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Article *in* Algebras and Representation Theory · January 2002

DOI: 10.1023/A:1016554601277

2 authors:


Matyas Domokos
Alfréd Rényi Institute of Mathematics


A. N. Zubkov
Omsk State Technical University

Algebras and Representation Theory **5:** 305–317, 2002.

Semisimple Representations of Quivers in Characteristic p

M. DOMOKOS[1] and A. N. ZUBKOV[2]
[1]*Rényi Institute of Mathematics, Hungarian Academy of Sciences, PO Box 127, 1364 Budapest, Hungary. e-mail: domokos@renyi.hu*
[2]*Omsk State Pedagogical University, Chair of Algebra, Tuhachevskogo embankment 14, Omsk-99, 644099 Russia. e-mail: zubkov@iitam.omsk.net.ru*

(Received: June 2000)

Presented by A. Verschoren

Abstract. We prove that the results of Le Bruyn and Procesi on the varieties parameterizing semisimple representations of quivers hold over an algebraically closed base field of arbitrary characteristic.

Mathematic Subject Classifications (2000): 13A50, 14L30, 16G20, 15A72.

Key words: quiver representation, étale slice, Luna stratification, polynomial invariant, module with good filtration, tilting module.

1. Introduction

Throughout this paper, the base field is assumed to be algebraically closed. The representation spaces of a quiver are affine spaces endowed with a regular action of a reductive group, such that the orbits are in bijection with the isomorphism classes of representations. Since closed orbits correspond to semisimple representations, the algebraic quotient of the representation space by the given group action is an affine variety parametrizing semisimple representations. When the base field is of characteristic zero, Le Bruyn and Procesi [LP] constructed the algebraic quotient by presenting generators of the subalgebra of polynomial invariants in the coordinate ring of the representation space. Moreover, they applied the theory of étale slices due to Luna [Lu] in order to study the quotient variety. They interpreted the Luna stratification into locally closed smooth irreducible subvarieties in terms of representation theory, and described combinatorially when a stratum lies on the closure of some other. They determined the dimension vectors of simple representations. They showed also that the local analytic structure can be described by another quiver setting, which is often simpler than the original.

Several analogous problems in the literature of representation theory are discussed with no restriction on the characteristic of the base field. Therefore it is desirable to develop the results of [LP] in a characteristic free manner, and this is the aim of the present note. The characteristic free description of the algebras of polynomial invariants of quivers is due to Donkin [D3]. Our main problem is that the existence of an étale slice at a closed orbit is not automatic in positive characteristic, and even if étale slices exist, certain properties of the resulting Luna strata are known only in characteristic zero. However, there is a criterion for the existence of an étale slice due to Bardsley and Richardson [BR]. Applying the theory of modules with good filtration we verify that this condition is fulfilled in the case of representation spaces of quivers. Thus by general arguments we obtain a stratification of the quotient variety into finitely many irreducible locally closed subvarieties belonging to conjugacy classes of stabilizers of points with closed orbit, and get also the closure relation for the strata. Studying the concrete situation and referring to the results of [D3], we are able to show the smoothness of the strata. The combinatorial description of the stratification of the representation space implies automatically that the results of [LP] on simple dimension vectors hold in any characteristic.

2. Étale Slices

We recall Luna's theory of étale slices from [Lu], for the characteristic free approach see [BR]. The book [PV] is a basic reference for the notions used below.

Throughout this section G is a (not necessarily connected) reductive affine algebraic group acting regularly on an affine algebraic variety X. (Sometimes we will restrict to the case when X is a finite dimensional vector space with a linear G-action.) The stabilizer of a point $x \in X$ is denoted by G_x, and X^G stands for the subset of G-fixpoints. We denote by X^{cl} for the subset of points $x \in X$ whose orbit Gx is closed. A point $x \in X$ is called separable if the differential of the map $\gamma\colon G \to Gx$, $g \mapsto gx$ is surjective, or equivalently, if γ induces an isomorphism $G/G_x \cong Gx$ (note that in characteristic zero any point is separable). Recall that for any $x \in X^{\mathrm{cl}}$ the stabilizer G_x is reductive (see [BR, 2.1.10. (b)]). The coordinate ring $k[X]$ is endowed with the natural linear G-action, and the subalgebra $k[X]^G$ of G-invariants is finitely generated. The injection $k[X]^G \to k[X]$ induces a surjective morphism $\pi = \pi_{X,G}\colon X \to X/_G$ of affine varieties, called the *algebraic quotient morphism*. Each fiber of π contains a unique closed orbit, and if Y is a G-stable closed subvariety of X then $\pi(Y)$ is closed in $X/_G$. By a *subvariety* of X we mean a locally closed subset with the induced structure of an algebraic variety. A *principal affine open subvariety* of X is $X_f = \{x \in X \mid f(x) \neq 0\}$, where $f \in k[X]$. We say that $Y \subseteq X$ is π*-saturated* provided $\pi^{-1}(\pi(y)) \subseteq Y$ holds for any $y \in Y$. Note that if Y is a π-saturated open subset of X, then $\pi(Y)$ is open in $X/_G$.

Fix $x \in X^{\mathrm{cl}}$, and let S be a G_x-stable affine subvariety of X with $x \in S$. Then the group $G \times G_x$ acts on $G \times S$ as

$$(g,h)\cdot(y,s) = (gyh^{-1}, hs) \quad (g, y \in G, h \in G_x, s \in S).$$

The algebraic quotient of $G \times S$ by the G_x-action is called the *homogeneous fiber space* $G \times^{G_x} S$, we denote by $[g, s]$ the element corresponding to (g, s). Recall that the algebraic quotient morphism $\pi_{G \times^{G_x} S, G}$: $G \times^{G_x} S \to (G \times^{G_x} S)/_G$ can be identified with ρ: $G \times^{G_x} S \to S/_{G_x}$, where $\rho([g, s]) = \pi_{S, G_x}(s)$. The morphism $G \times S \to X$, $(g, s) \mapsto gs$ factors through ϕ: $G \times^{G_x} S \to X$, $\phi([g, s]) = gs$. Summarizing, we have the following commutative diagram:

$$\begin{array}{ccc} G \times^{G_x} S & \xrightarrow{\phi} & X \\ \Big\downarrow{\scriptstyle \rho} & & \Big\downarrow{\scriptstyle \pi} \\ S/_{G_x} & \xrightarrow{\phi/_G} & X/_G \end{array} \tag{1}$$

DEFINITION 2.1. S is an étale slice at x, if

(i) $\phi/_G$ is an étale morphism;
(ii) the above diagram is a Cartesian square.

Note that it follows from (i) and (ii) that ϕ is also étale. (For the definition and basic properties of étale morphisms see, for example, the first chapter of [M], and for the notion of a Cartesian square see [BR, Definition 5.1.2].) We will use the following criterion for the existence of an étale slice.

PROPOSITION 2.2 ([BR, Proposition 7.4]). *Let X be a finite dimensional vector space with a linear G-action, and let $x \in X^{\mathrm{cl}}$ be a point with closed orbit. Then there is an étale slice at x if and only if x is separable and there is a G_x-stable linear subspace N of the tangent space $\mathrm{T}_x(X) = X$ such that $\mathrm{T}_x X = N \oplus \mathrm{T}_x(Gx)$. In this case for some $h \in k[N]^{G_x}$ the principal affine subvariety N_h of N is an étale slice at x.*

The following statement is a refinement of [BR, Proposition 8.6].

PROPOSITION 2.3. *Let X be an affine G-variety, and $x \in X^{\mathrm{cl}}$. Assume that there is an étale slice S at x, so we have the diagram* (1). *Then GS is an open π-saturated subset of X, which we will denote by U_x. The stabilizer of any point $gs \in U_x$ ($g \in G$, $s \in S$) is equal to $g(G_s \cap G_x)g^{-1}$. In particular, $G_s \leqslant G_x$ for any $s \in S$.*

Proof. $U_x = \mathrm{Im}(\phi)$ is open, because ϕ is an étale, hence open morphism. Take a point $\phi([g, s]) = gs \in U_x$. Since (1) is a Cartesian square, ϕ: $\rho^{-1}(\rho([g, s])) \to \pi^{-1}(\pi(gs))$ is an isomorphism of G-varieties (see [BR, Lemma 7.2]). It follows that U_x is π-saturated, and $G_{[g,s]} = G_{gs}$. On the other hand, for some $b \in G$ we have $b[g, s] = [g, s]$ if and only if there exists an $h \in G_x$ with $bg = gh^{-1}$, $s = hs$. This means that $G_{[g,s]} = g(G_s \cap G_x)g^{-1}$. $\square$

Denote by $\mathcal{T}$ the set of conjugacy classes of stabilizer subgroups G_x in G, as x ranges over X^{cl}. For $\xi \in X/_G$ denote by $T(\xi) \in \mathcal{T}$ the conjugacy class of stabilizers belonging to the unique closed orbit in $\pi^{-1}(\xi)$. $T(\xi)$ is called the *type* of ξ. We denote by the same symbol T the function $X \to \mathcal{T}$ defined by $T(x) = T(\pi(x))$, and call $T(x)$ the *type* of $x \in X$. For a fixed $\tau \in \mathcal{T}$ we put

$$X_\tau = \{x \in X \mid T(x) = \tau\} \quad \text{and} \quad (X/_G)_\tau = \{\xi \in X/_G \mid T(\xi) = \tau\}.$$

Clearly, $X_\tau = \pi^{-1}((X/_G)_\tau)$ by definition. For each $\tau \in \mathcal{T}$ we choose a stabilizer G_τ representing the conjugacy class τ, and we write $G_{\tau_1} \leqslant_{\mathrm{c}} G_{\tau_2}$ if G_{τ_1} is conjugate to a subgroup of G_{τ_2}. We define a partial order on $\mathcal{T}$ by the rule $\tau_1 \geqslant \tau_2$ if $G_{\tau_1} \leqslant_{\mathrm{c}} G_{\tau_2}$.

Next we collect some facts on the sets $(X/_G)_\tau$, which follow from the existence of étale slices at closed orbits. These facts are well known in characteristic zero, and they extend to arbitrary characteristic without any difficulty. For the sake of completeness we give their detailed proofs.

PROPOSITION 2.4. *Let X be an affine G-variety. Suppose that there is an étale slice S at any point $x \in X^{\mathrm{cl}}$.*

(i) *Then the Zariski closure of $(X/_G)_\tau$ is contained in $\bigcup_{\nu \leqslant \tau} (X/_G)_\nu$.*
(ii) *$X/_G = \bigcup_{\tau \in \mathcal{T}} (X/_G)_\tau$ is a finite stratification of $X/_G$ into locally closed subvarieties.*

Proof. For each $x \in X^{\mathrm{cl}}$ we put $U_x = GS$, where S is an étale slice at x.

(i) Take $\zeta \in (X/_G)_\tau$ and $\eta \in \overline{(X/_G)_\tau}$. Then $\eta = \pi(y)$ for some $y \in X^{\mathrm{cl}}$, and since U_y is a π-saturated open subset, its image $\pi(U_y)$ is also open. Hence, $\pi(U_y) \cap (X/_G)_\tau$ is nonempty, say it contains $\pi(z)$ for some $z \in X^{\mathrm{cl}}$. Then G_τ is conjugate to G_z, and $G_z \leqslant_{\mathrm{c}} G_y$ by Proposition 2.3, implying $\tau = T(z) \geqslant T(y) = T(\eta)$. This shows (i).

(ii) Since X is compact in the Zariski topology, it is possible to choose a finite subcovering from its open covering $\bigcup_{x \in X^{\mathrm{cl}}} U_x$, say $X = \bigcup_{i=1}^m U_{x_i}$. Therefore, by Proposition 2.3, for any $x \in X$ there exists an $i \in \{1, \ldots, m\}$ with $G_x \leqslant_{\mathrm{c}} G_{x_i}$. Let $G_1, \ldots, G_q$ be the elements of maximal dimension in $\{G_{x_i} \mid i = 1, \ldots, m\}$. Consider

$$\mathcal{S} = \{\tau \in \mathcal{T} \mid \exists i \in \{1, \ldots, q\} : \; G_i^0 \leqslant_{\mathrm{c}} G_\tau \leqslant_{\mathrm{c}} G_i\}$$

(here G_i^0 denotes the connected component of the identity in G_i). Clearly, $\mathcal{S}$ is a finite set.

We claim that if $\nu \in \mathcal{T}$ and $\nu \leqslant \tau$ for some $\tau \in \mathcal{S}$, then $\nu \in \mathcal{S}$. Indeed, there are elements $i \in \{1, \ldots, q\}$ and $j \in \{1, \ldots, m\}$ such that $G_i^0 \leqslant_{\mathrm{c}} G_\tau \leqslant_{\mathrm{c}} G_\nu \leqslant_{\mathrm{c}} G_{x_j}$. Therefore G_i^0 and G_{x_j} have the same dimension, implying that $G_{x_j} = G_l$, where $l \in \{1, \ldots, q\}$. Moreover, $G_i^0 \leqslant_{\mathrm{c}} G_l^0$ are connected algebraic groups of the same dimension, hence G_i^0 and G_l^0 are conjugate. So $G_l^0 \leqslant_{\mathrm{c}} G_\nu \leqslant_{\mathrm{c}} G_l$, and ν is contained in $\mathcal{S}$ by its definition.

The above property of $\mathcal{S}$ together with (i) implies that $\bigcup_{\tau \in \mathcal{S}} X_\tau$ is closed. Now take $X_1 = X \setminus \bigcup_{\tau \in \mathcal{S}} X_\tau$, $\mathcal{T}_1 = \mathcal{T} \setminus \mathcal{S}$, and repeat the same argument with X_1 and

$\mathcal{T}_1$ instead of X and $\mathcal{T}$ (note that X_1 is a G-stable π-saturated subvariety of X, and it has the open covering $\bigcup_{x \in X^{\mathrm{cl}} \cap X_1}(U_x \cap X_1)$). We obtain $X_2 \subseteq X_1$, $\mathcal{T}_2 \subseteq \mathcal{T}_1$, where X_2 is open and $\mathcal{T}_1 \setminus \mathcal{T}_2$ is finite. Continuing this process we obtain a chain $X \supseteq X_1 \supseteq X_2 \supseteq \cdots$ of open subvarieties, and a chain $\mathcal{T} = \mathcal{T}_0 \supseteq \mathcal{T}_1 \supseteq \mathcal{T}_2 \supseteq \cdots$, where $\mathcal{T}_j \setminus \mathcal{T}_{j+1}$ is finite for any j. Since in each step the maximal stabilizer dimension drops, the above process finishes in finitely many steps, i.e. for some r we get $X_{r+1} = \emptyset = \mathcal{T}_{r+1}$. This shows that $\mathcal{T}$ is finite.

Therefore (i) implies that both $\bigcup_{\nu \leqslant \tau}(X/_G)_\nu$ and $\bigcup_{\nu < \tau}(X/_G)_\nu$ are closed for any $\tau \in \mathcal{T}$, thus $(X/_G)_\tau$ is locally closed. □

The following statement is due to Schwarz [S, Lemma 5.5] in characteristic zero, and his arguments work in any characteristic.

PROPOSITION 2.5. *Let X be a finite-dimensional vector space with a linear G-action. Assume that there exists an étale slice at any point $x \in X^{\mathrm{cl}}$. Then for any $\tau \in \mathcal{T}$ the stratum $(X/_G)_\tau$ is irreducible, and its Zariski closure is $\bigcup_{\nu \leqslant \tau}(X/_G)_\nu$.*

Proof. For a stabilizer subgroup $H \in \tau$ define the subvariety $X(H) = X^H \cap X_\tau$.

We claim that $X(H)$ is open in X^H (implying in particular that $X(H)$ is irreducible). By definition of the stratum X_τ for any $x \in X(H)$ there is a $y \in X^{\mathrm{cl}} \cap \overline{Gx}$ such that G_y is conjugate to H. We may choose y with $G_y = H$. Consider the open neighbourhood U_y of y defined in Proposition 2.3. It is π-saturated, so contains x. Thus $x \in U_y \cap X^H$. Now take an arbitrary point $z \in U_y \cap X^H$. Then $G_z \geqslant H$, and by Proposition 2.3 $G_z \leqslant_{\mathrm{c}} G_y = H$, implying $G_z = H$. Since U_y is π-saturated, $\overline{Gz} \subseteq U_y$, hence $\dim G_w \leqslant \dim G_y = \dim H = \dim G_z$ for any $w \in \overline{Gz}$. On the other hand, it follows from the formula expressing the dimension of an orbit that for $w \in \overline{Gz}$ we have $\dim G_w \geqslant \dim G_z$, and equality holds if and only if $Gw = Gz$. So $Gw = Gz$ for any $w \in \overline{Gz}$. In other words, $U_y \cap X^H \subseteq X^{\mathrm{cl}} \cap X_\tau$. Summarizing, we have $X(H) \subseteq X^{\mathrm{cl}}$, and $X(H) = \bigcup_{x \in X(H)} U_x \cap X^H$ is a covering of $X(H)$ by open subvarieties of X^H. Therefore $X(H)$ is open in X^H.

Obviously, $\pi(X(H)) = (X/_G)_\tau$, so $(X/_G)_\tau$ is irreducible. Similarly, we have $\pi(X^H) \supseteq \bigcup_{\nu \leqslant \tau}(X/_G)_\nu$. Therefore

$$\overline{(X/_G)_\tau} = \overline{\pi(X(H))} \supseteq \pi(\overline{X(H)}) = \pi(X^H) \supseteq \bigcup_{\nu \leqslant \tau}(X/_G)_\nu.$$

The reverse inclusion was shown already in Proposition 2.4(i). □

PROPOSITION 2.6. *Let S be an étale slice at $x \in X^{\mathrm{cl}}$, so we have the diagram* (1). *Denote by τ the type containing G_x, i.e. $\tau = T(x) = T([1_G, x]) = T(\rho([1_G, x])) = T(\xi)$, where $\xi = \pi(x)$.*

(i) *Restricting $\phi/_G$ to $(S/_{G_x})_\tau$ we get an étale morphism*

$$\phi/_G|_{(S/_{G_x})_\tau} \colon (S/_{G_x})_\tau \to (X/_G)_\tau.$$

(ii) *The morphism $\bar{\rho}$: $s \mapsto \rho([1_G, s])$ maps S^{G_x} bijectively onto $(S/_{G_x})_\tau$.*

Proof. (i)

$$\phi/_G(S/_{G_x}) = \pi \circ \phi(G \times^{G_x} S) = \pi(GS) = U_\xi$$

is an open neighbourhood of ξ in $X/_G$. It follows from Propositions 2.3 and 2.4 that $\overline{(X/_G)_\tau} \cap U_\xi \subseteq (X/_G)_\tau$. Since diagram (1) is a Cartesian square, for any $s \in S/_{G_x}$ the G-varieties $\rho^{-1}(s)$ and $\pi^{-1}(\phi/_G(s))$ are isomorphic, hence $s \in (S/_{G_x})_\tau$ if and only if $\phi/_G(s) \in (X/_G)_\tau$. In particular, $\phi/_G^{-1}(\overline{(X/_G)_\tau}) = (S/_{G_x})_\tau$. Thus

$$\begin{array}{ccc} (S/_{G_x})_\tau & \longrightarrow & \overline{(X/_G)_\tau} \\ \downarrow & & \downarrow \\ S/_{G_x} & \xrightarrow{\phi/_G} & X/_G \end{array}$$

is a Cartesian square of affine varieties (where the vertical arrows are embeddings), and the assumption that $\phi/_G$ is étale implies that the upper horizontal morphism is étale (see [BR, 5.1.1]). We have seen already that $\phi/_G((S/_{G_x})_\tau) \subseteq (X/_G)_\tau$.

(ii) For any $[g, s] \in G \times^{G_x} S$ we have $G_{[g,s]} = g(G_s \cap G_x)g^{-1}$. Clearly, this group is conjugate to G_x if and only if $G_s \supseteq G_x$, i.e. if $s \in S^{G_x}$. This means that $\bar{\rho}(S^{G_x}) = (S/_{G_x})_\tau$. On the other hand, for any $s_1 \neq s_2 \in S^{G_x}$ the sets $\{[g, s_i] \mid g \in G\}$ $(i = 1, 2)$ are disjoint closed G-invariant subvarieties of $G \times^{G_x} S$, hence $\bar{\rho}$: $S^{G_x} \to (S/_{G_x})_\tau$ is injective. □

Remark. In characteristic zero the restriction of the algebraic quotient morphism $S \to S/_{G_x}$ to any G_x-stable closed subvariety Z of S gives the corresponding algebraic quotient morphism $Z \to Z/_{G_x}$. Applying this to the subvariety S^{G_x} we get that $\bar{\rho}$ is an isomorphism. This immediately implies that the strata $(X/G)_\tau$ in Proposition 2.5 are smooth. On the contrary, we will see in the proof of Theorem 5.1 that $\bar{\rho}$ is not always an isomorphism in positive characteristic. For this reason the general argument for the smoothness of the strata does not work in positive characteristic.

3. Preliminaries on Quivers and Tilting Modules

By a *quiver* we mean a finite oriented graph Q consisting of Q_0, the set of vertices, Q_1, the set of arrows, and the functions assigned to an arrow a its starting vertex a' and terminating vertex a''. A *representation* V of Q is a collection of vector spaces $V(i)$, $i \in Q_0$, and k-linear maps $V(a)$: $V(a') \to V(a'')$, $a \in Q_1$. We call $\alpha = (\alpha(i) \mid i \in Q_0)$ the *dimension vector* of V, where $\alpha(i) = \dim_k(V(i))$. A morphism ψ between two representations V and W is a collection of linear maps $\psi(i)$: $V(i) \to W(i)$, $i \in Q_0$, satisfying $W(a) \circ \psi(a') = \psi(a'') \circ V(a)$ for all $a \in Q_1$. The representations of Q form an Abelian category. For a fixed dimension vector α consider the affine space $R(Q, \alpha) = \bigoplus_{a\in Q_1} k^{\alpha(a'')\times\alpha(a')}$. This is called the *space of representations* of dimension vector α, since with each $x = (x(a) \mid a \in Q_1) \in R(Q, \alpha)$ we associate the representation V_x, where $V_x(i) = k^{\alpha(i)}$ is the space of column vectors ($i \in Q_0$), and $V_x(a)$ is the linear map operating by left multiplication by the matrix $x(a)$ ($a \in Q_1$). Clearly, any representation of dimension vector α is isomorphic to V_x with an appropriate $x \in R(Q, \alpha)$, and $V_x \cong V_y$ are isomorphic representations if and only if x and y lie in the same orbit of $GL(\alpha) = \times_{i \in Q_0} GL_{\alpha(i)}$, acting on $R(Q, \alpha)$ by the rule

$$(g \cdot x)(a) = g(a'')x(a)g(a')^{-1}, \quad a \in Q_1 \ (g \in GL(\alpha), x \in R(Q, \alpha)).$$

Next we recall the notion of modules with good filtration. Let G be a reductive group with a fixed Borel subgroup B. By a *good filtration* of the rational G-module M we mean an ascending chain $0 = M_0 \leqslant M_1 \leqslant M_2 \leqslant \cdots$ of submodules such that $M = \bigcup_i M_i$, and for any $i \in \mathbb{N}$ either M_i/M_{i-1} is a module induced from a one-dimensional B-module, or it is zero. (In the sequel we deal with finite-dimensional modules, hence finite filtrations only.) We say that M is a *tilting module*, if both M and its dual M^* have good filtration. We refer to [D4] for a survey on this concept and its role in the invariant theory of quivers.

We shall use the following known facts on modules with good filtration:

(i) The tensor product of modules with good filtration is also a module with good filtration with respect to the diagonal action (see [D1],[Ma]).
(ii) If $M \leqslant N$ are modules with good filtration, then N/M has also good filtration (cf. [D1, Proposition 3.2.4]).
(iii) Any direct summand of a module with good filtration has also good filtration (cf. [D1, Corollary 3.2.5]).

Let us introduce the following ad hoc definition: a subgroup G of the general linear group GL_n is called *standard*, if up to conjugation it has the form

$$G = \left\{ \operatorname{diag}(\underbrace{g_1, \ldots, g_1}_{b_1}, \ldots, \underbrace{g_r, \ldots, g_r}_{b_r}) \,\middle|\, g_i \in GL_{e_i} \right\}$$

for some positive integers $b_1, \ldots, b_r, e_1, \ldots, e_r$ with $\sum_{i=1}^{r} e_i b_i = n$.

LEMMA 3.1. *Let G be a standard subgroup of GL_n, and M a GL_n-module with good filtration. Then M has good filtration also as a G-module.*

Proof. Without loss of generality we may assume that M is a module induced from a one-dimensional module over some Borel subgroup of GL_n. Any such module is isomorphic to $L(\lambda) \otimes \det^{-m}$, where λ is a partition, $L(\lambda)$ is the corresponding Schur module, det denotes the one-dimensional representation given by the determinant, and m is a nonnegative integer (see [ABW]). The G-module $\det^{-m}$ clearly has good filtration, therefore we may assume that $M = L(\lambda)$. By [AB] it has a finite resolvent $\cdots \to \bigwedge^{\lambda}(k^n) \to L(\lambda) \to 0$, such that all members of this exact sequence are direct sums of tensor products of exterior powers of the standard GL_n-module k^n. It remains to prove our statement for $\bigwedge^t(k^n)$, $t = 1, \ldots, n$. Write

k^n as a direct sum $\bigoplus_{i=1}^r V_i^{\oplus b_i}$ (where $\dim_k(V_i) = e_i$), corresponding to the block diagonal structure of G. Then we have

$$\bigwedge{}^t(k^n) \cong \bigoplus_{t_1+\cdots+t_r=t} \bigotimes_{i=1}^r \bigwedge{}^{t_i}(V_i^{\oplus b_i}) \cong \bigoplus_{|\lambda_1|+\cdots+|\lambda_r|=t} \bigotimes_{i=1}^r \bigotimes_{j=1}^{b_i} \bigwedge{}^{\lambda_{ij}} V_i$$

(here λ_{ij} are nonnegative integers, $\lambda_i = (\lambda_{i1}, \ldots, \lambda_{ib_i})$, and $|\lambda_i| = \lambda_{i1} + \cdots + \lambda_{ib_i}$). The G-module V_i is obtained from the standard GL_{e_i}-module by extending the action trivially to the components GL_{e_j} $(j \neq i)$ of $G \cong \times_{m=1}^r GL_{e_m}$. Therefore any exterior power of V_i is an induced module, showing that $\bigwedge^t(k^n)$ is a G-module with good filtration. □

Remark. The above lemma will be used repeatedly in the proof of Proposition 4.2. We note that instead of referring to this lemma we could point out there that the modules under consideration are direct sums of G-modules of the form $V_i^* \otimes V_j$ (with the notation of the above proof), hence they are clearly tilting modules.

LEMMA 3.2. *Let G be a reductive group and* $0 \to U \to V \to W \to 0$ *an exact sequence of rational G-modules. If U and W^* have good filtration then this exact sequence splits.*

Proof. We have the isomorphisms

$$\mathrm{Ext}_G^1(W, U) \cong \mathrm{H}^1(G, \mathrm{Hom}(W, U)) \cong \mathrm{H}^1(G, W^* \otimes U).$$

By assumption, W^* and U are modules with good filtration, hence so is $W^* \otimes U$, implying that the Hochschild cohomology group $\mathrm{H}^1(G, W^* \otimes U)$ vanishes (see, for example, [D4] or [D1, 2.1.4]). □

4. Étale Slices in Representation Spaces

For the rest of the paper we fix the notation $X = R(Q, \alpha)$ and $G = GL(\alpha)$. Then X is endowed with the action of G introduced in Section 3, and we apply the results collected in Section 2 for the G-variety X.

In this situation an orbit Gx is closed if and only if V_x is a semisimple representation, i.e. $V_x \cong S_1^{\oplus e_1} \oplus \cdots \oplus S_r^{\oplus e_r}$, where S_i is simple of dimension vector β_i, and $S_i^{\oplus e_i}$ denotes the direct sum of e_i copies of S_i (we assume that $S_i \cong S_j$ implies $i = j$). Thus the variety $X/_G$ parametrizes the semisimple representations. Moreover, $x, y \in X$ belong to the same fiber of π if and only if V_x and V_y have the same composition factors (cf. [A]). By Schur's lemma in the category of representations of Q we have $\mathrm{End}_Q(S_i) = k$ and $\mathrm{Hom}_Q(S_i, S_j) = 0$ for $i \neq j$. Therefore $G_x \cong GL(\underline{e}) = \times_{i=1}^r GL_{e_i}$. Moreover, as is pointed out in [LP], the $GL(\alpha)$-conjugacy class of G_x is determined by the unordered r-tuple $(e_1, \beta_1; \ldots; e_r, \beta_r)$, called the *representation type* of V_x. Thus we may identify the type $\tau = T(x)$ of x (see Section 2) with the representation type of V_x.

Following [LP] we associate a new quiver Q_τ to the representation type τ. Recall that the *Euler bilinear form* on $\mathbb{Z}^{Q_0}$ is defined by

$$\langle \beta, \gamma\rangle_Q = \sum_{i\in Q_0} \beta(i)\gamma(i) - \sum_{a\in Q_1} \beta(a')\gamma(a'').$$

The vertex set of Q_τ is $\{1, \dots, r\}$, and for any pair (i, j) of vertices Q_τ has $\delta^i_j - \langle \beta_i, \beta_j\rangle_Q$ arrows from i to j (here δ^i_j is 1, if $i = j$, and 0 otherwise). Note that by standard properties of the Euler form the number of arrows from i to j in Q_τ is precisely the dimension of the space $\mathrm{Ext}^1_Q(S_i, S_j)$.

Given a representation $V = V_x$ the group G_x is naturally identified with the group of units in $\mathrm{End}_Q(V)$. Recall that $\mathrm{Ext}^1_Q(V, V)$ has a natural $\mathrm{End}_Q(V) \otimes \mathrm{End}_Q(V)^{\mathrm{op}}$-module structure (see, for example, [ARS, Chapter I.5]), and the embedding $G_x \to G_x \times G_x^{\mathrm{op}}$, $g \mapsto (g, g^{-1})$ makes $\mathrm{Ext}^1_Q(V, V)$ a G_x-module. The following quiver-version of Voigt's lemma is due to [LP], [L]. (Though it is stated there only in characteristic zero, the result is obviously independent of the characteristic.)

LEMMA 4.1. *Let $V = V_x$ be a semisimple representation of type $\tau = (e_1, \beta_1; \dots; e_r, \beta_r)$. Then we have the isomorphisms*

$$\mathrm{T}_x X/\mathrm{T}_x(Gx) \cong \mathrm{Ext}^1_Q(V, V) \cong R(Q_\tau, \underline{e})$$

of $G_x \cong GL(\underline{e})$-modules.

PROPOSITION 4.2. *Let $V = V_x$ be a semisimple representation. Then $x \in X$ is separable, and $\mathrm{T}_x(Gx)$ has a G_x-stable direct complement N_x in $\mathrm{T}_x X$. Moreover, the G_x-variety N_x is isomorphic to the $GL(\underline{e})$-variety $R(Q_\tau, \underline{e})$, where $\tau = (e_1, \beta_1; \dots; e_r, \beta_r)$ is the representation type of V.*

Proof. Denote by M_d the space of $d\times d$ matrices over k, where $d = \sum_{i\in Q_0} \alpha(i)$. Embed X into $\mathrm{M}_d^{Q_1}$ by identifying the matrix $x(a)$ ($x \in X$, $a \in Q_1$) with a $d \times d$ matrix partitioned into $|Q_0| \times |Q_0|$ number of blocks, containing the block $x(a)$ in the (a'', a') position, and zero blocks everywhere else. Accordingly, G is identified with the subgroup of block diagonal matrices of GL_d, and the Lie algebra $\mathrm{M}(\alpha)$ of G with the subspace of block diagonal matrices in M_d. The action of G on X ($\mathrm{M}(\alpha)$, respectively) is induced by the conjugation action of GL_d on $\mathrm{M}_d^{Q_0}$ (M_d, respectively). Since M_d is the tensor product of the standard GL_d-module and its dual, M_d and $\mathrm{M}_d^{Q_0}$ are tilting GL_d-modules, hence they are tilting modules over the standard subgroup G of GL_d by Lemma 3.1, and their direct summands $\mathrm{M}(\alpha)$, X are also tilting G-modules. Thus we have shown that the representation space of a quiver is a tilting module over the corresponding product of general linear groups; in particular, the Lie algebra of a product of general linear groups is a tilting module. This applies for the Lie subalgebra $\mathrm{Lie}(G_x) \leqslant \mathrm{M}(\alpha)$ of G_x. Moreover, since G_x is also a standard subgroup of GL_d, Lemma 3.1 implies that M_d and hence $\mathrm{M}(\alpha)$ are tilting G_x-modules as well.

Recall that any $y \in \mathrm{M}(\alpha)$ (resp. x) is identified with a $d \times d$ matrix (resp. $|Q_1|$-tuple of $d \times d$ matrices). Consider the map $\mathrm{M}(\alpha) \to \mathrm{T}_x(Gx)$ given by $y \mapsto yx - xy$. Its kernel obviously coincides with $\mathrm{Lie}(G_x)$. Hence we have the exact sequence $0 \to \mathrm{Lie}(G_x) \to \mathrm{M}(\alpha) \to \mathrm{T}_x(Gx) \to 0$ of G_x-modules. In particular, x is separable. Moreover, since the first two members of this exact sequence are G_x-modules with good filtration, $\mathrm{T}_x(Gx)$ is also a G_x-module with good filtration. By Lemma 4.1 we have the isomorphism $\mathrm{T}_x X/\mathrm{T}_x(Gx) \cong R(Q_\tau, \underline{e})$ of $G_x \cong GL(\underline{e})$-modules. Applying Lemma 3.2 to the short exact sequence E: $0 \to \mathrm{T}_x(Gx) \to \mathrm{T}_x X \to R(Q_\tau, \underline{e}) \to 0$ we get that $\mathrm{T}_x X = \mathrm{T}_x(Gx) \oplus N_x$ for a G_x-submodule N_x of $\mathrm{T}_x X$ with $N_x \cong R(Q_\tau, \underline{e})$. (Alternatively, dualizing E and using that $\mathrm{T}_x^* X$ and $R(Q_\tau, \underline{e})^*$ are modules with good filtration we get that $\mathrm{T}_x^*(Gx)$ is a module with good filtration. Thus E is a short exact sequence of tilting modules and, hence, splits by [D2, p. 46].) $\square$

By Proposition 2.2 we have the following corollary.

COROLLARY 4.3. *Let $x \in X$ be a point with V_x semisimple. Let $N = N_x$ denote a subspace of $X = \mathrm{T}_x X$ whose existence is guaranteed by the previous proposition. Then there exists an $h \in k[N]^{G_x}$ such that the principal affine open subvariety N_h of N is an étale slice in X at x. In particular, there is an étale morphism from a neighbourhood of the origin in the quotient variety $R(Q_\tau, \underline{e})/_{GL(\underline{e})}$ onto a neighbourhood of $\xi = \pi(x)$ in $X/_G$.*

5. Stratification of the Variety of Semisimple Representations

We keep the notation of Section 4. Denote by $\mathcal{T}$ the finite set of all representation types of representations of dimension vector α. Consider the partial order on $\mathcal{T}$ introduced in Section 2 in general. In the case of representation spaces it is described combinatorially in [LP]. Namely, for $\tau = (e_1, \beta_1; \ldots; e_r, \beta_r)$ we have $\tau' < \tau$ and τ' is a direct predecessor of τ if and only if one of the following alternatives holds:

(i) $\tau' = (e_1, \gamma; e_1, \delta; e_2, \beta_2; \ldots; e_r, \beta_r)$ with $\beta_1 = \gamma + \delta$ (for an appropriate indexing of the unordered r-tuple τ);

(ii) $\beta_1 = \beta_2$ and $\tau' = (e_1 + e_2, \beta_1; e_3, \beta_3; \ldots; e_r, \beta_r)$ (for an appropriate indexing of the unordered r-tuple τ).

We defined the subsets $(X/_G)_\tau = \{\xi \in X/_G \mid T(\xi) = \tau\}$ in Section 2.

THEOREM 5.1. *$\bigcup_{\tau \in \mathcal{T}} (X/_G)_\tau$ is a finite stratification of $X/_G$ into smooth, irreducible, locally closed subvarieties. The dimension of the stratum $(X/_G)_\tau$ equals the number of loops in the quiver Q_τ. Furthermore, $(X/_G)_\nu$ lies in the closure of $(X/_G)_\tau$ if and only if $\nu \leqslant \tau$.*

Proof. By Corollary 4.3 we can apply Propositions 2.4 and 2.5, and get all statements except the smoothness of the strata in positive characteristic. Assume that k is of characteristic $p > 0$. To show that $(X/_G)_\tau$ is smooth we use the results of [D3].

Take $x \in X_\tau$ with Gx closed, and the étale slice N_h in X at x (see Corollary 4.3). We have the bijective morphism $\bar{\rho}$: $N_h^{G_x} \to (N_h/_{G_x})_\tau$ defined in Proposition 2.6(ii). Denote by $\mathcal{L}$ the set of loops in the quiver Q_τ. We make the identifications $N = R(Q_\tau, \underline{e})$ and $G_x = GL(\underline{e})$ (see Proposition 4.2). For $a \in \mathcal{L}$ denote by $I_a \in N$ the representation of Q_τ assigning the identity matrix with the loop a, and the zero matrix with any other arrow. Clearly, $\{I_a \mid a \in \mathcal{L}\}$ is a k-basis of N^{G_x}. Thus $N_h^{G_x}$ consists of points $y = \sum_{a\in\mathcal{L}} y_a I_a$, where $(y_a \mid a \in \mathcal{L})$ ranges over a principal affine open neighbourhood of the origin in the affine space $\mathbb{A}^{|\mathcal{L}|}$. Recall that $\bar{\rho}$ is the restriction to $N_h^{G_x}$ of the quotient morphism $N_h \to N_h/_{G_x}$, and since h is contained in $k[N]^{G_x}$, by standard properties of affine quotients (see, for example, [BR, 2.1.8.]) $\bar{\rho} = \mu\big|_{N_h^{G_x}}$, where $\mu = \pi_{N,G_x}$: $N \to N/_{G_x}$ is the quotient morphism. Since N is the representation space of a quiver, we can refer to [D3] for an explicit description of μ. Namely, $k[N]^{G_x}$ is generated by finitely many elements of the form $\sigma_i(a_m \cdots a_1)$, where $a_1 \cdots a_m$ is an oriented cycle in the quiver Q_τ starting and terminating at some vertex v, $i = 1, \dots, e_v$, and for $z \in N$ $\sigma_i(a_m \cdots a_1)(z)$ is the ith coefficient of the characteristic polynomial of the $e_v \times e_v$ matrix $z(a_m) \cdots z(a_1)$. Obviously, if $\sigma_i(a_m \cdots a_1)(y)$ is nonzero for some $y = \sum_{a\in\mathcal{L}} y_a I_a \in N_h^{G_x}$, then $a_1, \dots, a_m$ are all loops at the same vertex v, and $i = p^{f_v} j$, $j \in \{1, \dots, q_v\}$, where $e_v = p^{f_v} q_v$ such that p is coprime to q_v. Moreover, in this case $\sigma_i(a_m \cdots a_1)(y) = \binom{q_v}{j}(y_{a_m} \cdots y_{a_1})^i$, where $\binom{q_v}{j}$ is a binomial coefficient. It follows that the restriction of $\sigma_i(a_m \cdots a_1)$ to $N_h^{G_x}$ is contained in the polynomial algebra

$$k[\sigma_{p^{f_v}}(a)|_{N_h^{G_x}} \mid a \in \mathcal{L},\ v = a' = a''].$$

(Note that for $a \in \mathcal{L}$, $v = a' = a''$ we have $\sigma_{p^{f_v}}(a)(y) = q_v y_a^{p^{f_v}}$.) This shows explicitly that $\bar{\rho}(N_h^{G_x})$ is a principal affine open subvariety of the affine space $\mathbb{A}^{|\mathcal{L}|}$ (though $\bar{\rho}$ is not an isomorphism, if there is a loop in Q_τ at some vertex v such that e_v is divisible by p), hence $(N_h/_{G_x})_\tau = \bar{\rho}(N_h^{G_x})$ is smooth of dimension $|\mathcal{L}|$. By Proposition 2.6(i) $\phi/_G$: $(N_h/_{G_x})_\tau \to (X/_G)_\tau \cap U_\xi$ is étale and surjective, implying that $(X/_G)_\tau$ is also smooth of dimension $|\mathcal{L}|$. $\square$

6. Simple Dimension Vectors

We would like to point out that it follows from Theorem 5.1 that the description in [LP] of the set of dimension vectors of simple representations is valid in any characteristic.

Let us recall the result. Let Q be a quiver, α a dimension vector, and denote by $\text{supp}(\alpha)$ the full subquiver spanned by the vertices $\{i \in Q_0 \mid \alpha(i) > 0\}$. With each vertex $i \in Q_0$ we assign the dimension vector ε_i defined by $\varepsilon_i(i) = 1$, and $\varepsilon_i(j) = 0$ for $j \neq i$. A quiver is said to be *strongly connected* if any pair of its vertices belongs to an oriented cycle. Denote by $\tilde{A}_n$ the extended Dynkin quiver with cyclic orientation whose underlying graph is a circle with $n+1$ vertices.

THEOREM 6.1. *There is a simple representation of Q of dimension vector α if and only if either* $\text{supp}(\alpha)$ *is $\tilde{A}_n$ for some n and $\alpha|_{\text{supp}(\alpha)} = (1, \ldots, 1)$, or* $\text{supp}(\alpha)$ *is strongly connected, different from $\tilde{A}_n$ for all n, and $\langle \alpha, \varepsilon_i \rangle_Q \leqslant 0$, $\langle \varepsilon_i, \alpha \rangle_Q \leqslant 0$ for all $i \in Q_0$.*

Proof. We can proceed in the same way as [LP] until we arrive at the case when $\alpha = \beta + \varepsilon_i$, where β is the dimension vector of a simple representation, $\text{supp}(\alpha)$ is different from $\tilde{A}_n$ for all n, and we have to show that α is a simple dimension vector. Of course, there exists a simple representation of dimension vector ε_i. Consider the representation type $\tau = (1, \beta; 1, \varepsilon_i)$. The only possible representation type ν with $\nu > \tau$ is $(1, \alpha)$, when α is the dimension vector of a simple representation. Otherwise it follows from Theorem 5.1 that $R(Q, \alpha) = \overline{R(Q, \alpha)_\tau}$. Since $T(x) = \tau$ means that V_x has a composition series whose factors have dimension vector (β, ε_i), it follows that V_x has either a subrepresentation or a factor representation of dimension vector ε_i for a general x, i.e. for all x from an open subset of $R(Q, \alpha)$. It is easy to see by elementary linear algebra arguments that a general representation of dimension vector α has a subrepresentation of dimension vector ε_i if and only if either $\text{supp}(\alpha)$ consists of a single loop at i, or there is no loop at i in Q, and $\alpha(i) > \sum_{a \in Q_1:\, a'=i} \alpha(a'')$ (i.e. $\langle \varepsilon_i, \alpha \rangle_Q > 0$). Dually, if a general representation of dimension vector α has a factor representation of dimension vector ε_i, and $\text{supp}(\alpha)$ is not the one-loop quiver, then $\langle \alpha, \varepsilon_i \rangle_Q > 0$. Thus the assumption $R(Q, \alpha) = \overline{R(Q, \alpha)_\tau}$ leads to a contradiction. This shows that α is a simple dimension vector. $\square$

We note finally that following the algorithm given in [L] one can determine the generic representation type in $R(Q, \alpha)$ for given α, and by Theorem 5.1 obtain the dimension of $R(Q, \alpha)/_{GL(\alpha)}$.

Acknowledgement

Part of this research was done during the second author's visit to Budapest within the framework of the exchange program between the Hungarian and Russian Academies of Sciences. We are thankful for this support. M.D. was supported by OTKA No. F032325 and a Bolyai Research Fellowship, and A.N.Z. was supported by Russian Fundamental Research Fund No. 980100932.

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