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Rashid Zaare-Nahandi
Institute for Advanced Studies in Basic Sciences

On isomorphism of simplicial complexes and their related rings

Rashid Zaare-Nahandi
Institute for Advanced Studies in Basic Sciences, Zanjan, Iran
E-mail: rashidzn@iasbs.ac.ir

Abstract

In this paper, we provide a simple proof for the fact that two simplicial complexes are isomorphic if and only if their associated Stanley-Reisner rings, or their associated facet rings are isomorphic as K-algebras. As a consequence, we show that two graphs are isomorphic if and only if their associated edge rings are isomorphic as K-algebras. Based on an explicit K-algebra isomorphism of two Stanley-Reisner rings, or facet rings or edge rings, we present a fast algorithm to find explicitly the isomorphism of the associated simplicial complexes, or graphs.

Introduction

Let X be a finite nonempty set. A simplicial complex Δ on X is a set of subsets of X such that, for any $x \in X$, $\{x\} \in \Delta$, and if $E \in \Delta$ and $F \subseteq E$, then $F \in \Delta$. A set in Δ is called a face and a maximal face in Δ is called a facet.

Let $X = \{x_1, \dots, x_n\}$, and Δ be a simplicial complex on X. Let $R = K[x_1, \dots, x_n]$ be the polynomial ring in n indeterminates and with coefficients in a field K. Let $I(\Delta)$ be the ideal in R generated by all square-free monomials $x_{i_1} \dots x_{i_s}$, provided that $\{x_{i_1}, \dots, x_{i_s}\} \notin \Delta$. The quotient ring $R/I(\Delta)$ is called the Stanley-Reisner ring of the simplicial complex Δ.

It is easy to see that any quotient of a polynomial ring over an ideal generated by square-free monomials of degree greater than 1, is the Stanley-Reisner ring of a simplicial complex. A natural question arises: If two Stanley-Reisner rings are isomorphic, are their corresponding simplicial complexes isomorphic? In 1996, W. Bruns and J. Gubeladze proved that if the isomorphism of the rings is a K-algebra isomorphism, then the corresponding simplicial complexes are isomorphic [1]. In this paper we provide a simple proof for this result.

In 2002, S. Faridi in [2] defined the notion of facet ideal for a simplicial complex which is a generalization of the notion of edge ideal for a graph defined by R. Villarreal [6].

Definition 1. Let Δ be a simplicial complex on the set $X = \{x_1, \ldots, x_n\}$. Let $F(\Delta)$ be the ideal of $R = K[x_1, \ldots, x_n]$ generated by all square-free monomials $x_{i_1} \ldots x_{i_s}$, provided that $\{x_{i_1}, \ldots, x_{i_s}\}$ is a facet in Δ. The quotient ring $R/F(\Delta)$ is called the facet ring of the simplicial complex Δ.

Definition 2. Let G be a finite, simple and undirected graph with vertex set $\{x_1, \ldots, x_n\}$. The ideal $E(G)$ of $R = K[x_1, \ldots, x_n]$ generated by all square-free monomials $x_i x_j$, provided that x_i is adjacent to x_j in G, is called the edge ideal of G. The quotient ring $R/E(G)$ is called the edge ring of the graph G.

A question similar to the case of simplicial complexes can be stated for facet rings and edge rings. In 1997, H. Hajiabolhassan and M. L. Mehrabadi in [3] proved that two graphs are isomorphic if and only if their corresponding edge rings are isomorphic as K-algebras. In this paper we will prove the statement for the facet rings and conclude it for the edge rings.

Finally, we will present a fast algorithm which admits a K-algebra isomorphism of two Stanley-Reisner rings, or two facet rings, or two edge rings as input and returns explicitly the isomorphism of corresponding simplicial complexes or graphs, as output.

The isomorphism

First we prove some lemmas.

Lemma 1. Let I and J be ideals in $R = K[x_1, \ldots, x_n]$ and $S = K[y_1, \ldots, y_m]$, respectively, both generated by monomials of degree greater than 1. Let $\phi : R/I \to S/J$ be a K-algebra isomorphism. Let f be a monomial in R and denote the image of f in R/I by $\bar{f}$. If $\bar{f}$ is a zero-divisor in R/I, then the constant term of $\phi(\bar{f})$ in S/J is zero.

Proof. Let $\bar{g}$ be a nonzero monomial in R/I and $\bar{f}\bar{g} = 0$. Let $\phi(\bar{f}) = \bar{h} = h + J$, $\phi(\bar{g}) = \bar{l} = l + J$. Let $\bar{h} = \bar{h}_0 + \bar{h}_1 + \cdots + \bar{h}_r$ and $\bar{l} = \bar{l}_i + \bar{l}_{i+1} + \cdots + \bar{l}_t$ be homogeneous decompositions of $\bar{h}$ and $\bar{l}$ with $\bar{l}_i \neq 0$. If $\bar{h}_0 \neq 0$, then $\bar{h}\bar{l} = \bar{h}_0\bar{l}_i +$ monomials of higher degree. But $\bar{f}\bar{g} = 0$ and then, $0 = \phi(\bar{f}\bar{g}) = \bar{h}\bar{l}$. That is, $h.l \in J$, and hence $l_i \in J$, which is a contradiction. $\square$

For any i, $1 \leq i \leq n$, let $L(x_i)$ denote the set of y_j's such that $\bar{y}_j$ appears in the linear part of $\phi(\bar{x}_i)$ with nonzero coefficient.

Lemma 2. With the above notations, if $\phi : R/I \to S/J$ is a K-algebra isomorphism, then $m = n$ and $L(x_i) \neq \varnothing$, for each i, $1 \leq i \leq n$.

Proof. The ideal J is generated by monomials of degree greater than one, therefore, $(S/J)_1$, the degree one homogeneous component of S/J, is an m-dimensional K-vector space with basis $\{\bar{y}_1, \ldots, \bar{y}_m\}$. The map ϕ is surjective, thus the set $\{(\phi(\bar{x}_1))_1, \ldots, (\phi(\bar{x}_n))_1\}$ generates $(S/J)_1$ as a vector space over K. Therefore, $n \geq m$. The map ϕ is isomorphism thus ϕ^{-1} is surjective too and therefore, $m \geq n$. For the last claim, note that if for some i, $1 \leq i \leq n$, $L(x_i) = \varnothing$, then $(\phi(\bar{x}_i))_1 = 0$ and the set $\{(\phi(\bar{x}_1))_1, \ldots, (\phi(\bar{x}_n))_1\}$ can not generate an n-dimensional vector space. $\square$

Let Δ_1 and Δ_2 be two simplicial complexes on sets $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_m\}$, respectively. Let $K[\Delta_1] = K[x_1, \ldots, x_n]/I(\Delta_1)$ and $K[\Delta_2] = K[y_1, \ldots, y_m]/I(\Delta_2)$ be the Stanley-Reisner rings associated to Δ_1 and Δ_2.

Theorem 1. With the above notations, Δ_1 and Δ_2 are isomorphic as simplicial complexes if and only if $K[\Delta_1]$ and $K[\Delta_2]$ are isomorphic as K-algebras.

Proof. It is obvious that if Δ_1 and Δ_2 are isomorphic as simplicial complexes, then $K[\Delta_1]$ and $K[\Delta_2]$ are isomorphic as K-algebras. For the converse, assume that $\phi : K[\Delta_1] \to K[\Delta_2]$ is a K-algebra isomorphism. By Lemma 2, $m = n$. An isomorphism as ϕ, can be uniquely determined by images of $\bar{x}_i$, $i = 1, \ldots, n$, and $\phi(\bar{f}(x_1, \ldots, x_n)) = \phi(f(\bar{x}_1, \ldots, \bar{x}_n)) = f(\phi(\bar{x}_1), \ldots, \phi(\bar{x}_n))$ for any polynomial $\bar{f}$ in $K[\Delta_1]$. Let $\phi(\bar{x}_i) = f_i(\bar{y}_1, \ldots, \bar{y}_n)$, $i = 1, \ldots, n$. By Lemma 1 and Lemma 2, for each i, f_i has no nonzero constant term and has nonzero linear part. Suppose ϕ^{-1} is inverse of ϕ and let $\phi^{-1}(\bar{y}_i) = g_i(\bar{x}_1, \ldots, \bar{x}_n)$, $i = 1, \ldots, n$. Let f_{i1} and g_{i1} denote the linear homogeneous component of f_i and g_i, respectively:

$$\begin{aligned} f_{i1} &= a_{i1}y_1 + \cdots + a_{in}y_n, \qquad i = 1, \ldots, n \\ g_{j1} &= b_{j1}x_1 + \cdots + b_{jn}x_n, \qquad j = 1, \ldots, n. \end{aligned}$$

The equalities $\phi \circ \phi^{-1}(y_i) = y_i$ and $\phi^{-1} \circ \phi(x_i) = x_i$ for $i = 1, \ldots, n$, imply that

$$\begin{bmatrix} a_{11} & \ldots & a_{1n} \\ a_{21} & \ldots & a_{2n} \\ \vdots & & \vdots \\ a_{n1} & \ldots & a_{nn} \end{bmatrix} \begin{bmatrix} b_{11} & \ldots & b_{1n} \\ b_{21} & \ldots & b_{2n} \\ \vdots & & \vdots \\ b_{n1} & \ldots & b_{nn} \end{bmatrix} = \begin{bmatrix} b_{11} & \ldots & b_{1n} \\ b_{21} & \ldots & b_{2n} \\ \vdots & & \vdots \\ b_{n1} & \ldots & b_{nn} \end{bmatrix} \begin{bmatrix} a_{11} & \ldots & a_{1n} \\ a_{21} & \ldots & a_{2n} \\ \vdots & & \vdots \\ a_{n1} & \ldots & a_{nn} \end{bmatrix} = I$$

where I is the identity matrix of order n. Therefore, in the following matrix, sum of entries of each row and each column is 1.

$$M(\phi) = \begin{bmatrix} a_{11}b_{11} & a_{21}b_{12} & \cdots & a_{n1}b_{1n} \\ a_{12}b_{21} & a_{22}b_{22} & \cdots & a_{n2}b_{2n} \\ \vdots & \vdots & & \vdots \\ a_{1n}b_{n1} & a_{2n}b_{n2} & \cdots & a_{nn}b_{nn} \end{bmatrix}$$

It is well known that, there is a transversal of length n with nonzero elements in the matrix $M(\phi)$. See for instance [4]. By a transversal we mean a sequence of entries of the matrix with no common columns or rows. In other words, a transitive is a term in expansion of determinant of the matrix. Let $1j_1, 2j_2, \ldots, nj_n$ be indices of a transversal with nonzero elements in $M(\phi)$. By a change of indices of y_i's and permuting corresponding columns of $M(\phi)$, suppose that the nonzero transversal is the main diagonal. Under this assumption, $y_i \in L(x_i)$ and $x_i \in L^{-1}(y_i)$, for $i = 1, \ldots, n$, where $L^{-1}(y_i)$ is the set of variables with nonzero coefficient in the linear part of $\phi^{-1}(y_i)$. For a subset of $\{x_1, \ldots, x_n\}$ as F, which is not in Δ_1, there is a minimal set $E \subseteq F$, where $E \notin \Delta_1$ and any proper subset of E belongs to Δ_1. Let $\{x_{i_1}, \ldots, x_{i_r}\}$ be a minimal set not belonging to Δ_1. Then, $x_{i_1} \cdots x_{i_r}$ is a generator in $I(\Delta_1)$, that is, $\bar{x}_{i_1} \cdots \bar{x}_{i_r} = 0$ in $K[\Delta_1]$ and $\phi(\bar{x}_{i_1} \cdots \bar{x}_{i_r}) = 0$ in $K[\Delta_2]$. This means that $\phi(\bar{x}_{i_1}) \cdots \phi(\bar{x}_{i_r}) \in I(\Delta_2)$. The ideal $I(\Delta_2)$ is a homogeneous and monomial ideal and therefore each homogeneous component and each monomial of $\phi(\bar{x}_{i_1}) \cdots \phi(\bar{x}_{i_r})$ belongs to $I(\Delta_2)$. Therefore, the product $f_{i_1 1} \cdots f_{i_r 1}$ is in $I(\Delta_2)$. In this case, there are two possibilities:

- $y_{i_1} \cdots y_{i_r} \in I(\Delta_2)$, which means that $\{y_{i_1}, \cdots, y_{i_r}\} \notin \Delta_2$;
- $y_{i_1} \cdots y_{i_r}$ is canceled by another monomial in $f_{i_1 1} \cdots f_{i_r 1}$.

We prove that the second case is not possible. The condition in case 2, is equal to say that, for some j, $1 \leq j \leq r$, y_{i_j} appears with nonzero coefficient in more than one $f_{i_l 1}$. Let $y_{t_1}^{\alpha_1} \cdots y_{t_s}^{\alpha_s}$ be a smallest monomial with lexicographic order in the set of monomials with nonzero coefficient in the expansion of $f_{i_1 1} \cdots f_{i_r 1}$ with $\{t_1, \ldots, t_s\} \subseteq \{i_1, \ldots, i_r\}$ and s is as small as possible. This monomial appears once in the expansion and so can not be canceled by another monomial. Therefore, $y_{t_1}^{\alpha_1} \cdots y_{t_s}^{\alpha_s} \in I(\Delta_2)$ and since (Δ_2) is a radical ideal, then $y_{t_1} \cdots y_{t_s} \in I(\Delta_2)$ and so, $\bar{y}_{i_1} \cdots \bar{y}_{i_r} \in I(\Delta_2)$. Therefore, $\{x_{i_1}, \ldots, x_{i_r}\} \notin \Delta_1$ implies that $\{y_{i_1}, \ldots, y_{i_r}\} \notin \Delta_2$. With a similar argument for ϕ^{-1}, $\{y_{i_1}, \ldots, y_{i_r}\} \notin \Delta_2$ implies $\{x_{i_1}, \ldots, x_{i_r}\} \notin \Delta_1$, that is, $\Delta_1 \cong \Delta_2$. $\square$

Note that, for any simplicial complex Δ, the ideal $I(\Delta)$ has no monomial of degree one, but in the case of facet ideals, zero dimensional facets correspond to degree one monomials in $F(\Delta)$. To use Lemma 1 in the proof of the next theorem, we will assume that there is not any zero dimensional facet in simplicial complexes. This does not reduce the generality of the theorem. Because, two simplicial complexes are isomorphic if and only if they have the same number of zero dimensional facets and the parts without any zero dimensional facet are isomorphic.

Theorem 2. Any two simplicial complexes Δ_1 and Δ_2 are isomorphic if and only if their corresponding facet rings are isomorphic as K-algebras.

Proof. The “only if” part is obvious. To prove the “if” part, let

$$K[x_1,\ldots,x_n]/F(\Delta_1) \xrightarrow{\phi} K[y_1,\ldots,y_m]/F(\Delta_2)$$

be a K-algebra isomorphism between the facet rings corresponding to Δ_1 and Δ_2. Similar to the proof of Theorem 1, it follows that $m=n$. By an appropriate change of indices, we may assume that $\{y_1,\ldots,y_n\}$ is the set corresponding to the main diagonal which we assumed to be a transversal with nonzero elements in the matrix $M(\phi)$. Then, $y_i \in L(i)$ and $x_i \in L^{-1}(y_i)$, for $i=1,\ldots,n$. Let $\{x_{i_1},\ldots,x_{i_s}\}$ be a facet in Δ_1. Then, $x_{i_1}\cdots x_{i_s}$ is in the minimal generating set of $F(\Delta_1)$ and $\phi(\bar{x}_{i_1})\cdots\phi(\bar{x}_{i_s}) \in F(\Delta_2)$. The same argument as the proof of Theorem 1, implies that $y_{i_1}\cdots y_{i_s} \in F(\Delta_2)$. If $y_{i_1}\cdots y_{i_s} \in F(\Delta_2)$ is not in the minimal generating set of $F(\Delta_2)$, then, without loss of generality, we may assume that $y_{i_1}\cdots y_{i_s-t} \in F(\Delta_2)$, for some t, $1 \le t \le s-1$. This means that, $\phi^{-1}(y_{i_1})\cdots\phi^{-1}(y_{i_s-t}) \in F(\Delta_1)$ and therefore, $x_{i_1}\cdots x_{i_s-t} \in F(\Delta_1)$, which is a contradiction to minimality of $x_{i_1}\cdots x_{i_s}$ in $F(\Delta_1)$. Therefore, $\{y_{i_1},\ldots,y_{i_s}\}$ is a facet in Δ_2, and this gives a bijection between Δ_1 and Δ_2 as an isomorphism of simplicial complexes. $\square$

A simple and undirected graph G can be regarded as a simplicial complex with facets $\{x_i,x_j\}$, where x_i is adjacent to x_j in G. With this interpretation, the edge ideal of G is the same as its facet ideal. Therefore, we have the following result.

Corollary. Let G_1 and G_2 be two graphs. G_1 and G_2 are isomorphic as graphs if and only if their edge rings are isomorphic as K algebras. $\square$

The algorithm

In this section, we present a fast algorithm to construct explicitly an isomorphism between two simplicial complexes or two graphs, when a K-algebra isomorphism of their associated rings is given.

Algorithm 1. Let $R = K[x_1,\ldots,x_n]/I$ and $S = K[y_1,\ldots,y_n]/J$ be two K-algebras and I and J be ideals generated by some square-free monomials of degree greater than one. Any K-algebra homomorphism $\phi : R \to S$ can be uniquely determined by the images of $\bar{x}_1,\ldots,\bar{x}_n$. In the following algorithm, we assume that R and S are Stanley-Reisner rings of some simplicial complexes.
Input: K-algebras R and S, and a K-algebra isomorphism $\phi : R \to S$,
Output: A simplicial complex Δ_1 associated to R and a simplicial complex Δ_2

associated to S as their Stanley-Reisner rings and a bijection $\psi : \{x_1, \dots, x_n\} \to \{y_1, \dots, y_n\}$ which determines an isomorphism of Δ_1 and Δ_2.

Step 1. Construct a simplicial complex Δ_1 with the underlying set $\{x_1, \dots, x_n\}$ and faces $\{x_{i_1}, \dots, x_{i_r}\}$ where $x_{i_1} \cdots x_{i_r}$ is not divided by any of generators of I. Construct Δ_2 on the set $\{y_1, \dots, y_n\}$ and faces $\{y_{i_1}, \dots, y_{i_s}\}$ where $y_{i_1} \cdots y_{i_s} \notin J$

Step 2. Find the matrix $M(\phi)$

Step 3. Find a transversal with nonzero elements in $M(\phi)$

Step 4. Use the transversal in step 3 to construct the map $\psi : \Delta_1 \to \Delta_2$.

The proof of Theorem 1, guaranties the correctness of the algorithm. It is known that finding a nonzero transversal in a matrix which has such a transversal, has a polynomial time algorithm [5], and so, the above algorithm is polynomial time too.

Algorithm 2. In Algorithm 1, we may consider the K-algebras R and S as facet rings of simplicial complexes, or if I and J are generated by square-free monomials of degree 2, as edge rings of graphs. Then, in Step 1, we must construct a simplicial complexes Δ_1 and Δ_2 such that R and S are their facet rings, respectively. Following steps 2, 3, and 4, we finally obtain an isomorphism of simplicial complexes or graphs.

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