \emptyset$, we have $\tau \cap \dot{\sigma} = \tau$ and so $\tau \subseteq \dot{\sigma}$.

Step 3. If $P(q,r)$ for all r, then $P(q + 1, r)$ for all r.

Proof: By Step 1, $P(q + 1,r)$ is true for all $r \geq q+1$. By Step 2, $P(q+1,q)$ is true. Let σ be a $(q+1)$-cell of K. Define A to be the union of the closures of the q-cells which intersect $\dot{\sigma}$. Since $\dot{\sigma}$ is compact, it meets only finitely many cells, by I.4.4. Thus A is a finite union of closed cells and so A is closed. Since $P(q+1,q)$ is true, each of the q-cells which meets $\dot{\sigma}$ is contained in $\dot{\sigma}$. Thus $A \subseteq \dot{\sigma}$. By construction $\dot{\sigma} \subseteq A \cup K_{q-1}$. We assert that $\dot{\sigma} \subseteq A \cup K_r$ for all r. This is proved by a descending induction on r. Suppose that $r \leq q-1$ and that $\dot{\sigma} \subseteq A \cup K_r$. We show that $\dot{\sigma} \subseteq A \cup K_{r-1}$. Let τ be an r-cell of K. If τ interesects A then τ intersects a closed q-cell contained in A, and so, by $P(q,r)$, $\tau \subseteq A \subseteq \dot{\sigma}$. Suppose that $\tau \cap A = \emptyset$. We claim that $\tau \cap \dot{\sigma} = \emptyset$. Since τ is open in K_r, and A is closed, τ is open in $A \cup K_r$. Let $f: S^q \longrightarrow \dot{\sigma}$ be a homeomorphism. Then $f^{-1}\tau$ is open in S^q. As in Step 2, f restricted to $f^{-1}\tau$ is an embedding of $f^{-1}\tau$ in τ. But τ is homeomorphic to R^r, and $r \leq q-1 < q$. By invariance of domain (3.10), $f^{-1}\tau$ is empty. Thus $\tau \cap \dot{\sigma} = \emptyset$, and so $\dot{\sigma} \subseteq A \cup K_{r-1}$. Thus $\dot{\sigma} \subseteq A \cup K_r$ for all r. So $\dot{\sigma} \subseteq A$. Thus $\dot{\sigma} = A$. Let τ be an r-cell of K such that $\tau \cap \dot{\sigma} \neq \emptyset$. Then since $\dot{\sigma}$ is a union of closed q-cells, τ meets some closed q-cell $\bar{\rho}$ contained in A. By $P(q,r)$, τ is contained in $\bar{\rho}$ and so is contained in $\dot{\sigma}$. Thus $P(q+1,r)$ is true for all r.

The proof of R.R.1 is completed by an induction on q. $P(0,r)$ is true for all r by Step 1. Step 3 provides the inductive step. Q.E.D

4.1. LEMMA. Let K be a regular complex, σ a cell of K. Let U be the union of all cells τ of K such that $\bar{\tau} \supseteq \sigma$. Then U is open.

Proof: We show that the complement of U is a subcomplex and thus is closed. Let ρ be a cell of $|K|-U$. Then $\bar{\rho}$ does not contain σ. By R.R.1, $\bar{\rho}$ does not intersect σ. If $\rho_0 < \rho$ then $\bar{\rho}_0 \subseteq \bar{\rho}$ and so $\bar{\rho}_0$ does not intersect σ. Thus $\rho_0 \subseteq |K|-U$. By R.R.1, $\dot{\rho}$ is the union of the proper **faces of ρ**. Since each proper face is contained in $|K|-U$, $\dot{\rho} \subseteq |K|-U$. So $|K|-U$ is a union of the cells of a collection satisfying I.4.2. Therefore $|K|-U$ is a subcomplex. Thus U is open as desired.

We next prove R.R.2. Now that we have proved R.R.1, I.4.2 implies that for each cell σ of a regular complex, $\dot{\sigma}$ is a regular subcomplex. Thus the following theorem implies R.R.2:

4.2. THEOREM. Let K be a regular complex on the n-sphere. Each q-cell of K is a face of at least one n-cell, and each (n-1)-cell is a face of exactly two n-cells.

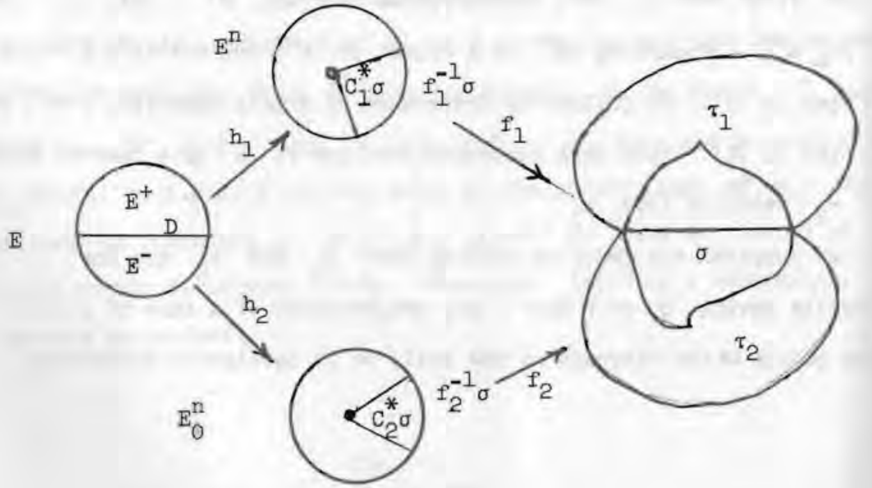
Proof: Let σ be a q-cell of K. Among those cells of K whose closures contain σ, choose a cell τ of maximal dimension. We show that τ is open in $|K|$. Suppose ρ is an arbitrary cell of K such that τ meets $\bar{\rho}$. Using R.R.1 it follows that $\tau \subseteq \bar{\rho}$. Thus $\bar{\rho}$ contains σ. Since τ is of maximal dimension among the cells of K whose closures contain σ, $\dim \tau = \dim \rho$, and so $\tau = \rho$. Thus τ meets $\bar{\rho}$ in an open set, and so τ is open in $|K|$. The theorem on invariance of domain implies that $\dim \tau = n$.

Now let σ be an (n-1)-cell of K. We have already proved that there is an n-cell τ_1 of K such that $\sigma < \tau_1$. We show that there is at least one other n-cell τ_2 of K such that $\sigma < \tau_2$. Suppose, to the contrary, that σ is a face only of the n-cell τ_1. By 4.1, $\tau_1 \cup \sigma$ is open in $|K|$. Let $f: E^n \longrightarrow \bar{\tau}_1$ be a homeomorphism carrying S^{n-1} onto $\dot{\tau}_1$. Let $U = f^{-1}\sigma \subseteq S^{n-1}$. Regarding E^n as a subset of R^n, the set $(E^n - S^{n-1}) \cup U$ is not open in R^n. It follows by invariance of domain that $f[(E^n - S^{n-1}) \cup U] = \tau_1 \cup \sigma$ is not open in $|K|$. This is a contradiction, and so σ is a face of some n-cell τ_2 distinct from τ_1.

We complete the proof by showing that τ_1 and τ_2 are the only n-cells having σ as a face. Any neighborhood of a face of a cell contains points in the interior of the cell, so it is clearly sufficient to show that $\tau_1 \cup \sigma \cup \tau_2$ is a neighborhood of σ. Let $f_2 : E_0^n \longrightarrow \bar{\tau}_2$ be a homeomorphism carrying S_0^{n-1} onto τ_2. (Here E_0^n is a copy of E^n.) We define a homeomorphism h sending an open n-ball E into $\tau_1 \cup \sigma \cup \tau_2$ such that $\sigma \subseteq h(E)$. Let E^+ be the upper half of E together with the open equatorial disc D, and let E^- be the lower half of E together with D. Finally, define $C_1\sigma$ to be the cone on $f_1^{-1}(\sigma)$ with apex at the origin of E^n, and let $C_2\sigma$ be the cone on $f_2^{-1}(\sigma)$ with apex at the origin of E_0^n. Then the set $C_1^*\sigma = C_1\sigma - \{\text{origin}\}$ is homeomorphic to the product $(0,1] \times \sigma$, and so there exists a homeomorphism $h_1 : E^+ \longrightarrow C_1^*\sigma$ carrying D onto $f_1^{-1}\sigma$. Similarly, there exists a homeomorphism $h_2 : E^- \longrightarrow C_2^*\sigma$ carryi[ng] D onto $f_2^{-1}\sigma$ and such that $f_2h_2|D = f_1h_1|D$. We define $h : E \longrightarrow \tau_1 \cup \sigma \cup \tau_2$ by

$$hx = \begin{cases} f_1h_1x & \text{if } x \in E^+ \\ f_2h_2x & \text{if } x \in E^- \end{cases}$$

Then h maps E homeomorphically into $\tau_1 \cup \sigma \cup \tau_2$ and carries D onto σ. Thus $\sigma \subseteq h(E)$, and by invariance of domain, $h(E)$ is open in $|K|$. The proof is complete.



4.1. THEOREM. (Lemma II.5.3) <u>Let K be a regular complex on S^n. Then K is an orientable n-circuit.</u>

Proof: The preceding proposition states that every (n-1)-cell of K is the face of precisely two n-cells of K. Since K_{n-2} is of dimension n-2, we know from the proof of 2.6 and from Borsuk's Theorem (3.7) that K_{n-2} does not separate S^n. Consequently the union of the n- and (n-1)-cells of K is connected. From this it follows very quickly that any two n-cells of K can be joined by a path of n-cells. To see this, let σ be an n-cell of K. Set A equal to the union of the closures of all n-cells which can be connected to σ by a path of n-cells. Then $A - K_{n-2}$ is both open and closed in $K-K_{n-2}$. Consequently $A = K-K_{n-2}$.

In proving that K is orientable we are going to use VI.4.9 (Topological Invariance) and so we have to be a little bit careful.

Let C_n be the statement that Theorem 4.1 is satisfied in dimensions $\leq n$. Let D_n be the statement that all regular complexes of dimension $\leq n$ can be oriented (R.R.3). By Theorem II.5.6, $C_n \Rightarrow D_{n+1}$. For in proving that for any complex K of dimension at most n+1 there exists an incidence function for K, we only need to know that the boundary of each q-cell of K is an orientable (q-1)-circuit for $q \leq n+1$. Using the theorem of topologica[l] invariance, we see that $D_{n+1} \Rightarrow C_{n+1}$. In detail, if K is a complex on the (n+1)-sphere, then by R.R.1, R.R.2, and D_{n+1}, we can define homology groups for K, and by topological invariance, $H_{n+1}(K)$ is isomorphic to $H_{n+1}(S^{n+1}) \approx Z$. Thus K is indeed orientable. Since C_1 is obviously true, it follows that C_n is true for all n. This proves 4.1. As in II.5.6, 4.1 implies R.R.3. The proofs of the redundant restrictions on regular complexes are complete.

5. Regular Quasi Complexes

On this section we prove that a regular quasi complex is a complex. The key step in the proof, Lemma 5.3 below, is due to Dennis Sullivan.

5.1. LEMMA. Let K be a regular quasi complex. Let σ be a cell of K such that the number of cells of K contained in $\bar{\sigma}$ is finite. Then $\bar{\sigma}$ is a union of finitely many cells of K.

Proof: The proof is by induction on the dimension of σ. If σ is of dimension 0 or 1, the lemma is obviously true. Suppose that the lemma is true for cells of dimension $<q$. Let σ be a q-cell such that the number of cells of K contained in $\bar{\sigma}$ is finite. Recalling Step 2 of the proof of R.R.1 in §4, we note that property 7) of Definition I.1.1 is never used. (Property 6) is not used either.) Thus any (q-1)-cell which meets $\bar{\sigma}$ must be contained in $\bar{\sigma}$. Let A be the union of the closures of the finitely many (q-1)-cells which are contained in $\bar{\sigma}$. It follows, just as in Step 3 of the proof of R.R.1, that $\dot{\sigma} = A$. Let τ be a (q-1)-cell in A. Then $\bar{\tau} \subsetneq \dot{\sigma}$ and so any cell contained in $\bar{\tau}$ is contained in $\bar{\sigma}$. Since the number of cells contained in $\bar{\sigma}$ is finite, the same is true for $\bar{\tau}$. By the inductive hypothesis, $\bar{\tau}$ is a union of finitely many cells. Since there are only finitely many (q-1)-cells contained in $\bar{\sigma}$, and $\dot{\sigma} = A$ is the union of their closures, $\bar{\sigma}$ itself is a union of finitely many cells. This completes the proof.

5.2. LEMMA. Let K be a regular quasi complex, and let σ be a q-cell of K. Then the number of r-cells which intersect $\bar{\sigma}$ is countable.

Proof: Let σ be a q-cell of K, and suppose there are uncountably many r-cells, τ_α, α ranging over an index set A, which intersect $\bar{\sigma}$. Choose a point $x_\alpha \in \tau_\alpha \cap \bar{\sigma}$ for each $\alpha \in A$. We claim that some point x_α is a limit point of the set $S = \{x_\alpha | \alpha \in A\}$. Let L be the set of limit points of S. Let $d(x,y)$ be a metric for $\bar{\sigma}$. Then if U_n, n=1,2,..., denotes the set of points in $\bar{\sigma}$ whose distance from L is less than $\frac{1}{n}$, we have $\bigcap_n U_n = L$, since L is closed. Each U_n must contain all but a finite number of the x_α's, because $\bar{\sigma}$ is compact and so any infinite subset has a limit point. Thus the intersection of the U_n's contains all but countably many of the x_α's. For some α, $x_\alpha \in L$. But $x_\alpha \in \tau_\alpha$, and τ_α is an open set in K_r. τ_α is thus a neighborhood of x_α which contains only one element of S. So x_α cannot be a limit point of S. This contradiction establishes the lemma.

5.3. LEMMA. Let K be a regular quasi complex. Let σ be a cell of K. Then the number of cells of K contained in $\bar{\sigma}$ is finite.

Proof: The proof is by a double induction. Let $Q(q,r)$ be the statement that the number of r-cells contained in the closure of each q-cell is finite.

Step 1. $Q(0,r)$ is clearly true for all r.

Step 2. $Q(q,0)$ is true for all q because the closure of a cell is compact and the zero skeleton K_0 is discrete.

Step 3. Suppose that $Q(s,t)$ is true for all pairs (s,t) with $s < q$. Suppose that $Q(q,t)$ is true for all $t < r$. Then $Q(q,r)$ is true.

Proof: Let σ be a q-cell of K. Let τ be an r-cell contained in $\bar{\sigma}$. Since $r < q$, $Q(r,t)$ is true for all t, and so the number of cells contained in $\bar{\tau}$ is finite. By Lemma 5.1, $\dot{\tau}$ is a finite union of cells. Assume that

(*) The number of r-cells contained in $\bar{\sigma}$ is infinite.

We will show that (*) leads to a contradiction. This will prove $Q(q,r)$. Steps 1,2 and 3 prove the lemma by double induction.

Since $Q(q,t)$ is true for all $t < r$, the collection C of all cells of dimension $< r$ which are contained in $\bar{\sigma}$ is finite. For each r-cell $\tau \subsetneq \bar{\sigma}$, $\dot{\tau}$ is a finite union of cells of the collection C. There are only finitely many such unions. Thus (*) implies that there are infinitely many r-cells $\tau_1, \tau_2, \ldots$ contained in $\bar{\sigma}$, all with a common boundary S^{r-1}. We show that this is impossible. Let $\{x_i\}$ be a sequence of points with $x_i \epsilon \tau_i$ for each i. Such a sequence we call a fundamental sequence in the cells τ_i. We define the <u>limit superior</u> of the cells τ_i, written $\limsup \tau_i$, to be the set of all limit points of fundamental sequences. Since the r-skeleton of K is closed, $\limsup \tau_i \subsetneq K_r$. Since each r-cell of K is open in K_r, and any fundamental sequence in the cells τ_i intersects any (open) r-cell in at most one point, no fundamental sequence has a limit point in any r-cell. Thus $\limsup \tau_i \subsetneq K_{r-1}$. Since $\bar{\sigma}$ is closed, $\limsup \tau_i \subsetneq K_{r-1} \cap \bar{\sigma}$. By lemma 5.2, $K_{r-1} \cap \bar{\sigma}$ is the union of countably many closed sets of dimension less than r. Thus, by the sum theorem for dimension (Hurewicz and Wallman, <u>Dimension Theory</u>, p. 30), the dimension of $K_{r-1} \cap \bar{\sigma}$ is $\leq r-1$. This implies that $\dim(\limsup \tau_i) \leq r-1$. But this is impossible, as shown by the following lemma.

5.4. LEMMA. <u>The statement (*) implies that the dimension of $\limsup \tau_i$ is $\geq r$.</u>

Proof: The common boundary S^{r-1} of the cells τ_i is a closed subset of $\limsup \tau_i$. Suppose that the dimension of $\limsup \tau_i$ is $\leq r-1$. Theorem 3.3 of this chapter can be generalized as follows:

THEOREM. <u>A separable metric space X has dimension $\leq r-1$ and only if for each closed subset C and each mapping $f: C \longrightarrow S^{r-1}$, there is an extension of f over all of X.</u>

The proof is given on page 83 of <u>Dimension Theory</u>. Applying this theorem with $X = \limsup \tau_i$, $C = S^{r-1}$, and f = identity: $S^{r-1} \longrightarrow S^{r-1}$, it follows that S^{r-1} is a retract of $\limsup \tau_i$.

Next, an easy argument shows that $\limsup \tau_i$ is closed in $\bar{\sigma}$. We now use the following theorem, proved on page 82 of <u>Dimension Theory</u>.

THEOREM. <u>Let C be a closed subset of the separable metric space X, and f a mapping of C to S^{r-1}. Then there is an open set containing C over which f can be extended.</u>

Applying this theorem with $C = \limsup \tau_i$, $X = \bar{\sigma}$, we may extend the retraction obtained above to a retraction r of some open set U containing $\limsup \tau_i$. Suppose that $\bar{\tau}_i \subsetneq U$ for some i. The restriction of r to $\bar{\tau}_i$ gives a retraction of $\bar{\tau}_i$ onto S^{n-1}, which is impossible by 3.1. Thus for each i there exists a point $x_i \epsilon \bar{\tau}_i - U$. The sequence $\{x_i\}$ converges to some point y since $\bar{\sigma}$ is compact. By definition, $y \epsilon \limsup \tau$ But each x_i lies outside of U, and since U is open, $y \notin U$. This contradicts the fact that $\limsup \tau_i \subsetneq U$. Thus the proof of 5.4, and hence that of 5.3, is complete.

5.5. COROLLARY TO LEMMA 5.3. <u>A regular quasi complex satisfies R.R.1. Each closed cell is a union of finitely many cells.</u>

Proof: This follows immediately from 5.3 together with 5.1.

5.6. THEOREM. <u>A regular quasi complex is a complex.</u>

Proof: Let K be a regular quasi complex. We have to prove that for any given q, if $A \subsetneq K_q$ meets every closed cell of dimension $\leq q$ in a closed set, then A is closed in K. Thus we must prove that A meets any closed cell of K in a closed set. Let σ be a cell of K. Then $\bar{\sigma}$ is a union of finitely many cells, by 5.5. In particular, $\bar{\sigma} \cap K_q$ is a union of finitely many closed cells. But A intersects each of these closed cells in a closed set. Thus $A \cap \bar{\sigma}$ is a finite union of closed sets, and so is closed. The proof is complete.

Chapter IX

SKELETAL HOMOLOGY

In this chapter we introduce the concepts of skeletal decomposition of a space and of skeletal homology groups. We show that there is a natural isomorphism of skeletal homology groups and singular homology groups of the underlying space. We also justify the methods and results of Chapter III.

1. Skeletal Homology.

1.1. DEFINITION. A <u>filtration</u> of a topological space $|X|$ is an increasing sequence $X = \{X_r : r = 0, 1, \ldots\}$ of closed subspaces of $|X|$ such that $\bigcup_r X_r = |X|$. A filtration X is called a <u>skeletal</u> <u>decomposition</u> of the space $|X|$ if the following two conditions are satisfied:

i) For all $r \geqslant 0$ and for all $q \neq r$, $H_q^S(X_r, X_{r-1}) = 0$; we set $X_{-1} = \emptyset$ for convenience.

ii) $|X|$ has the weak topology with respect to the subspaces X_r.

1.2. PROPOSITION. <u>If K is a (not necessarily regular) complex, then $H_q^S(K_r, K_{r-1}) = 0$ for $q \neq r$, and $H_q^S(K_q, K_{q-1})$ is isomorphic to the free abelian group generated by the q-cells of K.</u>

Proof: Let r be given. For each r-cell σ_i of K choose a relative homeomorphism $f_i: (E^r, S^{r-1}) \longrightarrow (\bar{\sigma}_i, \dot{\sigma}_i)$. Define $U_n = \bigcup_i f_i\left[\{x \in E^r \mid |x| < 1/n\}\right]$. Let $V = K_r - \bar{U}_3$ and let $W = K_r - \bar{U}_2$. Then V and W are open in K_r, and the closure of W is contained in V. The inclusion $(K_r, K_{r-1}) \subseteq (K_r, \bar{V})$ is a homotopy equivalence of pairs--a homotopy inverse maps $\bar{V}$ by radial projection to K_{r-1} and dilates $\bar{U}_3$ along radius vectors onto K_r--and so induces isomorphisms of singular homology groups. The inclusion $(K_r-W, \bar{V}-W) \subseteq (K_r, \bar{V})$ is a proper excision, and thus induces isomorphisms of singular homology. Finally, the pair $(K_r-W, \bar{V}-W) = (\bar{U}_2, \bar{U}_2-U_3)$ is homotopy equivalent as a pair to $(\bar{U}_2, \dot{U}_2)$. Thus $H_q^S(K_r, K_{r-1})$ is isomorphic to $H_q^S(\bar{U}_2, \dot{U}_2)$, and the conclusion of the proposition now follows from the fact that $\bar{U}_2$ is a union of disjoint closed r-cells, one for each r-cell of K.

We next construct an explicit isomorphism ξ mapping the free abelian group on the q-cells of K to $H_q^S(K_q, K_{q-1})$. For each q-cell σ of K, choose a relative homeomorphism $f_\sigma: (E^q, S^{q-1}) \longrightarrow (\bar{\sigma}, \dot{\sigma})$ and a generator $u_\sigma \in H_q(E^q, S^{q-1})$. Then we define $\xi(\sigma)$ to be the class in $H_q^S(K_q, K_{q-1})$ represented by $[u_\sigma, i_\sigma f_\sigma] \in H_q^{i_\sigma f_\sigma}(E^q, S^{q-1})$, where i_σ is the inclusion map of the pair $(\bar{\sigma}, \dot{\sigma})$ in (K_q, K_{q-1}). It follows from the proof of 1.2 that ξ so defined is an isomorphism.

1.3. COROLLARY. <u>The sequence of skeletons of a complex K is a skeletal decomposition of the underlying space |K|.</u>

<u>Exercise:</u> Give a proof of 1.2 using VIII.2.12.

Let X be a filtration of the space |X| satisfying condition ii) of Definition 1.1. Let $\{H_q^S(X_n): q \text{ fixed}, n = 0, 1, \ldots\}$ be the direct system with admissible homomorphisms induced by the inclusions $X_m \subseteq X_n$, $m \leq n$.

1.4. PROPOSITION. <u>For every q,</u> $H_q^S(|X|) \approx \operatorname{Lim}_n H_q^S(X_n)$.

Proof: For each n we have a homomorphism $h_n: H_q^S(X_n) \longrightarrow H_q^S(|X|)$ induced by inclusion. Suppose $u \in H_q^S(X_m)$ has the successor $v \in H_q^S(X_n)$. If we let i denote the inclusion of X_m in X_n, this means that $i_*^S u = v$. It follows, using property II of singular homology theory, that $h_n(v) = h_n(i_*^S u) = h_m(u)$. Thus the h_n's induce a homomorphism $h: \operatorname{Lim}_n H_q^S(X_n) \longrightarrow H_q^S(|X|)$.

We show that h is an isomorphism. First note that if Y is a compact set contained in |X|, then Y is contained in X_n for some n. (One proves this using sequential compactness, as in the first part of the proof of I.4.4.) Next, let $x \in H_q^S(|X|)$ be represented by $[u, f]$, where $u \in H_q(K)$ and f is a map of the finite complex K into |X|. Since f(K) is compact, f maps K into X_n for some n. Thus $[u, f]$ represents a class y in $H_q^S(X_n)$. Clearly $h_n y = x$, and so h is onto. Next, suppose that $x \in H_q^S(X_n)$ is such that $h_n x = 0$. Choose a representative $[u, f]$ for x, where u is in $H_q(K)$ and f maps the finite complex K into X_n. The class $h_n x$ is represented by $[u, jf]$, where j denotes the inclusion of X_n in $|X|$. To say that $h_n x = 0$ means that there exists an embedding $k\colon K \subseteq L$, where L is a finite complex, such that $k_* u = 0$, and a map $g\colon L \longrightarrow |X|$ such that $gk = jf$. Since L is compact, g maps L into X_m for some m, and we may assume that $m \geqslant n$. If i' denotes the inclusion of X_n in X_m, then it follows that $(i')^S_* x$ = class of $[k_* u, g] = 0$. Thus x represents the zero of $\operatorname{Lim}_n H_q^S(X_n)$ and so h is a monomorphism. The proof of 1.4 is complete.

1.5. COROLLARY. Let X be a skeletal decomposition of the space $|X|$. Then for each pair of integers q, r, with $q < r$, $(j_r)^S_*\colon H_q^S(X_r) \approx H_q^S(|X|)$, where j_r denotes the inclusion of X_r in $|X|$.

Proof: Let k be a non-negative integer, and consider the portion

$$H^S_{q+1}(X_{r+k+1}, X_{r+k}) \longrightarrow H_q^S(X_{r+k}) \xrightarrow{i_*} H_q^S(X_{r+k+1}) \longrightarrow H_q^S(X_{r+k+1}, X_{r+k})$$

of the homology sequence of the pair (X_{r+k+1}, X_{r+k}). Since $q < r$, the two relative groups are zero, and so i_* is an isomorphism. Thus the limit group of the direct system $\{H_q^S(X_n)\colon q \text{ fixed}, n = 0, 1, \dots\}$ is isomorphic to $H_q^S(X_r)$, and we apply 1.4 to complete the proof.

With each skeletal decomposition X we associate the chain complex $C(X) = (\{C_q(X)\}, \partial)$, defined as follows. For each q, $C_q(X) = H_q^S(X_q, X_{q-1})$, and $\partial_q\colon C_q(X) \longrightarrow C_{q-1}(X)$ is the boundary operator of the triple (X_q, X_{q-1}, X_{q-2}). That is, if $\partial_*\colon H_q^S(X_q, X_{q-1}) \longrightarrow H^S_{q-1}(X_{q-1})$ is the boundary operator for the pair (X_q, X_{q-1}) and $j_*\colon H^S_{q-1}(X_{q-1}) \longrightarrow H^S_{q-1}(X_{q-1}, X_{q-2})$ is the homomorphism induced by the inclusion, then $\partial_q = j_* \partial_*$. See the diagram below:

$$\begin{tikzcd}
C_q(X) = H^S_q(X_q, X_{q-1}) \arrow[dd, "\partial_q"'] \arrow[dr, "\partial_*"] & \\
& H^S_{q-1}(X_{q-1}) \arrow[dl, "j_*"] \\
C_{q-1}(X) = H^S_{q-1}(X_{q-1}, X_{q-2}) \arrow[dd, "\partial_{q-1}"'] \arrow[dr, "\partial_*"] & \\
& H^S_{q-2}(X_{q-2}) \arrow[dl, "j_*"] \\
C_{q-2}(X) = H^S_{q-2}(X_{q-2}, X_{q-3}) &
\end{tikzcd}$$

Since $\partial_* j_* = 0$ in the homology sequence of the pair (X_{q-1}, X_{q-2}), $\partial_{q-1}\partial_q = 0$. $C(X)$ is called the skeletal chain complex of X; the homology groups of C(X) are called the skeletal homology groups of X and are denoted by $H_*(X)$.

After looking at an example, we will prove that the skeletal homology of X is isomorphic to the singular homology of the underlying space $|X|$. (See Theorem 1.7 below.) This isomorphism, together with the conclusions of 1.2 and 1.3, give a method of computing the homology groups of the space underlying an arbitrary irregular complex.

Example: The homology of a torus $S^p \times S^q$, where p and q are positive integers. $S^p \times S^q$ is the underlying space of an irregular complex with four cells of dimensions 0, p, q, and p+q. Consider the skeletal decomposition K of $S^p \times S^q$ given by skeletons of the complex. We claim that the boundary operator in C(K) is zero. In dimension p+q, there is a commutative diagram:

$$\begin{tikzcd} C_{p+q}(K) \approx H^S_{p+q}(S^p \times S^q, S^p \vee S^q) \arrow[d, "\partial_*^S"] \arrow[dd, bend right, "\partial"'] \\ H^S_{p+q-1}(S^p \vee S^q) \arrow[d, "j_*"] \\ C_{p+q-1}(K) \approx H^S_{p+q-1}(S^p \vee S^q, K_{p+q-2}) \end{tikzcd}$$

The proposition below asserts in particular that $i_*: H^S_{p+q-1}(S^p \vee S^q) \longrightarrow H^S_{p+q-1}(S^p \times S^q)$ is a monomorphism. By the exactness of the singular homology sequence of the pair $(S^p \times S^q, S^p \vee S^q)$, $\partial^S_*: H^S_{p+q}(S^p \times S^q, S^p \vee S^q) \longrightarrow H^S_{p+q-1}(S^p \vee S^q)$ is 0, and thus also $\partial: C_{p+q}(K) \longrightarrow C_{p+q-1}(K)$. We leave the rest of the proof that $\partial = 0$ as an exercise to the reader. The skeletal homology of the decomposition K is then free abelian on generators corresponding to the four cells of $S^p \times S^q$.

1.6. PROPOSITION. Let K and L be regular complexes. Choose vertices $v \in K$ and $w \in L$. Then the inclusion $i: K \vee L = K \times w \cup v \times L \subseteq K \times L$ induces a monomorphism of homology groups.

Proof: In the proof we use cellular homology. Any cycle on $K \vee L$ is a sum of a cycle on K and a cycle on L. But K and L

are retracts of $K \times L$. Thus the homology groups of $K \vee L$, which in dimensions greater than zero split as the direct sum of the homology groups of K and those of L, must map monomorphically into the homology groups of $K \times L$.

Let X and Y be skeletal decompositions of spaces $|X|$ and $|Y|$. A map $f: |X| \longrightarrow |Y|$ is called a skeletal mapping from X to Y if f maps X_q into Y_q for each q. A skeletal mapping from X to Y induces homomorphisms from $H^S_q(X_q, X_{q-1})$ to $H^S_q(Y_q, Y_{q-1})$ for all q, and it follows from properties II and III of singular homology that these homomorphisms define a chain map from C(X) to C(Y). This chain map induces a homomorphism of skeletal homology, denoted by f_*.

We now show that if X is a skeletal decomposition of a space $|X|$ then the skeletal homology of X is naturally isomorphic to the singular homology of $|X|$. For each q, inclusions induce homomorphisms of singular homology:

$$\begin{tikzcd} H^S_q(X_q) \arrow[r, "\alpha"] \arrow[d, "\beta"] & H^S_q(X_q, X_{q-1}) = C_q(X) \\ H^S_q(X_{q+1}) \end{tikzcd}$$

1.7. THEOREM. The homomorphisms α and β defined above induce an isomorphism $\psi_X: H_q(X) \longrightarrow H^S_q(|X|)$ for each q. This isomorphism is natural under skeletal mappings: i.e., if $f: X \rightarrow Y$ is a skeletal mapping then the diagram below is commutative for each q.

$$\begin{array}{ccc} H_q(X) & \xrightarrow{\psi_X} & H_q^S(|X|) \\ \downarrow f_* & & \downarrow f_*^S \\ H_q(Y) & \xrightarrow{\psi_Y} & H_q^S(|Y|) \end{array}$$

Proof: First note that $H_q^S(X_r) = 0$ for $r < q$. This is proved by induction on r, using condition i) of Definition 1.1 and the exact homology sequences of the pairs (X_r, X_{r-1}). In particular, $H_q^S(X_{q-1}) = 0$, and so α is a monomorphism. Next, by 1.5, the inclusion of X_{q+1} in $|X|$ induces an isomorphism of singular homology, and by a similar argument $\beta: H_q^S(X_q) \to H_q^S(X_{q+1})$ is onto.

We now construct ψ_X. Let $u \in H_q(X)$ be represented by the skeletal cycle $z \in H_q^S(X_q, X_{q-1})$. We then have $\partial z = \alpha\partial_* z = 0$. Since α is a monomorphism, $\partial_* z = 0$. Thus $z = \alpha y$ for some $y \in H_q(X_q)$, and since α is a monomorphism, y is well-defined. Denote the inclusion of X_{q+1} in $|X|$ by j_{q+1}. Then we set $\psi_X(u) = (j_{q+1})_*\beta y$. If u is also represented by z', so that $z' - z = \partial x = \alpha\partial_* x$ for some $x \in H_{q+1}^S(X_{q+1}, X_q)$, then $z' = z + \alpha\partial_* x = \alpha(y + \partial_* x)$. Since

$$(j_{q+1})_*\beta(y + \partial_* x) = (j_{q+1})_*\beta y + (j_{q+1})_*\beta\partial_* x$$
$$= (j_{q+1})_*\beta y, \qquad (\beta\partial_* = 0),$$

it follows that $\psi_X(u)$ is well-defined.

It is clear that ψ_X as defined above is a homomorphism. Suppose that $\psi_X(u) = 0$. If u is represented by $z = \alpha y$, then

$\psi_X(u) = (j_{q+1})_*\beta y = 0$. Since $(j_{q+1})_*$ is an isomorphism, $\beta y = 0$, and so $y = \partial_* x$ for some $x \in H_{q+1}^S(X_{q+1}, X_q)$. But then $z = \alpha y = \alpha\partial_* x = \partial x$, and so z is a boundary and $u = 0$. Finally, the fact that β is onto implies immediately that ψ_X is onto, and so ψ_X is an isomorphism.

The proof that ψ_X is natural with respect to skeletal mappings is an easy exercise in the application of properties II and III of singular homology and is left to the reader.

1.8. COROLLARY. (Theorem 5.1 of Chapter III) <u>Let X be a topological space underlying a complex K with the property that for each q the topological boundary of each q-cell lies in the (q-2)-skeleton of K. Then for each q $H_q^S(X)$ is isomorphic to the free abelian group on the q-cells of K.</u>

Proof: The sequence of skeletons of K is a skeletal decomposition of X by 1.3, and we shall denote this decomposition by K. By 1.7, the singular homology groups of X are isomorphic to the skeletal homology groups $H_*(K)$. We show that the boundary operator in C(K) is identically zero. According to 1.2 and the remarks following, $C_q(K) = H_q^S(K_q, K_{q-1})$ is isomorphic to the free abelian group on the q-cells of K, and the generator $\xi(\sigma)$ corresponding to a given q-cell σ is represented by $[u_\sigma, i_\sigma f_\sigma] \in H_q^{i_\sigma f_\sigma}(E^q, S^{q-1})$. The class $\partial\xi(\sigma)$ in $C_{q-1}(K) = H_{q-1}^S(K_{q-1}, K_{q-2})$ is represented by

$\left[\partial_* u_\sigma,\ j(f_\sigma|S^{q-1})\right] \in H_{q-1}^{j(f_\sigma|S^{q-1})}(S^{q-1})$, where $j\colon \dot\sigma \longrightarrow (K_{q-1}, K_{q-2})$ is the inclusion. By hypothesis $\dot\sigma \subseteq K_{q-2}$. Thus the composition $j(f_\sigma|S^{q-1})$ sends S^{q-1} into K_{q-2}, and so induces a map h of the pair (S^{q-1}, S^{q-1}) to (K_{q-1}, K_{q-2}). If $k\colon S^{q-1} \subseteq (S^{q-1}, S^{q-1})$ denotes inclusion, then $\left[\partial_* u_\sigma,\ j(f_\sigma|S^{q-1})\right]$ has the successor $\left[k_*\partial_* u_\sigma,\ h\right] \in H_{q-1}^{h}(S^{q-1}, S^{q-1})$. But this latter group is zero, and so $\partial\xi(\sigma) = 0$. Thus the boundary in $C(K)$ is identically zero, and so

$$\begin{aligned} H_q^S(X) &= H_q(K) \\ &= C_q(K) \\ &= \text{free abelian group on the} \\ &\quad\ \text{q-cells of K, by 1.2.} \end{aligned}$$

For the final theorem of this section we suppose that K is a regular complex oriented by the incidence function α. Let $C(K)$ denote the cellular chain complex for the oriented regular complex K and let $D(K)$ denote the skeletal chain complex associated with the skeletal decomposition of K given by the skeletons of K.

1.9. THEOREM. <u>The chain complexes</u> $C(K)$ <u>and</u> $D(K)$ <u>are chain isomorphic.</u>

Proof: We define a chain isomorphism ϕ mapping $C(K)$ to $D(K)$. Let σ be a q-cell of K. Then $\phi_q\colon C_q(K) \longrightarrow D_q(K) = H_q^S(K_q, K_{q-1})$ maps the generator σ of $C_q(K)$ to the singular homology class

represented by $\left[u_\sigma,\ i_\sigma\right] \in H_q^{i_\sigma}(\bar\sigma, \dot\sigma)$, where $i_\sigma\colon (\bar\sigma, \dot\sigma) \subseteq (K_q, K_{q-1})$ denotes inclusion and $u_\sigma \in H_q(\bar\sigma, \dot\sigma)$ is a generator chosen inductively as follows. In dimension zero we demand only that in each connected component of K the classes $u_A \in H_0(A)$, A a vertex of K, be homologous in K. In dimension 1, we choose u_σ so as to satisfy $\partial u_\sigma = [\sigma : A]_\alpha i_* u_A + [\sigma : B]_\alpha j_* u_B$, where A and B are the vertices of σ and $i\colon A \subsetneq \dot\sigma$ and $j\colon B \subsetneq \dot\sigma$ are the inclusions. Now assume that $u_\tau \in H_r(\bar\tau, \dot\tau)$ has been chosen for each r-cell τ of dimension less than q, where $q \geqslant 2$. Let σ be a q-cell, and suppose that τ is a (q-1)-face of σ. The inclusion $i\colon (\bar\tau, \dot\tau) \subseteq (\dot\sigma, \dot\sigma - \tau)$ is an excision of regular complexes and so induces isomorphisms of cellular homology groups. We next consider the inclusion $\dot\sigma \subseteq (\dot\sigma, \dot\sigma - \tau)$.

1.10. LEMMA. <u>The regular complex</u> $\dot\sigma - \tau$ <u>is acyclic.</u>

Proof: By comparing $\dot\sigma - \tau$ with $\dot\sigma$ we see that $\dot\sigma - \tau$ has no non-bounding cycles except perhaps in dimension q-1. In dimension q-1, any cycle in $\dot\sigma - \tau$ must also be a cycle in $\dot\sigma$ and hence a multiple of the fundamental cycle in $\dot\sigma$. But the fundamental cycle in $\dot\sigma$ has coefficient ± 1 on the cell τ, and so there are no (q-1)-cycles in $\dot\sigma - \tau$.

Thus the inclusion $k\colon \dot\sigma \subseteq (\dot\sigma, \dot\sigma - \tau)$ induces isomorphisms of cellular homology groups in dimensions greater than zero. It follows that we have isomorphisms:

$$(1)\quad H_{q-1}(\bar{\tau},\dot{\tau}) \xrightarrow[\approx]{i_*} H_{q-1}(\dot{\sigma},\dot{\sigma}-\tau) \xleftarrow[\approx]{k_*} H_{q-1}(\dot{\sigma}) \xleftarrow[\approx]{\partial_*} H_q(\bar{\sigma},\dot{\sigma})$$

We denote the composition by $\Theta(\sigma,\tau) : H_{q-1}(\bar{\tau},\dot{\tau}) \longrightarrow H_q(\bar{\sigma},\dot{\sigma})$.

Finally, we set $u_\sigma = [\sigma : \tau]_\alpha \Theta(\sigma,\tau)\, u_\tau$.

1.11. LEMMA. The definition of u_σ given above is independent of the choice of τ.

Proof: If τ' is another $(q-1)$-face of σ then, since $\dot{\sigma}$ is an orientable $(q-1)$-circuit, there is a chain of $(q-1)$-cells from τ to τ'. Therefore it is sufficient to show that the definitions of u_σ are the same if we start from adjacent cells τ and τ'. We let ρ be a $(q-2)$-face of τ and τ'. We have defined isomorphisms

$$\Theta(\sigma,\tau) : H_{q-1}(\bar{\tau},\dot{\tau}) \longrightarrow H_q(\bar{\sigma},\dot{\sigma})$$

$$\Theta(\sigma,\tau') : H_{q-1}(\bar{\tau}',\dot{\tau}') \longrightarrow H_q(\bar{\sigma},\dot{\sigma})$$

$$\Theta(\tau,\rho) : H_{q-2}(\bar{\rho},\dot{\rho}) \longrightarrow H_{q-1}(\bar{\tau},\dot{\tau})$$

$$\Theta(\tau',\rho) : H_{q-2}(\bar{\rho},\dot{\rho}) \quad H_{q-1}(\bar{\tau}',\dot{\tau}').$$

We want to show that

$$(2)\qquad [\sigma : \tau]_\alpha \Theta(\sigma,\tau)\, u_\tau = [\sigma : \tau']_\alpha \Theta(\sigma,\tau')\, u_{\tau'}.$$

Substituting expressions in terms of u_ρ for u_τ and $u_{\tau'}$, we replace (2) by

$$[\sigma{:}\tau][\tau{:}\rho]\,\Theta(\sigma,\tau)\,\Theta(\tau,\rho)\,u_\rho = [\sigma{:}\tau'][\tau'{:}\rho]\,\Theta(\sigma,\tau')\,\Theta(\tau',\rho)\,u_\rho.$$

Since $[\sigma{:}\tau][\tau{:}\rho] + [\sigma{:}\tau'][\tau'{:}\rho] = 0$ it is sufficient to prove that

$$(3)\qquad \Theta(\sigma,\tau)\,\Theta(\tau,\rho) = -\,\Theta(\sigma,\tau')\,\Theta(\tau',\rho).$$

In the proof of (3) we will use the following "hexagonal lemma", which is proved in Eilenberg and Steenrod, Foundations of Algebraic Topology, p. 38.

1.12. LEMMA. Consider the diagram of groups and homomorphisms below:

$$\begin{tikzcd}
 & G_1 \arrow[dl, "j_1"'] \arrow[dr, "i_1"] \arrow[dd, "k_1"] & \\
G_6 & & G_2 \\
 & G_7 \arrow[ul, "g_2"] \arrow[ur, "f_2"'] \arrow[dd, "k_2"] & \\
G_5 \arrow[uu, "j_2"] \arrow[ur, "f_1"] \arrow[dr, "j_3"'] & & G_3 \arrow[uu, "i_2"'] \arrow[ul, "g_1"'] \arrow[dl, "i_3"] \\
 & G_4 &
\end{tikzcd}$$

Assume that

a) Each triangle is commutative

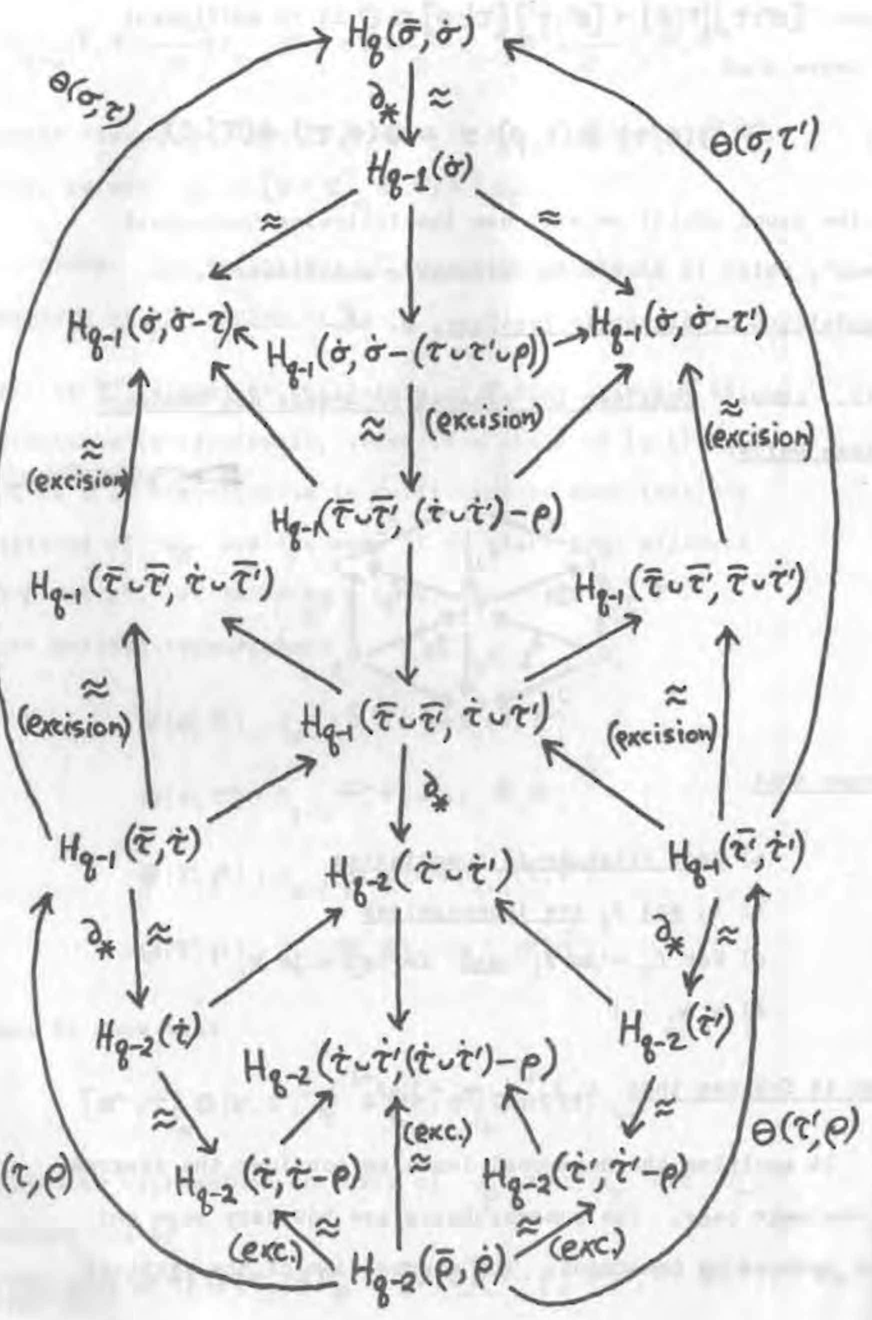
b) i_2 and j_2 are isomorphisms

c) $\operatorname{Ker} f_2 = \operatorname{Im} f_1$ and $\operatorname{Ker} g_2 = \operatorname{Im} g_1$

d) $k_2 k_1 = 0$

Then it follows that $i_3 i_2^{-1} i_1 = -\,j_3 j_2^{-1} j_1$.

In applying the hexagonal lemma we consider the diagram on the next page. The homomorphisms are boundary maps and maps induced by inclusion. The composition of the vertical



maps from $H_q(\bar\sigma,\dot\sigma)$ to $H_{q-2}(\bar\rho,\dot\rho)$ via $H_{q-1}(\bar\tau\cup\bar\tau',\dot\tau\cup\dot\tau')$ is zero because the map from $H_{q-1}(\bar\tau\cup\bar\tau',(\dot\tau\cup\dot\tau')-\rho)$ to $H_{q-2}(\dot\tau\cup\dot\tau')$ factors through $H_{q-2}((\dot\tau\cup\dot\tau')-\rho)$, and so two successive maps of the exact homology sequence of the pair $(\dot\tau\cup\dot\tau',(\dot\tau\cup\dot\tau')-\rho)$ occur. The sequence

$$H_{q-1}(\bar\tau,\dot\tau)\longrightarrow H_{q-1}(\bar\tau\cup\bar\tau',\dot\tau\cup\dot\tau')\longrightarrow H_{q-1}(\bar\tau\cup\bar\tau',\bar\tau\cup\dot\tau')$$

is easily seen to be exact--in fact it is isomorphic to the split exact sequence $Z\longrightarrow Z\oplus Z\longrightarrow Z$. The commutativity of all the necessary triangles follows by naturality and the boundary axiom of cellular homology. Thus the hexagonal lemma applies to yield (3) as desired.

According to 1.2 and the remarks before 1.3, the map ϕ_q so defined is an isomorphism for each q. Thus the proof of 1.9 will be complete when we show that the diagram below is commutative for each q:

$$\begin{tikzcd}
C_q(K) \arrow[r, "\phi_q"] \arrow[dd, "\partial"'] & H_q^S(K_q, K_{q-1}) \arrow[d, "\partial_*^S"] \\
& H_{q-1}^S(K_{q-1}) \arrow[d, "k_*^S\ (\text{inclusion})"] \\
C_{q-1}(K) \arrow[r, "\phi_{q-1}"] & H_{q-1}^S(K_{q-1}, K_{q-2})
\end{tikzcd}$$

Let σ be a q-cell of K. (We assume that $q\geqslant 2$; the proof for $q = 0$ or 1 is easy.) Then $\phi_q(\sigma)$ is represented by $[u_\sigma, i_\sigma]\in H_q^{i_\sigma}(\bar\sigma,\dot\sigma)$. Thus $k_*^S\partial_*^S\phi_q(\sigma)$ is represented by $[\partial_* u_\sigma, kj]\in H_{q-1}^{kj}(\dot\sigma)$, where $j:\dot\sigma\subseteq K_{q-1}$ is the inclusion.

Let L denote the (q-2)-skeleton of $\bar{\sigma}$. The composition kj carries L into the (q-2)-skeleton of K and so $\left[\partial_* u_\sigma,\ kj\right]$ has the successor $\left[h_*\partial_* u_\sigma,\ g\right] \in H^{g}_{q-1}(\dot{\sigma}, L)$, where $g\colon (\dot{\sigma}, L) \subseteq (K_{q-1}, K_{q-2})$ and $h\colon \dot{\sigma} \subseteq (\dot{\sigma}, L)$ are inclusions.

On the other hand, $\partial\sigma = \sum_{\tau}[\sigma:\tau]\tau$. For each (q-1)-face τ of σ, $\phi_{q-1}(\tau)$ is represented by $[u_\tau,\ i_\tau] \in H^{i_\tau}_{q-1}(\bar{\tau}, \dot{\tau})$. The inclusion $(\bar{\tau}, \dot{\tau}) \subseteq (K_{q-1}, K_{q-2})$ factors through $(\dot{\sigma}, L)$ and so $[u_\tau,\ i_\tau]$ has the successor $\left[(j_\tau)_* u_\tau,\ g\right]$ $H^{g}_{q-1}(\dot{\sigma}, L)$, where $j_\tau\colon (\bar{\tau}, \dot{\tau}) \subseteq (\dot{\sigma}, L)$ is the inclusion. Thus $\phi_{q-1}\partial\sigma$ is represented by $\left[\sum_{\tau}[\sigma:\tau](j_\tau)_* u_\tau,\ g\right]$ $H^{g}_{q-1}(\dot{\sigma}, L)$. We complete the proof that the diagram above is commutative by showing that

(4) $$h_*\partial_* u_\sigma = \sum_{\tau}[\sigma:\tau](j_\tau)_* u_\tau .$$

Since L is the (q-2)-skeleton of $\dot{\sigma}$, 1.2 implies that $H_{q-1}(\dot{\sigma}, L)$ is the direct sum of the images of $H_{q-1}(\bar{\tau}, \dot{\tau})$ under $(j_\tau)_*$, where τ varies over the (q-1)-faces of σ. To show that (4) holds it is sufficient to check that (4) holds under projection onto each direct summand of $H_{q-1}(\dot{\sigma}, L)$. Consider the maps:

$$(\bar{\tau}, \dot{\tau}) \xrightarrow[j_\tau]{} (\dot{\sigma}, L) \xrightarrow[p]{} (\dot{\sigma}, \dot{\sigma}-\tau) \xleftarrow[k]{} \dot{\sigma}.$$

The homomorphism p_* maps each direct summand of $H_{q-1}(\dot{\sigma}, L)$ to zero except for $(j_\tau)_* H_{q-1}(\bar{\tau}, \dot{\tau})$, which is mapped isomorphically (by the excision axiom) onto $H_{q-1}(\dot{\sigma}, \dot{\sigma}-\tau)$. But by definition (See (1) above) $k_*\partial_* u_\sigma = [\sigma:\tau]\, p_*(j_\tau)_* u_\tau$. This is true for each (q-1)-face τ of σ, and so equation (4) is proved. This completes the proof of 1.9.

2. Complexes with Identifications.

Let K be a regular complex, F a family of identifications on K. (See Chapter III for definitions.) Let $s\colon |K| \longrightarrow K/F$ denote projection onto the identification space. In general, K/F is an irregular complex with $(K/F)_q = s(K_q)$. Choose an incidence function α for K which is invariant under F. Let $C(K/F)$ denote the chain complex defined as in III.2 in terms of the incidence function α. Finally, let $D(K/F)$ denote the skeletal chain complex associated to the decomposition of K/F as a complex.

2.1. THEOREM. <u>The homology groups of C(K/F) are isomorphic to the singular homology groups of the space K/F.</u>

Proof: Theorem 1.7 implies that it is sufficient to show that C(K/F) and D(K/F) are chain isomorphic. Consider the diagram:

$$\begin{tikzcd}
C_q(K/F) \arrow[r, "\delta'"] \arrow[dddd, bend right=60, "\xi_q"'] & C_{q-1}(K/F) \arrow[dddd, bend left=60, "\xi_{q-1}"] \\
C_q(K) \arrow[r, "\partial"] \arrow[u, "s_\#"] \arrow[d, "\phi"] & C_{q-1}(K) \arrow[u, "s_\#"'] \arrow[d, "\phi"'] \\
H^S_q(K_q, K_{q-1}) \arrow[r, "\partial"] \arrow[d, "s_*"] & H^S_{q-1}(K_{q-1}, K_{q-2}) \arrow[d, "s_*"'] \\
H^S_q((K/F)_q, (K/F)_{q-1}) \arrow[r, "\partial"] \arrow[d, leftrightarrow] & H^S_{q-1}((K/F)_{q-1}, (K/F)_{q-2}) \arrow[d, leftrightarrow] \\
D_q(K/F) \arrow[r, "\partial"] & D_{q-1}(K/F)
\end{tikzcd}$$

The map ϕ is the chain isomorphism of 1.9. The only freedom of choice in the definition of ϕ occurs in dimension zero. We stipulate that for each pair of vertices A and B mapping by s to the same vertex of K/F $s_* u_A = s_* u_B$.

The first or uppermost square of the diagram is commutative by construction. (See Chapter III.) The second square is commutative because ϕ is a chain map. The third square is commutative because s is a skeletal mapping.

We define a chain map $\xi : C(K/F) \longrightarrow D(K/F)$ as follows. Let σ be a q-cell of K/F. Suppose that τ is a q-cell of K which is mapped by s onto σ. Then $\xi(\sigma) \in H_q^S((K/F)_q, (K/F)_{q-1})$ is represented by $[u_\tau, s\, i_\tau] \in H_q^{s\, i_\tau}(\bar{\tau}, \dot{\tau})$. It is easy to show by induction on q that the definition of $\xi(\sigma)$ is independent of the choice of τ. Also, if τ is mapped onto σ by s, then $\xi(\sigma) = s_* \phi(\tau)$, and so ξ is a chain map since the diagram on page 255 is commutative. The remarks following the proof of 1.2 imply that ξ is an isomorphism in each dimension, and so ξ is a chain isomorphism. This completes the proof of Theorem 2.1.