Lecture 1

Homological Algebra of Spectral Sequences

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1 Introduction

> If we want information, we must be prepared to go where the geometry is
>
> Adams 1978

We'll begin our adventure with spectral sequences in the driest way possible: talking about homological algebra with little geometric motivation. The reason for this is simple: most of the attendees of this seminar have experienced spectral sequences with motivation, but without rigid formalism. In particular, it is assumed that the reader understands *why* spectral sequences might be helpful; throughout this semester, we will explore specifically how to use spectral sequences for computations and proofs, ultimately focusing on applications to topology.

One difficulty in learning about spectral sequences for the first time is keeping track of the myriad indices it seems one has to use. A purpose of this lecture is to convince you that it is possible to avoid being encumbered by notation while working with spectral sequences, and that it is actually not too difficult to parse.

For prerequisites, we use categorical language freely throughout. We also assume the reader has experience with some form of co/homology, and knows enough homological algebra to, say, prove the snake lemma via a diagram chase. The most important prerequisite, however, is a willingness to *go slowly* through proofs and definitions, understanding where all the maps are defined, what certain bidegrees are, and keeping track of indices. I believe in you!

Overview of the Seminar

I foresee the seminar proceeding as follows: we will start with this lecture on homological algebra and spectral sequences in general. The most natural next step, from a topological perspective, is to consider the Serre spectral sequence. This will occupy 3 lectures: first, the setup and an initial few examples; second, the Steenrod algebra and cohomology of $K(G, n)$s; and third, the Kudo transgression theorem and some computations of unstable homotopy groups ($\pi_4(S^2)$).

We then take a detour to the world of spectra and generalized cohomology theories, in which we can introduce the Atiyah-Hirzebruch spectral sequence, a specific example of a general class of spectral sequences that includes the Serre SS. If wanted, we can take a detour to discuss complex oriented cohomology theories, and show (for instance) that complex K-theory is one (by exhibiting the collapse of the AHSS on the E_2-page).

2 The Objects

There are some objects used in the theory of spectral sequences that may be unfamiliar: graded and bigraded abelian groups/rings, exact couples, filtered complexes and their associated gradeds, and double complexes. We present their basic definitions here.

2.1 Gradings and Bigradings

In this section, we recall terminology regarding graded and bigraded objects in a category (often abelian).

Definition 2.1. Let $\mathscr{C}$ be a category. A GRADED OBJECT $X^\bullet$ in $\mathscr{C}$ is a $\mathbb{Z}$-indexed family $\{X^p\}_{p\in\mathbb{Z}}$ of objects in $\mathscr{C}$. A BIGRADED OBJECT $E^{\bullet,\bullet}$ is a $(\mathbb{Z}\times\mathbb{Z})$-indexed family $\{E^{p,q}\}_{(p,q)\in\mathbb{Z}\times\mathbb{Z}}$ of objects in $\mathscr{C}$. Often, the bullets are suppressed, and we refer to X as a graded object, or E as a bigraded object.

Definition 2.2. Let X and Y be graded objects in a category $\mathscr{C}$. A GRADED MAP OF DEGREE d for an integer $d \in \mathbb{Z}$, written $f : X \to Y$, is a family of maps $f_p : X^p \to Y^{p+d}$ in $\mathscr{C}$ indexed by $p \in \mathbb{Z}$. We refer to d as the DEGREE of f, and write $\deg(f) = d$.

Similarly, if D and E are *bi*graded objects in $\mathscr{C}$, we can define a BIGRADED MAP of *bidegree* (c,d). This is a $(\mathbb{Z}\times\mathbb{Z})$-indexed collection of maps $f_{p,q} : D^{p,q} \to E^{p+c,q+d}$, which is written $f : D \to E$. The pair (c,d) is called the BIDEGREE of f, written $\deg(f) = (c,d)$.

Remark 2.3. Sometimes, we will write the grading/bigrading of an object as a subscript instead of a superscript. In context, this should present no confusion. Furthermore, we often consider objects graded not by $\mathbb{Z}$, but by the non-negative integers $\mathbb{N}$.

It is clear that graded/bigraded objects in a category $\mathscr{C}$ form categories on their own, denoted $\mathrm{gr}\mathscr{C}$ and $\mathrm{bigr}\mathscr{C}$, respectively.

Examples 2.4. A polynomial ring $R[x]$ can be viewed as a graded abelian group, indexed by the non-negative integers. Indeed, the grading is by degree: $R[x]_n = R \cdot x^n$. The polynomial ring $R[x,y]$ in two variables carries the structure of a bigraded abelian group: a monomial ax^ny^m is considered as being in $R[x,y]_{n,m}$. The reader can formulate for him/herself the notion of an n-graded object in a category, and see that the polynomial ring $R[x_1,\ldots,x_n]$ is an example of an n-graded abelian group.

Perhaps the canonical examples of graded abelian groups come from co/homology theories: recall that a cohomology theory is a sequence of contravariant functors $\{h^n : \mathsf{Top} \to \mathsf{Ab}\}_{n\in\mathbb{Z}}$ that satisfy certain (i.e., Eilenberg-Steenrod) axioms. In the language of graded objects, a cohomology theory is a single contravariant functor

$$h^* : \mathsf{Top} \to \mathrm{gr}\mathsf{Ab},$$

valued in the category of graded abelian groups.

Warning 2.5. The reader may recall that singular cohomology theory, in addition to many others, has a natural *cup product*, making it into a "graded ring." This notion of graded ring does *not* agree with the notion of graded object in the category Ring: rather, it is a graded abelian group equipped with a ring structure that plays with the grading in a suitable way (i.e., $R_n \cdot R_m \subseteq R_{n+m}$).

For instance, the polynomial ring $R[x]$ is a graded ring, but its structure as a graded abelian group does not respect its ring structure—$x \cdot x$ is no longer in $R[x]_1$.

Definition 2.6. Let's now fix a commutative ring R, and consider the categories of graded and bigraded R-modules, $\mathrm{gr}\mathsf{Mod}_R$ and $\mathrm{bigr}\mathsf{Mod}_R$. A SUBMODULE M' of a graded module M is a graded module such that M'_n is a submodule of M_n for each n. Submodules of bigraded modules are similarly defined bidegree-wise.

If M' is a submodule of a graded (resp. bigraded) module M, we define the QUOTIENT MODULE M/M' degree-wise: $(M/M')_n = M_n/M'_n$ (respectively, $(M/M')_{p,q} = M_{p,q}/M'_{p,q}$.

It should be clear that degree-wise construction allows us to define for the category $\mathrm{gr}\mathsf{Mod}_R$ all the usual notions we know and love in Mod_R, and similarly for $\mathrm{bigr}\mathsf{Mod}_R$: for instance, we say that a sequence

$$M \xrightarrow{f} N \xrightarrow{g} K$$

is EXACT if, for each $n \in \mathbb{Z}$, the corresponding sequence of R-modules

$$M_n \xrightarrow{f_n} N_n \xrightarrow{g_n} K_n$$

is exact. A similar formulation (but with bidegrees) works for bigraded modules.

2.2 Exact Couples

Suppose we have a short exact sequence

$$0 \longrightarrow C_1^\bullet \xrightarrow{i} C_2^\bullet \xrightarrow{j} C_3^\bullet \longrightarrow 0$$

of chain complexes. A standard diagram chase gives a long exact sequence in homology:

$$\begin{tikzcd}
 & \cdots \arrow[r] & H_{i+1}(C_3^\bullet) \arrow[dll, out=-30, in=150, "\partial_*"'] \\
H_i(C_1^\bullet) \arrow[r, "i_*"] & H_i(C_2^\bullet) \arrow[r, "j_*"] & H_i(C_3^\bullet) \arrow[dll, out=-30, in=150, "\partial_*"'] \\
H_{i-1}(C_1^\bullet) \arrow[r] & \cdots &
\end{tikzcd}$$

where the "going around" maps ∂_* are defined in part by the snake lemma. A more succinct way of presenting this information is saying that the triangle of graded abelian groups

$$\begin{tikzcd}
H_*(C_3^\bullet) \arrow[rr, "\partial_*"] & & H_*(C_1^\bullet) \arrow[dl, "i_*"] \\
 & H_*(C_2^\bullet) \arrow[ul, "j_*"] &
\end{tikzcd}$$

is exact at each point, where the maps i_* and j_* have degree 0, while ∂_* has degree -1. In the case where $C_1^\bullet = C_3^\bullet$ (perhaps, level-wise, one has $C_2^n = C_1^n \oplus C_3^n$), the picture looks like

$$\begin{tikzcd}
E \arrow[rr, "\partial_*"] & & E \arrow[dl, "i_*"] \\
 & D. \arrow[ul, "j_*"] &
\end{tikzcd}$$

The data present here can be encapsulated in the following definition:

Definition 2.7. Let $\mathscr{A}$ be an abelian category[1], E and D objects in $\mathscr{A}$, and $\alpha : E \to E$, $\beta : E \to D$, and $\gamma : D \to E$ morphisms. The data $(E, D, \alpha, \beta, \gamma)$ form an EXACT COUPLE if the diagram

$$\begin{tikzcd}
E \arrow[rr, "\alpha"] & & E \arrow[dl, "\beta"] \\
 & D \arrow[ul, "\gamma"] &
\end{tikzcd}$$

is exact at all three points.

[1]By the Freyd-Mitchell embedding theorem, we can (and will!) assume that $\mathscr{A}$ is a full subcategory of Mod_R, for some R. Thus, we may work with "elements" of objects in $\mathscr{A}$ without further thought.

Given an exact couple

$$\begin{tikzcd} E \arrow[rr, "\alpha"] & & E \arrow[dl, "\beta"] \\ & D, \arrow[ul, "\gamma"] & \end{tikzcd}$$

note that the composite $\gamma \circ \beta$ is zero. Thus we have $(\beta \circ \gamma)^2 = 0$, so there is a differential $d = \beta \circ \gamma : D \to D$. We can then take homology of D with respect to this differential d, to get

$$H(E, d) = \ker(d)/\operatorname{im}(d).$$

Definition 2.8. Let $(E, D, \alpha, \beta, \gamma)$ be an exact couple

$$\begin{tikzcd} E \arrow[rr, "\alpha"] & & E \arrow[dl, "\beta"] \\ & D \arrow[ul, "\gamma"] & \end{tikzcd}$$

The DERIVED COUPLE of $(E, D, \alpha, \beta, \gamma)$ is the couple

$$\begin{tikzcd} E' \arrow[rr, "\alpha'"] & & E' \arrow[dl, "\beta'"] \\ & D', \arrow[ul, "\gamma'"] & \end{tikzcd}$$

where

$$\begin{aligned} E' &= \operatorname{im}(\alpha) \subseteq E & D' &= H(E, d) \\ \alpha' &= \alpha|_{E'} & \beta'(\alpha(x)) &= \beta(x) \in \ker(d)/\operatorname{im}(d) \\ \gamma'(z + \operatorname{im}(d)) &= \gamma(z) \end{aligned}$$

Proposition 2.9. *The maps α', β', and γ' defined in Definition 2.8 are well-defined, and make the derived couple $(E', D', \alpha', \beta', \gamma')$ into an exact couple.*

Proof. The map $\alpha' : E' \to E'$ is clearly well-defined, as it is the restriction of a well-defined map. If $\alpha(x) = \alpha(y)$ for some $x, y \in E$, then $x - y \in \operatorname{im}(\gamma)$, by exactness of

$$D \xrightarrow{\gamma} E \xrightarrow{\alpha} E.$$

Thus $\beta(x - y)$ is in the image of d, and as $\beta(x) = \beta(y) + \beta(x - y)$, we get that $\beta(x) = \beta(y)$ upon passage to homology. If $z, z' \in D$, with $z' \in \operatorname{im}(d)$, we have $\gamma(z + z') = \gamma(z) + \gamma(z')$. By exactness at

$$E \xrightarrow{\beta} D \xrightarrow{\gamma} E,$$

we see that $\gamma(z') = 0$, so γ' is well-defined.

Now, we check exactness. I'll leave this as a string of equalities for the sake of being concise;

the reader should make sure he/she understands them fully.

$$\begin{aligned}
\ker(\alpha') &= \ker(\alpha) \cap \operatorname{im}(\alpha) = \operatorname{im}(\gamma) \cap \ker(\beta) \\
&= \gamma(\gamma^{-1}(\ker \beta)) = \gamma(\ker d) = \gamma'(\ker d / \operatorname{im} d) \\
&= \operatorname{im} \gamma'. \\
\ker(\beta') &\overset{*}{=} \beta^{-1}(\operatorname{im} d) / \ker i = \beta^{-1}(\beta(\operatorname{im} \gamma)) \\
&= (\operatorname{im} \gamma + \ker \beta) / \operatorname{im} \alpha = (\ker \alpha + \ker \beta) / \ker \alpha \\
&\overset{*}{=} \alpha(\ker \beta) = \alpha(\operatorname{im} \alpha) = \operatorname{im} \alpha'. \\
\ker(\gamma') &= \ker \gamma / \operatorname{im} d = \operatorname{im} \beta / \operatorname{im} d = \beta E / \operatorname{im} d \\
&= \operatorname{im} \beta'.
\end{aligned}$$

The two starred equalities use the fact that if $i : M \to M$ is a self-map of modules, then $\operatorname{im} i \cong M / \ker i$. □

2.3 Filtered Complexes

Spectral sequences most naturally arise when we filter a complex; the resulting object is unsurprisingly called a *filtered complex*.

Definition 2.10. Let $\mathscr{C}$ be a category, and $X \in \mathscr{C}$ an object.

(a) A DESCENDING FILTRATION for X is a sequence of morphisms (usually monomorphisms)

$$\cdots \longrightarrow X_{n+1} \longrightarrow X_n \longrightarrow \cdots \longrightarrow X.$$

(b) An INCREASING FILTRATION for X is a sequence of (mono)morphisms

$$X \longrightarrow \cdots \longrightarrow X_n \longrightarrow X_{n+1} \longrightarrow \cdots.$$

An object X equipped with a filtration is called a FILTERED OBJECT in $\mathscr{C}$.

Definition 2.11. A FILTERED COMPLEX is a filtered object in the category Ch_R of chain complexes of R-modules for a commutative ring R. More precisely, it is the following data:

1. A chain complex $C_\bullet = (\cdots \to C_n \to C_{n-1} \to \cdots)$ in Mod_R, with differential maps $\partial_n : C_n \to C_{n-1}$.

2. For each integer n, there is a filtration $F_\bullet C_n$ (either ascending or descending) on C_n, compatible with the differentials ∂_*, in that

$$\partial(F_p C_n) \subset F_p C_{n-1}.$$

3 Spectral Sequences

3.1 In General

We'll start with the general definition of a spectral sequence, and then talk about how they arise "in nature."

Definition 3.1. Let R be a ring (commutative, with unity). A DIFFERENTIAL BIGRADED MODULE over R is a $\mathbb{Z} \times \mathbb{Z}$-indexed collection $\{E^{p,q}\}$ of R-modules, equipped with an R-linear map $d : E^{*,*} \to E^{*,*}$ of total degree $+1$ (for a bigraded module of *cohomological type*) or -1 (for one of *homological type*)[2]. The map d is required to be DIFFERENTIAL in that $d \circ d = 0$.

The HOMOLOGY of the differential bigraded module $(E^{*,*}, d)$ is defined to be

$$H^{p,q}(E^{*,*}, d) := \frac{\ker d : E^{p,q} \to E^{p \pm s, q \mp (s-1)}}{\operatorname{im} d : E^{p \mp s, q \pm (s-1)} \to E^{p,q}}$$

Definition 3.2. A SPECTRAL SEQUENCE is a collection $\{E_r^{*,*}, d_r\}$ of differential bigraded R-modules, where the differentials are either (i) all of bidegree $(-r, r-1)$—in which case we say the spectral sequence is of HOMOLOGICAL TYPE—or (ii) all of bidegree $(r, 1-r)$—in which case we say it is of COHOMOLOGICAL TYPE. We furthermore require that for all r, p, q,

$$E_{r+1}^{p,q} \cong H^{p,q}(E_r^{*,*}, d_r).$$

The bigraded module $E_r^{*,*}$ is called the E_r-PAGE of the spectral sequence. We may also refer to it as the r-th page or the E_r-term of the spectral sequence.

Note that though knowledge of E_1 and d_1 tells us $E_2^{*,*}$, it does not give us any information about the d_2 differential. This is an issue; many deep theorems in topology revolve around determining certain facts about differentials. We often picture spectral sequences as modules living in lattice points of the x-y-plane:

In Figure 3.1, the solid dots represent generators of R-modules; it is assumed that there are no other non-zero classes. This is a special case of a spectral sequence, as it is concentrated in the first quadrant. Such gadgets (creatively called "first-quadrant spectral sequences") have a certain nice property: suppose that $E_r^{p,q}$ is nonzero, where $r > \max(p, q+1)$. Because $q+1-r$ is negative, $E_r^{p+r,q+1-r}$ is trivial; so the d_r differential starting at $E_r^{p,q}$ is trivial as well. The same holds for the potential d_r differential ending at $E_r^{p,q}$. Thus $E_{r+1}^{p,q} = E_r^{p,q}$, and the cycle has stabilized. We call this common module $E_\infty^{p,q}$.

Definition 3.3. Let $\{E_r^{p,q}, d_r\}$ be a first-quadrant spectral sequence of cohomological type. We say that the spectral sequence CONVERGES to a graded module H^* if there is a filtration F^* on H^* such that

$$E_\infty^{p,q} \cong F^p H^{p+q} / F^{p+1} H^{p+q}.$$

Very frequently, a spectral sequence will be presented to us in a theorem of the form "*there is a spectral sequence with E_2-page given by [something computable] and converging to H^**". We encapsulate this data by writing

$$E_2^{*,*} \cong \text{"something computable"} \Rightarrow H^*;$$

this is called the SIGNATURE of the spectral sequence.

[2]Thus the bidegree of the differential is $(\pm s, \mp(s-1))$.

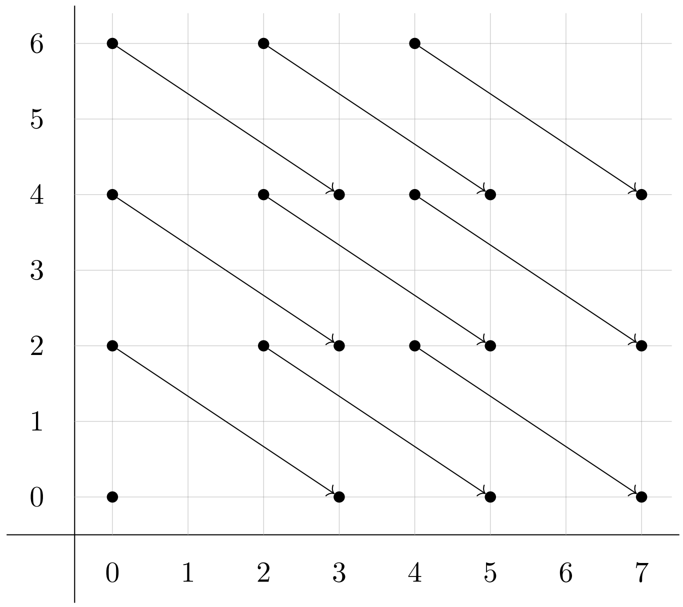


Figure 1: The E_3-page of a cohomological spectral sequence

Often "in the wild," a spectral sequence will be one not only of modules, but of algebras; that is, there is some further multiplicative structure to take into account. The following definitions can make this rigorous:

Definition 3.4. Let H^* be a graded R-module. Recall that the tensor product $H^* \otimes_R H^*$ is given, as a graded module, by

$$(H^* \otimes_R H^*))^n = \bigoplus_{i+j=n} H^i \otimes_R H^j.$$

We say that H^* is a GRADED R-ALGEBRA if there is a map of graded R-modules $\varphi : H^* \otimes_R H^* \to H^*$, called the PRODUCT, such that the diagram

$$\begin{tikzcd}
H^* \otimes_R H^* \otimes_R H^* \arrow[r, "\varphi\otimes 1"] \arrow[d, "1\otimes\varphi"] & H^* \otimes_R H^* \arrow[d, "\varphi"] \\
H^* \otimes_R H^* \arrow[r, "\varphi"] & H^*
\end{tikzcd}$$

commutes. We may define the notion of a BIGRADED R-ALGEBRA analogously; the tensor product $E^{*,*} \otimes_R E^{*,*}$ is defined by

$$(E^{*,*} \otimes_R E^{*,*})^{p,q} = \bigoplus_{\substack{i+j=p \\ k+\ell=q}} E^{i,k} \otimes_R E^{j,\ell}.$$

Either notion may be made DIFFERENTIAL if it comes equipped with a degree-1 (bi)graded[3]

[3]In the case of a bigraded module, degree is taken to mean total degree

R-linear endomorphism that commutes with the product and satisfies a LEIBNIZ RULE:

$$d(a \cdot b) = d(a) \cdot b + (-1)^{\deg a} a \cdot d(b).$$

In that case, we call (H^*, d) a DIFFERENTIAL GRADED R-ALGEBRA, or $(E^{*,*}, d)$ a DIFFERENTIAL BIGRADED R-ALGEBRA. For short, we'll call these DGA's and DBA's, respectively.

Definition 3.5 (Differential bigraded tensor product of differential graded algebras.)**.** Suppose we are given two DGA's, (A^*, d) and (B^*, δ), with products φ and ψ, respectively. Set $E^{p,q} = A^p \otimes_R B^q$; we can put the structure of a bigraded algebra on $E^{*,*}$ via the composite

$$\Phi : E^{p,q} \otimes E^{r,s} = A^p \otimes B^q \otimes A^r \otimes B^s \xrightarrow{1 \otimes T \otimes 1} A^p \otimes A^r \otimes B^q \otimes B^s \xrightarrow{\varphi \otimes \psi} A^{p+r} \otimes B^{q+s} = E^{p+r,q+s}.$$

There is a natural choice for differential on $E^{*,*}$ as well, with a sign convention coming from topology:

$$d_{\otimes}(a \otimes b) = d(a) \otimes b + (-1)^{\deg a} a \otimes \delta(b).$$

One can check that $d_{\otimes} : E^{*,*} \to E^{*,*}$ is a differential, and that it respects the product Φ.

These lead us to the following definition:

Definition 3.6. A SPECTRAL SEQUENCE OF ALGEBRAS is a spectral sequence $\{E_r^{*,*}, d_r\}$ where each $E_r^{*,*}$ is a DBA with respect to product φ_r, such that φ_{r+1} is the composite

$$H(E_r^{*,*}, d_r) \otimes H^(E_r^{*,*}, d_r) \xrightarrow{\quad \simeq \quad} H(E_r^{*,*} \otimes E_r^{*,*}, d_r \otimes 1 \pm 1 \otimes d_r) \xrightarrow{H(\varphi_r)} H(E_r^{*,*}, d_r).$$

The spectral sequence is said to CONVERGE to the DGA H^* if there is a filtration F^* on H^*, respecting the differential, such that

$$E_\infty^{p,q} \cong F^p H^{p+q} / F^{p+1} H^{p+q}.$$

Warning 3.7. You might object to my saying that the above spectral sequence can converge, as I have not specified that it is first-quadrant. Hold your horses; a solution will come.

3.2 Convergence Matters

Unfortunately, not all spectral sequences are handed to us in a first-quadrant form. In this section, we explain how to handle convergence for an arbitrary spectral sequence, as well as give a different presentation of the data a spectral sequence contains. This section will be tough to follow because of the proliferation of indexing; if you find yourself lost, don't pay attention to the sub- and super-scripts.

In what follows, suppose we are working with a spectral sequence of R-modules, of cohomological type (so that the differentials d_r are of bidegree $(r, 1-r)$). The corresponding facts for spectral sequences with multiplicative structure or of homological type may be easily intuited and developed if you so wish. For your sake, I hope you don't. We'll start by looking at the E_2-page. Let us define

$$Z_2^{p,q} := \ker\left(d_2 : E_2^{p,q} \to E_2^{p+2,q-1}\right)$$
$$B_2^{p,q} := \operatorname{im}\left(d_2 : E_2^{p-2,q+1} \to E_{p,q}\right)$$

to be the groups of (p,q)-2-CYCLES and (p,q)-2-BOUNDARIES, respectively (this notation is chosen to mimic homology). As $d^2 = 0$, we have $B_2^{p,q} \subset Z_2^{p,q} \subset E_2^{p,q}$. Now we do what it seems we have set up to do: definition tells us that $E_3^{p,q} \cong Z_2^{p,q}/B_2^{p,q}$. Continuing along our merry way, let's define

$$\widetilde{Z}_3^{p,q} := \ker\left(d_3 : E_3^{p,q} \to E_3^{p+3,q-2}\right)$$
$$\widetilde{B}_3^{p,q} := \operatorname{im}\left(d_3 : E_3^{p-3,q+2} \to E_3^{p,q}\right)$$

As $\widetilde{Z}_3^{p,q} \subset E_3^{p,q}$, there is some submodule $Z_3^{p,q} \subset Z_2^{p,q}$ such that $\widetilde{Z}_3^{p,q} \cong Z_3^{p,q}/B_2^{p,q}$. Similarly, there is some $B_3^{p,q} \subset Z_2^{p,q}$ such that $\widetilde{B}_3^{p,q} \cong B_3^{p,q}/B_2^{p,q}$. It follows that

$$E_4^{p,q} \cong \widetilde{Z}_3^{p,q}/\widetilde{B}_3^{p,q} \cong (Z_3^{p,q}/B_2^{p,q})/(B_3^{p,q}/B_2^{p,q}) \cong Z_3^{p,q}/B_3^{p,q}.$$

Note that we have the tower of inclusions

$$B_2^{p,q} \subset B_3^{p,q} \subset Z_3^{p,q} \subset Z_2^{p,q} \subset E_2^{p,q}.$$

We can iterate this construction to get an infinite tower of submodules of $E_2^{p,q}$:

$$B_2^{p,q} \subset B_3^{p,q} \subset \cdots \subset B_n^{p,q} \subset \cdots \subset Z_n^{p,q} \subset \cdots \subset Z_3^{p,q} \subset Z_2^{p,q} \subset E_2^{p,q},$$

where $E_{n+1}^{p,q} \cong Z_n^{p,q}/B_n^{p,q}$, and $d_{n+1} : Z_{n+1}^{p,q}/B_{n+1}^{p,q} \to Z_{n+1}^{p+n+1,q-n}/B_{n+1}^{p+n+1,q-n}$ has kernel $Z_{n+1}^{p,q}/B_n^{p,q}$ and image $B_{n+1}^{p+n+1,q-n}/B_n^{p+n+1,q-n}$.

The data of these submodule towers and differentials, for every (p,q), completely encapsulate that contained in the spectral sequence. It thus makes sense for us to talk about convergence in these terms:

Definition 3.8. With notation as above, let

$$Z_\infty^{p,q} := \bigcap_n Z_n^{p,q},$$

and

$$B_\infty^{p,q} := \bigcup_n B_n^{p,q}$$

be the elements that stay cycles forever, and those that eventually become boundaries (we call $Z_\infty^{p,q}$ the PERMANENT CYCLES and $B_\infty^{p,q}$ the EVENTUAL BOUNDARIES). Then the "E_∞-page" of the spectral sequence is given by

$$E_\infty^{p,q} = Z_\infty^{p,q}/B_\infty^{p,q}.$$

We say that the spectral sequence CONVERGES to the graded module H^* if there exists a filtration F^* on H^*, perhaps respecting some additional structure (i.e., that of a DGA), such that

$$E_\infty^{p,q} \cong F^p H^{p+q} / F^{p+1} H^{p+q}.$$

3.3 The Spectral Sequence of a Filtered Complex

With all that index-bashing out of the way, we can describe how to extract a spectral sequence from the (commonly-occurring) notion of a filtered complex. It will be helpful to think of a filtered complex as a single *filtered differential graded module*; do that.

Definition 3.9. Let F^* be a filtration on an R-module A. The ASSOCIATED GRADED module $E_0^*(A)$ is the graded module given by

$$E_0^p(A) = \begin{cases} F^p A / F^{p+1} A & \text{if } F^* \text{ is decreasing} \\ F^p A / F^{p-1} A & \text{if } F^* \text{ is increasing.} \end{cases}$$

If $A = H^*$ is a graded module already, then we set $E_0^{p,q}(H^*, F) = E_0^p(H^q)$.

If A is a differential graded module and the filtration F^* commutes with the differential, then there is an induced filtration on the homology $H(A, d)$ of A:

$$F^p H(A, d) = \operatorname{im}\left(H(F^p A, d) \xrightarrow{H(i)} H(A, d) \right),$$

where $i : F^p A \hookrightarrow A$ is the inclusion.

Theorem 3.10. *Suppose (A, d, F^*) is a filtered differential graded module, where d has degree $+1$. Then there exists a spectral sequence $\{E_r^{*,*}\}$ of cohomological type such that*

$$E_1^{p,q} \cong H^{p+q}(F^p A / F^{p+1} A).$$

If furthermore the filtration F^* on A is bounded, then the spectral sequence converges to $H(A, d)$, with the filtration induced by F^*:

$$E_\infty^{p,q} \cong F^p H^{p+q}(A, d) / F^{p+1} H^{p+q}(A, d).$$

Proof. See McCleary 2001, §2.2. The proof is unenlightening and messy; we do not recommend going through it unless absolutely deemed necessary. The basic idea is to construct a tower of submodules akin to that in the previous subsection; these allow you to reconstruct a spectral sequence. □

3.4 The Spectral Sequence of an Exact Couple

I mentioned earlier that an exact couple can give rise to a spectral sequence; in this subsection, I explain how. I'll begin with something this document has been lacking: a little bit of motivation. I attribute this example to Eric Peterson, because he first taught me it.

When working with CW-complexes (i.e., the only spaces—up to weak homotopy equivalence—

one should ever work with), one is often presented with a space X as an "iterated gluing diagram":

$$\begin{tikzcd}
X_1 \arrow[r] & X_2 \arrow[r] & X_3 \arrow[r] & \cdots \arrow[r] & X = \operatorname{colim} X_i \\
A_1 \arrow[u] & A_2 \arrow[u] & A_3 \arrow[u] & &
\end{tikzcd}$$

where each A_n is a bouquet of disks, and each X_{n+1} is the homotopy cofiber (pushout)

$$\begin{tikzcd}
\bigvee_\alpha S^n_\alpha \arrow[r, hook, "i"] \arrow[d] & \bigvee_\alpha D^n_\alpha \arrow[d] \\
X_n \arrow[r] & X_{n+1} \arrow[ul, phantom, very near start, "\ulcorner"].
\end{tikzcd}$$

If we were interested in the homology of X, we might hope to determine it by knowing the (easy-to-compute) homology of the spaces A_n. Because each "L-shape"

$$\begin{tikzcd}
X_n \arrow[r] & X_{n+1} \\
A_n \arrow[u] &
\end{tikzcd}$$

is a cofibration, applying homology gives a diagram of the form

$$\begin{tikzcd}
H_*X_1 \arrow[r] & H_*X_2 \arrow[r] \arrow[dl, "(-1)"'] & H_*X_3 \arrow[r] \arrow[dl, "(-1)"'] & \cdots \arrow[r] \arrow[dl, dotted, "(-1)"'] & H_*X_n \arrow[r] \arrow[dl, dashed] & H_*X_{n+1} \arrow[r] \arrow[dl, "(-1)"'] & \cdots \arrow[r] \arrow[dl, dotted, "(-1)"'] & H_*X \\
H_*A_1 \arrow[u] & H_*A_2 \arrow[u] & H_*A_3 \arrow[u] & \cdots & H_*A_n \arrow[u] & H_*A_{n+1} \arrow[u] & &
\end{tikzcd}$$

Each triangle

$$\begin{tikzcd}
H_*X_n \arrow[r] & H_*X_{n+1} \arrow[dl, "(-1)"'] \\
H_*A_n \arrow[u] &
\end{tikzcd}$$

is secretly the long exact sequence in homology

$$\cdots \longrightarrow H_{i+1}A_n \longrightarrow H_{i+1}X_n \longrightarrow H_{i+1}X_{n+1} \xrightarrow{\partial} H_iA_n \longrightarrow \cdots,$$

only "rolled up" into a triangle of graded abelian groups. We can "roll" this diagram up even further, as there is a grading on the levels of the attaching maps. That is, we can present our data concisely in the form of a triangle

$$\begin{tikzcd}
H_*X_* \arrow[rr, "i"] & & H_*X_* \arrow[dl, "\partial"] \\
& H_*A_* \arrow[ul, "j"] &
\end{tikzcd}$$

where i has bidegree $(0,1)$, ∂ has bidegree $(-1,-1)$, and j has bidegree $(0,0)$. By exactness of the long exact sequence in homology associated to a cofibration, this is an exact triangle. So one should expect a spectral sequence starting from $E^2_{*,*} = H_* A_*$ and converging to $H_* X$ with a level-wise filtration.

Switching conventions to be cohomological, we have the following theorem:

Theorem 3.11 (McCleary 2001, Theorem 2.8)**.** *Suppose*

$$\begin{tikzcd} D^{*,*} \arrow[rr, "i"] & & D^{*,*} \arrow[dl, "j"] \\ & E^{*,*} \arrow[ul, "k"] & \end{tikzcd}$$

is an exact triangle of bigraded R-modules, where i, j, and k are homomorphisms of bidegrees $(-1, 1)$, $(0, 0)$, and $(1, 0)$, respectively. These data give rise to a spectral sequence $\{E_r^{*,*}, d_r\}$, where $E_r^{*,*} = (E^{*,*})^{(r-1)}$, the $(r-1)$-st derived module (i.e., the corresponding module in the derived couple) of $E^{*,*}$, and $d_r = j^{(r-1)} \circ k^{(r-1)}$.

Proof. By definition of the derived couple of an exact couple, we know that E_{r+1} is the homology of E_r with respect to d_r. It thus suffices to check that the derived differentials d_r have the correct bidegree, $(r, 1-r)$, which we show by induction. For the base case, let $E_1 = E^{*,*}$ and $d_1 = j \circ k$, so d_1 has bidegree $(1, 0)$ (bidegree is additive). Now suppose by induction that $j^{(r-1)}$ has bidegree $(r-2, 2-r)$ and $k^{(r-1)}$ has bidegree $(1, 0)$. By definition,

$$j^{(r)}(i^{(r-1)}(x)) = j^{(r-1)}(x) + d^{(r-1)}E^{(r-1)},$$

so the image of $j^{(r)}$ in $(E^{p,q})^{(r)}$ must come from

$$i^{(r-1)}(D^{p-r+2,q+r-2})^{(r-1)} = (D^{p-r+1,q+r-1})^{(r)},$$

so $j^{(r)}$ has bidegree $(r-1, 1-r)$. Furthermore, as

$$k^{(r)}(e + d^{(r-1)}E^{(r-1)}) = k^{(r-1)}(e)$$

and $k^{(r-1)}$ has bidegree $(1, 0)$, so does $k^{(r)}$. Thus by induction, we find that $d^{(r)}$ has bidegree $(r, 1-r)$, and so $\{E_r, d_r\}$ is a cohomological spectral sequence. $\square$

4 Examples

We'll now use our newfangled terminology to compute the cohomology ring $H^*(\Omega S^n)$, of loops on the n-sphere. Next time, we will meet the *Serre spectral sequence*, a gadget which assigns to a fibration $F \hookrightarrow E \to B$ a spectral sequence of signature

$$E_2^{p,q} \cong H^p(B; \mathcal{H}^q(F; R)) \Rightarrow H^*(E; R).$$

This spectral sequence, furthermore, is one of algebras.

There is a path-loops fibration

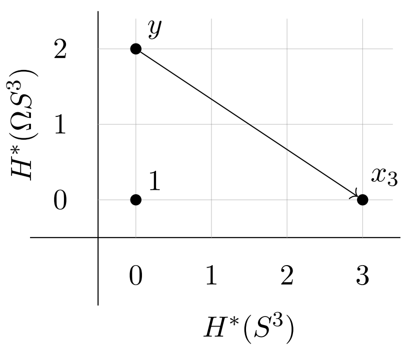
$$\Omega S^n \hookrightarrow * \longrightarrow S^n,$$

where all spaces are pointed, and maps are basepoint-preserving. The homology of the base is torsion-free in all degrees, so the universal coefficient theorem gives the identification

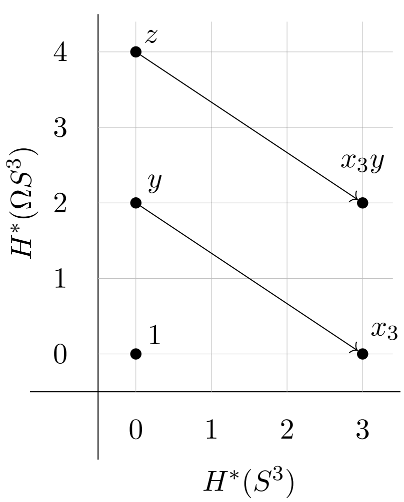
$$E_2^{p,q} \cong H^p(B;R) \otimes H^q(F;R). \tag{1}$$

Case 1: $H^*(\Omega S^n)$ for odd n[4]. We know that the cohomology of S^{2n+1} is $\mathbb{Z}$ in degree zero and degree $2n+1$; the ring structure gives an isomorphism of graded $\mathbb{Z}$-modules $H^*(S^{2n+1}) \cong \Lambda[x_{2n+1}]$. When the degree of x_{2n+1} is clear, we may leave out the subscript.

Because the spectral sequence converges to $\mathbb{Z}$ in degree $(0,0)$, there must be some class that kills x_{2n+1}. Classes are never created past the E_2-page, so this class must already be there; but the identification (1) implies that the only classes live in horizontal degree 0 and $2n+1$. Thus, x_{2n+1} must be hit by a class y in degree $2n$, via a d_{2n+1} differential.



By the multiplicative structure of the spectral sequence (being one of algebras), we must now admit to the existence of a class yx_{2n+1} in bidegree $(2n+1, 2n)$. As before, this must be hit by a d_{2n+1} differential. Let us call the class that hits it z, living in degree $4n$.



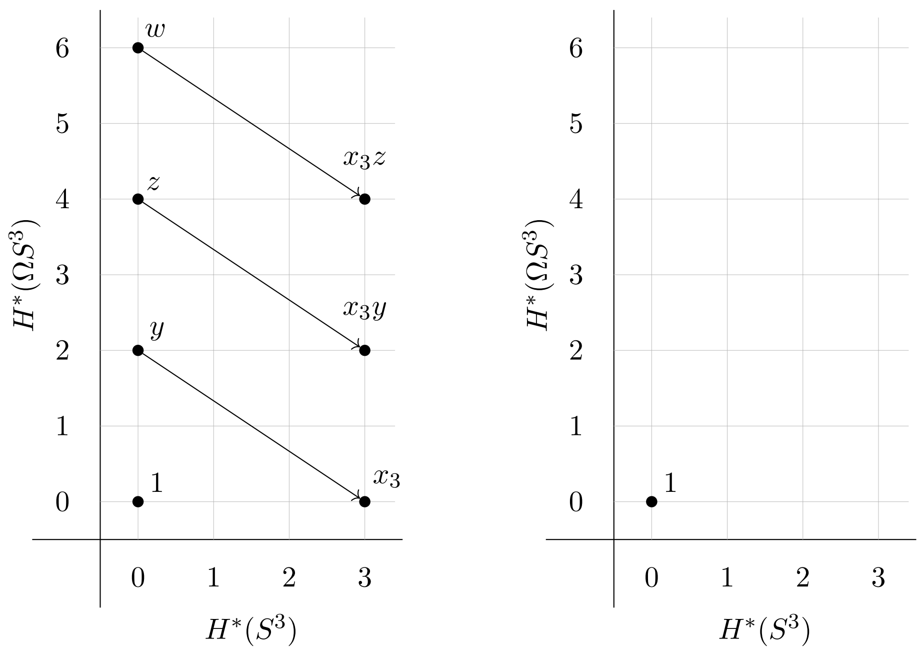
We can ask: does $z = y^2$? The d_{2n+1} differential is an isomorphism, so we can determine this

[4]The figures depict the computation for $n = 3$.

by applying the differential to both classes. By definition, $d_{2n+1}(z) = yx$. By the Leibniz rule,

$$\begin{aligned} d_{2n+1}(y^2) &= d_{2n+1}(y) \cdot y + (-1)^{\deg y} d_{2n+1}(y) \cdot y \\ &= x \cdot y + (-1)^{2n} x \cdot y \\ &= 2x \cdot y = 2z \end{aligned}$$

Thus $z = \frac{1}{2}y$; we can think of this as a formal fraction. Continuing as before, we admit to the existence of xz in bidegree $(2n+1, 4n)$, and thus to some w in bidegree $(0, 6n)$; the process continues (the $E_4 \cong E_\infty$-page is shown on the right):



We ask again: does $w = y^3$? Let's compute differentials:

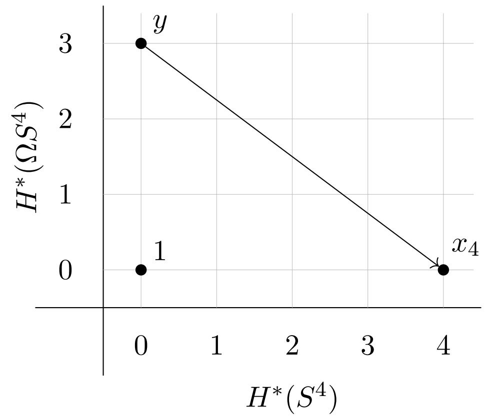
$$\begin{aligned} d_{2n+1}(y^3) &= d_{2n+1}(y) \cdot y^2 + (-1)^{4n} d_{2n+1}(y^2) \cdot y \\ &= xy^2 + 2xy^2 = 3xy^2 \end{aligned}$$

As $2z = y^2$, we see that $w = 6y^3$. This pattern repeats in general: the cohomology $H^*(\Omega S^{2n+1})$ is a DIVIDED POWER ALGEBRA on the generator y in degree $2n$; we denote such an algebra by $\Gamma[y]$. Informally speaking, $\Gamma[y]$ is what results when we allow terms like $y^n/n!$.

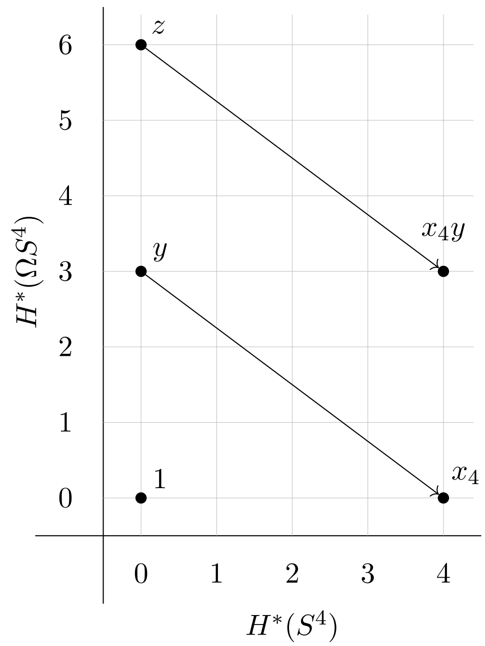
CASE 2: $H^*(\Omega S^n)$ for even n.[5] Again, it is known that the cohomology ring of S^{2n} is the exterior algebra on one class in degree $2n$: $H^*(S^{2n}) \cong \Lambda[x_{2n}]$. This allows us to put down two classes on the E_2-page of the Serre spectral sequence associated to the fibration $\Omega S^{2n} \hookrightarrow * \to S^{2n}$, at bidegrees $(0, 0)$ and $(4, 0)$.

As the SSS converges to $\mathbb{Z}$, where the sole class is in bidegree $(0, 0)$, there must be some class and some differential hitting x_{2n}. By degree considerations, this can only be in bidegree $(0, 2n-1)$; call this class y–the differential connecting the classes is a d_{2n} differential. Its existence requires us to admit to the existence of the product class $x_{2n}y$ in bidegree $(2n, 2n-1)$.

[5]Pictures show the computation for ΩS^4.



The computation should now be expected: there must be a class called z in bidegree $(0, 4n-2)$ that hits xy with a d_{2n} differential.



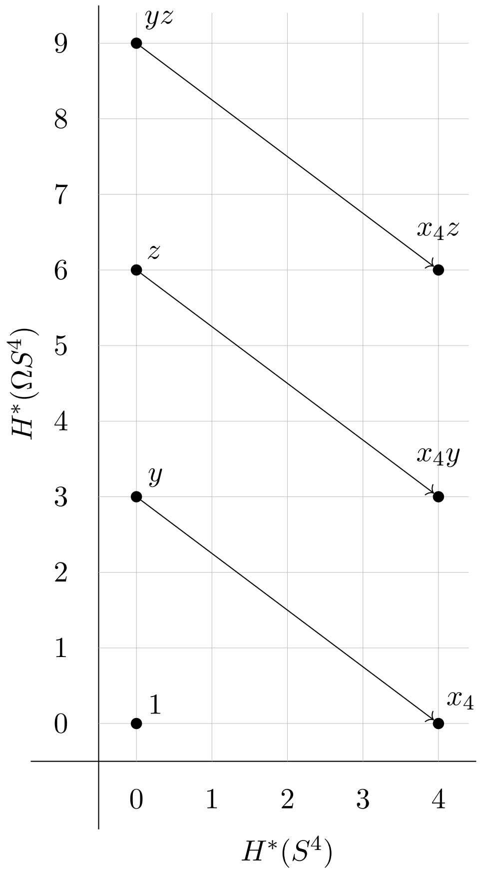
Ask if $y^2 = z$: by the Leibniz rule, we have $d_{2n}(y^2) = d_{2n}(y) \cdot y + (-1)^{2n-1} d_{2n}(y) \cdot y = xy - xy = 0$. It is thus impossible that $y^2 = z$; furthermore, as the differentials are maps of $\mathbb{Z}$-module isomorphisms[6], it follows that $y^2 = 0$, and similarly with all higher powers of y. But now we admit to the existence of xz, in bidegree $(2n, 4n-2)$, and to a class w in bidegree $(0, 6n-3)$ that hits xz under a d_{2n} differential.

It is not possible that $w = y^3$, but it is possible that $w = yz$; so let's check if this is the case.

$$\begin{aligned} d_{2n}(yz) &= d_{2n}(y)z + (-1)^{4n-2} y d_{2n}(z) \\ &= xz + yxy \\ &= xz, \end{aligned}$$

[6] All of this discussion holds for cohomology with coefficients in an arbitrary commutative ring with unity.

so we conclude that, indeed, $w = yz$.



It may be difficult to discern any pattern here; have no fear. The classes in non-zero horizontal degree can only be of the form $x_{2n}y^iz^j$, where i is either zero or 1 and j is a natural number. The numbers i and j furthermore determine a unique bidegree; the corresponding classes thus *must* be the ones appearing in that degree in the SSS, up to some scalar. Just focusing on the classes with a "z" appearing in them, we can compute that $H^*(\Omega S^{2n})$ contains a copy of $\Gamma[z]$; similarly, we've seen already that it contains $\Lambda[y]$. It is then not difficult to see that $H^*(\Omega(S^{2n}))$ is the tensor product $\Gamma[z] \otimes_{\mathbb{Z}} \Lambda[y]$. In summary,

Theorem 4.1. *Let ΩS^n denote the loop space on the n-sphere. The cohomology $H^*(\Omega S^n; R)$ with coefficients in the ring R thus falls into two categories, based on the parity of n:*

$$H^*(\Omega S^n; R) \cong \begin{cases} \Gamma[y], \deg(y) = n - 1 & \text{if } n \text{ is odd;} \\ \Lambda[y] \otimes_R \Gamma[z], \deg(y) = n - 1, \deg(z) = 2n - 2 & \text{if } n \text{ is even.} \end{cases}$$

References

[1] J. Frank Adams. *Infinite loop spaces*. Annals of Mathematics Studies 90. Princeton University Press, 1978.

[2] John McCleary. *A User's Guide to Spectral Sequences*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2001. ISBN: 9780521567596. URL: https://books.google.com/books?id=NijkPwesh-EC.

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