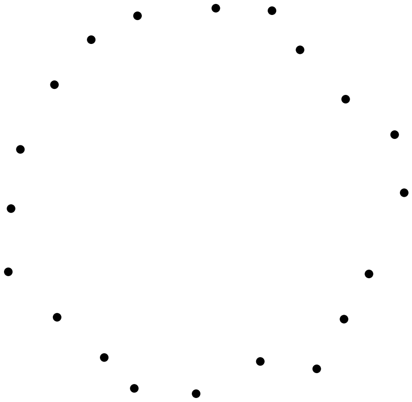
# Invariants for multiparameter persistence

Oliver Gäfvert

KTH Royal Institute of Technology

oliverg@kth.se

# Persistent homology



# Persistent homology

Given data $X := \{x_1, \ldots, x_m\} \subset \mathbb{R}^d$ we want to study **geometrical and topological features**.

# Persistent homology

Given data $X := \{x_1, \ldots, x_m\} \subset \mathbb{R}^d$ we want to study **geometrical and topological features**.

**Idea:** Thicken $X$ by $\epsilon$ and study its features.

# Persistent homology

Given data $X := \{x_1, \ldots, x_m\} \subset \mathbb{R}^d$ we want to study **geometrical and topological features**.

**Idea:** Thicken $X$ by $\epsilon$ and study its features.

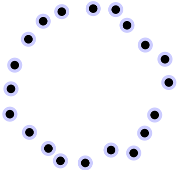
**Question:** How do we choose $\epsilon$?

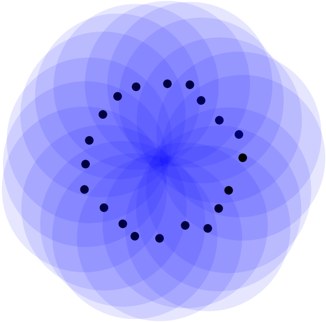
# Persistent homology

Given data $X := \{x_1, \dots, x_m\} \subset \mathbb{R}^d$ we want to study **geometrical and topological features**.

**Idea:** Thicken $X$ by $\epsilon$ and study its features.

**Question:** How do we choose $\epsilon$?





# Multiparameter persistence

**Pipeline**

# Multiparameter persistence

**Pipeline**

- Thicken $X$ using one or more metrics $d_1, \ldots, d_n$.

# Multiparameter persistence

**Pipeline**

- Thicken $X$ using one or more metrics $d_1, \dots, d_n$.
- Compute the nerve of $X_\epsilon$, where $\epsilon \in \mathbb{R}^n$.

# Multiparameter persistence

**Pipeline**

- Thicken $X$ using one or more metrics $d_1, \ldots, d_n$.
- Compute the nerve of $X_\epsilon$, where $\epsilon \in \mathbb{R}^n$.
- Compute $i$th homology of the resulting simplicial complex.

# Multiparameter persistence

**Pipeline**

- Thicken $X$ using one or more metrics $d_1, \dots, d_n$.
- Compute the nerve of $X_\epsilon$, where $\epsilon \in \mathbb{R}^n$.
- Compute *i*th homology of the resulting simplicial complex.
- The result is a functor $F \colon \mathbb{R}^n \to \mathrm{Vect}_k$ called the ***i*th persistence module** of $X$.

# Multiparameter persistence

**Pipeline**

- Thicken $X$ using one or more metrics $d_1, \ldots, d_n$.
- Compute the nerve of $X_\epsilon$, where $\epsilon \in \mathbb{R}^n$.
- Compute $i$th homology of the resulting simplicial complex.
- The result is a functor $F\colon \mathbb{R}^n \to \mathrm{Vect}_k$ called the ***i*th persistence module** of $X$.

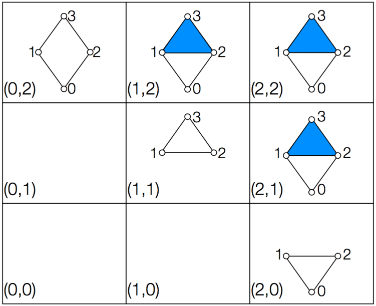



Figure: A bifiltered simplicial complex.

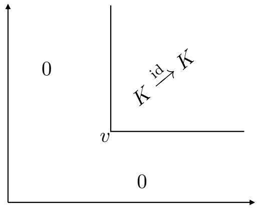
# Preliminaries

*Persistence modules* are functors $F\colon \mathbb{R}^n \to \mathsf{Vect}_k$.

# Preliminaries

*Persistence modules* are functors $F\colon \mathbb{R}^n \to \mathsf{Vect}_k$.

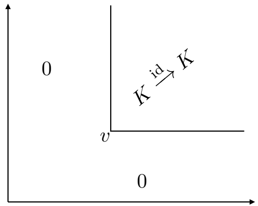
The free functor $K(v,-)\colon \mathbb{R}^2 \to \mathsf{Vect}_k$ on one generator:



# Preliminaries

*Persistence modules* are functors $F \colon \mathbb{R}^n \to \mathrm{Vect}_k$.

The free functor $K(v, -) \colon \mathbb{R}^2 \to \mathrm{Vect}_k$ on one generator:

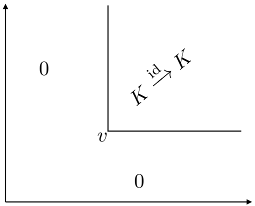



- $F$ is *finitely gen.* if there is a surjection $\oplus_{i=1}^m K(v_i, -) \xrightarrow{\phi} F$.

# Preliminaries

*Persistence modules* are functors $F\colon \mathbb{R}^n \to \mathsf{Vect}_k$.

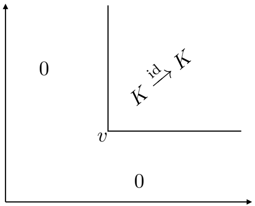
The free functor $K(v,-)\colon \mathbb{R}^2 \to \mathsf{Vect}_k$ on one generator:




- $F$ is *finitely gen.* if there is a surjection $\oplus_{i=1}^m K(v_i,-) \overset{\phi}{\twoheadrightarrow} F$.
- The minimal such $m$ is called the *rank* of $F$.

*Persistence modules* are functors $F\colon \mathbb{R}^n \to \mathsf{Vect}_k$.

The free functor $K(v,-)\colon \mathbb{R}^2 \to \mathsf{Vect}_k$ on one generator:




- $F$ is *finitely gen.* if there is a surjection $\oplus_{i=1}^{m} K(v_i,-) \overset{\phi}{\twoheadrightarrow} F$.
- The minimal such $m$ is called the *rank* of $F$.
- $F$ is *finitely presented* if it is f.g and the kernel of $\phi$ is also f.g.

# Multiparameter persistence

*The categories of:*

- *finitely presented persistence modules*
- *f.g n-graded $k[x_1, \ldots, x_n]$-modules*

*have the same formal properties:*

# Multiparameter persistence

*The categories of:*

- *finitely presented persistence modules*
- *f.g n-graded $k[x_1, \ldots, x_n]$-modules*

*have the same formal properties:*

- *enough projectives*

# Multiparameter persistence

*The categories of:*

- *finitely presented persistence modules*
- *f.g n-graded $k[x_1, \ldots, x_n]$-modules*

*have the same formal properties:*

- *enough projectives*
- *all projectives are free*

# Multiparameter persistence

*The categories of:*

- *finitely presented persistence modules*
- *f.g n-graded $k[x_1, \ldots, x_n]$-modules*

*have the same formal properties:*

- *enough projectives*
- *all projectives are free*
- *any object has a minimal resolution of length at most n.*

# Multiparameter persistence

*The categories of:*

- *finitely presented persistence modules*
- *f.g $n$-graded $k[x_1, \ldots, x_n]$-modules*

*have the same formal properties:*

- *enough projectives*
- *all projectives are free*
- *any object has a minimal resolution of length at most $n$.*

**Theorem (Carlsson and Zomorodian [4])**

*We can identify a finitely presented persistence module with a f.g $n$-graded $k[x_1, \ldots, x_n]$-module by restricting to a small enough grid $\mathbb{N}^n \subset \mathbb{R}^n$.*

**Sketch of proof**

**Sketch of proof**

- The dataset $X$ is finite,

# Multiparameter persistence

**Sketch of proof**

- The dataset $X$ is finite, which means that $F \colon \mathbb{R}^n \to \mathsf{Vect}_k$ is constant except at a finite number of critical points where it changes value.

# Multiparameter persistence

**Sketch of proof**

- The dataset $X$ is finite, which means that $F\colon \mathbb{R}^n \to \mathrm{Vect}_k$ is constant except at a finite number of critical points where it changes value.
- We can write it as $\tilde{F}\colon \mathbb{N}^n \to \mathrm{Vect}_k$.

# Multiparameter persistence

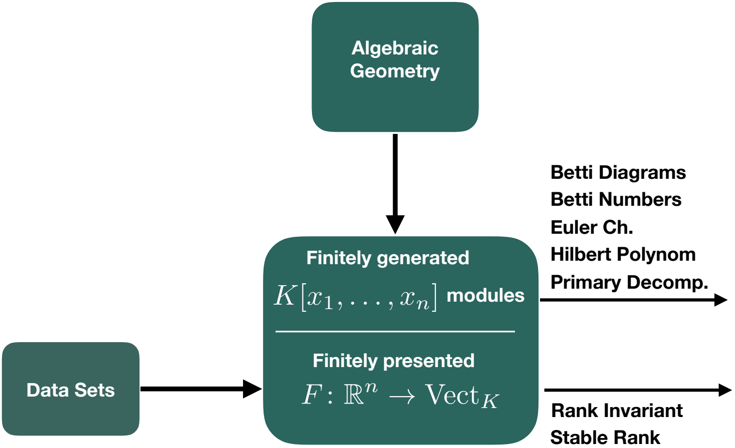
**Sketch of proof**

- The dataset $X$ is finite, which means that $F\colon \mathbb{R}^n \to \mathrm{Vect}_k$ is constant except at a finite number of critical points where it changes value.
- We can write it as $\tilde{F}\colon \mathbb{N}^n \to \mathrm{Vect}_k$.
- The maps $\tilde{F}(u) \to \tilde{F}(v)$ for $u \leq v$ correspond to multiplication by $x^{v-u}$ in the $n$-graded module

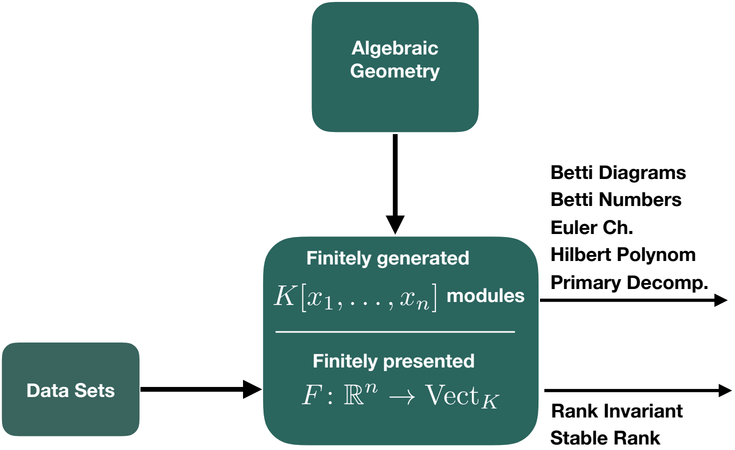
$$\alpha(\tilde{F}) = \bigoplus_{v \in \mathbb{N}^n} \tilde{F}(v)$$

# Invariants

# Invariants


Algebraic Geometry
Finitely generated
$K[x_1, \ldots, x_n]$ modules
Finitely presented
$F\colon \mathbb{R}^n \to \mathrm{Vect}_K$
Data Sets
Betti Diagrams
Betti Numbers
Euler Ch.
Hilbert Polynom
Primary Decomp.
Rank Invariant
Stable Rank

# Invariants




**Need for stable invariants!**

# List of invariants

- For $n = 1$ there is a *complete* invariant called the **barcode** [16].

# List of invariants

- For $n = 1$ there is a *complete* invariant called the **barcode** [16].
- Decomposing the module into indecomposables [6, 1, 10].

# List of invariants

- For $n = 1$ there is a *complete* invariant called the **barcode** [16].
- Decomposing the module into indecomposables [6, 1, 10].
- The rank invariant [4].

# List of invariants

- For $n = 1$ there is a *complete* invariant called the **barcode** [16].
- Decomposing the module into indecomposables [6, 1, 10].
- The rank invariant [4].
- Persistence landscapes [2, 15].

# List of invariants

- For $n = 1$ there is a *complete* invariant called the **barcode** [16].
- Decomposing the module into indecomposables [6, 1, 10].
- The rank invariant [4].
- Persistence landscapes [2, 15].
- The stable rank [14, 7].

# List of invariants

- For $n = 1$ there is a *complete* invariant called the **barcode** [16].
- Decomposing the module into indecomposables [6, 1, 10].
- The rank invariant [4].
- Persistence landscapes [2, 15].
- The stable rank [14, 7].
- Classical invariants from commutative algebra [9].

# Metrics on Persistence Modules

Most common distance is the **interleaving distance** [11].

Most common distance is the **interleaving distance** [11].

$F$ and $G$ are *$\epsilon$-interleaved* if there are maps s.t the following diagram commutes for all $v$:

$$
\begin{tikzcd}
F(v) \arrow[r] \arrow[dr] & F(v+\epsilon) \arrow[r] \arrow[dr] & F(v+2\epsilon) \\
G(v) \arrow[r] \arrow[ur] & G(v+\epsilon) \arrow[r] \arrow[ur] & G(v+2\epsilon)
\end{tikzcd}
$$

# Metrics on Persistence Modules

Most common distance is the **interleaving distance** [11].

$F$ and $G$ are *$\epsilon$-interleaved* if there are maps s.t the following diagram commutes for all $v$:

$$
\begin{tikzcd}
F(v) \arrow[r] \arrow[dr] & F(v+\epsilon) \arrow[r] \arrow[dr] & F(v+2\epsilon) \\
G(v) \arrow[r] \arrow[ur] & G(v+\epsilon) \arrow[r] \arrow[ur] & G(v+2\epsilon)
\end{tikzcd}
$$

$$d_{\bowtie} = \inf\{\epsilon \mid F \text{ and } G \text{ are } \epsilon\text{-interleaved}\}$$

# Barcode

For $n = 1$ there is a complete stable invariant called the **barcode**.

# Barcode

For $n = 1$ there is a complete stable invariant called the **barcode**.

Note that $k[x]$ is a PID.

# Barcode

For $n = 1$ there is a complete stable invariant called the **barcode**.

Note that $k[x]$ is a PID. By the structure theorem for f.g graded modules over a PID there is a unique decomposition:

# Barcode

For $n = 1$ there is a complete stable invariant called the **barcode**.

Note that $k[x]$ is a PID. By the structure theorem for f.g graded modules over a PID there is a unique decomposition:

$$F \cong \bigoplus_{i=1}^{l_1} x^{a_i} k[x]/x^{b_i} k[x] \oplus \bigoplus_{i=1}^{l_2} x^{c_i} k[x]$$

# Barcode

For $n = 1$ there is a complete stable invariant called the **barcode**.

Note that $k[x]$ is a PID. By the structure theorem for f.g graded modules over a PID there is a unique decomposition:

$$F \cong \bigoplus_{i=1}^{l_1} x^{a_i} k[x]/x^{b_i} k[x] \oplus \bigoplus_{i=1}^{l_2} x^{c_i} k[x]$$

The numbers $a_i, b_i$ and $c_i$ yield intervals

$$\{[a_i, b_i)\}_{i=1}^{l_1} \cup \{[c_i, \infty)\}_{i=1}^{l_2}$$

which constitute the invariant.

# Example

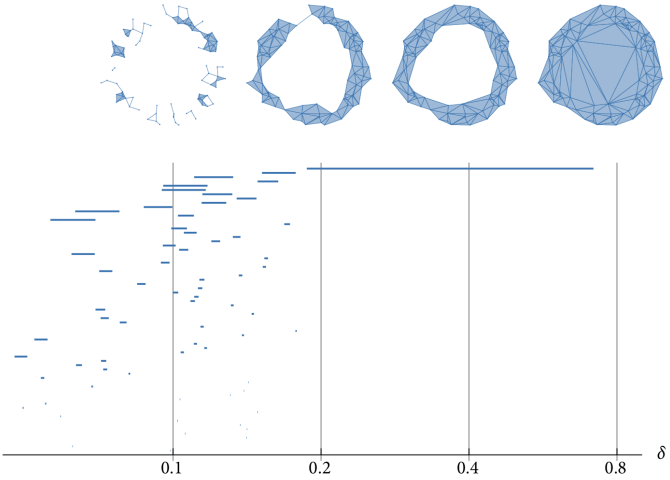



Figure: Illustration by U. Bauer depicting the barcode of the first persistence module of $X$.

# Decomposing $\tilde{F}\colon \mathbb{N}^n \to \mathrm{Vect}_k$

- The decomposition for $n = 1$ is a stable process [5].

# Decomposing $\tilde{F}\colon \mathbb{N}^n \to \mathsf{Vect}_k$

- The decomposition for $n = 1$ is a stable process [5].
- The barcode is a multisubset of a countable set of indecomposables

$$\{[a, b)\}_{a<b\in\mathbb{N}} \cup \{[c, \infty)\}_{c\in\mathbb{N}}$$

# Decomposing $\tilde{F}\colon \mathbb{N}^n \to \mathrm{Vect}_k$

- The decomposition for $n = 1$ is a stable process [5].
- The barcode is a multisubset of a countable set of indecomposables

$$\{[a, b)\}_{a<b\in\mathbb{N}} \cup \{[c, \infty)\}_{c\in\mathbb{N}}$$

- By the Krull-Schmidt theorem there is a unique decomposition of $\tilde{F}$ even for $n > 1$.

# Decomposing $\tilde{F}\colon \mathbb{N}^n \to \mathsf{Vect}_k$

- The decomposition for $n = 1$ is a stable process [5].
- The barcode is a multisubset of a countable set of indecomposables

$$\{[a, b)\}_{a<b\in\mathbb{N}} \cup \{[c, \infty)\}_{c\in\mathbb{N}}$$

- By the Krull-Schmidt theorem there is a unique decomposition of $\tilde{F}$ even for $n > 1$. But it is not stable.

# Decomposing $\tilde{F} \colon \mathbb{N}^n \to \mathrm{Vect}_k$

- The decomposition for $n = 1$ is a stable process [5].
- The barcode is a multisubset of a countable set of indecomposables

$$\{[a, b)\}_{a<b\in\mathbb{N}} \cup \{[c, \infty)\}_{c\in\mathbb{N}}$$

- By the Krull-Schmidt theorem there is a unique decomposition of $\tilde{F}$ even for $n > 1$. But it is not stable.
- A small perturbation of $X$ can change the decomposition of $\tilde{F}$ by a lot.

# Decomposing $\tilde{F}\colon \mathbb{N}^n \to \mathrm{Vect}_k$

- The decomposition for $n = 1$ is a stable process [5].
- The barcode is a multisubset of a countable set of indecomposables

$$\{[a, b)\}_{a<b\in\mathbb{N}} \cup \{[c, \infty)\}_{c\in\mathbb{N}}$$

- By the Krull-Schmidt theorem there is a unique decomposition of $\tilde{F}$ even for $n > 1$. But it is not stable.
- A small perturbation of $X$ can change the decomposition of $\tilde{F}$ by a lot.
- The set of indecomposables is no longer countable and the structure of the indecomposables can be very complicated [4].

# Decomposing $\tilde{F}\colon \mathbb{N}^n \to \mathrm{Vect}_k$

- The decomposition for $n = 1$ is a stable process [5].
- The barcode is a multisubset of a countable set of indecomposables

$$\{[a,b)\}_{a<b\in\mathbb{N}} \cup \{[c,\infty)\}_{c\in\mathbb{N}}$$

- By the Krull-Schmidt theorem there is a unique decomposition of $\tilde{F}$ even for $n > 1$. But it is not stable.
- A small perturbation of $X$ can change the decomposition of $\tilde{F}$ by a lot.
- The set of indecomposables is no longer countable and the structure of the indecomposables can be very complicated [4].
- Therefore there is no invariant analogous to the barcode for $n > 1$.

# Rank invariant

- Introduced by Carlsson and Zomorodian in [4].

# Rank invariant

- Introduced by Carlsson and Zomorodian in [4].
- The **rank invariant** is a function $\rho_F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{N}$ taking $(u, v)$ to the rank of the map $F(u) \to F(v)$.

# Rank invariant

- Introduced by Carlsson and Zomorodian in [4].
- The **rank invariant** is a function $\rho_F \colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{N}$ taking $(u, v)$ to the rank of the map $F(u) \to F(v)$.
- For $n = 1$, $\rho_F$ contains the same information as the barcode.

# Rank invariant

- Introduced by Carlsson and Zomorodian in [4].
- The **rank invariant** is a function $\rho_F \colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{N}$ taking $(u, v)$ to the rank of the map $F(u) \to F(v)$.
- For $n = 1$, $\rho_F$ contains the same information as the barcode.
- Extracting multivariate persistent information from $\rho_F$ is likely hard.
- Software for visualizing the rank invariant RIVET [12].

# Example of RIVET

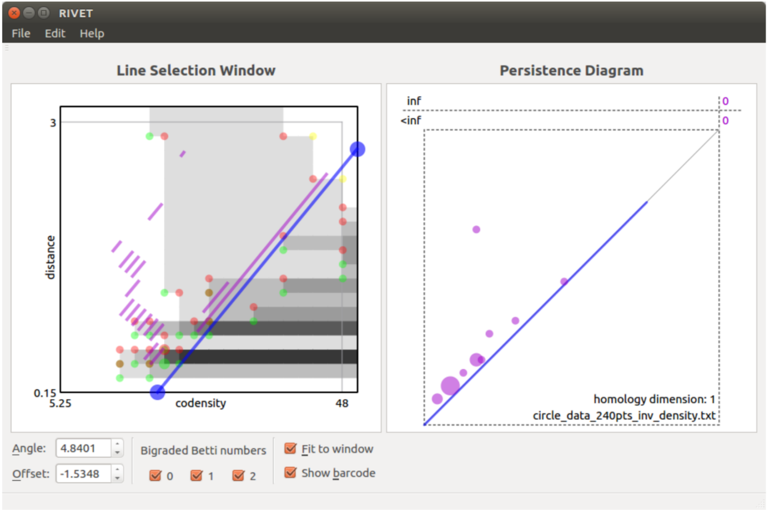



Figure: Illustration of the RIVET software [12] depicting the first bigraded persistence module of $X$.

# Persistence landscapes

- Introduced by P. Bubenik [2] and generalized to multipersistence by O. Vipond [15].

# Persistence landscapes

- Introduced by P. Bubenik [2] and generalized to multipersistence by O. Vipond [15].
- **Goal:** to do statistics on barcodes (compute mean and variance etc).

# Persistence landscapes

- Introduced by P. Bubenik [2] and generalized to multipersistence by O. Vipond [15].
- **Goal:** to do statistics on barcodes (compute mean and variance etc).
- The **persistence landscape** is the function $\lambda \colon \mathbb{N} \times \mathbb{R} \to \mathbb{R}_\infty$ defined as

$$\lambda(k, t) = \sup\{s \in \mathbb{R} \mid \mathrm{rank}(F(t-s) \to F(t+s)) \geq k\}$$

# Persistence landscapes

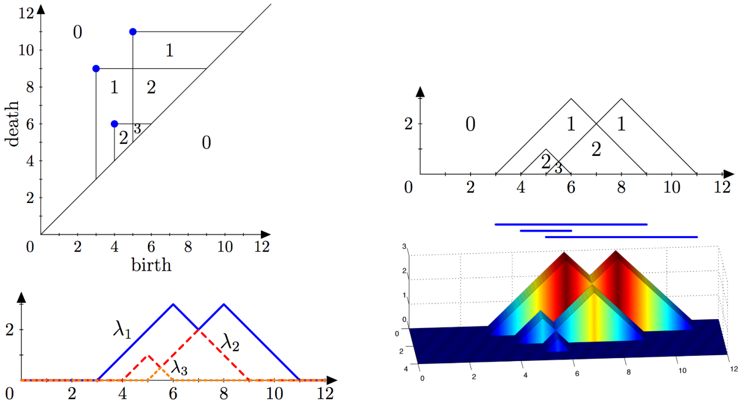



Figure: Illustration by P. Bubenik [2] depicting a persistence landscape and how it is constructed from the barcode and persistence diagram.

# Stable Rank

- Introduced by Scolamiero et. al. in [14].

# Stable Rank

- Introduced by Scolamiero et. al. in [14].
- The **stable rank** is the function $\widehat{\beta}_0(F)\colon \mathbb{R} \to \mathbb{N}$ defined as

$$\widehat{\beta}_0(F)(\tau) = \min\{\mathrm{rank}(G) \mid d(F, G) \leq \tau\}$$

# Stable Rank

- Introduced by Scolamiero et. al. in [14].
- The **stable rank** is the function $\widehat{\beta}_0(F)\colon \mathbb{R} \to \mathbb{N}$ defined as

$$\widehat{\beta}_0(F)(\tau) = \min\{\mathrm{rank}(G) \mid d(F, G) \leq \tau\}$$

- This is a stabilization of a classical invariant, namely the zeroth Betti number.

# Stable Rank

- Introduced by Scolamiero et. al. in [14].
- The **stable rank** is the function $\widehat{\beta}_0(F)\colon \mathbb{R} \to \mathbb{N}$ defined as

$$\widehat{\beta}_0(F)(\tau) = \min\{\mathrm{rank}(G) \mid d(F, G) \leq \tau\}$$

- This is a stabilization of a classical invariant, namely the zeroth Betti number.
- Can be computed using the Topcat software [8].

# Stable Rank

- Introduced by Scolamiero et. al. in [14].
- The **stable rank** is the function $\widehat{\beta}_0(F)\colon \mathbb{R} \to \mathbb{N}$ defined as

$$\widehat{\beta}_0(F)(\tau) = \min\{\mathrm{rank}(G) \mid d(F, G) \leq \tau\}$$

- This is a stabilization of a classical invariant, namely the zeroth Betti number.
- Can be computed using the Topcat software [8].

Using *persistence contours*, we can show:

**Theorem (G. and Chachólski, 2017, [7])**

*For $n \geq 2$, computing $\widehat{\beta}_0(F)$ is NP-hard.*

# Persistence Contours

(Metrics on persistence modules)

# Persistence Contours

A *persistence contour* is a functor $C\colon U \times \mathbb{R} \to U$ s.t $U \subseteq \mathbb{R}^n_\infty$ and for any $v \in U$ and any $\epsilon, \tau \in \mathbb{R}$:

1. $v \leq C(v, \epsilon)$,
2. $C(C(v, \epsilon), \tau) \leq C(v, \epsilon + \tau)$.

# Persistence Contours

A *persistence contour* is a functor $C\colon U \times \mathbb{R} \to U$ s.t $U \subseteq \mathbb{R}^n_\infty$ and for any $v \in U$ and any $\epsilon, \tau \in \mathbb{R}$:

1. $v \leq C(v, \epsilon)$,
2. $C(C(v, \epsilon), \tau) \leq C(v, \epsilon + \tau)$.

Similar to superlinear families by Bubenick et. al. [3].

# Persistence Contours

A *persistence contour* is a functor $C \colon U \times \mathbb{R} \to U$ s.t $U \subseteq \mathbb{R}^n_\infty$ and for any $v \in U$ and any $\epsilon, \tau \in \mathbb{R}$:

1. $v \leq C(v, \epsilon)$,
2. $C(C(v, \epsilon), \tau) \leq C(v, \epsilon + \tau)$.

Similar to superlinear families by Bubenick et. al. [3].

- The *$\epsilon$-shift*,

$$F[\epsilon] = \langle \{ F(v_i \leq C(v_i, \epsilon))(g_i) \in F(C(v_i, \epsilon)) \} \rangle$$

for any generating set $\{g_i \in F(v_i)\}$ of $F$.

# Persistence Contours

A *persistence contour* is a functor $C\colon U \times \mathbb{R} \to U$ s.t $U \subseteq \mathbb{R}^n_\infty$ and for any $v \in U$ and any $\epsilon, \tau \in \mathbb{R}$:

1. $v \leq C(v, \epsilon)$,
2. $C(C(v, \epsilon), \tau) \leq C(v, \epsilon + \tau)$.

Similar to superlinear families by Bubenick et. al. [3].

- The *$\epsilon$-shift*,

  $$F[\epsilon] = \langle \{F(v_i \leq C(v_i, \epsilon))(g_i) \in F(C(v_i, \epsilon))\} \rangle$$

  for any generating set $\{g_i \in F(v_i)\}$ of $F$.

- $$\begin{tikzcd} & H \arrow[dl, "\varphi"'] \arrow[dr, "\psi"] & \\ F & & G \end{tikzcd}$$

  $F$ and $G$ are *$\epsilon$-close* if $\ker \varphi[a] = \operatorname{coker} \varphi[b] = \ker \psi[c] =$
  $= \operatorname{coker} \psi[d] = 0$ and $a + b + c + d \leq \epsilon$.

# Examples

- Let $C\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ be the contour defined by

$$C(v, \epsilon) = v + \epsilon\mathbf{1}$$

It's called the *standard persistence contour*.

# Examples

- Let $C\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ be the contour defined by

$$C(v, \epsilon) = v + \epsilon\mathbf{1}$$

  It's called the *standard persistence contour*.

- Let $C\colon \mathbb{R}^n_{\geq 0} \times \mathbb{R} \to \mathbb{R}^n_{\geq 0}$ be defined by

$$C(v, t) = ve^{kt}$$

  This is called the *exponential persistence contour*.

# Examples

- Let $C\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ be the contour defined by

$$C(v, \epsilon) = v + \epsilon\mathbf{1}$$

  It's called the *standard persistence contour*.

- Let $C\colon \mathbb{R}^n_{\geq 0} \times \mathbb{R} \to \mathbb{R}^n_{\geq 0}$ be defined by

$$C(v, t) = ve^{kt}$$

  This is called the *exponential persistence contour*.

- Let $w \in \mathbb{R}$ and let $C\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by:

$$C(v, t) = \min\left(v + t, \max\left(v, w\right)\right)$$

# Persistence Contours

Given a choice of persistence contour:

$$d(F, G) := \inf\{\epsilon \mid F \text{ and } G \text{ are } \epsilon\text{-close}\}$$

# Persistence Contours

Given a choice of persistence contour:

$$d(F, G) := \inf\{\epsilon \mid F \text{ and } G \text{ are } \epsilon\text{-close}\}$$

$\Longrightarrow$ a class of extended pseudometrics on persistence modules.

# Persistence Contours

Given a choice of persistence contour:

$$d(F, G) := \inf\{\epsilon \mid F \text{ and } G \text{ are } \epsilon\text{-close}\}$$

$\Longrightarrow$ a class of extended pseudometrics on persistence modules.

**Theorem**

*For the standard persistence contour:*

$$\frac{1}{6} d(F, G) \leq d_{\bowtie}(F, G) \leq d(F, G)$$

# Persistence Contours

Given a choice of persistence contour:

$$d(F, G) := \inf\{\epsilon \mid F \text{ and } G \text{ are } \epsilon\text{-close}\}$$

$\implies$ a class of extended pseudometrics on persistence modules.

**Theorem**

*For the standard persistence contour:*

$$\frac{1}{6} d(F, G) \leq d_{\bowtie}(F, G) \leq d(F, G)$$

We can define parametrized classes of metrics that can be optimized over to solve classification problems.

# Persistence Contours

Given a choice of persistence contour:

$$d(F, G) := \inf\{\epsilon \mid F \text{ and } G \text{ are } \epsilon\text{-close}\}$$

$\Longrightarrow$ a class of extended pseudometrics on persistence modules.

**Theorem**

*For the standard persistence contour:*

$$\frac{1}{6} d(F, G) \leq d_{\bowtie}(F, G) \leq d(F, G)$$

We can define parametrized classes of metrics that can be optimized over to solve classification problems. **Contour learning**

# Stabilization

# Hierarchical Stabilization

Given the following data:

- a set $T$,
- an extended pseudometric $d\colon T \times T \to \mathbb{R}$,
- an invariant $f\colon T \to \mathbb{R}$

# Hierarchical Stabilization

Given the following data:

- a set $T$,
- an extended pseudometric $d \colon T \times T \to \mathbb{R}$,
- an invariant $f \colon T \to \mathbb{R}$

Let $\mathrm{Fun}^{\geq}(\mathbb{R}, \mathbb{R})$ denote non-increasing functions from $\mathbb{R}$ to $\mathbb{R}$.

# Hierarchical Stabilization

Given the following data:

- a set $T$,
- an extended pseudometric $d\colon T \times T \to \mathbb{R}$,
- an invariant $f\colon T \to \mathbb{R}$

Let $\mathrm{Fun}^{\geq}(\mathbb{R}, \mathbb{R})$ denote non-increasing functions from $\mathbb{R}$ to $\mathbb{R}$. We define the *hierarchical stabilization* of $f$ w.r.t $d$, to be the function

$$\begin{aligned} \hat{f}\colon T &\to \mathrm{Fun}^{\geq}(\mathbb{R}, \mathbb{R}) \\ t &\mapsto (\tau \mapsto \min\{f(s) \mid d(s,t) < \tau\}) \end{aligned}$$

# Hierarchical Stabilization

Given the following data:

- a set $T$,
- an extended pseudometric $d\colon T \times T \to \mathbb{R}$,
- an invariant $f\colon T \to \mathbb{R}$

Let $\mathrm{Fun}^{\geq}(\mathbb{R},\mathbb{R})$ denote non-increasing functions from $\mathbb{R}$ to $\mathbb{R}$. We define the *hierarchical stabilization* of $f$ w.r.t $d$, to be the function

$$\begin{aligned} \hat{f}\colon T &\to \mathrm{Fun}^{\geq}(\mathbb{R},\mathbb{R}) \\ t &\mapsto (\tau \mapsto \min\{f(s) \mid d(s,t) < \tau\}) \end{aligned}$$

Define metric on $\mathrm{Fun}^{\geq}(\mathbb{R},\mathbb{R})$:

$$d_{\bowtie}(f,g) := \inf\left\{\epsilon \mid {\scriptstyle f(x+\epsilon)\leq g(x) \atop g(x+\epsilon)\leq f(x)} \text{ for all } x \in \mathbb{R}\right\}$$

# Hierarchical Stabilization

**Theorem (G. and Chachólski, 2017)**

*The function $\hat{f}$ is 1-Lipschitz, i.e for any $s, t \in T$,*

$$d(s,t) \geq d_{\bowtie}(\hat{f}(s), \hat{f}(t))$$

*where the metric on $Fun^{\geq}(\mathbb{R}, \mathbb{R})$ is the interleaving distance.*

# Hierarchical Stabilization

**Theorem (G. and Chachólski, 2017)**

*The function $\hat{f}$ is 1-Lipschitz, i.e for any $s, t \in T$,*

$$d(s,t) \geq d_{\bowtie}(\hat{f}(s), \hat{f}(t))$$

*where the metric on $Fun^{\geq}(\mathbb{R}, \mathbb{R})$ is the interleaving distance.*

Let

# Hierarchical Stabilization

**Theorem (G. and Chachólski, 2017)**

*The function $\hat{f}$ is 1-Lipschitz, i.e for any $s, t \in T$,*

$$d(s,t) \geq d_{\bowtie}(\hat{f}(s), \hat{f}(t))$$

*where the metric on $Fun^{\geq}(\mathbb{R}, \mathbb{R})$ is the interleaving distance.*

Let

- $T$ be the set of persistence modules

# Hierarchical Stabilization

**Theorem (G. and Chachólski, 2017)**

*The function $\hat{f}$ is 1-Lipschitz, i.e for any $s, t \in T$,*

$$d(s,t) \geq d_{\bowtie}(\hat{f}(s), \hat{f}(t))$$

*where the metric on $Fun^{\geq}(\mathbb{R}, \mathbb{R})$ is the interleaving distance.*

Let

- $T$ be the set of persistence modules
- $d$ be the a distance from a persistence contour

# Hierarchical Stabilization

**Theorem (G. and Chachólski, 2017)**

*The function $\hat{f}$ is 1-Lipschitz, i.e for any $s, t \in T$,*

$$d(s,t) \geq d_{\bowtie}(\hat{f}(s), \hat{f}(t))$$

*where the metric on $Fun^{\geq}(\mathbb{R}, \mathbb{R})$ is the interleaving distance.*

Let

- $T$ be the set of persistence modules
- $d$ be the a distance from a persistence contour
- the invariant $f$ be the zeroth Betti number, $\beta_0$.

# Hierarchical Stabilization

**Theorem (G. and Chachólski, 2017)**

*The function $\hat{f}$ is 1-Lipschitz, i.e for any $s, t \in T$,*

$$d(s,t) \geq d_{\bowtie}(\hat{f}(s), \hat{f}(t))$$

*where the metric on $Fun^{\geq}(\mathbb{R}, \mathbb{R})$ is the interleaving distance.*

Let

- $T$ be the set of persistence modules
- $d$ be the a distance from a persistence contour
- the invariant $f$ be the zeroth Betti number, $\beta_0$.

Then the hierarchical stabilization of $\beta_0$ is the stable rank, $\widehat{\beta_0}$.

# Hierarchical Stabilization

**Theorem (G. and Chachólski, 2017)**

*The function $\hat{f}$ is 1-Lipschitz, i.e for any $s, t \in T$,*

$$d(s,t) \geq d_{\bowtie}(\hat{f}(s), \hat{f}(t))$$

*where the metric on $Fun^{\geq}(\mathbb{R}, \mathbb{R})$ is the interleaving distance.*

Let

- $T$ be the set of persistence modules
- $d$ be the a distance from a persistence contour
- the invariant $f$ be the zeroth Betti number, $\beta_0$.

Then the hierarchical stabilization of $\beta_0$ is the stable rank, $\widehat{\beta_0}$.

Using hierarchical stabilization we can *stabilize classical invariants*,

# Hierarchical Stabilization

**Theorem (G. and Chachólski, 2017)**

*The function $\hat{f}$ is 1-Lipschitz, i.e for any $s, t \in T$,*

$$d(s,t) \geq d_{\bowtie}(\hat{f}(s), \hat{f}(t))$$

*where the metric on $Fun^{\geq}(\mathbb{R}, \mathbb{R})$ is the interleaving distance.*

Let

- $T$ be the set of persistence modules
- $d$ be the a distance from a persistence contour
- the invariant $f$ be the zeroth Betti number, $\beta_0$.

Then the hierarchical stabilization of $\beta_0$ is the stable rank, $\widehat{\beta_0}$.

Using hierarchical stabilization we can *stabilize classical invariants*, define mean, variance and do statistics.

# Approximating the stable rank

# Approximations

What problem are we approximating?

# Approximations

What problem are we approximating?

Observation:

**Theorem**

$$\widehat{\beta}_0(F)(\epsilon) = min\{rank(G) \mid F[\epsilon] \subseteq G \subseteq F \ and \ d(F, G) \leq \epsilon\}$$

# Approximations

What problem are we approximating?
Observation:

**Theorem**

$$\widehat{\beta}_0(F)(\epsilon) = min\{rank(G) \mid F[\epsilon] \subseteq G \subseteq F \; and \; d(F,G) \leq \epsilon\}$$

It suffices to minimize over the subfunctors of $F$.

# Approximations

What problem are we approximating?
Observation:

**Theorem**

$$\widehat{\beta}_0(F)(\epsilon) = min\{rank(G) \mid F[\epsilon] \subseteq G \subseteq F \text{ and } d(F, G) \leq \epsilon\}$$

It suffices to minimize over the subfunctors of $F$.

| | |
|---|---|
| $\{g_i \in F(v_i)\}$ | $\{F(v_i \leq C(v_i, \epsilon))(g_i) \in F(C(v_i, \epsilon))\}$ |
| Generating set for $F$ | Generating set for $F[\epsilon]$ |

# Approximations

What problem are we approximating?
Observation:

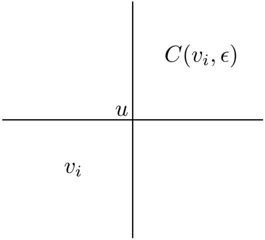
**Theorem**

$$\widehat{\beta}_0(F)(\epsilon) = min\{rank(G) \mid F[\epsilon] \subseteq G \subseteq F \text{ and } d(F,G) \leq \epsilon\}$$

It suffices to minimize over the subfunctors of $F$.

| | |
|---|---|
| $\{g_i \in F(v_i)\}$ | $\{F(v_i \leq C(v_i,\epsilon))(g_i) \in F(C(v_i,\epsilon))\}$ |
| Generating set for $F$ | Generating set for $F[\epsilon]$ |

For $\epsilon$ large enough, there is $u$ s.t:



# Approximations

Want to find minimal number of elements in:

$$\bigcup_i F(u \leq v_i)^{-1}(F(v_i \leq C(v_i, \epsilon))(g_i)) \subseteq F(u)$$

that generate $F[\epsilon]$.

# Approximations

Want to find minimal number of elements in:

$$\bigcup_i F(u \le v_i)^{-1}(F(v_i \le C(v_i, \epsilon))(g_i)) \subseteq F(u)$$

that generate $F[\epsilon]$.

Can be phrased as the following *matrix rank minimization problem with an affine constraint set*:

**Input** Vectors $x_1, \dots, x_m \in k^n$ and subspaces $L_1, \dots, L_m \subseteq k^n$
**Output** $\min\{\operatorname{rank}(\begin{bmatrix} c_1 & \dots & c_m \end{bmatrix}) \mid c_i - x_i \in L_i\}$

# Approximations

Want to find minimal number of elements in:

$$\bigcup_i F(u \leq v_i)^{-1}(F(v_i \leq C(v_i, \epsilon))(g_i)) \subseteq F(u)$$

that generate $F[\epsilon]$.

Can be phrased as the following *matrix rank minimization problem with an affine constraint set*:

**Input** Vectors $x_1, \ldots, x_m \in k^n$ and subspaces $L_1, \ldots, L_m \subseteq k^n$
**Output** $\min\{\operatorname{rank}(\begin{bmatrix} c_1 & \ldots & c_m \end{bmatrix}) \mid c_i - x_i \in L_i\}$

This is NP-hard, by reduction from a graph colouring problem [7].

# Example

Calculating $\widehat{\beta}_0(F)(2)$ for the following functor requires you to solve such a rank minimization problem:

| | | | | | | |
|---|---|---|---|---|---|---|
| 5 | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| 4 | $k$ | $k^2$ | $k^3/L_2$ | $0$ | $0$ | $0$ |
| 3 | $k$ | $k^2$ | $k^3$ | $k^3/L_1$ | $0$ | $0$ |
| 2 | $k$ | $k^2$ | $k^3$ | $k^3$ | $k^3/L_0$ | $0$ |
| 1 | $0$ | $k$ | $k^2$ | $k^2$ | $k^2$ | $0$ |
| 0 | $0$ | $0$ | $k$ | $k$ | $k$ | $0$ |
| | 0 | 1 | 2 | 3 | 4 | 5 |

# Approximations

$$\min_{A \in \mathcal{C} \subseteq k^{m \times m}} \quad \mathrm{rank}(A)$$

# Approximations

$$\min_{A \in \mathcal{C} \subseteq k^{m \times m}} \quad \operatorname{rank}(A)$$

**Case $k = \mathbb{R}$:** The problem can be efficiently approximated by the nuclear norm [13]:

# Approximations

$$\min_{A \in \mathcal{C} \subseteq k^{m \times m}} \quad \text{rank}(A)$$

**Case** $k = \mathbb{R}$**:** The problem can be efficiently approximated by the nuclear norm [13]:

$$\min_{A \in \mathcal{C}} \quad \text{rank}(A) \quad \sim \quad \min_{A \in \mathcal{C}} \quad ||A||_*$$

where $||A||_* = \sum \sigma_i(A)$.

# Approximations

$$\min_{A \in \mathcal{C} \subseteq k^{m\times m}} \quad \operatorname{rank}(A)$$

**Case** $k = \mathbb{R}$**:** The problem can be efficiently approximated by the nuclear norm [13]:

$$\min_{A \in \mathcal{C}} \quad \operatorname{rank}(A) \quad \sim \quad \min_{A \in \mathcal{C}} \quad ||A||_*$$

where $||A||_* = \sum \sigma_i(A)$.

**Case** $k = \mathbb{Z}/2\mathbb{Z}$**:** For each $l$, we can write the determinants of the $l \times l$ minors as Boolean clauses. Can be solved using a modern SAT solver, which can handle millions of variables and clauses.

# Approximations

$$\min_{A \in \mathcal{C} \subseteq k^{m\times m}} \quad \operatorname{rank}(A)$$

**Case $k = \mathbb{R}$:** The problem can be efficiently approximated by the nuclear norm [13]:

$$\min_{A \in \mathcal{C}} \quad \operatorname{rank}(A) \quad \sim \quad \min_{A \in \mathcal{C}} \quad ||A||_*$$

where $||A||_* = \sum \sigma_i(A)$.

**Case $k = \mathbb{Z}/2\mathbb{Z}$:** For each $l$, we can write the determinants of the $l \times l$ minors as Boolean clauses. Can be solved using a modern SAT solver, which can handle millions of variables and clauses.

**Case $F = H_0(X(-); k)$:** The $F$ is a functor $F\colon \mathbb{R}^n \to$ Sets and the matrix rank minimization turns into a vertex covering problem (which can be approximated).

**Thank you!**

# References I


Ulrich Bauer, Magnus B. Botnan, Steffen Oppermann, and Johan Steen.
Cotorsion torsion triples and the representation theory of filtered hierarchical clustering.
*arXiv e-prints*, page arXiv:1904.07322, Apr 2019.

Peter Bubenik.
Statistical topological data analysis using persistence landscapes.
*J. Mach. Learn. Res.*, 16(1):77–102, January 2015.

Peter Bubenik, Vin de Silva, and Jonathan Scott.
Metrics for generalized persistence modules.
*Foundations of Computational Mathematics*, 15(6):1501–1531, Dec 2015.

Gunnar Carlsson and Afra Zomorodian.
The theory of multidimensional persistence.
*Discrete & Computational Geometry*, 42(1):71–93, 2009.

David Cohen-Steiner, Herbert Edelsbrunner, and John Harer.
Stability of persistence diagrams.
*Discrete & Computational Geometry*, 37(1):103–120, Jan 2007.

Tamal K. Dey and Cheng Xin.
Generalized Persistence Algorithm for Decomposing Multi-parameter Persistence Modules.
*arXiv e-prints*, page arXiv:1904.03766, Apr 2019.

O. Gäfvert and W. Chachólski.
Stable Invariants for Multidimensional Persistence.
*ArXiv e-prints*, March 2017.

# References II


Oliver Gäfvert.
Topcat: a multiparameter persistence toolbox.
*https://github.com/olivergafvert/topcat*.

Heather A. Harrington, Nina Otter, Hal Schenck, and Ulrike Tillmann.
Stratifying multiparameter persistent homology.
*arXiv e-prints*, page arXiv:1708.07390, Aug 2017.

Kevin P. Knudson.
A refinement of multi-dimensional persistence.
*arXiv e-prints*, page arXiv:0706.2608, Jun 2007.

Michael Lesnick.
The theory of the interleaving distance on multidimensional persistence modules.
*Found. Comput. Math.*, 15(3):613–650, June 2015.

Michael Lesnick and Matthew Wright.
Interactive Visualization of 2-D Persistence Modules.
*arXiv e-prints*, page arXiv:1512.00180, Dec 2015.

B. Recht, W. Xu, and B. Hassibi.
Necessary and sufficient conditions for success of the nuclear norm heuristic for rank minimization.
In *2008 47th IEEE Conference on Decision and Control*, pages 3065–3070, Dec 2008.

Martina Scolamiero, Wojciech Chachólski, Anders Lundman, Ryan Ramanujam, and Sebastian Öberg.
Multidimensional persistence and noise.
*Foundations of Computational Mathematics*, pages 1–40, 2016.

# References III


Oliver Vipond.
Multiparameter Persistence Landscapes.
*arXiv e-prints*, page arXiv:1812.09935, Dec 2018.

Afra Zomorodian and Gunnar Carlsson.
Computing persistent homology.
*Discrete & Computational Geometry*, 33(2):249–274, Feb 2005.